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The trace of the curvature determines similarity

by

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Abstract. We prove that the quantity that appears in a recent similarity characterization for Cowen–Douglas operators is the trace of the curvature of the eigenvector bundle. This gives the first geometric interpretation of the similarity of operators.

1. Introduction. The main objects of this paper are Cowen–Douglas operators—bounded linear operators with a Hermitian holomorphic vector bundle constructed over the set of their eigenvalues (which is an open set). M. J. Cowen and R. G. Douglas [CD] completed the unitary classification of these operators by showing that the curvature and the covariant derivatives of these bundles are the unitary invariants. Their result demonstrates a close relationship between the similarity problem of operator theory and the bundle equivalence problem of geometry. Unlike its unitary equivalence counterpart, it is much more difficult to obtain a similarity classification of Cowen–Douglas operators involving a geometric concept such as curvature. Recent work in this direction has been restricted to the backward shift operator on various spaces, perhaps the best known example of a Cowen–Douglas operator.

In [DKS] and [KS], by considering a function whose values are projections onto the fiber of the eigenvector bundle, the authors gave a necessary and a sufficient condition for a Cowen–Douglas operator to be similar to the backward shift in the Hardy and weighted Bergman spaces. The Hilbert–Schmidt norm of the partial derivative of this function appears in the characterization. Although this Hilbert–Schmidt norm seemed to be related to the curvature of the bundle, a precise identification has been absent. In [S], it is proven that for a line bundle, i.e., a bundle of rank one, the norm equals the curvature of the bundle.

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In this paper, we generalize this result to a bundle of arbitrary rank and show that the trace of the curvature for the bundle is the correct description of this Hilbert–Schmidt norm. Our result gives the first characterization of similarity in terms of geometry, which has been an open problem since the introduction of Cowen–Douglas operators.

2. Preliminaries

2.1. Notation. As usual, we let $\mathcal{L}(\mathcal{E}_1, \mathcal{E}_2)$ be the algebra of bounded linear operators from a Hilbert space $\mathcal{E}_1$ to another Hilbert space $\mathcal{E}_2$. If $\mathcal{E}_1 = \mathcal{E}_2 = \mathcal{E}$, then we write $\mathcal{L}(\mathcal{E})$. $\ker T$ and $\operatorname{ran} T$ denote the kernel and the range of an operator $T \in \mathcal{L}(\mathcal{E}_1, \mathcal{E}_2)$, respectively. Moreover, $L^\infty_{\mathcal{E}_1 \to \mathcal{E}_2}$ is the class of bounded functions on the unit circle $\mathbb{T}$ whose values are operators from $\mathcal{E}_1$ to $\mathcal{E}_2$, and $H^\infty_{\mathcal{E}_1 \to \mathcal{E}_2} \subset L^\infty_{\mathcal{E}_1 \to \mathcal{E}_2}$ is the corresponding Hardy class of bounded, analytic functions.

2.2. Analytic function spaces. For each positive integer n, we define the Hilbert space $\mathcal{M}_n$ of analytic functions on the unit disk $\mathbb{D}$ as

$$\mathcal{M}_n := \left\{ f = \sum_{k=0}^{\infty} \hat{f}(k) z^k : \sum_{k=0}^{\infty} |\hat{f}(k)|^2 \frac{1}{\binom{n+k-1}{k}} < \infty \right\}.$$

Note that $\mathcal{M}_1$ and $\mathcal{M}_2$ are the well-known Hardy and Bergman spaces, respectively, and for all other n, we obtain weighted Bergman spaces. Of particular interest is the fact that $\mathcal{M}_n$ is a reproducing kernel Hilbert space with reproducing kernel

$$k_{n,\lambda} = (1 - \bar{\lambda} z)^{-n},$$

so that for all $f \in \mathcal{M}_n$ and $\lambda \in \mathbb{D}$, we have

$$\langle f, k_{n,\lambda} \rangle = f(\lambda).$$

For a Hilbert space $\mathcal{E}$, $\mathcal{M}_{n,\mathcal{E}}$ will denote the space $\mathcal{M}_n \otimes \mathcal{E}$.

2.3. Basic complex geometry. The following is the definition of the Cowen–Douglas class:

Definition 2.1 ([CD]). Let $\mathcal{H}$ be a separable Hilbert space. If Ω is an open connected set of the complex plane $\mathbb{C}$ and m is a positive integer, then the *Cowen–Douglas class* $B_m(\Omega)$ consists of operators $T \in \mathcal{L}(\mathcal{H})$ satisfying

- $\Omega \subset \sigma(T) = \{\lambda \in \mathbb{C} : T - \lambda \text{ is not invertible}\}$;
- $\operatorname{ran}(T - \lambda)$ is closed for every $\lambda \in \Omega$;
- $\bigvee_{\lambda \in \Omega} \ker(T - \lambda) = \mathcal{H}$; and
- $\dim \ker(T - \lambda) = m$ for every $\lambda \in \Omega$.

