

Solutions to certain linear equations in Piatetski-Shapiro sequences

by

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1. Introduction. A *Piatetski-Shapiro sequence* is a sequence of the form $(\lfloor n^\alpha \rfloor)_{n \in \mathbb{N}}$ for non-integral $\alpha > 1$, where $\lfloor x \rfloor$ is the integer part of $x \in \mathbb{R}$ and \mathbb{N} is the set of positive integers. Denote by $\text{PS}(\alpha)$ the image of $n \mapsto \lfloor n^\alpha \rfloor$. We will say that the linear equation

$$(1) \quad y = ax + b, \quad a, b \in \mathbb{R},$$

is *solvable in* $\text{PS}(\alpha)$ if there exist infinitely many distinct pairs $(x, y) \in \text{PS}(\alpha) \times \text{PS}(\alpha)$ satisfying (1), and *unsolvable* otherwise. This terminology extends as expected to solving equations and systems of equations in other subsets of \mathbb{N} .

THEOREM 1. *Suppose that (1) is solvable in \mathbb{N} . For Lebesgue-a.e. $\alpha > 1$:*

- (i) *if $\alpha < 2$ and $0 \leq b < a$, then (1) is solvable in $\text{PS}(\alpha)$;*
- (ii) *if $\alpha > 2$ and $(a, b) \neq (1, 0)$, then (1) is unsolvable in $\text{PS}(\alpha)$.*

Piatetski-Shapiro sequences get their name from Ilya Piatetski-Shapiro, who proved a Prime Number Theorem for $(\lfloor n^\alpha \rfloor)_{n \in \mathbb{N}}$ for all $1 < \alpha < 12/11$; see [13]. Similar results regarding the distribution of $(\lfloor n^\alpha \rfloor)_{n \in \mathbb{N}}$ in arithmetic progressions and the square-free numbers hold for various ranges of α in both metrical and complete versions; see [1] for recent results in this direction and further references.

The motivation for this work comes from another line of thought. Since $\text{PS}(\alpha)$ is the (rounded) image of \mathbb{N} under the *Hardy field function* $(^1) x \mapsto x^\alpha$,

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⁽¹⁾ Any subfield of the ring of germs at $+\infty$ of continuous real-valued functions on \mathbb{R} which is closed under differentiation is a *Hardy field*, and its members are *Hardy field functions*.

it is known to be a so-called *set of multiple recurrence* in ergodic theory (see [8]); thus, for example, every $E \subseteq \mathbb{N}$ with $\limsup_{N \rightarrow \infty} |E \cap \{1, \dots, N\}|/N > 0$ contains arbitrarily long arithmetic progressions with step size in $\text{PS}(\alpha)$. That $\text{PS}(\alpha)$ is a set of multiple recurrence follows from it containing “many divisible polynomial patterns” (see [8, Section 5]); in particular, when $1 < \alpha < 2$, the set $\text{PS}(\alpha)$ contains arbitrarily long arithmetic progressions and arithmetic progressions of every sufficiently large step.

Another class of sets which enjoy strong recurrence properties are those which possess *IP-structure*. A *finite sums* set in \mathbb{N} is a set of the form

$$(2) \quad \text{FS}((x_i)_{i=1}^n) = \left\{ \sum_{i \in I} x_i \mid I \subseteq \{1, \dots, n\}, I \neq \emptyset \right\},$$

where $(x_i)_{i=1}^n \subseteq \mathbb{N}$, and a set $A \subseteq \mathbb{N}$ is called IP_0 if it contains arbitrarily large finite sums sets. Finite sums sets define “linear” IP-structure; we define a higher order analogue, VIP-structure, in Section 5. Sets with VIP-structure are known to be sets of multiple recurrence in both topological and measure-theoretic dynamical systems; see, for example, [3, 5, 6, 9].

It would be interesting, therefore, to identify IP-structure in sequences arising from Hardy field functions, and the Piatetski-Shapiro sequences provide an ideal first candidate in this search: they are easily described and already known to form sets of multiple recurrence. We elaborate on this further in Section 5.

In attempting to find IP-structure, a more basic question arises that does not seem to have been addressed in the literature: which linear equations are solvable in $\text{PS}(\alpha)$? In some cases, this question can already be easily answered. For example, for all $1 < \alpha < 2$, because $\text{PS}(\alpha)$ contains arbitrarily long arithmetic progressions, it contains infinitely many solutions to balanced, homogeneous linear equations: $\mathbf{a} \cdot \mathbf{x} = 0$ where $\sum_i a_i = 0$. Because $\text{PS}(\alpha)$ contains progressions of every sufficiently large step, it contains solutions to linear equations such as $x + y = z$ as well.

Whether other simple linear equations, such as $y = 2x$, are solvable in $\text{PS}(\alpha)$ does not follow from the aforementioned results. Theorem 1 serves as a partial answer to the question of which bivariate linear equations are solvable in $\text{PS}(\alpha)$. As a corollary, we find sets of the form $\text{FS}((x_i)_{i=1}^3)$ in $\text{PS}(\alpha)$ for Lebesgue-a.e. $1 < \alpha < 2$; this is a famous open problem in the set of squares, $\text{PS}(2)$.

