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# Iterates of systems of operators in spaces of $\omega$-ultradifferentiable functions

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**Abstract.** Given two systems $P = (P_j(D))_{j=1}^N$ and $Q = (Q_j(D))_{j=1}^M$ of linear partial differential operators with constant coefficients, we consider the spaces $\mathcal{E}_\omega^P$ and $\mathcal{E}_\sigma^Q$ of weighted-ultradifferentiable functions with respect to the iterates of the systems $P$ and $Q$ respectively. We find necessary and sufficient conditions, on the systems and on the weights $\omega(t)$ and $\sigma(t)$, for the inclusion $\mathcal{E}_\omega^P \subseteq \mathcal{E}_\sigma^Q$. As a consequence we obtain a generalization of the classical Theorem of the Iterates.

**1. Introduction.** The problem of iterates was first introduced by Komatsu [K1] in the 60's, when he characterized analytic functions $u$ on an open subset $\Omega \subseteq \mathbb{R}^n$ in terms of the behaviour of successive iterates $P^j(D)u$ for an elliptic linear partial differential operator $P(D)$ with constant coefficients. He proved that if $P(D)$ is an elliptic operator of order $m$, then a $C^\infty$ function $u$ is real analytic in $\Omega$ if and only if for every compact $K \subset\subset \Omega$ there is a constant $C > 0$ such that

$$\|P^j(D)u\|_{L^2(K)} \leq C^{j+1}(j!)^m, \quad \forall j \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}, \tag{1.1}$$

where $\|\cdot\|_{L^2(K)}$ is the $L^2$ norm on $K$. This is known as the *Theorem of the Iterates*.

Moreover, the condition that $P(D)$ is elliptic is sufficient and also necessary (cf. [M], [LW]) for the above mentioned result, so that, given a linear partial differential operator $P(D)$ of order $m$ with constant coefficients, the ellipticity growth condition

$$|\xi|^{2m} \leq C(1 + |P(\xi)|^2), \quad \forall \xi \in \mathbb{R}^n, \tag{1.2}$$

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for a constant $C > 0$, is equivalent to the equality

$$\mathcal{A}(\Omega) = \mathcal{A}^P(\Omega),$$

where $\mathcal{A}(\Omega)$ is the space of real analytic functions on $\Omega$ and $\mathcal{A}^P(\Omega)$ is the space of real analytic functions on $\Omega$ with respect to the iterates of $P$, i.e. the space of $C^\infty$ functions $u$ on $\Omega$ satisfying (1.1).

This problem was generalized by Newberger and Zielezny [NZ] to the class of Gevrey functions proving, more generally, that, for a pair of hypoelliptic linear partial differential operators $P(D)$ and $Q(D)$ with constant coefficients, of order $m$ and $r$ respectively, the condition

$$|Q(\xi)|^2 \leq C(1 + |P(\xi)|^2)^h, \quad \forall \xi \in \mathbb{R}^n, \tag{1.3}$$

for some $h > 0$, is equivalent to an inclusion of the form

$$\mathcal{E}^P_{\{t^{1/s}\}}(\Omega) \subseteq \mathcal{E}^Q_{\{t^{r/(smh)}\}}(\Omega)$$

if $s$ is large enough, where $\mathcal{E}^P_{\{t^{1/s}\}}(\Omega)$ is the space of Gevrey functions of order $s$ with respect to the iterates of $P = P(D)$, as defined in (2.5) for the Gevrey weight $\omega(t) = t^{1/s}$.

This result was generalized to the class of $\omega$-ultradifferentiable functions in the sense of [BMT] by [JH], and was considered in the case of systems of operators in the Gevrey setting by [BC1]. Here we improve both papers [JH] and [BC1], considering the case of systems in the spaces of $\omega$-ultradifferentiable functions.

In Section 2 we define the spaces of $\omega$-ultradifferentiable functions $\mathcal{E}^P_\omega(\Omega)$ with respect to the iterates of the system $P = (P_j(D))_{j=1}^N$, both in the Beurling and in the Roumieu setting.

