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Elementary methods for incidence problems in finite fields

by

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1. Introduction. Throughout, $\mathbb{F}_q$ will denote a finite field with characteristic strictly greater than 2. We establish that for a point set $P \subset \mathbb{F}_q^d$ and a family $\mathcal{S}$ of *spheres* in $\mathbb{F}_q^d$, the number of incidences between the points and the spheres, which is denoted by $I(P,\mathcal{S}) := |\{(p,S) \in P \times \mathcal{S} : p \in S\}|$, satisfies the bound

$$\frac{|P|\,|\mathcal{S}|}{q} - |P|^{1/2}|\mathcal{S}|^{1/2}q^{d/2} < I(P,\mathcal{S}) < \frac{|P|\,|\mathcal{S}|}{q} + |P|^{1/2}|\mathcal{S}|^{1/2}q^{d/2}.$$

Many results on incidence problems in finite fields have appeared in recent years; see for example [3, 8, 11, 14]. For relatively large sets of points and surfaces in $\mathbb{F}_q^d$, Fourier analysis and spectral graph theory have been the main tools to deal with these problems. For example, Vinh [14] used the spectral method to prove that, for the sets P and $\mathcal{L}$ of points and lines respectively in $\mathbb{F}_q^2$,

$$\frac{|P|\,|\mathcal{L}|}{q} - (|P|\,|\mathcal{L}|q)^{1/2} \le I(P,\mathcal{L}) \le \frac{|P|\,|\mathcal{L}|}{q} + (|P|\,|\mathcal{L}|q)^{1/2}. \tag{1.1}$$

The result was extended to incidences between points and hyperplanes in $\mathbb{F}_q^d$, and can also be proven using discrete Fourier analysis.

In [5], the first author found an elementary method to prove some results on combinatorial problems in finite fields, including an alternative proof of (1.1). Here we follow that elementary approach to give an estimate on incidences between points and spheres in $\mathbb{F}_q^d$.

Before saying any more about spheres in finite fields, it is necessary to define what is meant by such an object. We follow the standard definition

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of the distance: given a pair of points $x = (x_1, \dots, x_d)$ and $y = (y_1, \dots, y_d)$ in $\mathbb{F}_q^d$, the *distance* between x and y is given by

$$\|x - y\| := (x_1 - y_1)^2 + \cdots + (x_d - y_d)^2.$$

As one might expect, a *sphere* in $\mathbb{F}_q^d$ (a *circle* for $d = 2$) is a set of points which are the same distance λ from a given central point $(\alpha_1, \dots, \alpha_d)$. That is, a sphere is the set of points $x = (x_1, \dots, x_d)$ which satisfy an equation of the form

$$(x_1 - \alpha_1)^2 + \cdots + (x_d - \alpha_d)^2 = \lambda. \tag{1.2}$$

The sphere defined by (1.2) is said to have radius λ and centre $(\alpha_1, \dots, \alpha_d)$. These spheres have many natural properties which are analogous to spheres in $\mathbb{R}^d$. For example, it is easy to check that any two circles in $\mathbb{F}_q^2$ with radii not equal to zero intersect in at most two points.

We illustrate our incidence theorem with two applications. The first is a pinned distance result. We prove that, for any point set $P \subset \mathbb{F}_q^d$ such that $|P| \geq 3q^{(d+1)/2}$, the bound

$$|\{\|x - y\| : y \in P\}| \geq q/2$$

holds for at least half of the elements $x \in P$ (see Corollary 3.2 for a more general statement). This result was established previously in [4], with a different proof, and is known to be tight (up to constant factors) when $d \geq 3$ is odd (see Hart et al. [7]). The result was recently improved by Hanson et al. [6] in the case $d = 2$, with the condition relaxed to $|P| = \Omega(q^{4/3})$.

The second application of our incidence theorem gives a version of Beck's Theorem for points and circles in $\mathbb{F}_q^2$ which is tight up to multiplicative constants. Given three non-collinear points in $\mathbb{F}_q^2$ there is a unique circle which contains them. Therefore, we say that a circle is *determined by* P if it contains three or more points from P. In Theorem 3.5 below, it is established that any set $P \subset \mathbb{F}_q^2$ such that $|P| \geq 5q$ determines at least $4q^3/9$ circles.

On the other hand, if one takes a set P of q points lying on a single non-isotropic line in the plane, then P does not determine any circles. Similarly, if P consists of, say, $q + 1$ points on the same circle with non-zero radius, then P determines only one circle. These degenerate examples are in a sense 1-dimensional, and illustrate the tightness of Theorem 3.5.

