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Topological radicals of nest algebras

by

G. Andreolas and M. Anoussis (Karlovassi)

Abstract. Let $\mathcal{N}$ be a nest on a Hilbert space H and $\operatorname{Alg}\mathcal{N}$ the corresponding nest algebra. We determine the hypocompact radical of $\operatorname{Alg}\mathcal{N}$. Other topological radicals are also characterized.

1. Introduction. Let $\mathcal{A}$ be a Banach algebra. The *Jacobson radical* of $\mathcal{A}$ is defined as the intersection of the kernels of the algebraically irreducible representations of $\mathcal{A}$. A *topologically irreducible representation* of $\mathcal{A}$ is a continuous homomorphism of $\mathcal{A}$ into the Banach algebra of bounded linear operators on a Banach space X for which no nontrivial closed subspace of X is invariant. It has been shown in [5] that the intersection of the kernels of all such representations is, in a reasonable sense, a new radical that can be strictly smaller than the Jacobson radical.

The theory of topological radicals of Banach algebras was originated by Dixon [5] in order to study this new radical as well as other radicals associated with various types of representations.

Shulman and Turovskii have further developed the theory of topological radicals in a series of papers [8–13] and applied it to the study of various problems of operator theory and Banach algebras. They introduced many new topological radicals. Among them are the hypocompact radical, the hypofinite radical and the scattered radical. These radicals are closely related to the theory of elementary operators on Banach algebras [3, 10].

Let us recall Dixon's definition of topological radicals.

Definition 1.1. A *topological radical* is a map $\mathcal{R}$ associating with each Banach algebra $\mathcal{A}$ a closed ideal $\mathcal{R}(\mathcal{A}) \subseteq \mathcal{A}$ in such a way that:

(1) $\mathcal{R}(\mathcal{R}(\mathcal{A})) = \mathcal{R}(\mathcal{A})$.
(2) $\mathcal{R}(\mathcal{A}/\mathcal{R}(\mathcal{A})) = \{0\}$, where $\{0\}$ denotes the zero coset in $\mathcal{A}/\mathcal{R}(\mathcal{A})$.

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(3) If $\mathcal{A}$, $\mathcal{B}$ are Banach algebras and $\phi : \mathcal{A} \to \mathcal{B}$ is a continuous epimorphism, then $\phi(\mathcal{R}(\mathcal{A})) \subseteq \mathcal{R}(\mathcal{B})$.
(4) If $\mathcal{I}$ is a closed ideal of $\mathcal{A}$, then $\mathcal{R}(\mathcal{I})$ is a closed ideal of $\mathcal{A}$ and $\mathcal{R}(\mathcal{I}) \subseteq \mathcal{R}(\mathcal{A}) \cap \mathcal{I}$.

An element a of a Banach algebra $\mathcal{A}$ is said to be *compact* if the map $M_{a,a} : \mathcal{A} \to \mathcal{A}$, $x \mapsto axa$ is compact. Following Shulman and Turovskii [12] we will call a Banach algebra $\mathcal{A}$ *hypocompact* if any nonzero quotient $\mathcal{A}/\mathcal{J}$ by a closed ideal $\mathcal{J}$ contains a nonzero compact element. Shulman and Turovskii [12, Corollary 3.10 and Theorem 3.13] have proved that any Banach algebra $\mathcal{A}$ has a largest hypocompact ideal, denoted by $\mathcal{R}_{\mathrm{hc}}(\mathcal{A})$, and that the map $\mathcal{A} \mapsto \mathcal{R}_{\mathrm{hc}}(\mathcal{A})$ is a topological radical. The ideal $\mathcal{R}_{\mathrm{hc}}(\mathcal{A})$ is called the *hypocompact radical* of $\mathcal{A}$.

If X is a Banach space, we shall denote by $\mathcal{B}(X)$ the Banach algebra of all bounded operators on X, and by $\mathcal{K}(X)$ the Banach algebra of all compact operators on X. Vala [14] has shown that if X is a Banach space, then $a \in \mathcal{B}(X)$ is a compact element if and only if $a \in \mathcal{K}(X)$. Since by [3, Lemma 8.2] the compact elements are always contained in the hypocompact radical, we obtain $\mathcal{K}(X) \subseteq \mathcal{R}_{\mathrm{hc}}(\mathcal{B}(X))$. It follows that if H is a separable Hilbert space, the hypocompact radical of $\mathcal{B}(H)$ is $\mathcal{K}(H)$. Indeed, the ideal $\mathcal{K}(H)$ is the only proper ideal of $\mathcal{B}(H)$ and the Calkin algebra $\mathcal{B}(H)/\mathcal{K}(H)$ has no nonzero compact elements [6, Section 5].