In Sections 3 and 4 we prove that, given two systems $P = (P_j(D))_{j=1}^N$ and $Q = (Q_j(D))_{j=1}^M$ of order $m$ and $r$ respectively, the condition

$$\sum_{j=1}^M |Q_j(\xi)| \leq C\Big(1 + \sum_{j=1}^N |P_j(\xi)|\Big)^h, \quad \forall \xi \in \mathbb{R}^n,$$

is necessary and sufficient for an inclusion of the form

$$\mathcal{E}^P_{\omega'}(\Omega) \subseteq \mathcal{E}^Q_{\sigma'}(\Omega),$$

under assumptions weaker than hypoellipticity (condition $(\mathscr{H})$ for the sufficiency in Theorem 3.8 and condition $(\mathscr{C})$ for the necessity in Theorem 4.4), where $\sigma'(t) = \omega'(t^{r/(mh)})$ with $\omega'(t) = \omega(t^{1/s})$ and $s$ large enough, both in the Beurling and in the Roumieu setting, for a non-quasianalytic weight $\omega$.

In particular, if $P = (P_j(D))_{j=1}^N$ is an elliptic system, we obtain the Theorem of the Iterates (see Corollary 3.10), i.e.

$$\mathcal{E}^P_{\omega'}(\Omega) = \mathcal{E}_{\omega'}(\Omega).$$

Moreover, we prove that the ellipticity of the system $P$ is also necessary (see Corollary 4.6).

In Example 3.11 we give an application of the above results.

Let us finally recall that the Theorem of the Iterates has also been generalized to the case of variable coefficients, for a single elliptic operator $P(x, D)$. It has been proved in the class of real analytic functions by Kotake and Narasimhan [KN]; in the case of Denjoy–Carleman classes of Roumieu type by Lions and Magenes [LM] and of Beurling type with some loss of regularity with respect to the coefficients by Oldrich [O]; and in the classes of $\omega$-ultradifferentiable functions of Roumieu type, or of Beurling type but with some loss of regularity with respect to the coefficients, by Boiti and Jornet [BJ3].

For a microlocal version of the Theorem of the Iterates see, for instance, [BCM], [BJJ], [BJ1], [BJ2]. For anisotropic Gevrey classes we refer to [Z], [BC2].

**2. Spaces of $\omega$-ultradifferentiable functions with respect to the iterates of a system of operators.** Let us first recall, from [BMT], the notion of weight functions and of spaces of $\omega$-ultradifferentiable functions of Beurling and Roumieu type:

DEFINITION 2.1. A *non-quasianalytic weight function* is a continuous increasing function $\omega : [0, \infty) \to [0, \infty)$ with the following properties:

($\alpha$) there exists $L > 0$ such that $\omega(2t) \le L(\omega(t) + 1)$ for all $t \ge 0$;
($\beta$) $\int_1^\infty (\omega(t)/t^2)\, dt < \infty$;
($\gamma$) $\log t = o(\omega(t))$ as $t \to \infty$;
($\delta$) $\varphi_\omega(t) := \omega(e^t)$ is convex.

For $z \in \mathbb{C}^n$ we write $\omega(z)$ for $\omega(|z|)$, where $|z| = \sum_{j=1}^n |z_j|$. We write $\varphi$ for $\varphi_\omega$ when it is clear from the context.

REMARK 2.2. Condition ($\beta$) is the condition of non-quasianalyticity and it will ensure the existence of non-trivial $\omega$-ultradifferentiable functions with compact support.

In the Beurling setting, condition ($\gamma$) may be weakened (cf. [BG], [Bj]) to the following:

($\gamma$)$'$ there are $a \in \mathbb{R}$ and $b > 0$ such that

$$\omega(t) \ge a + b \log(1 + t) \quad \text{for all } t \ge 0.$$

The *Young conjugate* $\varphi^*$ of $\varphi$ is defined by

$$\varphi^*(s) := \sup_{t \ge 0} \{st - \varphi(t)\}, \quad s \ge 0.$$

Assuming, without any loss of generality, that $\omega$ vanishes on $[0, 1]$, we find that $\varphi^*$ has only non-negative values, it is convex and increasing, $\varphi^*(0) = 0$, $\varphi^*(s)/s$ is increasing and $(\varphi^*)^* = \varphi$ (cf. [BMT]).