1.1. Work of Phoung, Thang and Vinh. Shortly after an earlier draft of this paper was made available online, we became aware of its overlap with a concurrent work of Phoung, Thang and Vinh [13]. Those authors give an independent proof of Theorem 2.3 via graph-theoretic methods similar to those used in [14]. Further applications of the incidence result are also given in [13].

2. Incidences between spheres and points. Given finite sets A, B in a finite group $(G,+)$ we use the notation

$$r_{A+B}(x) = |\{(a,b) \in A \times B : a+b = x\}|.$$

We recall the following elementary and well known identities:

$$\sum_{x\in G} r_{A+B}(x) = |A|\,|B|, \tag{2.1}$$

$$\sum_{x\in G} r^2_{A+B}(x) = \sum_{x\in G} r_{A-A}(x) r_{B-B}(x). \tag{2.2}$$

The quantity in (2.2) is called the *additive energy* of A and B.

Lemma 2.1. *Define*

$$A := \{(a_1,\dots,a_d, a_1^2+\dots+a_d^2) : a_1,\dots,a_d \in \mathbb{F}_q\} \subset \mathbb{F}_q^{d+1}. \tag{2.3}$$

Then, for all $x = (x_1,\dots,x_{d+1}) \neq (0,\dots,0)$,

$$r_{A-A}(x) \le q^{d-1}. \tag{2.4}$$

Proof. This can be calculated directly. Indeed, the quantity $r_{A-A}(x)$ is the number of solutions

$$(a_1,\dots,a_d,b_1,\dots,b_d) \in \mathbb{F}_q^{2d}$$

to the system of equations

$$\begin{aligned} a_1 - b_1 &= x_1,\\ a_2 - b_2 &= x_2,\\ &\cdots\\ a_d - b_d &= x_d,\\ a_1^2+\dots+a_d^2 - b_1^2 - \dots - b_d^2 &= x_{d+1}. \end{aligned}$$

The b_i variables can be eliminated, and this system of equations reduces to

$$2a_1x_1 + \dots + 2a_dx_d - x_1^2 - \dots - x_d^2 = x_{d+1}. \tag{2.5}$$

If $x \neq 0$ then there is some $1 \le i \le d$ such that $x_i \neq 0$. Otherwise $x_{d+1} = 0$ and $x = 0$ is the only choice which admits solutions to (2.5). Without loss of generality, we may take $i = 1$. If we fix $a_2,\dots,a_d$, then since the characteristic of the field is not equal to 2, we have $2x_1 \neq 0$ and the value of a_1 is uniquely determined. This gives $r_{A-A}(x) \le q^{d-1}$.

If $x = 0$, then trivially $r_{A-A}(0) = |A| = q^d$. ∎

Lemma 2.2. *Let A be as defined in* (2.3), *and let $B, C \subset \mathbb{F}_q^{d+1}$. Then*

$$|\{(b,c) \in B \times C : b - c \in A\}| = \frac{|B|\,|C|}{q} + \theta |B|^{1/2}|C|^{1/2}q^{d/2}$$

for some $\theta \in \mathbb{R}$ such that $|\theta| < 1$.

Proof. Note that

$$|\{(b,c) \in B \times C : b - c \in A\}| - \frac{|B|\,|C|}{q}$$
$$= \sum_{b\in B}\left(|\{c \in C : b - c \in A\}| - \frac{|C|}{q}\right) = \sum_{b \in B}\left(r_{A+C}(b) - \frac{|C|}{q}\right) := E.$$

By the Cauchy–Schwarz inequality

$$|E|^2 \le |B| \sum_{b\in B}\left(r_{A+C}(b) - \frac{|C|}{q}\right)^2 \le |B| \sum_{x\in\mathbb{F}_q^{d+1}}\left(r_{A+C}(x) - \frac{|C|}{q}\right)^2.$$

Using (2.1), (2.2) and the fact that $|A| = q^d$ we have

$$\begin{aligned}\sum_{x\in\mathbb{F}_q^{d+1}}\left(r_{A+C}(x) - \frac{|C|}{q}\right)^2 &= \sum_{x\in\mathbb{F}_q^{d+1}} r_{A+C}^2(x) - q^{d-1}|C|^2\\ &= \sum_{x\in\mathbb{F}_q^{d+1}} r_{A-A}(x) r_{C-C}(x) - q^{d-1}|C|^2\\ &\le |A|\,|C| + q^{d-1}\sum_{x\ne 0} r_{C-C}(x) - q^{d-1}|C|^2\\ &= |A|\,|C| + q^{d-1}(|C|^2 - |C|) - q^{d-1}|C|^2\\ &= |C|q^{d-1}(q-1).\end{aligned}$$