Shulman and Turovskii [12, p. 298] observe that there exist Banach spaces X such that the hypocompact radical $\mathcal{R}_{\mathrm{hc}}(\mathcal{B}(X))$ contains all weakly compact operators and strictly contains the ideal $\mathcal{K}(X)$ of compact operators.

Argyros and Haydon [2] construct a Banach space X such that every operator in $\mathcal{B}(X)$ is a scalar multiple of the identity plus a compact operator. In that case, $\mathcal{B}(X)/\mathcal{K}(X)$ is finite-dimensional, and hence $\mathcal{R}_{\mathrm{hc}}(\mathcal{B}(X)) = \mathcal{B}(X)$.

In this paper we characterize the hypocompact radical of a nest algebra. The nest algebras form a class of nonselfadjoint operator algebras that generalize the block upper triangular matrices to an infinite-dimensional Hilbert space context. They were introduced by Ringrose [7], and since then they have been studied by many authors. The monograph of Davidson [4] is recommended as a reference.

The ideal structure has an important part in the development of the theory of nest algebras. Ringrose [7, Theorem 5.3] characterized the Jacobson radical of a nest algebra. Moreover, it follows from [7, Theorems 4.9 and 5.3] that the intersection of the kernels of all the topologically irreducible representations of a nest algebra coincides with the Jacobson radical.

We now introduce some definitions and notation that we will use in the following. A *nest* $\mathcal{N}$ is a totally ordered family of closed subspaces of a Hilbert space H containing $\{0\}$ and H, which is closed under intersection and closed span. If H is a Hilbert space and $\mathcal{N}$ a nest on H, then the *nest algebra* $\operatorname{Alg}\mathcal{N}$ is the algebra of all operators $T \in \mathcal{B}(H)$ such that $T(N) \subseteq N$ for all $N \in \mathcal{N}$. We shall usually denote both the subspaces belonging to a nest and the corresponding orthogonal projections by the same symbol. If $(N_\lambda)_{\lambda\in\Lambda}$ is a family of subspaces of a Hilbert space, we denote by $\bigvee\{N_\lambda : \lambda \in \Lambda\}$ their closed linear span and by $\bigwedge\{N_\lambda : \lambda \in \Lambda\}$ their intersection. If $\mathcal{N}$ is a nest and $N \in \mathcal{N}$, then $N_- = \bigvee\{N' \in \mathcal{N} : N' < N\}$. Similarly we define $N_+ = \bigwedge\{N' \in \mathcal{N} : N' > N\}$. The subspaces $N \cap N_-^\perp$ are called the *atoms* of $\mathcal{N}$.

If e, f are elements of a Hilbert space H, we denote by $e \otimes f$ the rank one operator on H defined by $(e \otimes f)(h) = \langle h, e\rangle f$. We shall frequently use the fact that a rank one operator $e \otimes f$ belongs to a nest algebra, $\operatorname{Alg}\mathcal{N}$, if and only if there exists $N \in \mathcal{N}$ such that $e \in N_-^\perp$ and $f \in N$ [4, Lemmas 2.8 and 3.7].

Throughout the paper we denote by $\mathcal{N}$ a nest acting on a Hilbert space H and by $\mathcal{K}(\mathcal{N})$ the ideal of compact operators of $\operatorname{Alg}\mathcal{N}$. In addition, all ideals are considered to be closed. The radical of a nest algebra $\operatorname{Alg}\mathcal{N}$ will be denoted by $\operatorname{Rad}(\mathcal{N})$. The following is [7, Theorem 5.3].