NOTATION. For $x \in \mathbb{R}$, denote the distance to the nearest integer by $\|x\|$, the fractional part by $\{x\}$, the integer part (or floor) by $\lfloor x \rfloor$, and the ceiling by $\lceil x \rceil := -\lfloor -x \rfloor$. Denote the Lebesgue measure on \mathbb{R} by λ , and denote the set of those points belonging to infinitely many of the sets in the sequence $(E_n)_{n \in \mathbb{N}}$ by $\limsup_{n \rightarrow \infty} E_n$. Given two positive-valued functions f and g , we write $f \ll_{a_1, \dots, a_k} g$ or $g \gg_{a_1, \dots, a_k} f$ if there exists a constant $K > 0$

depending only on the quantities a_1, \dots, a_k for which $f(x) \leq Kg(x)$ for all x in the domain common to both f and g .

2. Reduction to Diophantine approximation. We prove Theorem 1 by reducing it to the following theorem in Diophantine approximation.

THEOREM 2. *Let $I \subseteq [0, 1)$ be a set with non-empty interior, $a, c > 0$, $a \neq 1$, and $\gamma \in \mathbb{R} \setminus \{0\}$. For Lebesgue-a.e. $\alpha > 1$, the system*

$$(3) \quad \begin{cases} \|a^{1/\alpha}n\| \leq \frac{c}{n^{\alpha-1}}, \\ \{\gamma n^\alpha\} \in I \end{cases}$$

is solvable or unsolvable in \mathbb{N} according as $\alpha < 2$ or $\alpha > 2$.

This theorem can be seen as “twisted” Diophantine approximation. Indeed, when $\alpha - 1 < 1$, the first inequality in (3) is solvable in \mathbb{N} by Dirichlet’s Theorem; more concretely, any sufficiently large denominator of a continued fraction convergent of $a^{1/\alpha}$ will yield a solution. The second condition in (3) provides the twist.

Proof of Theorem 1 assuming Theorem 2. Note that (1) is solvable in \mathbb{N} if and only if

$$a, b \in \mathbb{Q}, \quad a = a_1/a_2, \quad a_1, a_2 \in \mathbb{N}, \quad (a_1, a_2) = 1, \quad \text{and} \quad a_2 b \in \mathbb{Z}.$$

Combined with the fact that consecutive differences in $\text{PS}(\alpha)$ tend to infinity, this shows the theorem holds when $a = 1$. Assume $a \neq 1$. Note that there exists an integer $0 \leq d \leq a_2 - 1$ for which

$$a[n^\alpha] + b \in \mathbb{Z} \Leftrightarrow [n^\alpha] \equiv d \pmod{a_2} \Leftrightarrow \left\{ \frac{n^\alpha}{a_2} \right\} \in \left[\frac{d}{a_2}, \frac{d+1}{a_2} \right).$$

It follows that

$$(4) \quad \begin{aligned} a[n^\alpha] + b \in \text{PS}(\alpha) &\Leftrightarrow \exists k \in \mathbb{N}, a[n^\alpha] + b = [k^\alpha] \\ &\Leftrightarrow \begin{cases} a[n^\alpha] + b \in \mathbb{Z} \text{ and} \\ \exists k \in \mathbb{N}, a[n^\alpha] + b \leq k^\alpha < a[n^\alpha] + b + 1 \end{cases} \\ &\Leftrightarrow \left\{ \frac{n^\alpha}{a_2} \right\} \in \left[\frac{d}{a_2}, \frac{d+1}{a_2} \right) \text{ and } J_n \cap \mathbb{N} \neq \emptyset, \end{aligned}$$

where, by the Mean Value Theorem,

$$\begin{aligned} J_n &= [(a[n^\alpha] + b)^{1/\alpha}, (a[n^\alpha] + b + 1)^{1/\alpha}] = a^{1/\alpha}n + [L_n, R_n), \\ L_n &= -\frac{a}{\alpha} \left(\{n^\alpha\} - \frac{b}{a} \right) l_n^{-1+1/\alpha}, \quad l_n \text{ between } an^\alpha \text{ and } a[n^\alpha] + b, \\ R_n &= \frac{a}{\alpha} \left(\frac{b+1}{a} - \{n^\alpha\} \right) r_n^{-1+1/\alpha}, \quad r_n \text{ between } an^\alpha \text{ and } a[n^\alpha] + b + 1. \end{aligned}$$

Note that $J_n, L_n, R_n, l_n,$ and r_n all depend on α . This shows so far that (1) is solvable in $\text{PS}(\alpha)$ if and only if the system in (4) is solvable in \mathbb{N} .

We proceed by showing that solutions to (3) yield solutions to (4) and vice versa when $I, c,$ and γ are chosen appropriately. To this end, for $i = 1, 2,$ let

$$\begin{aligned} A &= \{\alpha > 1 \mid (4) \text{ is solvable in } \mathbb{N}\}, \\ B_i &= \{\alpha > 1 \mid (3) \text{ is solvable in } \mathbb{N} \text{ for } I_i, c_i, \gamma_i, a\}. \end{aligned}$$

To prove Theorem 1, it suffices by Theorem 2 to find $I_i, c_i, \gamma_i, i = 1, 2,$ for which

$$(5) \quad B_1 \cap (1, 2) \subseteq A \subseteq B_2.$$

We begin with the first containment in (5), which we will show under the assumption that $0 \leq b < a$. Let I_0 be the middle-third subinterval of the interval $(b/a, \min(1, (b+1)/a))$. Let $I_1 = d/a_2 + I_0/a_2, \gamma_1 = 1/a_2,$ and c_1 be a constant depending only on a and b to be specified momentarily.