Thus, $|E| < (|B|\,|C|)^{1/2}q^{d/2}$, which completes the proof. ∎

Theorem 2.3. *Let $P \subset \mathbb{F}_q^d$ and let $\mathcal{S}$ be a family of spheres in $\mathbb{F}_q^d$. Then*

$$\frac{|P|\,|\mathcal{S}|}{q} - |P|^{1/2}|\mathcal{S}|^{1/2}q^{d/2} < I(P,\mathcal{S}) < \frac{|P|\,|\mathcal{S}|}{q} + |P|^{1/2}|\mathcal{S}|^{1/2}q^{d/2}.$$

Proof. We denote by $S_{\alpha_1,\dots,\alpha_d,\lambda}$ the sphere

$$\{(x_1,\dots,x_d) : (x_1-\alpha_1)^2 + \dots + (x_d - \alpha_d)^2 = \lambda\}.$$

Define

$$B = \{(p_1,\dots,p_d,0) : (p_1,\dots,p_d) \in P\},$$
$$C = \{(\alpha_1,\dots,\alpha_d,-\lambda) : S_{\alpha_1,\dots,\alpha_d,\lambda} \in \mathcal{S}\}.$$

Note that $|B| = |P|$ and $|C| = |\mathcal{S}|$.

Now,

$$\begin{aligned}
&|\{(b,c)\in B\times C : b-c\in A\}| \\
&\quad = |\{((p_1,\dots,p_d,0),(\alpha_1,\dots,\alpha_d,-\lambda))\in B\times C : \\
&\qquad\qquad (p_1-\alpha_1,\dots,p_d-\alpha_d,\lambda)\in A\}| \\
&\quad = |\{((p_1,\dots,p_d,0),(\alpha_1,\dots,\alpha_d,-\lambda))\in B\times C : \\
&\qquad\qquad (p_1-\alpha_1)^2+\dots+(p_d-\alpha_d)^2=\lambda)\}| \\
&\quad = I(P,\mathcal{S}).
\end{aligned}$$

An application of Lemma 2.2 completes the proof. ∎

3. Applications of the incidence bound

3.1. Pinned distances. Let P be a set of points in $\mathbb{F}_q^d$, and $y\in\mathbb{F}_q^d$. Following the notation of Chapman et al. [4], let $\triangle_y(P)$ denote the set of distances between the point y and the set P, that is,

$$\triangle_y(P) := \{\|x-y\| : x\in P\}.$$

It was established in [4, Theorem 2.3] that a sufficiently large set of points determines many pinned distances, for many different pins. Here, we use Theorem 2.3 to give an alternative proof.

COROLLARY 3.1. *Let $P\subset\mathbb{F}_q^d$ be such that $|P|\ge \epsilon^{-1}(1-\epsilon)^{1/2}q^{(d+1)/2}$ for some $0<\epsilon<1$. Then*

$$\frac{1}{|P|}\sum_{p\in P}|\triangle_p(P)| > (1-\epsilon)q. \tag{3.1}$$

Proof. Fix a point $p\in P$, and construct a family $\mathcal{S}_p$ of spheres by minimally covering P by concentric spheres around p. Note that $|\mathcal{S}_p| = |\triangle_p(P)|$, and that $I(P,\mathcal{S}_p)=|P|$. Repeating this process for each point in P, we generate a family $\mathcal{S}$ of spheres defined by the disjoint union

$$\mathcal{S} := \bigcup_{p\in P}\mathcal{S}_p.$$

Observe that $I(P,\mathcal{S}) = \sum_{p\in P} I(P,\mathcal{S}_p) = |P|^2$. On the other hand, Theorem 2.3 implies that

$$\begin{aligned}
|P|^2 = I(P,\mathcal{S}) &< \frac{|P|\,|\mathcal{S}|}{q} + |P|^{1/2}|\mathcal{S}|^{1/2}q^{d/2} \\
&= \frac{|P|\sum_{p\in P}|\triangle_p(P)|}{q} + |P|^{1/2}\Big(\sum_{p\in P}|\triangle_p(P)|\Big)^{1/2}q^{d/2}.
\end{aligned}$$

If $\frac{1}{|P|}\sum_{p\in P}|\triangle_p(P)| \le (1-\epsilon)q$ then the last inequality would imply that $|P| < \epsilon^{-1}(1-\epsilon)^{1/2}q^{(d+1)/2}$. ∎

Corollary 3.2. *Let $P \subset \mathbb{F}_q^d$ be such that $|P| \geq \alpha^{-2}(1-\alpha^2)^{1/2}q^{(d+1)/2}$ for some $0 < \alpha < 1$. Then*

$$|\triangle_p(P)| > (1-\alpha)q$$

for more than $(1-\alpha)|P|$ points $p \in P$.