Theorem 1.2 (Ringrose's Theorem). *Let $\mathcal{N}$ be a nest on a Hilbert space H. The Jacobson radical of $\operatorname{Alg}\mathcal{N}$ coincides with the set of operators $a \in \operatorname{Alg}\mathcal{N}$ for which the quantities $\inf\{\|PQ^\perp aPQ^\perp\| : P \in \mathcal{N},\ P > Q\}$ and $\inf\{\|QP^\perp aQP^\perp\| : P \in \mathcal{N},\ P < Q\}$ are zero for all $Q \in \mathcal{N}$.*

2. Main result

Lemma 2.1. *Let $\mathcal{N}$ be a nest on a Hilbert space H and $Q \in \mathcal{N}$ be such that $Q_- = Q$. Suppose that $a, b \in \operatorname{Alg}\mathcal{N}$ are such that $\|QP^\perp aQP^\perp\| \geq 2\varepsilon$ and $\|QP^\perp bQP^\perp\| \geq 2\varepsilon$ for some $\varepsilon > 0$ and for all $P < Q$, $P \in \mathcal{N}$. Then, there exist orthonormal sequences (e_n), (f_n) such that $e_n \otimes f_n \in \operatorname{Alg}\mathcal{N}$ and $\|QP^\perp a(\sum_{n=1}^\infty e_{k_n} \otimes f_{k_n})bQP^\perp\| \geq \varepsilon^2$ for all $P < Q$, $P \in \mathcal{N}$ and for any strictly increasing sequence $(k_n) \subseteq \mathbb{N}$.*

Proof. Let $(R_n)_{n\in\mathbb{N}}$ be a strictly increasing sequence that is sot-convergent to Q. We set $P_1 = R_1$. Then $\|QP_1^\perp aQP_1^\perp\| \geq 2\varepsilon$. We choose a norm one vector $f_1' \in Q$ such that $\|QP_1^\perp aQP_1^\perp(f_1')\| \geq \frac{3}{2}\varepsilon$. Then, we choose a projection $P_2 = R_{k_2}$ with $k_2 > 1$ such that

$$\|QP_1^\perp aQP_1^\perp P_2(f_1')\| \geq \varepsilon.$$

We set $f_1 = \frac{1}{\|P_1^\perp P_2 f_1'\|} P_1^\perp P_2 f_1'$. Then, $f_1 \in P_1^\perp P_2$, $\|f_1\| = 1$ and

$$\|P_2P_1^\perp aP_2P_1^\perp(f_1)\| \geq \varepsilon.$$

Suppose that there exist $P_3 = R_{k_3}, \ldots, P_n = R_{k_n}$, where $k_2 < \cdots < k_n$, such that $\|P_iP_{i-1}^\perp aP_iP_{i-1}^\perp(f_{i-1})\| \geq \varepsilon$ for some orthonormal vectors $(f_i)_{i=3}^n$, where $f_{i-1} \in P_i P_{i-1}^{\perp}$, $i \in \{3, \dots, n\}$. Assuming that $\|QP_n^{\perp} a QP_n^{\perp}\| \geq 2\varepsilon$, we apply the arguments of the first step of the proof to obtain a projection $P_{n+1} = R_{k_{n+1}}$ for some $k_{n+1} > k_n$ and a norm one vector $f_n \in P_{n+1}P_n^{\perp}$ such that

$$\|P_{n+1}P_n^{\perp} a P_{n+1} P_n^{\perp}(f_n)\| \geq \varepsilon.$$

Note that $(P_n)_{n\in\mathbb{N}}$ is sot-convergent to Q as a subsequence of $(R_n)_{n\in\mathbb{N}}$.

In the same way, we can find a subsequence $(S_n)_{n\in\mathbb{N}}$ of $(R_n)_{n\in\mathbb{N}}$ such that $S_n > P_{n+1}$ for all $n \in \mathbb{N}$ and an orthonormal sequence $(e_n)_{n\in\mathbb{N}} \subseteq H$ such that $e_n \in S_{n+1}S_n^{\perp}$ and

$$\|S_{n+1}S_n^{\perp} b^* S_{n+1}S_n^{\perp}(e_n)\| \geq \varepsilon,$$

for all $n \in \mathbb{N}$. It follows that $e_n \otimes f_n \in \operatorname{Alg}\mathcal{N}$. Let $P \in \mathcal{N}$ and $i \in \mathbb{N}$ be such that $P_i > P$. Then