An easy computation shows that, for every $a > 0$,

$$\sigma(t) = \omega(t^a) \;\Rightarrow\; \varphi^*_\sigma(s) = \varphi^*_\omega(s/a). \tag{2.1}$$

For a compact set $K \subset \mathbb{R}^n$ which coincides with the closure of its interior, and $\lambda > 0$ we consider the seminorm

$$p_{K,\lambda}(u) = \sup_{\alpha\in\mathbb{N}_0^n} \sup_{x\in K} |D^\alpha u(x)| e^{-\lambda\varphi^*(|\alpha|/\lambda)};$$

then

$$\mathcal{E}_{\omega,\lambda}(K) := \{u \in C^\infty(K) : p_{K,\lambda}(u) < \infty\} \tag{2.2}$$

is a Banach space endowed with the norm $p_{K,\lambda}$.

Let us now recall from [BMT] the definition of the space of *$\omega$-ultradifferentiable functions of Beurling type* in an open set $\Omega \subseteq \mathbb{R}^n$:

$$\mathcal{E}_{(\omega)}(\Omega) := \operatorname*{proj}_{\substack{\longleftarrow \\ K\subset\subset\Omega}} \operatorname*{proj}_{\substack{\longleftarrow \\ \lambda>0}} \mathcal{E}_{\omega,\lambda}(K).$$

This is a Fréchet space.

The space of *$\omega$-ultradifferentiable functions of Roumieu type* is defined by

$$\mathcal{E}_{\{\omega\}}(\Omega) := \operatorname*{proj}_{\substack{\longleftarrow \\ K\subset\subset\Omega}} \operatorname*{ind}_{\substack{\longrightarrow \\ m\in\mathbb{N}}} \mathcal{E}_{\omega,1/m}(K).$$

Let us now consider a system $P = (P_j(D))_{j=1}^N$ of linear partial differential operators with constant coefficients. For $\beta \in \mathbb{N}_0^N$ we define the iterates of the system $P$ as

$$P^\beta := P_1^{\beta_1}(D) \circ P_2^{\beta_2}(D) \circ \cdots \circ P_N^{\beta_N}(D),$$

where $P_j^{\beta_j}(D)$ is the $\beta_j$th iterate of the operator $P_j(D)$, i.e.

$$P_j^{\beta_j}(D) = \underbrace{P_j(D) \circ \cdots \circ P_j(D)}_{\beta_j},$$

and $P^0(D)u = u$.

We shall say that the system $P = (P_j(D))_{j=1}^N$ has order $m$ if each operator $P_j(D)$ has order $m$. In that case, for a compact $K \subset \mathbb{R}^n$ which coincides with the closure of its interior and $\lambda > 0$ we consider the seminorm

$$p^P_{K,\lambda}(u) := \sup_{\beta\in\mathbb{N}_0^N} \|P^\beta u\|_{L^2(K)} e^{-\lambda\varphi^*(|\beta|m/\lambda)}$$

and define

$$\mathcal{E}^P_{\omega,\lambda}(K) := \{u \in C^\infty(K) : p^P_{K,\lambda}(u) < \infty\}. \tag{2.3}$$

For an open set $\Omega \subseteq \mathbb{R}^n$ we define the space of *ω-ultradifferentiable functions of Beurling type with respect to the iterates of the system* $P = (P_j(D))_{j=1}^N$ by

$$\mathcal{E}^P_{(\omega)}(\Omega) := \operatorname*{proj}_{\overleftarrow{K\subset\subset\Omega}} \operatorname*{proj}_{\overleftarrow{\lambda>0}} \mathcal{E}^P_{\omega,\lambda}(K). \tag{2.4}$$

Analogously, we define the space of *ω-ultradifferentiable functions of Roumieu type with respect to the iterates of the system* $P$ by

$$\mathcal{E}^P_{\{\omega\}}(\Omega) := \operatorname*{proj}_{\overleftarrow{K\subset\subset\Omega}} \operatorname*{ind}_{\overrightarrow{\ell\in\mathbb{N}}} \mathcal{E}^P_{\omega,1/\ell}(K). \tag{2.5}$$

Notation. In the following we shall write $\mathcal{E}^P_\omega(\Omega)$ if the statement holds both in the Beurling case $\mathcal{E}^P_{(\omega)}(\Omega)$ and in the Roumieu case $\mathcal{E}^P_{\{\omega\}}(\Omega)$.