Suppose $\alpha \in B_1 \cap (1, 2)$ and that n is a solution to (3); we will show that if n is sufficiently large, then it solves the system in (4). By (3),

$$\left\{ \frac{n^\alpha}{a_2} \right\} = \{\gamma_1 n^\alpha\} \in I_1 \subseteq \left[\frac{d}{a_2}, \frac{d+1}{a_2} \right).$$

This also implies that $\{n^\alpha\} \in I_0$, so

$$\{n^\alpha\} - \frac{b}{a} \gg_{a,b} 1, \quad \frac{b+1}{a} - \{n^\alpha\} \gg_{a,b} 1.$$

Combining these estimates with the facts that $\alpha \in (1, 2)$ and, for n sufficiently large, $a[n^\alpha] + b + 1 \leq 2an^\alpha$, we get

$$\begin{aligned} -L_n &= \frac{a}{\alpha} \left(\{n^\alpha\} - \frac{b}{a} \right) l_n^{-1+1/\alpha} \gg_{a,b} \frac{1}{n^{\alpha-1}}, \\ R_n &= \frac{a}{\alpha} \left(\frac{b+1}{a} - \{n^\alpha\} \right) r_n^{-1+1/\alpha} \gg_{a,b} \frac{1}{n^{\alpha-1}}. \end{aligned}$$

Set c_1 to be half the minimum of the constants implicit in these two expressions. It follows that J_n contains an open interval centered at $a^{1/\alpha}n$ of length $2c_1/n^{\alpha-1}$. By (3), $\|a^{1/\alpha}n\| \leq c_1/n^{\alpha-1}$, so the interval J_n contains the nearest integer to $a^{1/\alpha}n$; in particular, $J_n \cap \mathbb{N} \neq \emptyset$, so n solves (4).

The second containment in (5) is handled similarly. Let $I_2 = [0, 1), \gamma_2 = 1,$ and c_2 be a constant depending only on a to be specified momentarily. Suppose that $\alpha \in A$ and n solves (4); we will show that n satisfies (3). The second condition in (3) is satisfied automatically by our choice of I_2 . For n sufficiently large, $a[n^\alpha] + b \geq an^\alpha/2$, whereby $|L_n|, |R_n| \leq c_2/2n^{\alpha-1}$, where c_2 is chosen (depending only on a) to satisfy both inequalities. Since

J_n contains an integer, it must be that $\|a^{1/\alpha}n\| \leq c_2/n^{\alpha-1}$, meaning n satisfies (3). ■

3. Proof of Theorem 2. To prove Theorem 2, we first change variables under $t_a(x) = (\log_a x)^{-1}$ to arrive at the equivalent Theorem 3. Proof of the equivalence is a routine exercise using the fact that t_a is measure-theoretically non-singular. The lemmata used in the proof of the following theorem may be found in Section 4.

THEOREM 3. *Let $I \subseteq [0, 1)$ be a set with non-empty interior, $a, c > 0$, $a \neq 1$, and $\gamma \in \mathbb{R} \setminus \{0\}$. If $a < 1$, then for Lebesgue-a.e. $a < \theta < 1$, the system*

$$(6) \quad \begin{cases} \|\theta n\| \leq \frac{c}{n^{t_a(\theta)-1}}, \\ \{\gamma n^{t_a(\theta)}\} \in I \end{cases}$$

is solvable or unsolvable in \mathbb{N} according as $\theta < \sqrt{a}$ or $\theta > \sqrt{a}$. If $a > 1$, then for Lebesgue-a.e. $1 < \theta < a$, the system is solvable or unsolvable in \mathbb{N} according as $\theta > \sqrt{a}$ or $\theta < \sqrt{a}$.

Proof. Fix I , a , c , and γ . Suppose $a < 1$; the case $a > 1$ follows from the proof below with the obvious modifications. Without loss of generality, we may assume that I is an interval with non-empty interior. For brevity, we will suppress dependence on I , a , c , and γ in the asymptotic notation appearing in the proof.

Let $\Theta \subseteq (a, 1)$ be the set of those θ satisfying the conclusion of the theorem. We will show that Θ is of full Lebesgue measure by showing that it has full measure in the intervals (a, \sqrt{a}) and $(\sqrt{a}, 1)$.

To show that $\Theta \cap (a, \sqrt{a})$ is of full measure, it suffices by Lemma 1 to show that there exists a $\delta > 0$ such that for all $a < \theta_1 < \theta_2 < \sqrt{a}$,

$$(7) \quad \lambda(\Theta \cap (\theta_1, \theta_2)) \geq \delta(\theta_2 - \theta_1).$$

To this end, fix $a < \theta_1 < \theta_2 < \sqrt{a}$. In what follows, the phrase “for all sufficiently large n ” means “for all $n \geq n_0$,” where $n_0 \in \mathbb{N}$ may depend on any of the parameters introduced so far, including θ_1 and θ_2 .