$$\begin{aligned}
\Big\|QP^{\perp} a\Big(\sum_{n\in\mathbb{N}} e_n \otimes f_n\Big) bQP^{\perp}\Big\| &\geq \Big\|P_{i+1}P_i^{\perp} a\Big(\sum_{n\in\mathbb{N}} e_n \otimes f_n\Big) bS_{i+1}S_i^{\perp}\Big\| \\
&= \Big\|\sum_{n\in\mathbb{N}} S_{i+1}S_i^{\perp} b^*(e_n) \otimes P_{i+1}P_i^{\perp} a(f_n)\Big\| \\
&= \|S_{i+1}S_i^{\perp} b^*(e_i) \otimes P_{i+1}P_i^{\perp} a(f_i)\| \\
&= \|S_{i+1}S_i^{\perp} b^*(e_i)\|\, \|P_{i+1}P_i^{\perp} a(f_i)\| \geq \varepsilon^2.
\end{aligned}$$

The proof is identical for any strictly increasing sequence $(k_n) \subseteq \mathbb{N}$. ∎

In the proof of Theorem 2.4 we shall use the following fact which is a consequence of [4, Proposition 1.18] and Ringrose's Theorem.

Lemma 2.2. *Let $Q \in \mathcal{N}$, $Q = Q_-$ and $a \in \mathcal{K}(\mathcal{N}) + \operatorname{Rad}(\mathcal{N})$. Then, $\inf\{\|QP^{\perp} a QP^{\perp}\| : P \in \mathcal{N},\ P < Q\} = 0$.*

Remark 2.3. Similar statements to those of Lemmas 2.1 and 2.2 hold in the case $Q_+ = Q$.

Theorem 2.4. *The hypocompact radical of $\operatorname{Alg}\mathcal{N}$ is the ideal $\mathcal{K}(\mathcal{N}) + \operatorname{Rad}(\mathcal{N})$.*

Proof. The ideal generated by the compact elements of $\operatorname{Alg}\mathcal{N}$ is the ideal $\mathcal{K}(\mathcal{N}) + \operatorname{Rad}(\mathcal{N})$ [1, Theorem 3.2], and therefore $\mathcal{K}(\mathcal{N}) + \operatorname{Rad}(\mathcal{N}) \subseteq \mathcal{R}_{\mathrm{hc}}(\mathcal{N})$ [3, Lemma 8.2]. Let $\mathcal{J}$ be an ideal of $\operatorname{Alg}\mathcal{N}$ strictly larger than $\mathcal{K}(\mathcal{N}) + \operatorname{Rad}(\mathcal{N})$ and pick $a \in \mathcal{J} \setminus (\mathcal{K}(\mathcal{N}) + \operatorname{Rad}(\mathcal{N}))$. It suffices to show that the element $\varphi(a) \in \mathcal{J}/(\mathcal{K}(\mathcal{N}) + \operatorname{Rad}(\mathcal{N}))$ is not compact, where $\varphi : \mathcal{J} \to \mathcal{J}/(\mathcal{K}(\mathcal{N}) + \operatorname{Rad}(\mathcal{N}))$ is the quotient map. We consider the sets

$$\mathfrak{Q}_- = \{Q \in \mathcal{N} : Q \neq \{0\},\ \exists \epsilon_Q > 0\ \forall P \in \mathcal{N},\ P < Q : \|QP^{\perp} a QP^{\perp}\| \geq 2\epsilon_Q\}$$

and

$$\mathfrak{Q}_+ = \{Q \in \mathcal{N} : Q \neq H,\ \exists \epsilon_Q > 0\ \forall P \in \mathcal{N},\ P > Q : \|Q^{\perp} P a Q^{\perp} P\| \geq 2\epsilon_Q\}.$$

From Ringrose's Theorem it follows that the set $\mathfrak{Q} = \mathfrak{Q}_- \cup \mathfrak{Q}_+$ is not empty. We distinguish two cases:

(1) First, suppose there exists a projection $Q \in \mathfrak{Q}_-$ such that $Q_- = Q$; if we assume there exists $Q \in \mathfrak{Q}_+$ such that $Q = Q_+$, the proof is similar (see Remark 2.3). Applying Lemma 2.1 we obtain orthonormal sequences (h_n) and (g_n) such that $h_n \otimes g_n \in \operatorname{Alg}\mathcal{N}$ for all $n \in \mathbb{N}$ and $\|QP^\perp a(\sum_{n\in\mathbb{N}} h_n \otimes g_n) a QP^\perp\| \geq \epsilon_Q^2$ for all $P < Q$. We set $x = (\sum_{n\in\mathbb{N}} h_n \otimes g_n)a \in \mathcal{J}$. Let $\epsilon > 0$ be such that $2\epsilon = \min\{2\epsilon_Q, \epsilon_Q^2\}$. Again applying Lemma 2.1 to the operators $ax = a(\sum_{n\in\mathbb{N}} h_n \otimes g_n)a$ and a, we obtain orthonormal sequences (e_n) and (f_n) such that $e_n \otimes f_n \in \operatorname{Alg}\mathcal{N}$ and $\|QP^\perp ax(\sum_{n\in\mathbb{N}} e_{k_n} \otimes f_{k_n}) a QP^\perp\| \geq \epsilon^2$ for all $P < Q$ and any strictly increasing sequence $(k_n) \subseteq \mathbb{N}$. Let $(A_n)_{n\in\mathbb{N}}$ be a partition of $\mathbb{N}$ such that A_n is an infinite set for all $n \in \mathbb{N}$. We set $B_n = \bigcup_{i=1}^n A_i$. Note that $\|\sum_{i\in C} e_i \otimes x(f_i)\| \leq \|a\|$ for any subset C of $\mathbb{N}$.

Now, we shall prove that the sequence

$$\Big(\varphi\Big(a\Big(\sum_{i\in B_n} e_i \otimes x(f_i)\Big)a\Big)\Big)_{n\in\mathbb{N}} \subseteq \mathcal{J}/(\mathcal{K}(\mathcal{N}) + \operatorname{Rad}(\mathcal{N}))$$

has no Cauchy subsequence. Indeed, for any $l, m \in \mathbb{N}$ with $l > m$,

$$\begin{aligned}
&\Big\|\varphi(a)\varphi\Big(\sum_{i\in B_l} e_i \otimes x(f_i)\Big)\varphi(a) \\
&\qquad\qquad - \varphi(a)\varphi\Big(\sum_{j\in B_m} e_j \otimes x(f_j)\Big)\varphi(a)\Big\|_{\mathcal{J}/(\mathcal{K}(\mathcal{N}) + \operatorname{Rad}(\mathcal{N}))} \\
&\qquad\qquad = \inf_{r\in(\mathcal{K}(\mathcal{N}) + \operatorname{Rad}(\mathcal{N}))} \Big\|a\Big(\sum_{i\in B_l - B_m} e_i \otimes x(f_i)\Big)a + r\Big\| \\
&\qquad\qquad \geq \Big\|a\Big(\sum_{i\in B_l - B_m} e_i \otimes x(f_i)\Big)a + r_\epsilon\Big\| - \epsilon^2/4,
\end{aligned}$$

for some $r_\epsilon \in \mathcal{K}(\mathcal{N}) + \operatorname{Rad}(\mathcal{N})$. There exists a projection $P < Q$ such that $\|QP^\perp r_\epsilon QP^\perp\| < \epsilon^2/4$ (Lemma 2.2). Therefore,

$$\begin{aligned}
&\Big\|a\Big(\sum_{i\in B_l - B_m} e_i \otimes x(f_i)\Big)a + r_\epsilon\Big\| - \epsilon^2/4 \\
&\qquad \geq \Big\|QP^\perp\Big(a\Big(\sum_{i\in B_l - B_m} e_i \otimes x(f_i)\Big)a + r_\epsilon\Big)QP^\perp\Big\| - \epsilon^2/4 \\
&\qquad \geq \Big\|QP^\perp a\Big(\sum_{i\in B_l - B_m} e_i \otimes x(f_i)\Big)aQP^\perp\Big\| - \|QP^\perp r_\epsilon QP^\perp\| - \epsilon^2/4 \\
&\qquad \geq \epsilon^2 - \epsilon^2/4 - \epsilon^2/4 = \epsilon^2/2.
\end{aligned}$$

Thus, $\varphi(a)$ is a noncompact element of $\mathcal{J}/(\mathcal{K}(\mathcal{N}) + \operatorname{Rad}(\mathcal{N}))$.