Proof. Corollary 3.1 implies that

$$\sum_{p\in P} |\triangle_p(P)| > (1-\alpha^2)q|P|.$$

On the other hand let

$$P' = \{p \in P : |\triangle_p(P)| > (1-\alpha)q\}$$

and suppose that $|P'| \leq (1-\alpha)|P|$. Then we would have

$$\begin{aligned}\sum_{p\in P} |\triangle_p(P)| &= \sum_{p\in P\setminus P'} |\triangle_p(P)| + \sum_{p\in P'} |\triangle_p(P)| \\ &\leq (1-\alpha)q(|P|-|P'|) + q|P'| = (1-\alpha)q|P| + \alpha q|P'| \\ &\leq (1-\alpha^2)q|P|. \ \blacksquare\end{aligned}$$

3.2. A version of Beck's Theorem for circles. A result which is closely related to the Szemerédi–Trotter Theorem and incidence geometry is Beck's Theorem [2]. This result states that a set of N points in $\mathbb{R}^2$ determines $\Omega(N^2)$ distinct lines by connecting pairs of points, provided that the set of points does not contain a single line which supports cN points, where c is a small but fixed constant. We say that P *determines a line* l if there exist two points p_1 and p_2 belonging to P which both lie on l. In finite fields, an analogue of Beck's Theorem was proven by Alon [1], in the following form:

Theorem 3.3. *Let $\epsilon > 0$ and let P be a set of points in the projective plane $\mathbb{PF}_q^2$ such that $|P| > (1+\epsilon)(q+1)$. Then P determines at least*

$$\frac{\epsilon^2(1-\epsilon)}{2+2\epsilon}(q+1)^2$$

distinct straight lines.

Iosevich, Rudnev and Zhai [10] used Fourier-analytic techniques to establish a similar result. So, a sufficiently large set of points in the plane determines a positive proportion of all possible lines. The aim here is to establish an analogue of Theorem 3.3 with the role of lines replaced by circles. Since there are q^2 choices for the location of a circle's centre, and q choices for the radius, we want to generate $\Omega(q^3)$ circles. We first need an obvious definition of what it means for a circle to be generated by a set of points.

Given three non-collinear points in $\mathbb{R}^2$, there exists a unique circle which passes through each of the three points. The same is true of three points in $\mathbb{F}_q^2$:

LEMMA 3.4. *Let (x_1, y_1), (x_2, y_2) and (x_3, y_3) be three distinct non-collinear points in $\mathbb{F}_q^2$. Then there exists a unique circle supporting them.*

Proof. We will show that there exists a unique triple of elements $a, b, c \in \mathbb{F}_q$ such that the system of equations

$$\begin{cases} (x_1 - a)^2 + (y_1 - b)^2 = c, \\ (x_2 - a)^2 + (y_2 - b)^2 = c, \\ (x_3 - a)^2 + (y_3 - b)^2 = c \end{cases} \tag{3.2}$$

can be realised. Note that we cannot have $x_1 = x_2 = x_3$, since this would contradict the hypothesis that the three points are non-collinear. Therefore, it is assumed without loss of generality that $x_1 \neq x_2$ and $x_1 \neq x_3$.

Subtracting the first equation in (3.2) from the second and third yields

$$\begin{cases} 2(x_1 - x_2)a + 2(y_1 - y_2)b = x_1^2 + y_1^2 - x_2^2 - y_2^2, \\ 2(x_1 - x_3)a + 2(y_1 - y_3)b = x_1^2 + y_1^2 - x_3^2 - y_3^2. \end{cases} \tag{3.3}$$

It cannot be the case that both $y_2 = y_1$ and $y_3 = y_1$ (otherwise the three points would be collinear), and so at least one of the b coefficients is non-zero. Therefore, this is a system of two linear equations with two variables (a and b). This system has a unique solution (a, b), unless it is degenerate. This solution can then be plugged into (3.2) to give a unique solution to the system as required. It remains to show that (3.3) is non-degenerate.