For $n \in \mathbb{N}$, define

$$\begin{aligned} E_n &= \{\theta \in (\theta_1, \theta_2) \mid \|\theta n\| \leq \psi(n)\}, & \psi(n) &= 1/n, \\ F_n &= \{\theta \in (\theta_1, \theta_2) \mid \{\gamma n^{t_a(\theta)}\} \in I\}, & G_n &= E_n \cap F_n. \end{aligned}$$

For $\theta \in (\theta_1, \theta_2)$, we have $t_a(\theta) < t_a(\theta_2) < 2$, so for sufficiently large n , $\psi(n) \leq c/n^{t_a(\theta)-1}$. It follows that $\limsup_{n \rightarrow \infty} G_n \subseteq \Theta \cap (\theta_1, \theta_2)$. Therefore, in order to show (7), it suffices to prove that there exists a $\delta > 0$, independent

of θ_1, θ_2 , for which

$$(8) \quad \lambda\left(\limsup_{\substack{p \rightarrow \infty \\ p \text{ prime}}} G_p\right) \geq \delta(\theta_2 - \theta_1).$$

Passing to primes here makes parts of the argument technically easier. To ease notation, any sum indexed over p or q will be understood to be a sum over prime numbers.

To prove (8), it suffices by Lemma 2 to prove that

$$(9) \quad \sum_{p=2}^{\infty} \lambda(G_p) = \infty$$

and that there exists a $\delta > 0$, independent of θ_1, θ_2 , for which

$$(10) \quad \limsup_{N \rightarrow \infty} \left(\sum_{p=2}^N \lambda(G_p) \right)^2 \left(\sum_{p,q=2}^N \lambda(G_p \cap G_q) \right)^{-1} \geq \delta(\theta_2 - \theta_1).$$

First we show (9) using Lemma 4. Fix $0 < \eta < \min(\theta_1, 1 - \theta_2, (\theta_2 - \theta_1)/3)$. For $n \in \mathbb{N}$, let

$$(11) \quad \begin{aligned} S_n &= \{m \in \mathbb{Z} \mid \theta_1 + \eta < m/n < \theta_2 - \eta\}, \\ T_n &= \{m \in \mathbb{Z} \mid \theta_1 - \eta < m/n < \theta_2 + \eta\}, \end{aligned}$$

and note that for n sufficiently large,

$$(12) \quad (\theta_2 - \theta_1)n \ll |S_n| < |T_n| \ll (\theta_2 - \theta_1)n.$$

For n sufficiently large, the set E_n may be approximated by a disjoint union of intervals:

$$\bigcup_{m \in S_n} E_{n,m} \subseteq E_n \subseteq \bigcup_{m \in T_n} E_{n,m},$$

where

$$E_{n,m} = [m/n - \psi(n)/n, m/n + \psi(n)/n].$$

Let $I_0 \subseteq I$ be the middle-third subinterval of I .

CLAIM 1. *For n sufficiently large and $m \in S_n$, if $\{\gamma n^{t_a(m/n)}\} \in I_0$, then $E_{n,m} \subseteq F_n$.*

Proof. Note first that $|t'_a(x)| \ll 1$ for $x \in (a, \sqrt{a})$. For $\theta \in E_{n,m}$, by applying the Mean Value Theorem twice, we see that for some ξ_1 between $t_a(\theta)$ and $t_a(m/n)$ and some ξ_2 between θ and m/n ,

$$\begin{aligned} |\gamma n^{t_a(\theta)} - \gamma n^{t_a(m/n)}| &= |\gamma| |t_a(\theta) - t_a(m/n)| (\log n) n^{\xi_1} \\ &\leq |\gamma| \left| \theta - \frac{m}{n} \right| |t'_a(\xi_2)| (\log n) n^{t_a(\theta_2)} \\ &\ll \frac{\psi(n)}{n} (\log n) n^{t_a(\theta_2)} \leq \frac{\log n}{n^{2-t_a(\theta_2)}}. \end{aligned}$$

Since $2 - t_a(\theta_2) > 0$, for n sufficiently large,

$$|\gamma n^{t_a(\theta)} - \gamma n^{t_a(m/n)}| < \lambda(I)/3.$$

Therefore, if $\{\gamma n^{t_a(m/n)}\} \in I_0$, then for all $\theta \in E_{n,m}$, $\{\gamma n^{t_a(\theta)}\} \in I$, i.e. $E_{n,m} \subseteq F_n$. ■

By the equidistribution result in Lemma 4, for n sufficiently large,

$$\frac{|\{m \in S_n \mid \{\gamma n^{t_a(m/n)}\} \in I_0\}|}{(\theta_2 - \theta_1 - 2\eta)n} \geq \frac{\lambda(I_0)}{2}.$$

Combining this with Claim 1 and the bounds on η shows that there are $\gg (\theta_2 - \theta_1)n$ integers $m \in S_n$ for which $E_{n,m} \subseteq F_n$. It follows that for n sufficiently large,

$$(13) \quad \lambda(G_n) \gg (\theta_2 - \theta_1)n \frac{\psi(n)}{n} = (\theta_2 - \theta_1) \frac{1}{n}.$$

This proves (9).