Remark 2.3. When the system is given by a single operator $P = P(D)$, the above defined spaces $\mathcal{E}^P_\omega(\Omega)$ coincide with the corresponding ones defined in [BJJ] (see [JH] for the original, slightly different, definition).

Analogously to [BC1], we give the following:

Definition 2.4. We say that the system $P = (P_j(D))_{j=1}^N$ satisfies *condition* $(\mathscr{C})$ if, for every $\lambda > 0$ and for every compact subset $K$ of $\Omega$ which coincides with the closure of its interior, the space $\mathcal{E}^P_{\omega,\lambda}(K)$ defined in (2.3) is a Banach space endowed with the norm $p^P_{K,\lambda}$.

Remark 2.5. Condition $(\mathscr{C})$ was introduced in [BC1] in the Gevrey setting, in order to improve the results of [NZ] on the theorem of iterates, by weakening the assumption of hypoellipticity on the operators (see also Remark 2.8 below). We shall use it in the proof of the necessity part, i.e. of Theorem 4.4.

Example 2.6. If $P = (P_j(D))_{j=1}^n = (D_j)_{j=1}^n$, for $D_j = -i\partial x_j$, and $K$ is a connected compact set which coincides with the closure of its interior, then by Sobolev's lemma (cf. [K1, Lemma 2]) the space $\mathcal{E}^P_{\omega,\lambda}(K)$ defined in (2.3) coincides with the space $\mathcal{E}_{\omega,\lambda}(K)$ defined in (2.2), which is a Banach space. Therefore $P$ satisfies condition $(\mathscr{C})$.

More generally, we can take $P_j(D) = \sum_{h=1}^n c_{hj} D_h$ for a constant invertible matrix $(c_{hj})_{1\le h,j\le n}$ and infer that $P = (P_j(D))_{j=1}^n$ satisfies condition $(\mathscr{C})$.

Example 2.7. If $P_j(D) = D_1$ for all $1 \le j \le N$, then the space $\mathcal{E}^P_{\omega,\lambda}(K)$ is not a Banach space, so that the system $P = (P_j(D))_{j=1}^N$ does not satisfy condition $(\mathscr{C})$.

For further comments about condition $(\mathscr{C})$ and its relation to hypoellipticity, we consider a compact exhaustion $\{K_\ell\}_{\ell\in\mathbb{N}}$ of $\Omega$, i.e. a sequence of compact subsets of $\Omega$ with $K_\ell \subset \mathring{K}_{\ell+1}$ and $\bigcup_\ell K_\ell = \Omega$. We can write

$$\mathcal{E}^P_{(\omega)}(\Omega) = \operatorname*{proj}_{\overleftarrow{\ell\in\mathbb{N}}} \operatorname*{proj}_{\overleftarrow{m\in\mathbb{N}}} \mathcal{E}^P_{\omega,m}(K_\ell) = \operatorname*{proj}_{\overleftarrow{\ell\in\mathbb{N}}} \mathcal{E}^P_{\omega,\ell}(K_\ell). \tag{2.6}$$

Remark 2.8. If condition $(\mathscr{C})$ is satisfied, then $\mathcal{E}^P_{(\omega)}(\Omega)$, endowed with the metrizable local convex topology defined by the fundamental system of seminorms $\{p^P_{K_\ell,\ell}\}_{\ell\in\mathbb{N}}$, is a Fréchet space. On the other hand, condition $(\mathscr{C})$ does not guarantee that $\mathcal{E}^P_{\{\omega\}}(\Omega)$ is complete.

However, if $P = (P_j(D))_{j=1}^N$ is a system of hypoelliptic operators, then it can be proved, as in [JH, Thm. 3.3], that both $\mathcal{E}^P_{(\omega)}(\Omega)$ and $\mathcal{E}^P_{\{\omega\}}(\Omega)$ are complete.