Here, $\bigvee$ stands for the closed linear span.

It is shown in [CD] that each operator $T \in B_m(\Omega)$ induces a Hermitian holomorphic eigenvector bundle, i.e., the complex bundle

$$\mathcal{E}_T := \{(\lambda, x) \in \Omega \times \mathcal{H} : x \in \ker(T - \lambda)\}$$

over Ω. Since $\dim\ker(T-\lambda) = m$ for every $\lambda \in \Omega$, $\mathcal{E}_T$ is of rank m, so we let $\{e_i(\lambda)\}_{i=1}^m$ be its holomorphic frame and form the matrix of inner products

$$h(\lambda) := (\langle e_j(\lambda), e_i(\lambda)\rangle)_{m\times m}$$

for each $\lambda \in \Omega$. The *curvature function* $\mathcal{K}_T$ of $\mathcal{E}_T$ is defined as

$$\mathcal{K}_T = -\overline{\partial}(h^{-1}\partial h).$$

In particular, for $T \in B_1(\Omega)$, this is equivalent to

$$\mathcal{K}_T(\lambda) = -\partial\bar{\partial}\log\|\gamma(\lambda)\|^2 \quad \text{for all } \lambda \in \Omega, \tag{2.1}$$

where $\gamma(\lambda) \in \ker(T - \lambda)$ is a cross section of $\mathcal{E}_T$ [CD].

2.4. Similarity to the backward shift operator. The forward shift operator $S_{n,\mathcal{E}}$ is defined as

$$S_{n,\mathcal{E}}f(z) = zf(z) \quad \text{for } f \in \mathcal{M}_{n,\mathcal{E}},$$

and the backward shift operator $S^*_{n,\mathcal{E}}$ is the adjoint of the forward shift,

$$\langle S_{n,\mathcal{E}}f, g\rangle = \langle f, S^*_{n,\mathcal{E}}g\rangle \quad \text{for } f, g \in \mathcal{M}_{n,\mathcal{E}}.$$

It is easy to see that the set of eigenvalues for $S^*_{n,\mathcal{E}}$ is the entire unit disk $\mathbb{D}$.

The following result characterizes Cowen–Douglas operators that are similar to a backward shift operator on one of the spaces $\mathcal{M}_{n,\mathcal{E}}$. Recall that an *n-hypercontraction* is an operator T with

$$\sum_{j=0}^{k}(-1)^j\binom{k}{j}(T^*)^jT^j \geq 0$$

for all $1 \leq k \leq n$, a generalization of the concept of a contraction [A1], [A2]. Note that $S^*_{n,\mathcal{E}}$ is an n-hypercontraction in $B_{\dim\mathcal{E}}(\mathbb{D})$.

Let $T \in \mathcal{L}(\mathcal{H})$ belong to the Cowen–Douglas class $B_m(\Omega)$. Denote by $\Pi : \Omega \to \mathcal{L}(\mathcal{H})$ the projection-valued holomorphic function such that for each $\lambda \in \Omega$, $\Pi(\lambda)$ is the orthogonal projection onto $\ker(T - \lambda)$.

THEOREM 2.2 ([DKS]). *Let $T \in B_m(\mathbb{D})$ be an n-hypercontraction. Then T is similar to the backward shift operator $S^*_{n,\mathbb{C}^m}$ on $\mathcal{M}_{n,\mathbb{C}^m}$ if and only if there exists a bounded subharmonic function ψ defined on $\mathbb{D}$ such that*

$$\partial\bar{\partial}\psi(\lambda) \geq \|\partial\Pi(\lambda)\|^2_{\mathfrak{S}_2} - \frac{mn}{(1-|\lambda|^2)^2} \quad \textit{for all } \lambda \in \mathbb{D}.$$

Here, $\mathfrak{S}_2$ stands for the Hilbert–Schmidt class of operators.

The following theorem constitutes the main result of the paper, which implies that similarity can be described in terms of the difference of the traces of the curvatures of the eigenvector bundles of the operators involved.