Now we show (10) by estimating the “overlaps” between the G_p ’s. It suffices to prove that there exists a constant K (which may depend on any of the parameters introduced so far) such that for all sufficiently large primes p and for all $N \geq p$,

$$(14) \quad \sum_{q>p}^N \lambda(E_p \cap E_q) \ll (\theta_2 - \theta_1) \sum_{q>p}^N \psi(p)\psi(q) + K\psi(p).$$

Indeed, suppose (13) and (14) both hold for all primes p greater than some sufficiently large $p_0 \in \mathbb{N}$. Using the trivial bound $\lambda(G_p \cap G_q) \leq \lambda(E_p \cap E_q)$, it follows that

$$\begin{aligned} \sum_{p,q=2}^N \lambda(G_p \cap G_q) &\leq 2 \left(\sum_{\substack{p \geq p_0 \\ q > p}}^N \lambda(G_p \cap G_q) + \sum_{\substack{p < p_0 \\ q > p}}^N \lambda(G_q) \right) + \sum_{q=2}^N \lambda(G_q) \\ &\ll (\theta_2 - \theta_1) \sum_{\substack{p \geq p_0 \\ q > p}}^N \psi(p)\psi(q) + K \sum_{p \geq p_0}^N \psi(p) + \sum_{\substack{p < p_0 \\ q \geq p}}^N \lambda(G_q) \\ &\ll \frac{1}{\theta_2 - \theta_1} \sum_{p,q=2}^N \lambda(G_p)\lambda(G_q) + p_0 \left(\frac{K}{\theta_2 - \theta_1} + 1 \right) \sum_{q=2}^N \lambda(G_q), \end{aligned}$$

where the last line follows from (13). This combines with (9) to yield (10).

To show (14), note that the set E_p is covered by a union of intervals $E_{p,r}$, each of length $2\psi(p)/p$. If $p < q$ and $E_{p,r} \cap E_{q,s} \neq \emptyset$, then by estimating the

distance between the midpoints of the intervals,

$$|sp - rq| < pq \, 3 \max\left(\frac{\psi(p)}{p}, \frac{\psi(q)}{q}\right) = 3q\psi(p),$$

$$\lambda(E_{p,r} \cap E_{q,s}) \leq \min\left(2\frac{\psi(p)}{p}, 2\frac{\psi(q)}{q}\right) = 2\frac{\psi(q)}{q}.$$

The left hand side of (14) is then

$$\sum_{q>p}^N \lambda(E_p \cap E_q) = \sum_{q>p}^N \sum_{\substack{r \in T_p \\ s \in T_q}} \lambda(E_{p,r} \cap E_{q,s}) \ll \sum_{q>p}^N \frac{\psi(q)}{q} \sum_{\substack{r \in T_p, s \in T_q \\ |sp-rq| < 3q\psi(p)}} 1.$$

Now (14) will follow by partitioning the range of the sum on q and applying Lemma 3. Indeed, the right hand side of the previous expression is equal to

$$\sum_{\ell=0}^{\infty} \sum_{\substack{2^\ell p < q < 2^{\ell+1} p \\ q \leq N}} \frac{\psi(q)}{q} \sum_{\substack{r \in T_p, s \in T_q \\ |sp-rq| < 3q\psi(p)}} 1 \leq \sum_{\ell=0}^{\infty} \frac{\psi(2^\ell p)}{2^\ell p} \sum_{\substack{Q_\ell < q < \min(2Q_\ell, N+1) \\ r \in T_p, s \in T_q \\ |sp-rq| < L_\ell}} 1,$$

where $L_\ell = 3 \cdot 2^{\ell+1} p \psi(p)$ and $Q_\ell = 2^\ell p$. For each ℓ , we apply Lemma 3 with N and p as they are, Q_ℓ as Q , L_ℓ as L , and $(\theta_1 - \eta, \theta_2 + \eta)$ as (η_1, η_2) : there exists a $K > 0$ depending only on θ_1, θ_2 (since η depends only on θ_1, θ_2) such that the right hand side of the previous expression is

$$\begin{aligned} &\ll \sum_{\ell=0}^{\infty} \frac{\psi(2^\ell p)}{2^\ell p} \left((\theta_2 - \theta_1 + 2\eta) L_\ell \sum_{\substack{2^\ell p < q < 2^{\ell+1} p \\ q \leq N}} 1 + K 2^\ell p \right) \\ &\ll (\theta_2 - \theta_1) \sum_{\ell=0}^{\infty} \sum_{\substack{2^\ell p < q < 2^{\ell+1} p \\ q \leq N}} \psi(p) \psi(2^\ell p) + K \sum_{\ell=0}^{\infty} \psi(2^\ell p) \\ &\leq (\theta_2 - \theta_1) \sum_{\ell=0}^{\infty} \sum_{\substack{2^\ell p < q < 2^{\ell+1} p \\ q \leq N}} \psi(p) \psi(1/2) \psi(q) + K \psi(p) \sum_{\ell=0}^{\infty} \psi(2^\ell) \\ &\ll (\theta_2 - \theta_1) \sum_{q>p}^N \psi(p) \psi(q) + K \psi(p), \end{aligned}$$

where the third and fourth lines follow by noting that $\psi(2^\ell p) \leq \psi(q/2)$, that ψ is multiplicative, and that $\sum_{\ell=0}^{\infty} \psi(2^\ell)$ converges. This shows (14), completing the proof of (10).