In the case of a single operator $P = P(D)$ it was proved in [JH, Prop. 3.1] that also the converse is valid: if $\mathcal{E}^P_\omega(\Omega)$ is complete, then $P(D)$ must be hypoelliptic. This is not true in the case of systems. Take, for instance, $P = (D_j)_{j=1}^n$ for $D_j = -i\partial_{x_j}$. Then $\mathcal{E}^P_\omega(\Omega) = \mathcal{E}_\omega(\Omega)$ is complete by [BMT, Prop. 4.9], but the operators $P_j(D) = D_j$ are not hypoelliptic.

Remark 2.9. It is possible to construct a finer locally convex topology that makes $\mathcal{E}^P_\omega(\Omega)$ always complete, without any assumption on the operators.

In the Beurling case we take a compact exhaustion $\{K_\ell\}_{\ell\in\mathbb{N}}$ of $\Omega$, set

$$p_\ell(u) := \sup_{|\alpha|\le\ell} \sup_{x\in K_\ell} |D^\alpha u(x)|,$$

and then consider the seminorm

$$\tau^P_\ell(u) := \max\{p^P_{K_\ell,\ell}(u), p_\ell(u)\}.$$

We see that $\mathcal{E}^P_{(\omega)}(\Omega)$, endowed with the locally convex topology defined by the fundamental system of seminorms $\{\tau^P_\ell\}_{\ell\in\mathbb{N}}$, is a Fréchet space. The proof is standard.

In the Roumieu case we consider, for $\ell \in \mathbb{N}$ and $K \subset\subset \Omega$, the fundamental system of seminorms $\{\tau^P_{K,\ell,m}\}_{m\in\mathbb{N}}$ defined by

$$\tau^P_{K,\ell,m}(u) := \max\Big\{p^P_{K,1/\ell}(u), \sup_{|\alpha|\le m} \sup_{x\in K} |D^\alpha u(x)|\Big\}. \tag{2.7}$$

This makes $\mathcal{E}^P_{\omega,1/\ell}(K)$ a Fréchet space. Considering then on $\mathcal{E}^P_{\{\omega\}}(\Omega)$ the topology induced by (2.5), we can prove, as in [JH, Prop. 3.5], that $\mathcal{E}^P_{\{\omega\}}(\Omega)$ is complete.

We now want to look for sufficient and necessary conditions in order to obtain the Theorem of the Iterates for systems $P = (P_j(D))_{j=1}^N$ of linear partial differential operators with constant coefficients in the classes of $\omega$-ultradifferentiable functions.

**3. A sufficient condition.** Analogously to [BC1], we give the following:

DEFINITION 3.1. Let $P = (P_j(D))_{j=1}^N$ be a system of linear partial differential operators with constant coefficients of order $m$. We say that $P$ satisfies *condition* $(\mathscr{H})$ if there exist $C > 0$ and $\gamma \geq m$ such that

$$\sum_{j=1}^{N} |P_j^{(\alpha)}(\xi)| \leq C\Big(1 + \sum_{j=1}^{N} |P_j(\xi)|\Big)^{1-|\alpha|/\gamma}, \quad \forall \alpha \in \mathbb{N}_0^n,\ \xi \in \mathbb{R}^n, \tag{3.1}$$

where $P_j^{(\alpha)}(\xi) = \partial_\xi^\alpha P_j(\xi)$.

REMARK 3.2. If the system $P = (P_j(D))_{j=1}^N$ satisfies condition $(\mathscr{H})$ for some $\gamma \geq m$, there exists a smallest $\gamma_P \geq m$ such that $P$ satisfies (3.1) for $\gamma = \gamma_P$; moreover $\gamma_P \in \mathbb{Q}$. Indeed, the inequality (3.1) implies that there exists $C' > 0$ such that

$$|\operatorname{grad} P_i(\xi)| \leq C'\Big(1 + \sum_{j=1}^{N} |P_j(\xi)|\Big)^{1-1/\gamma}, \quad \forall i = 1, \ldots, N. \tag{3.2}$$