Main Theorem 2.3. *Let $T \in B_m(\Omega)$ for some open, connected subset Ω of $\mathbb{C}$. Then*

$$\|\partial \Pi(\lambda)\|_{\mathfrak{S}_2}^2 = -\operatorname{trace} \mathcal{K}_T(\lambda) \quad \text{for all } \lambda \in \Omega.$$

Corollary 2.4. *If $\Pi(\lambda)$ denotes the orthogonal projection onto* $\ker(S^*_{n,\mathbb{C}^m} - \lambda)$ *for $\lambda \in \mathbb{D}$, then*

$$\|\partial \Pi(\lambda)\|_{\mathfrak{S}_2}^2 = \frac{mn}{(1-|\lambda|^2)^2}.$$

Proof. First note that $S_n^* \in \mathcal{M}_n$ is in the Cowen–Douglas class $B_1(\mathbb{D})$. Using formula (2.1) where one sets $\gamma(\lambda) = (1-\bar{\lambda}z)^{-n}$, the preferred cross section of $\mathcal{E}_T$ that is the reproducing kernel for $\mathcal{M}_n$, one easily obtains

$$\mathcal{K}_{S_n^*}(\lambda) = -\frac{n}{(1-|\lambda|^2)^2} \quad \text{for } \lambda \in \mathbb{D},$$

so that

$$\mathcal{K}_{S^*_{n,\mathbb{C}^m}}(\lambda) = -\frac{n}{(1-|\lambda|^2)^2} I_{m\times m}. \ \blacksquare$$

As another corollary to Theorem 2.3, we obtain the following result of J. Sarkar [S]:

Corollary 2.5. *Let $T \in B_1(\mathbb{D})$ and denote by $\Pi(\lambda)$ the orthogonal projection onto* $\ker(T-\lambda)$. *Then*

$$\|\partial \Pi(\lambda)\|_{\mathfrak{S}_2}^2 = -\mathcal{K}_T(\lambda) \quad \text{for } \lambda \in \mathbb{D}.$$

Remark 2.6. G. Misra [M] proved that $\mathcal{K}_T(\lambda) \le 0$ for all $T \in B_1(\mathbb{D})$ and $\lambda \in \mathbb{D}$.

3. Proof of Theorem 2.3. We first present some notation and a lemma. Let $\mathcal{H}$ be a Hilbert space and let

$$(\alpha_1,\dots,\alpha_m),(\beta_1,\dots,\beta_m) \in \bigoplus^m \mathcal{H}.$$

We let

$$(\alpha_1,\dots,\alpha_m)^T \cdot (\beta_1,\dots,\beta_m) := (\langle \beta_t,\alpha_s\rangle)_{1\le s,t\le m},$$

$$(\alpha_1,\dots,\alpha_m) \cdot (\beta_1,\dots,\beta_m)^T := \sum_{s=1}^m \langle \alpha_s,\beta_s\rangle.$$

LEMMA 3.1. *Let $\alpha_s, \beta_t \in \mathcal{H}$ and $x_{s,t}, y_{s,t} \in \mathbb{C}$ for $1 \le s,t \le m$. Denote by * the conjugate of the transpose operation ^{T}. Then*

$$(1)\quad \langle(\alpha_1,\dots,\alpha_m)(x_{s,t})_{1\le s,t\le m}, (\beta_1,\dots,\beta_m)(y_{s,t})_{1\le s,t\le m}\rangle = [(\alpha_1,\dots,\alpha_m)(x_{s,t})_{1\le s,t\le m}(y_{s,t})^*_{1\le s,t\le m}]\cdot(\beta_1,\dots,\beta_m)^T,$$

$$(2)\quad [(\alpha_1,\dots,\alpha_m)(x_{s,t})_{1\le s,t\le m}]\cdot(\beta_1,\dots,\beta_m)^T = \operatorname{trace}[(x_{s,t})_{1\le s,t\le m}((\alpha_1,\dots,\alpha_m)^T\cdot(\beta_1,\dots,\beta_m))^*].$$

Proof. For (1), we first note that

$$(x_{s,t})_{1\le s,t\le m}(y_{s,t})^*_{1\le s,t\le m} = \sum_{s=1}^m (x_{1,s},\dots,x_{m,s})^T(\overline{y_{1,s}},\dots,\overline{y_{m,s}}).$$

We then obtain

$$\begin{aligned}
&\langle(\alpha_1,\dots,\alpha_m)(x_{s,t})_{1\le s,t\le m}, (\beta_1,\dots,\beta_m)(y_{s,t})_{1\le s,t\le m}\rangle\\
&\quad= \sum_{s=1}^m \langle(\alpha_1,\dots,\alpha_m)(x_{1,s},\dots,x_{m,s})^T, (\beta_1,\dots,\beta_m)(y_{1,s},\dots,y_{m,s})^T\rangle\\
&\quad= \sum_{s=1}^m \langle(\alpha_1,\dots,\alpha_m)(x_{1,s},\dots,x_{m,s})^T(\overline{y_{1,s}},\dots,\overline{y_{m,s}}), (\beta_1,\dots,\beta_m)\rangle\\
&\quad= [(\alpha_1,\dots,\alpha_m)(x_{s,t})_{1\le s,t\le m}(y_{s,t})^*_{1\le s,t\le m}]\cdot(\beta_1,\dots,\beta_m)^T.
\end{aligned}$$