(2) Secondly, suppose that $Q_- < Q$ for all $Q \in \mathfrak{Q}_-$ and $Q_+ > Q$ for all $Q \in \mathfrak{Q}_+$. In that case, we shall consider the set $\mathfrak{Q}_-$ instead of $\mathfrak{Q}$ since $Q_+ \in \mathfrak{Q}_-$ for all $Q \in \mathfrak{Q}_+$. Then, the set $\mathfrak{E}_n = \{Q : \|QQ_-^\perp a QQ_-^\perp\| > 1/n\}$ is finite for all $n \in \mathbb{N}$. Observe that if $\mathfrak{E}_n$ were infinite for some $n \in \mathbb{N}$, then there would be a projection $Q \in \mathfrak{Q}$ which is an accumulation point, i.e. either $Q = Q_-$ or $Q = Q_+$, contrary to our assumption.

Now, we suppose that the operator $QQ_-^\perp a QQ_-^\perp$ is compact for all $Q \in \mathfrak{Q}_-$ and we shall arrive at a contradiction. Indeed, since $\mathfrak{E}_n$ is finite for all $n \in \mathbb{N}$, the series $\sum_{Q\in\mathfrak{Q}_-} QQ_-^\perp a QQ_-^\perp$ is norm convergent, and therefore its limit belongs to $\mathcal{K}(\mathcal{N})$. Hence $a - \sum_{Q\in\mathfrak{Q}_-} QQ_-^\perp a QQ_-^\perp \in \mathrm{Rad}(\mathcal{N})$ and therefore $a \in \mathcal{K}(\mathcal{N}) + \mathrm{Rad}(\mathcal{N})$, a contradiction. Thus, there exists a $Q \in \mathfrak{Q}_-$ such that $a_Q = QQ_-^\perp a QQ_-^\perp$ is not compact. Hence, $\mathcal{B}(QQ_-^\perp) \subseteq \mathcal{J}$. We define the map

$$\begin{gathered} i : \mathcal{B}(QQ_-^\perp)/\mathcal{K}(QQ_-^\perp) \to \mathcal{B}(QQ_-^\perp)/(\mathcal{K}(\mathcal{N}) + \mathrm{Rad}(\mathcal{N})), \\ x + \mathcal{K}(QQ_-^\perp) \mapsto x + \mathcal{K}(\mathcal{N}) + \mathrm{Rad}(\mathcal{N}). \end{gathered}$$

This map is obviously well defined. Now, i is an isometric isomorphism:

$$\begin{aligned} \|x + \mathcal{K}(QQ_-^\perp)\|_{\mathcal{B}(QQ_-^\perp)/\mathcal{K}(QQ_-^\perp)} &= \inf\{\|x + K\| : K \in \mathcal{K}(QQ_-^\perp)\} \\ &= \inf_{\substack{K\in\mathcal{K}(\mathcal{N}) \\ R\in\mathrm{Rad}(\mathcal{N})}} \|QQ_-^\perp(x + K + R)QQ_-^\perp\| \\ &\le \inf_{\substack{K\in\mathcal{K}(\mathcal{N}) \\ R\in\mathrm{Rad}(\mathcal{N})}} \|x + K + R\| \\ &= \|x + \mathcal{K}(\mathcal{N}) + \mathrm{Rad}(\mathcal{N})\|_{\mathcal{B}(QQ_-^\perp)/(\mathcal{K}(\mathcal{N}) + \mathrm{Rad}(\mathcal{N}))}, \end{aligned}$$

and the opposite inequality is immediate since $\mathcal{K}(QQ_-^\perp) \subseteq \mathcal{K}(\mathcal{N}) + \mathrm{Rad}(\mathcal{N})$. If $\varphi(a)$ is a compact element of $\mathcal{J}/(\mathcal{K}(\mathcal{N}) + \mathrm{Rad}(\mathcal{N}))$, then $\varphi(a_Q)$ is a compact element of $\mathcal{B}(QQ_-^\perp)/(\mathcal{K}(\mathcal{N}) + \mathrm{Rad}(\mathcal{N}))$. Since $i(a_Q + \mathcal{K}(QQ_-^\perp)) = \varphi(a_Q)$, it follows from the above that $a_Q + \mathcal{K}(QQ_-^\perp)$ is a compact element of $\mathcal{B}(QQ_-^\perp)/\mathcal{K}(QQ_-^\perp)$. From [6] we know that $\mathcal{B}(QQ_-^\perp)/\mathcal{K}(QQ_-^\perp)$ has no compact elements. Hence, $\phi(a)$ is not a compact element of $\mathcal{J}/(\mathcal{K}(\mathcal{N}) + \mathrm{Rad}(\mathcal{N}))$. ∎

Remark 2.5. The hypocompact radical of $\mathrm{Alg}\,\mathcal{N}$ coincides with the ideal generated by the compact elements of $\mathrm{Alg}\,\mathcal{N}$.

