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TWO INFINITE FAMILIES OF CONGRUENCES MODULO 9 FOR OVERCUBIC PARTITION PAIRS

BY

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Abstract. Let $\overline{b}(n)$ denote the number of overcubic partition pairs of n. Applying the theory of modular forms, Kim obtained two congruences for $\overline{b}(n)$ modulo 3 and 64. More congruences modulo 3 and 5 have been found by the first author of the present paper. In this paper, we proceed with the study of the congruence properties of $\overline{b}(n)$ and establish two infinite families of congruences modulo 9.

1. Introduction. In a series of papers [3, 4, 5], Chan studied congruence properties of the cubic partition function $a(n)$ whose generating function satisfies

$$\sum_{n=0}^{\infty} a(n)q^n = \frac{1}{(q;q)_\infty (q^2;q^2)_\infty}, \tag{1.1}$$

where $(a;q)_\infty = (1-a)(1-aq)(1-aq^2)\cdots$.

Inspired by Chan's work, many researchers engaged in investigating analogous partition functions. First, Kim [8] considered its overpartiton analog and introduced the overcubic partition function $\overline{a}(n)$ by

$$\sum_{n=0}^{\infty} \overline{a}(n)q^n = \frac{(-q;q)_\infty (-q^2;q^2)_\infty}{(q;q)_\infty (q^2;q^2)_\infty}. \tag{1.2}$$

Using the theory of modular functions, he showed that

$$\sum_{n=0}^{\infty} \overline{a}(3n+2)q^n = 6\frac{(q^3;q^3)_\infty^6 (q^4;q^4)_\infty^3}{(q;q)_\infty^8 (q^2;q^2)_\infty^3}, \tag{1.3}$$

from which $\overline{a}(3n+2) \equiv 0 \pmod 6$ follows immediately.

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Later, Zhao and Zhong [14] studied the partition function $b(n)$ defined by

$$\sum_{n=0}^{\infty} b(n)q^n = \frac{1}{(q;q)_\infty^2(q^2;q^2)_\infty^2}. \tag{1.4}$$

They established several Ramanujan type congruences modulo 5, 7 and 9:

$$b(5n+4) \equiv 0 \pmod{5}, \tag{1.5}$$

$$b(7n+i) \equiv 0 \pmod{7}, \tag{1.6}$$

$$b(9n+7) \equiv 0 \pmod{9}, \tag{1.7}$$

where $i = 2, 3, 4, 6$.

Since $b(n)$ counts pairs of cubic partitions, Kim [9] christened $b(n)$ the number of cubic partition pairs. Recently, Kim [10] introduced its overpartition analog $\overline{b}(n)$, which counts the number of overcubic partition pairs of n and satisfies

$$\sum_{n=0}^{\infty} \overline{b}(n)q^n = \frac{(-q;q)_\infty^2(-q^2;q^2)_\infty^2}{(q;q)_\infty^2(q^2;q^2)_\infty^2}. \tag{1.8}$$

Based on the arithmetic properties of quadratic forms and modular forms, he derived the congruences

$$\overline{b}(9n+3) \equiv 0 \pmod{3}, \tag{1.9}$$

$$\overline{b}(8n+7) \equiv 0 \pmod{64}. \tag{1.10}$$

More recently, Lin [11] established several congruences modulo 3 and 5 for $\overline{b}(n)$, including

$$\overline{b}(3^\alpha(3n+2)) \equiv 0 \pmod{3}, \tag{1.11}$$

$$\overline{b}(3^\alpha(4n+2)) \equiv 0 \pmod{3}, \tag{1.12}$$

$$\overline{b}(20n+10) \equiv 0 \pmod{5}, \tag{1.13}$$

for all $n \geq 0$ and $\alpha \geq 2$. Moreover, invoking the Ramanujan theta function

$$\varphi(q) = \sum_{n=-\infty}^{\infty} q^{n^2},$$

Lin established the congruence

$$\sum_{n=0}^{\infty} \overline{b}(10n)(-q)^n \equiv \varphi(-q^2) \sum_{n=0}^{\infty} \overline{p}(5n)q^{2n} \pmod{5},$$

where

$$\sum_{n=0}^{\infty} \overline{p}(n)q^n = \frac{(-q;q)_\infty}{(q;q)_\infty}.$$

Applying the following result due to Treneer [12]:

$$\sum_{n=0}^{\infty}\overline{p}(5n)q^n \equiv \varphi(-q)^3 \pmod 5,$$

we obtain the nice congruence relation

$$\sum_{n=0}^{\infty}\overline{b}(10n)q^n \equiv \varphi(-q^2)^4 \pmod 5, \tag{1.14}$$

which should have appeared in [11].