To show that the set $\Theta \cap (\sqrt{a}, 1)$ is of full measure, we will show that for all $\theta_3 > \sqrt{a}$, the set $(\theta_3, 1) \setminus \Theta$ has zero measure. Let

$$H_n = \left\{ \theta \in (\theta_3, 1) \mid \|\theta n\| \leq \frac{c}{n^{t_a(\theta_3)-1}} \right\}.$$

If $\theta \in (\theta_3, 1) \setminus \Theta$, then $\|\theta n\| \leq c/n^{t_a(\theta_3)-1}$ for infinitely many $n \in \mathbb{N}$, so

$$(15) \quad (\theta_3, 1) \setminus \Theta \subseteq \limsup_{n \rightarrow \infty} H_n.$$

Since H_n is a union of $\ll (1 - \theta_3)n$ intervals, each of length $2c/n^{t_a(\theta_3)}$, and since $t_a(\theta_3) > 2$, $\sum_{n=1}^{\infty} \lambda(H_n) < \infty$. By the first Borel–Cantelli Lemma, $\limsup_{n \rightarrow \infty} H_n$ has zero measure, so $(\theta_3, 1) \setminus \Theta$ has zero measure by (15). ■

4. Supporting lemmata. Here we collect some supporting lemmata. Recall that λ denotes the Lebesgue measure on \mathbb{R} and that $t_a(x) = (\log_a x)^{-1}$.

LEMMA 1 ([11, Lemma 1.6]). *Let $I \subseteq \mathbb{R}$ be an interval and $A \subseteq I$ be measurable. If there exists a $\delta > 0$ such that for every subinterval $I_0 \subseteq I$, $\lambda(A \cap I_0) \geq \delta \lambda(I_0)$, then A is of full measure in I : $\lambda(I \setminus A) = 0$.*

LEMMA 2 ([11, Lemma 2.3]). *Let (X, \mathcal{B}, μ) be a measure space with $\mu(X) < \infty$. If $(G_n)_{n \in \mathbb{N}} \subseteq \mathcal{B}$ is a sequence of subsets of X for which $\sum_{n=1}^{\infty} \mu(G_n) = \infty$, then*

$$\mu\left(\limsup_{n \rightarrow \infty} G_n\right) \geq \limsup_{N \rightarrow \infty} \left(\sum_{n=1}^N \mu(G_n) \right)^2 \left(\sum_{n,m=1}^N \mu(G_n \cap G_m) \right)^{-1}.$$

LEMMA 3. *Let $N, Q, p \in \mathbb{N}$, p prime with $p \leq Q$, $L > 0$, and $0 < \eta_1 < \eta_2 < 1$. There exists a constant $K > 0$ depending only on η_1, η_2 such that the number of triples $(q, r, s) \in \mathbb{N}^3$ satisfying*

$$Q < q < \min(2Q, N+1), \quad q \text{ prime}, \quad r/p, s/q \in (\eta_1, \eta_2), \quad |sp - rq| < L,$$

is

$$\ll (\eta_2 - \eta_1)L \sum_{\substack{Q < q < \min(2Q, N+1) \\ q \text{ prime}}} 1 + KQ.$$

Proof. This lemma follows immediately from [11, Lemma 6.2] by letting the set \mathcal{A} of integers be those primes strictly between Q and $\min(2Q, N+1)$ and taking the worst error KQ . ■

LEMMA 4. *Let $a > 0$, $a \neq 1$, $\gamma \in \mathbb{R} \setminus \{0\}$, and $\min(\sqrt{a}, a) < \eta_1 < \eta_2 < \max(\sqrt{a}, a)$. Then*

$$\frac{1}{n(\eta_2 - \eta_1)} \sum_{\substack{m \in \mathbb{Z} \\ m/n \in (\eta_1, \eta_2)}} \delta_{\{\gamma n^{t_a(m/n)}\}} \rightarrow \lambda|_{[0,1]} \quad \text{weakly as } n \rightarrow \infty,$$

where δ_x denotes the point mass at $x \in [0, 1)$.

Proof. Let $N_n = |\{m \in \mathbb{Z} \mid m/n \in (\eta_1, \eta_2)\}|$, and note that $N_n/(n(\eta_2 - \eta_1)) \rightarrow 1$ as $n \rightarrow \infty$. For $n, h \in \mathbb{N}$, let

$$g_n(x) = \gamma n^{t_a((x + \lfloor \eta_1 n \rfloor)/n)} \quad \text{and} \quad g_{n,h}(x) = g_n(x+h) - g_n(x).$$

In this notation, we must show

$$(16) \quad \frac{1}{N_n} \sum_{i=1}^{N_n} \delta_{\{g_n(i)\}} \rightarrow \lambda|_{[0,1]} \quad \text{weakly as } n \rightarrow \infty.$$

Since t_a is either increasing ($a < 1$) or decreasing ($a > 1$) between \sqrt{a} and a , we can fix σ_1, σ_2 such that for all $x \in (\eta_1, \eta_2)$,

$$1 < \sigma_1 < t_a(x) < \sigma_2 < 2.$$

To handle the exponential sum estimates that follow, we will show that there exist positive constants C_1 and C_2 (depending on a and γ) such that for all $h \in \mathbb{N}$, all sufficiently large $n \in \mathbb{N}$, and all $x \in [1, N_n - h]$,

$$(17) \quad C_1 h \frac{(\log n)^3}{n^{3-\sigma_1}} \leq |g''_{n,h}(x)| \leq C_2 h \frac{(\log n)^3}{n^{3-\sigma_2}}.$$