For (2), we have

$$\begin{aligned}
&[(\alpha_1,\dots,\alpha_m)(x_{s,t})_{1\le s,t\le m}]\cdot(\beta_1,\dots,\beta_m)^T\\
&\quad= \Big(\sum_{s=1}^m x_{s,1}\alpha_s,\dots,\sum_{s=1}^m x_{s,m}\alpha_s\Big)\cdot(\beta_1,\dots,\beta_m)^T\\
&\quad= \sum_{s=1}^m\sum_{t=1}^m x_{s,t}\langle\alpha_s,\beta_t\rangle,
\end{aligned}$$

and

$$\begin{aligned}
&(x_{s,t})_{1\le s,t\le m}((\alpha_1,\dots,\alpha_m)^T\cdot(\beta_1,\cdots,\beta_m))^*\\
&\quad= (x_{s,t})_{1\le s,t\le m}(\langle\beta_t,\alpha_s\rangle)^*_{1\le s,t\le m}\\
&\quad= \begin{pmatrix} \sum_{t=1}^m x_{1,t}\langle\alpha_1,\beta_t\rangle & * & * \\ * & \ddots & * \\ * & * & \sum_{t=1}^m x_{m,t}\langle\alpha_m,\beta_t\rangle \end{pmatrix}. \blacksquare
\end{aligned}$$

Let $\{e_j\}_{j=1}^\infty$ be an orthonormal basis for $\mathcal{H}$ and suppose that $\ker(T-\lambda)$ is spanned by a holomorphic frame $\alpha_l(\lambda), 1\le l\le m$. Since $\Pi(\lambda)$ is the orthogonal projection onto $\ker(T-\lambda)$, one can find, for each $j\ge 1$, $x_l^j\in\mathbb{C}$

such that

$$\Pi(\lambda)e_j := \sum_{l=1}^{m} x_l^j \alpha_l(\lambda).$$

Moreover, since $\Pi(\lambda)\alpha_l(\lambda) = \alpha_l(\lambda)$, we have

$$\langle \Pi(\lambda)e_j, \alpha_l(\lambda)\rangle = \langle e_j, \alpha_l(\lambda)\rangle$$

for all $j \geq 1$ and $1 \leq l \leq m$. Hence,

$$\begin{pmatrix} \langle e_j, \alpha_1\rangle \\ \vdots \\ \langle e_j, \alpha_m\rangle \end{pmatrix} = h \begin{pmatrix} x_1^j \\ \vdots \\ x_m^j \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} x_1^j \\ \vdots \\ x_m^j \end{pmatrix} = h^{-1} \begin{pmatrix} \langle e_j, \alpha_1\rangle \\ \vdots \\ \langle e_j, \alpha_m\rangle \end{pmatrix},$$

where $h = (\langle \alpha_t, \alpha_s\rangle)_{1\leq s,t\leq m}$.

Therefore, letting

$$A(\lambda) := (\alpha_1(\lambda), \ldots, \alpha_m(\lambda)), \qquad Y_j(\lambda) := \begin{pmatrix} \langle e_j, \alpha_1(\lambda)\rangle \\ \vdots \\ \langle e_j, \alpha_m(\lambda)\rangle \end{pmatrix},$$

one has

$$\Pi(\lambda)e_j = A(\lambda)h^{-1}(\lambda)Y_j(\lambda),$$

and

$$(A^T \cdot A)^* = h, \quad ((\partial A)^T \cdot \partial A)^* = \overline{\partial}\partial h, \quad ((\partial A)^T \cdot A)^* = \partial h, \quad (A^T \cdot \partial A)^* = \overline{\partial} h.$$