The goal of this paper is to prove the following three congruences:

$$\overline{b}(3^\alpha(3n+2)) \equiv 0 \pmod 9, \tag{1.15}$$

$$\overline{b}(3^\alpha(4n+2)) \equiv 0 \pmod 9, \tag{1.16}$$

$$\overline{b}(36n+12) \equiv 0 \pmod 9, \tag{1.17}$$

for all $n \geq 0$ and $\alpha \geq 2$.

2. Preliminaries. For notational convenience, we set

$$f_t := (q^t;q^t)_\infty.$$

We now list some preliminary results, to be used later.

LEMMA 2.1. *We have*

$$\varphi(-q) = \frac{f_1^2}{f_2}, \tag{2.1}$$

$$\varphi(q) = \frac{f_2^5}{f_1^2 f_4^2}, \tag{2.2}$$

$$\varphi(-q) = \varphi(-q^9) - 2q\frac{f_3 f_{18}^2}{f_6 f_9}. \tag{2.3}$$

Proof. By Jacobi's triple product identity [2, Entry 19, p. 35], it is easy to obtain the product formulae for $\varphi(-q)$ and $\varphi(q)$. For the proof of (2.3), see [2, p. 49]. ∎

The following result of Hirschhorn [6] plays a key role in the proofs of our main results.

LEMMA 2.2. *We have*

$$\frac{(-q;q)_\infty(-q^2;q^2)_\infty}{(q;q)_\infty(q^2;q^2)_\infty} = A_0(q^3) + qA_1(q^3) + q^2A_2(q^3),$$

where

$$A_0(q) = \frac{f_4^4 f_6^7}{f_2^8 f_3^2 f_{12}^3} + 12q\frac{f_3^3 f_4^3 f_6^3}{f_1^7 f_2^4},$$
$$A_1(q) = 2\frac{f_3^9 f_4^3}{f_1^9 f_2^2 f_6^3} + 8q\frac{f_4^3 f_6^6}{f_1^6 f_2^5},$$
$$A_2(q) = 6\frac{f_3^6 f_4^3}{f_1^8 f_2^3}.$$

In [1], Andrews, Hirschhorn and Sellers studied the arithmetic properties of a partition function $\mathrm{ped}(n)$, whose generating function satisfies

$$\sum_{n=0}^{\infty} \mathrm{ped}(n) q^n = \frac{f_4}{f_1}.$$

They established the following nice 3-dissection formula for $\mathrm{ped}(n)$.

Lemma 2.3.

$$\sum_{n=0}^{\infty} \mathrm{ped}(3n) q^n = \frac{f_4 f_6^4}{f_1^3 f_{12}^2},$$
$$\sum_{n=0}^{\infty} \mathrm{ped}(3n+1) q^n = \frac{f_2^2 f_3^3 f_{12}}{f_1^4 f_6^2},$$
$$\sum_{n=0}^{\infty} \mathrm{ped}(3n+2) q^n = 2\frac{f_2 f_6 f_{12}}{f_1^3}.$$

We also need the following four 2-dissection formulae for f_1^2, f_1^{-4}, $f_3^3 f_1^{-1}$, $f_1 f_3^{-3}$.

Lemma 2.4 (see [13, (2.4) and (2.10)]).

$$f_1^2 = \frac{f_2 f_8^5}{f_4^2 f_{16}^2} - 2q\frac{f_2 f_{16}^2}{f_8},$$
$$\frac{1}{f_1^4} = \frac{f_4^{14}}{f_2^{14} f_8^4} + 4q\frac{f_4^2 f_8^4}{f_2^{10}}.$$

Lemma 2.5 (see [7]).

$$\frac{f_3^3}{f_1} = \frac{f_4^3 f_6^2}{f_2^2 f_{12}} + q\frac{f_{12}^3}{f_4},$$
$$\frac{f_1}{f_3^3} = \frac{f_2 f_4^2 f_{12}^2}{f_6^7} - q\frac{f_2^3 f_{12}^6}{f_4^2 f_6^9}.$$

The following two congruence relations are easy to establish from the binomial theorem, and we will frequently use them without explicit mention.