By the Mean Value Theorem, $g''_{n,h}(x) = h g'''_n(\xi_x)$ for some $\xi_x \in (x, x+h)$, so it suffices to show that for all $h \in \mathbb{N}$, all sufficiently large $n \in \mathbb{N}$, and all $x \in [1, N_n]$,

$$(18) \quad C_1 \frac{(\log n)^3}{n^{3-\sigma_1}} \leq |g'''_n(x)| \leq C_2 \frac{(\log n)^3}{n^{3-\sigma_2}}.$$

Writing $g'''_n(x)$ explicitly reveals that

$$g'''_n(x) = g_n(x) \left(\frac{\log n}{n} \right)^3 \left(\frac{x + \lfloor \eta_1 n \rfloor}{n} \right)^{-3} \left(\frac{t_a \left(\frac{x + \lfloor \eta_1 n \rfloor}{n} \right)^6}{-(\log a)^3} + \frac{r(x)}{\log n} \right),$$

where $|r(x)| \ll_a 1$ for $x \in [1, N_n]$ because $(x + \lfloor \eta_1 n \rfloor)/n \in (\eta_1, \eta_2)$. The double inequality in (18) follows for n sufficiently large since $|\gamma| n^{\sigma_1} \leq |g_n(x)| \leq |\gamma| n^{\sigma_2}$.

To prove (16), it suffices by Weyl's Criterion [12, Chapter 1, Theorem 2.1] to show that for all $b \in \mathbb{Z} \setminus \{0\}$,

$$\frac{1}{N_n} \sum_{i=1}^{N_n} e(bg_n(i)) \rightarrow 0 \quad \text{as } n \rightarrow \infty,$$

where $e(x) = e^{2\pi i x}$. By the van der Corput Difference Theorem [12, Chapter 1, Theorem 3.1] and another application of Weyl's Criterion, it suffices to prove that for all $h \in \mathbb{N}$ and all $b \in \mathbb{Z} \setminus \{0\}$,

$$(19) \quad \frac{1}{N_n - h} \sum_{i=1}^{N_n - h} e(bg_{n,h}(i)) \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

An exponential sum estimate [12, Chapter 1, Theorem 2.7] gives

$$\frac{1}{N_n - h} \left| \sum_{i=1}^{N_n - h} e(bg_{n,h}(i)) \right| \leq \left(\frac{|b| |g'_{n,h}(N_n - h) - g'_{n,h}(1)| + 2}{N_n - h} \right) \left(\frac{4}{\sqrt{|b|\rho}} + 3 \right),$$

where $\rho = C_1 h (\log n)^3 / n^{3-\sigma_1}$ from (17). By the Mean Value Theorem and the upper bound from (17), we see the right hand side is bounded from above for sufficiently large n by

$$\left(|b| C_2 h \frac{(\log n)^3}{n^{3-\sigma_2}} + \frac{2}{N_n - h} \right) \left(\frac{4n^{(3-\sigma_1)/2}}{\sqrt{|b| C_1 h (\log n)^3}} + 3 \right) \ll \frac{n^{(3-\sigma_1)/2}}{n \sqrt{|b| h (\log n)^3}},$$

where the implicit constant depends on a , γ , η_1 , and η_2 . The limit in (19) follows since $(3 - \sigma_1)/2 < 1$. ■

5. Remarks and conclusions. Here are a number of natural questions and further directions.

(i) Equations of the form $y = ax - b$ where $0 \leq b < 1$ are handled by Theorem 1 by rearranging, but there are still simple linear equations which the theorem does not handle. For example, is the equation $y = 2x + 2$ solvable in $\text{PS}(\alpha)$ for Lebesgue-a.e. $1 < \alpha < 2$?

QUESTION 1. *Does Theorem 1 hold with the assumption $0 \leq b < a$ in part (i) replaced by $a \notin \{0, 1\}$?*

The answer is likely “yes.” Using the technique above, we need infinitely many n ’s for which the interval $a^{1/\alpha}n + [L_n, R_n)$ contains an integer, where now L_n and R_n are both positive or both negative. Accounting for $\{n^\alpha\}$ and changing variables, this requires control on the set of n ’s for which $\{\theta n\}$ falls within a shrinking annulus about 0. These shrinking annuli are not nested, making them difficult to handle with the established theory.

(ii) Is there a non-metrical version of Theorem 1?

QUESTION 2. *Does Theorem 1 hold with “Lebesgue-a.e.” replaced by “all”?*

The inequality in (3) is solvable in \mathbb{N} when $a^{1/\alpha}$ is irrational by Dirichlet’s Theorem, and the whole system is solvable in \mathbb{N} when $a^{1/\alpha}$ is rational since $\{n^\alpha\}$ is uniformly distributed along arithmetic progressions containing 0. Therefore, exceptional α ’s would only arise because of the second condition, the “twist,” in (3) when $a^{1/\alpha}$ is irrational.