Next, note that for any $1 \leq s, t \leq m$,

$$\sum_{j=1}^{\infty} \langle e_j, \alpha_s\rangle\langle \alpha_t, e_j\rangle = \Big\langle \sum_{j=1}^{\infty} \langle \alpha_t, e_j\rangle e_j, \sum_{j=1}^{\infty} \langle \alpha_s, e_j\rangle e_j \Big\rangle = \langle \alpha_t, \alpha_s\rangle,$$

so that $\sum_{j=1}^{\infty} Y_j Y_j^* = h$. By Lemma 3.1, it then follows that

$$\begin{aligned}
\|\partial \Pi\|_{\mathfrak{S}_2}^2 &= \sum_{j=1}^{\infty} \|\partial \Pi e_j\|^2 = \sum_{j=1}^{\infty} \langle \partial(Ah^{-1})Y_j, \partial(Ah^{-1})Y_j\rangle \\
&= \sum_{j=1}^{\infty} [\langle \partial A h^{-1} Y_j, \partial A h^{-1} Y_j\rangle + \langle \partial A h^{-1} Y_j, A\partial h^{-1} Y_j\rangle \\
&\quad + \langle A\partial h^{-1} Y_j, \partial A h^{-1} Y_j\rangle + \langle A\partial h^{-1} Y_j, A\partial h^{-1} Y_j\rangle] \\
&= \sum_{j=1}^{\infty} [\partial A h^{-1} Y_j Y_j^* (h^{-1})^* \cdot (\partial A)^T] + \sum_{j=1}^{\infty} [\partial A h^{-1} Y_j Y_j^* (\partial h^{-1})^* \cdot A^T] \\
&\quad + \sum_{j=1}^{\infty} [A\partial h^{-1} Y_j Y_j^* (h^{-1})^* \cdot (\partial A)^T] + \sum_{j=1}^{\infty} [A\partial h^{-1} Y_j Y_j^* (\partial h^{-1})^* \cdot A^T]
\end{aligned}$$

$$\begin{aligned}&= \partial A h^{-1} h (h^{-1})^* \cdot (\partial A)^T + \partial A h^{-1} h (\partial h^{-1})^* \cdot A^T \\ &\quad + A \partial h^{-1} h (h^{-1})^* \cdot (\partial A)^T + A \partial h^{-1} h (\partial h^{-1})^* \cdot A^T.\end{aligned}$$

Now using $\overline{\partial} \alpha_j = 0$ for $1 \leq j \leq m$ and Lemma 3.1 again, we obtain the following four expressions:

$$\begin{aligned}\partial A (h^{-1})^* \cdot (\partial A)^T &= \operatorname{trace}[(h^{-1})^*((\partial A)^T \cdot \partial A)^*] = \operatorname{trace}(h^{-1}\overline{\partial}\partial h),\\ \partial A (\partial h^{-1})^* \cdot A^T &= \operatorname{trace}[(\partial h^{-1})^*((\partial A)^T \cdot A)^*] = \operatorname{trace}[(\partial h^{-1})^* \partial h],\\ A \partial h^{-1} h (h^{-1})^* \cdot (\partial A)^T &= \operatorname{trace}[\partial h^{-1} h (h^{-1})^* (A^T \cdot \partial A)^*],\\ &= \operatorname{trace}[\partial h^{-1} h (h^{-1})^* \overline{\partial} h],\\ A \partial h^{-1} h (\partial h^{-1})^* \cdot A^T &= \operatorname{trace}[\partial h^{-1} h (\partial h^{-1})^* (A^T \cdot A)^*]\\ &= \operatorname{trace}[\partial h^{-1} h (\partial h^{-1})^* h].\end{aligned}$$

Lastly, note that

$$\begin{aligned}\operatorname{trace}[(\partial h^{-1})^* \partial h] &= \operatorname{trace}(-h^{-1}\overline{\partial} h h^{-1} \partial h),\\ \operatorname{trace}[\partial h^{-1} h (h^{-1})^* \overline{\partial} h] &= \operatorname{trace}(\partial h^{-1} \overline{\partial} h) = \operatorname{trace}(-h^{-1} \partial h h^{-1} \overline{\partial} h)\\ &= \operatorname{trace}(-h^{-1}\overline{\partial} h h^{-1} \partial h),\\ \operatorname{trace}[\partial h^{-1} h (\partial h^{-1})^* h] &= \operatorname{trace}(h^{-1} \partial h h^{-1} \overline{\partial} h) = \operatorname{trace}(h^{-1} \overline{\partial} h h^{-1} \partial h).\end{aligned}$$

Hence,

$$\begin{aligned}\|\partial \Pi\|_{\mathfrak{S}_2}^2 &= \operatorname{trace}(h^{-1}\overline{\partial}\partial h - h^{-1}\overline{\partial} h h^{-1} \partial h - h^{-1}\overline{\partial} h h^{-1} \partial h + h^{-1}\overline{\partial} h h^{-1} \partial h)\\ &= -\operatorname{trace} \mathcal{K}_T.\end{aligned}$$

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