LEMMA 2.6.

$$f_1^3 \equiv f_3 \pmod 3, \qquad f_1^9 \equiv f_3^3 \pmod 9.$$

LEMMA 2.7.

$$\varphi(-q) \equiv \frac{f_3^2 f_{12}}{f_1^4 f_2 f_4^3} + 3q\frac{f_2 f_3 f_{12}^3}{f_1 f_4 f_6^2} \pmod 9.$$

Proof. By [7, (1.35)], we have

$$\frac{f_1^3}{f_3} = \frac{f_4^3}{f_{12}} - 3q\frac{f_2^2 f_{12}^3}{f_4 f_6^2}.$$

Squaring the above identity, we obtain

$$\frac{f_1^6}{f_3^2} \equiv \frac{f_4^6}{f_{12}^2} + 3q\frac{f_2^2 f_4^2 f_{12}^2}{f_6^2} \pmod 9.$$

Multiplying the above identity by $\frac{f_3^2}{f_1^4 f_2}$ yields

$$\begin{aligned}\frac{f_1^2}{f_2} &\equiv \frac{f_4^6 f_3^2}{f_{12}^2 f_1^4 f_2} + 3q\frac{f_2 f_4^2 f_{12}^2 f_3^2}{f_6^2 f_1^4} \\ &\equiv \frac{f_3^2 f_4^6}{f_1^4 f_2}\cdot\frac{f_{12}}{f_4^9} + 3q\frac{f_3^2 f_{12}^2}{f_6^2}\cdot\frac{f_2 f_{12}}{f_4 f_1 f_3} \\ &\equiv \frac{f_3^2}{f_1^4 f_2}\cdot\frac{f_{12}}{f_4^3} + 3q\frac{f_3 f_{12}^3 f_2}{f_1 f_4 f_6^2} \pmod 9.\end{aligned}$$

This completes the proof. ∎

3. Congruences modulo 9 for $\overline{b}(n)$.

We start by proving the following congruence relation for the generating function of $\overline{b}(3n)$.

LEMMA 3.1.

$$\begin{aligned}\sum_{n=0}^{\infty}\overline{b}(3n)q^n \equiv \frac{f_6^8}{f_3^4 f_{12}^3}\varphi(-q^2) + 6q\frac{f_6^6}{f_{12} f_3}\sum_{n=0}^{\infty}\mathrm{ped}(n)q^n + 6q\frac{f_3^8 f_{12}^2}{f_6^4}\varphi(-q)^2 \\ - 3q^2 f_6^4 f_{12}^2\varphi(-q)^2 \pmod 9.\end{aligned} \tag{3.1}$$

Proof. It is easy to see that

$$\sum_{n=0}^{\infty}\overline{b}(n)q^n = (A_0(q^3) + qA_1(q^3) + q^2A_2(q^3))^2.$$

Extracting the terms of q^{3n} on both sides of the above identity and replacing q^3 by q, we obtain

$$\sum_{n=0}^{\infty}\overline{b}(3n)q^n = A_0(q)^2 + 2qA_1(q)A_2(q).$$

It follows from Lemma 2.2 that

$$\begin{aligned}\sum_{n=0}^{\infty}\overline{b}(3n)q^n &\equiv \frac{f_4^8f_6^{14}}{f_2^{16}f_3^4f_{12}^6}+6q\frac{f_4^7f_6^{10}f_3}{f_2^{12}f_{12}^3f_1^7}+2q\left(3\frac{f_3^{15}f_4^6}{f_1^{17}f_2^5f_6^3}-6q\frac{f_3^6f_4^6f_6^6}{f_1^{14}f_2^8}\right)\\ &\equiv \frac{f_6^8}{f_3^4f_{12}^3}\cdot\frac{f_2^2}{f_4}+6q\frac{f_6^6}{f_{12}f_3}\cdot\frac{f_4}{f_1}+6q\frac{f_3^8f_{12}^2}{f_6^4}\cdot\frac{f_1^4}{f_2^2}-3q^2f_6^4f_{12}^2\cdot\frac{f_1^4}{f_2^2}\pmod 9.\end{aligned}$$

Substituting the product formula for $\varphi(-q)$ and the generating function for $\operatorname{ped}(n)$ yields the desired result. ∎

We are now in a position to prove our first two congruences.

THEOREM 3.2. *For $n\geq 0$ and $\alpha\geq 2$,*

$$\overline{b}(3^\alpha(3n+2))\equiv 0 \pmod 9, \tag{3.2}$$

$$\overline{b}(3^\alpha(4n+2))\equiv 0 \pmod 9. \tag{3.3}$$

Proof. Choosing those terms from (3.1) whose powers of q are multiples of 3, replacing q^3 by q, and employing Lemmas 2.1 and 2.3, we find that

$$\begin{aligned}\sum_{n=0}^{\infty}\overline{b}(9n)q^n \equiv{}& \frac{f_2^8}{f_1^4f_4^3}\varphi(-q^6)+6q\frac{f_2^6}{f_1f_4}\cdot 2\frac{f_2f_6f_{12}}{f_1^3}+6q\frac{f_1^8f_4^2}{f_2^4}\cdot 4\frac{f_1^2f_6^4}{f_2^2f_3^2}\\ &-3qf_2^4f_4^2\cdot\varphi(-q^3)\cdot(-4)\frac{f_1f_6^2}{f_2f_3}\pmod 9.\end{aligned}$$