Here are two thoughts for proving the result for all $1 < \alpha < 2$. The result would be immediate from Theorem 2 if $(n^\alpha)_{n \in \mathbb{N}}$ was known to be equidistributed (or even dense) modulo 1 along denominators of the continued fraction convergents of $a^{1/\alpha}$. Alternatively, perhaps the set $\text{PS}(\alpha)$ is

sufficiently pseudorandom to contain solutions to linear equations; see [7] for recent results regarding combinatorial structure in sparse random sets.

(iii) Asymptotics are known for the distribution of Piatetski-Shapiro sequences in arithmetic progressions, the square-free numbers, and the primes; it is feasible that analogous asymptotics hold for the number of solutions to linear equations as well.

QUESTION 3. *Is it true that for Lebesgue-a.e. or for all $1 < \alpha < 2$,*

$$|\{1 \leq m, n \leq N \mid \lfloor m^\alpha \rfloor = a \lfloor n^\alpha \rfloor + b\}| \sim_{a,b,\alpha} N^{2-\alpha} ?$$

It is not hard to verify that $N^{2-\alpha}$ is the correct asymptotic for $\alpha \leq 1$.

(iv) Which systems of linear equations are solvable in $\text{PS}(\alpha)$? Consider, for example, the system $y = 2x, z = 3x$. Just as is done in Section 2, this can be reduced after a change of variables to the system

$$\begin{cases} \|\theta n\| \leq \frac{c}{n^{t_a(\theta)-1}}, \\ \|\theta^{\log_3 2} n\| \leq \frac{c}{n^{t_a(\theta)-1}}, \\ \{\gamma n^{t_a(\theta)}\} \in I. \end{cases}$$

This is “twisted” Diophantine approximation on the curve $x \mapsto (x, x^{\log_3 2})$; see [2, Theorem 1]. Assuming the twist does not interfere with the approximation, it is conceivable that this system is solvable for Lebesgue-a.e. $1 < \alpha < 3/2$.

(v) Solving the system $y = 2x, z = 3x$ in $\text{PS}(\alpha)$ is the same as finding $\text{FS}((x, x, x))$ in $\text{PS}(\alpha)$; recall (2). It is an open problem [10, D18] to determine whether or not the set of squares contains a set of the form $\text{FS}((x_i)_{i=1}^3)$. We can use Theorem 1 to solve this problem in almost all $\text{PS}(\alpha)$.

COROLLARY 1. *For Lebesgue-a.e. $1 < \alpha < 2$, the set $\text{PS}(\alpha)$ contains infinitely many sets of the form $\text{FS}((x_i)_{i=1}^3)$.*

Proof. By Theorem 1, there are infinitely many $x \in \text{PS}(\alpha)$ for which $2x \in \text{PS}(\alpha)$. For x sufficiently large (depending on α), the set $\text{PS}(\alpha)$ contains an arithmetic progression of step x and length 3 [8, Proposition 5.1]. If z starts such a progression, then $\text{FS}((x, x, z)) \subseteq \text{PS}(\alpha)$. ■

QUESTION 4 (V. Bergelson). *Is $\text{PS}(\alpha)$ an IP_0 set for Lebesgue-a.e. or all $1 < \alpha < 2$?*

V. Bergelson remarked that while $\text{PS}(\alpha)$ may not always possess “linear structure,” it may contain higher order structure. Indeed, the set $\text{PS}(m/n)$ contains the set of m th powers, and this implies that $\text{PS}(m/n)$ is a set of multiple recurrence (see [4]).

The set $A \subseteq \mathbb{N}$ possesses VIP-structure ⁽²⁾ if it contains arbitrarily large subsets of the form

$$\left\{ f \left(\sum_{i \in I} x_i^{(1)}, \dots, \sum_{i \in I} x_i^{(k)} \right) \mid I \subseteq \{1, \dots, n\}, I \neq \emptyset \right\},$$

where $f \in \mathbb{Z}[z_1, \dots, z_k]$ has zero constant term and $(x_i^{(1)})_{i=1}^n, \dots, (x_i^{(k)})_{i=1}^n \subseteq \mathbb{N}$. Thus, the set of m th powers has VIP-structure. Note that when $\deg(f) = 1$, the set above is a finite sums set. Recall from Section 1 that any set with VIP-structure is a set of multiple recurrence; perhaps VIP-structure in $\text{PS}(\alpha)$ gives an alternative explanation of the set's recurrence properties.

QUESTION 5 (V. Bergelson). *Does $\text{PS}(\alpha)$ have VIP-structure for all $\alpha > 1$?*

(vi) Theorem 1 implies that for many linear equations, $\alpha = 2$ is a threshold value for being solvable or unsolvable in $\text{PS}(\alpha)$. Do other linear equations have such a threshold, and can we compute it?

QUESTION 6. *Does there exist an $\alpha_S > 1$ with the property that for Lebesgue-a.e. or all $\alpha > 1$, the equation $x + y = z$ is solvable or unsolvable in $\text{PS}(\alpha)$ according as $\alpha < \alpha_S$ or $\alpha > \alpha_S$?*

As mentioned in Section 1, the equation $x + y = z$ is solvable in $\text{PS}(\alpha)$ for all $\alpha < 2$; when $\alpha \geq 3$ is an integer, the equation is unsolvable in $\text{PS}(\alpha)$. What happens for α just larger than 2? The same question is meaningful and interesting for more general (systems of) linear equations.

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⁽²⁾ For a more general definition, see [3, 5].

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