The following definitions and results are taken from [13]. An element a of a Banach algebra $\mathcal{A}$ is said to be of *finite rank* if the map $M_{a,a} : \mathcal{A} \to \mathcal{A}$, $x \mapsto axa$, has finite rank. A Banach algebra $\mathcal{A}$ is called *hypofinite* if any nonzero quotient $\mathcal{A}/\mathcal{J}$ by a closed ideal $\mathcal{J}$ contains a nonzero finite rank element. A Banach algebra $\mathcal{A}$ has a largest hypofinite ideal, denoted by $\mathcal{R}_{\mathrm{hf}}(\mathcal{A})$, and the map $\mathcal{A} \mapsto \mathcal{R}_{\mathrm{hf}}(\mathcal{A})$ is a topological radical [13, 2.3.6]. The ideal $\mathcal{R}_{\mathrm{hf}}(\mathcal{A})$ is called the *hypofinite radical* of $\mathcal{A}$.

A Banach algebra is called *scattered* if the spectrum of every element $a \in \mathcal{A}$ is finite or countable. A Banach algebra $\mathcal{A}$ has a largest scattered ideal denoted by $\mathcal{R}_{\mathrm{sc}}(\mathcal{A})$ and the map $\mathcal{A} \mapsto \mathcal{R}_{\mathrm{sc}}(\mathcal{A})$ is a topological radical as well [13, Theorems 8.10, 8.11]. The ideal $\mathcal{R}_{\mathrm{sc}}(\mathcal{A})$ is called the *scattered radical* of $\mathcal{A}$.

COROLLARY 2.6. $\mathcal{R}_{\mathrm{hf}}(\operatorname{Alg}\mathcal{N}) = \mathcal{R}_{\mathrm{hc}}(\operatorname{Alg}\mathcal{N}) = \mathcal{R}_{\mathrm{sc}}(\operatorname{Alg}\mathcal{N})$.

Proof. For $P \in \mathcal{N}$ denote by $\mathcal{J}_P$ the ideal

$$\mathcal{J}_P = \{a \in \operatorname{Alg}\mathcal{N} : a = PaP^{\perp}\}$$

of $\operatorname{Alg}\mathcal{N}$. Since $\mathcal{J}_P$ has trivial multiplication, it is a hypofinite ideal and is contained in the hypofinite radical of $\operatorname{Alg}\mathcal{N}$. It follows from [7, Theorem 5.4] that the Jacobson radical of $\operatorname{Alg}\mathcal{N}$ is the closure of the linear span of the set $\bigcup_{P\in\mathcal{N}} \mathcal{J}_P$, and hence it is contained in the hypofinite radical of $\operatorname{Alg}\mathcal{N}$. The corollary now follows from [13, Theorem 8.15] and Theorem 2.4. ∎

COROLLARY 2.7. *The hypocompact radical of* $\operatorname{Alg}\mathcal{N}/\mathcal{K}(\mathcal{N})$ *coincides with its scattered radical, which in turn is equal to* $(\operatorname{Rad}(\mathcal{N}) + \mathcal{K}(\mathcal{N}))/\mathcal{K}(\mathcal{N})$.

Proof. It follows from [12, Corollary 3.9] and Theorem 2.4 that $(\operatorname{Rad}(\mathcal{N}) + \mathcal{K}(\mathcal{N}))/\mathcal{K}(\mathcal{N})$ is the hypocompact radical of $\operatorname{Alg}\mathcal{N}/\mathcal{K}(\mathcal{N})$.

It follows from [13, Corollary 8.13], Theorem 2.4 and Corollary 2.6 that the scattered radical of $\operatorname{Alg}\mathcal{N}/\mathcal{K}(\mathcal{N})$ is $(\operatorname{Rad}(\mathcal{N}) + \mathcal{K}(\mathcal{N}))/\mathcal{K}(\mathcal{N})$. ∎

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G. Andreolas, M. Anoussis
Department of Mathematics
University of the Aegean
832 00 Karlovassi, Samos, Greece
E-mail: gandreolas@aegean.gr
mano@aegean.gr