Applying the easily checked result

$$\frac{f_1^8f_4^2}{f_2^4}\cdot\frac{f_1f_6^2}{f_2f_3}\equiv f_2^4f_4^2\varphi(-q^3)\pmod 3,$$

we conclude that

$$\begin{aligned}\sum_{n=0}^{\infty}\overline{b}(9n)q^n &\equiv \frac{f_2^8}{f_1^4f_4^3}\cdot\frac{f_6^2}{f_{12}}+3q\frac{f_2^7f_6f_{12}}{f_4f_1^4}\\ &\equiv \frac{f_6^5}{f_1^4f_2f_4^3f_{12}}+3q\frac{f_2f_6^3f_{12}}{f_1f_3f_4}\pmod 9,\end{aligned}$$

which can be rewritten as

$$\sum_{n=0}^{\infty}\overline{b}(9n)q^n \equiv \frac{f_6^5}{f_3^2f_{12}^2}\left(\frac{f_3^2f_{12}}{f_1^4f_2f_4^3}+3q\frac{f_2f_3f_{12}^3}{f_1f_4f_6^2}\right)\pmod 9.$$

Invoking Lemmas 2.1 and 2.7, we arrive at

$$\sum_{n=0}^{\infty}\overline{b}(9n)q^n \equiv \varphi(q^3)\varphi(-q) \tag{3.4}$$

$$\equiv \varphi(q^3)\left(\varphi(-q^9)-2q\frac{f_3f_{18}^2}{f_6f_9}\right)\pmod 9. \tag{3.5}$$

Collecting the terms whose powers of q are multiples of 3 from (3.5), and replacing q^3 by q, we obtain

$$\sum_{n=0}^{\infty}\bar{b}(27n)q^n \equiv \varphi(q)\varphi(-q^3) \pmod 9.$$

Combining (3.4) with the above congruence, we find that

$$\sum_{n=0}^{\infty}\bar{b}(9n)q^n \equiv \sum_{n=0}^{\infty}\bar{b}(27n)(-q)^n \pmod 9,$$

which implies that for $n \geq 0$,

$$\bar{b}(9n) \equiv (-1)^n\bar{b}(27n) \pmod 9. \tag{3.6}$$

Equating the coefficients of q^{3n+2} on both sides of (3.5), we get

$$\bar{b}(27n+18) \equiv 0 \pmod 9. \tag{3.7}$$

By the definition of $\varphi(q)$, we can express (3.4) as

$$\sum_{n=0}^{\infty}\bar{b}(9n)q^n \equiv \sum_{k=-\infty}^{\infty}\sum_{\ell=-\infty}^{\infty}(-1)^{\ell}q^{k^2+3\ell^2} \pmod 9.$$

It is easy to see that there are no integers k and ℓ such that $k^2+3\ell^2 \equiv 2 \pmod 4$. This gives

$$\sum_{n=0}^{\infty}\bar{b}(36n+18)q^n \equiv 0 \pmod 9.$$

Consequently,

$$\bar{b}(36n+18) \equiv 0 \pmod 9. \tag{3.8}$$

Based on (3.6)–(3.8), and proceeding by induction on α, we obtain the desired results (3.2) and (3.3) immediately. ∎

Finally, we prove the last congruence.

THEOREM 3.3. *For $n \geq 0$, we have*

$$\bar{b}(36n+12) \equiv 0 \pmod 9. \tag{3.9}$$

Proof. Selecting those terms on each side of (3.1) whose powers of q are of the form $3n+1$, dividing by q, and replacing q^3 by q, we obtain

$$\begin{aligned}\sum_{n=0}^{\infty}\bar{b}(9n+3)q^n &\equiv \frac{f_2^8}{f_1^4f_4^3}\cdot 0 + 6\frac{f_2^6}{f_4f_1}\cdot\frac{f_4f_6^4}{f_1^3f_{12}^2} + 6\frac{f_1^8f_4^2}{f_2^4}\varphi^2(-q^3) - 3qf_2^4f_4^2\cdot 4\frac{f_1^2f_6^4}{f_2^2f_3^2}\\ &\equiv 6\frac{f_2^6f_6^4}{f_{12}^2}\cdot\frac{1}{f_1^4} + 6\frac{f_4^2}{f_2^4f_6^2}\cdot f_1^2f_9^2 + 6qf_2^2f_4^2f_6^4\cdot\frac{1}{f_1^4} \pmod 9.\end{aligned}$$