Suppose that (3.3) is degenerate. Then there exists $\lambda \in \mathbb{F}_q \setminus \{0\}$ such that

$$\begin{cases} \lambda(x_2 - x_1) = x_3 - x_1, \\ \lambda(y_2 - y_1) = y_3 - y_1, \end{cases} \tag{3.4}$$

Since it is known that at least one of $y_2 - y_1$ and $y_3 - y_1$ is non-zero, and λ is non-zero, it must be that both $y_2 - y_1 \neq 0$ and $y_3 - y_1 \neq 0$. Therefore,

$$\lambda = \frac{x_3 - x_1}{x_2 - x_1} = \frac{y_3 - y_1}{y_2 - y_1}.$$

Hence

$$y_3 - y_1 = (x_3 - x_1)\frac{y_2 - y_1}{x_2 - x_1},$$

and clearly

$$y_2 - y_1 = (x_2 - x_1)\frac{y_2 - y_1}{x_2 - x_1}.$$

This implies that $(x_1, y_1), (x_2, y_2)$ and (x_3, y_3) are supported on a line with

equation
$$y = \frac{y_2 - y_1}{x_2 - x_1} x + C, \quad \text{where} \quad C = \frac{y_1 x_2 - x_1 y_2}{x_2 - x_1}.$$
This is a contradiction, and the proof is complete. ∎

What about three collinear points? Here, we must distinguish circles with radius 0. If the three points are collinear with respect to an isotropic line, that is, a line with gradient i such that $i^2 = -1$, then there are q circles which pass through the three points. If the three points are collinear with respect to a non-isotropic line, then there is no circle which passes through all three points. Isotropic lines only occur $q = 1 \pmod 4$, since this is the only case when there exists such an $i \in \mathbb{F}_q$. These facts can be checked by calculations similar to those in the proof of Lemma 3.4. See also [6] for a more systematic approach.

We define a circle C to be *determined by* P if there exist three points from P which belong to C. One could also use a stricter definition of what it means for a circle to be determined by P, by saying that a circle is determined by P if there exist three points which belong to that circle and no other circle contains all three points. With this definition, we could obtain a version of the forthcoming theorem with a slightly weaker multiplicative constant.

We are now ready to state our version of Beck's Theorem for circles:

THEOREM 3.5. *Let $P \subset \mathbb{F}_q^2$ be such that $|P| \geq 5q$. Then P determines at least $4q^3/9$ distinct circles.*

Note that in the statement of Theorems 3.3 and 3.5, the conclusion is that we determine a positive proportion of all possible lines and circles respectively. If one asks how many points are needed to generate *all* lines (respectively circles), then the problem becomes rather different, since one can take a point set $P = \mathbb{F}_q^2 \backslash l$ where l is a line (respectively $P = \mathbb{F}_q^2 \backslash C$ where C is a circle), and the line l (respectively the circle C) is not determined by P. So, we cannot hope to show that a set of $o(q^2)$ points determines all possible lines or circles.

Proof of Theorem 3.5. At the outset, identify a subset $P' \subset P$ such that $|P'| = 5q$. The aim is to show that P', and hence also P, determines many circles.

Let $\mathcal{S}$ be the set of all circles which contain less than or exactly two points from P'. We will show that

$$|\mathcal{S}| < 5q^3/9, \tag{3.5}$$

and then since there are q^3 circles, there must be at least $4q^3/9$ circles which contain at least three points from P'. These $4q^3/9$ circles are therefore spanned by P.

It remains to prove (3.5), and to do so we will make use of the lower bound on $I(P', \mathcal{S})$ from Theorem 2.3. We have

$$5|\mathcal{S}| - |P'|^{1/2}|\mathcal{S}|^{1/2}q = \frac{|P'|\,|\mathcal{S}|}{q} - |P'|^{1/2}|\mathcal{S}|^{1/2}q < I(P', \mathcal{S}) \le 2|\mathcal{S}|,$$

a rearrangement of which gives $|\mathcal{S}| < |P'|q^2/9 = 5q^3/9$, as required. ∎

Arguments similar to the proof above can be found in [12]. Note that using the lower bound on the number of incidences between sets of points and lines in $\mathbb{F}_q^2$ in (1.1), it is straightforward to adapt the proof of Theorem 3.5 with lines in place of circles in order to prove a version of Theorem 3.3.

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