Extracting the terms with odd powers of q, dividing by q, replacing q^2 by q, and applying Lemma 2.4, we get

$$\begin{aligned}
&\sum_{n=0}^{\infty}\bar{b}(18n+12)q^n \\
&\qquad\equiv 6\frac{f_1^6f_3^4}{f_6^2}\cdot 4\frac{f_2^2f_4^4}{f_1^{10}}+6\frac{f_2^2}{f_1^4f_3^2}\left(q^4\cdot\frac{f_1f_4^5}{f_2^2f_8^2}\cdot\frac{f_9f_{72}^2}{f_{36}}+\frac{f_1f_8^2}{f_4}\cdot\frac{f_9f_{36}^5}{f_{18}^2f_{72}^2}\right)\\
&\qquad\quad+6f_1^2f_2^2f_3^4\cdot\frac{f_2^{14}}{f_1^{14}f_4^4}\\
&\qquad\equiv 6\frac{f_2^2f_4^4}{f_6^2}\cdot\frac{f_3^3}{f_1}+6q^4\frac{f_4^5f_{72}^2}{f_8^2f_{36}}+6\frac{f_2^2f_8^2f_{36}^5}{f_4f_{18}^2f_{72}^2}+6\frac{f_2^{16}}{f_4^4} \pmod 9.
\end{aligned}$$

If we now take those terms whose powers of q are even, and replace q^2 by q, by Lemma 2.5 we obtain

$$\begin{aligned}
\sum_{n=0}^{\infty}\bar{b}(36n+12)q^n &\equiv 6\frac{f_1^2f_2^4}{f_3^2}\cdot\frac{f_2^3f_3^2}{f_1^2f_6}+6q^2\frac{f_2^5f_{36}^2}{f_4^2f_{18}}+6\frac{f_1^2f_4^2f_{18}^5}{f_2f_9^2f_{36}^2}+6\frac{f_1^{16}}{f_2^4}\\
&\equiv 6f_2^4+6q^2\frac{f_4^{16}}{f_2^4}+6\frac{f_2^{44}}{f_4^{16}}\cdot\frac{1}{f_1^{16}}+6\frac{1}{f_2^4}\cdot f_1^{16}\\
&\equiv 6f_2^4+6q^2\frac{f_4^{16}}{f_2^4}+6\frac{f_2^{44}}{f_4^{16}}\cdot\left(\frac{f_1}{f_3^3}\right)^2+6\frac{1}{f_2^4}\cdot\left(\frac{f_3^3}{f_1}\right)^2\\
&\equiv 6f_2^4+6q^2\frac{f_4^{16}}{f_2^4}+6\frac{f_2^{44}}{f_4^{16}}\left(\frac{f_2^2f_4^4f_{12}^4}{f_6^{14}}+q^2\frac{f_2^6f_{12}^{12}}{f_4^4f_6^{18}}-2q\frac{f_2^4f_{12}^8}{f_6^{16}}\right)\\
&\quad+6\frac{1}{f_2^4}\left(\frac{f_4^6f_6^4}{f_2^4f_{12}^2}+q^2\frac{f_{12}^6}{f_4^2}+2q\frac{f_4^2f_6^2f_{12}^2}{f_2^2}\right) \pmod 9,
\end{aligned}$$

which can be simplified to

$$\begin{aligned}
\sum_{n=0}^{\infty}\bar{b}(36n+12)q^n &\equiv 6f_2^4+6q^2\frac{f_4^{16}}{f_2^4}+6\frac{f_2^{44}f_2^2f_4^4f_4^{12}}{f_4^{16}f_2^{42}}+6q^2\frac{f_2^{44}f_2^6f_4^{36}}{f_4^{16}f_4^4f_2^{54}}\\
&\quad-12q\frac{f_2^{44}f_2^4f_4^{24}}{f_4^{16}f_2^{48}}+6\frac{f_4^6f_2^{12}}{f_2^4f_2^4f_4^6}+6q^2\frac{f_4^{18}}{f_2^4f_4^2}+12q\frac{f_4^2f_2^6f_4^6}{f_2^4f_2^2}\\
&\equiv 18f_2^4+18q^2\frac{f_4^{16}}{f_2^4} \pmod 9.
\end{aligned}$$

Equating the coefficients of q^n yields the desired result. ∎

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