Applying then the Tarski–Seidenberg theorem to the semialgebraic function

$$M_i(\lambda) = \sup_{\sum_{j=1}^N |P_j(\xi)| = \lambda} |\operatorname{grad} P_i(\xi)|,$$

we can argue as in [H1, Thm. 3.1] to prove that for every $i \in \{1, \ldots, N\}$ there exists a smallest $\gamma_i$ such that

$$|P_i^{(\alpha)}(\xi)| \leq C\Big(1 + \sum_{j=1}^{N} |P_j(\xi)|\Big)^{1-|\alpha|/\gamma_i}, \quad \forall \alpha \in \mathbb{N}_0^n,\ \xi \in \mathbb{R}^n. \tag{3.3}$$

Then $\gamma_P := \max\{\gamma_1, \ldots, \gamma_N\}$ is the smallest $\gamma$ satisfying (3.1) and moreover $\gamma_P \in \mathbb{Q}$ and $\gamma_P \geq m$.

In what follows, for a system $P$ satisfying condition $(\mathscr{H})$, we shall always refer to $\gamma_P$ as defined in Remark 3.2.

REMARK 3.3. If $P = P(D)$ is a hypoelliptic operator, then condition $(\mathscr{H})$ is satisfied because of [H1, Thm. 3.1]. However, in general condition $(\mathscr{H})$ is weaker than hypoellipticity. Take for instance in $\mathbb{R}^2$ the operator $P(D) = P(D_1, D_2) = D_1^2$. It is trivially not hypoelliptic, but it satisfies condition $(\mathscr{H})$ for $\gamma = 2$.

More generally, if $P = (P_j(D))_{j=1}^N$ is a system of hypoelliptic operators, then $P$ satisfies condition $(\mathscr{H})$. If the system $P$ is elliptic, i.e.

$$|\xi|^m \leq C\Big(1 + \sum_{j=1}^{N} |P_j(\xi)|\Big), \quad \forall \xi \in \mathbb{R}^n, \tag{3.4}$$

then condition $(\mathscr{H})$ is satisfied for $\gamma_P = m$.

In order to compare, for two given systems $P = (P_j(D))_{j=1}^N$ and $Q = (Q_j(D))_{j=1}^M$, the corresponding spaces $\mathcal{E}_\omega^P(\Omega)$ and $\mathcal{E}_\sigma^Q(\Omega)$, we introduce the following:

Definition 3.4. Let $P = (P_j(D))_{j=1}^N$ and $Q = (Q_j(D))_{j=1}^M$ be systems of linear partial differential operators with constant coefficients. If there exist $C, h > 0$ such that

$$\sum_{j=1}^{M} |Q_j(\xi)| \leq C\Big(1 + \sum_{j=1}^{N} |P_j(\xi)|\Big)^h, \quad \forall \xi \in \mathbb{R}^n, \tag{3.5}$$

we say that $Q$ is *$h$-weaker* than $P$, and we write $Q \prec_h P$.

Remark 3.5. If $P = P(D)$ and $Q = Q(D)$ are single operators and $P(D)$ is hypoelliptic, then by [H1, Thm. 3.2] there is a smallest $h$ such that $Q$ is $h$-weaker than $P$, and moreover $h \in \mathbb{Q}$.

More generally, if $Q = (Q_j(D))_{j=1}^M$ is $h$-weaker than $P = (P_j(D))_{j=1}^N$, then there exists a smallest $h > 0$ such that (3.5) is satisfied and moreover $h \in \mathbb{Q}$. Indeed, we can argue as in [H1, Thm. 3.2] and Remark 3.2, taking the semialgebraic functions

$$M_i(\lambda) = \sup_{\sum_{j=1}^N |P_j(\xi)| = \lambda} |Q_i(\xi)|.$$

Definition 3.6. If $P = (P_j(D))_{j=1}^N$ and $Q = (Q_j(D))_{j=1}^M$ are systems with $P \prec_h Q$ and $Q \prec_h P$, we say that $P$ and $Q$ are *$h$-equally strong*, and we write $P \approx_h Q$.

Remark 3.7. Arguing as in [H1, pg 210], we can easily prove that if $P = (P_j(D))_{j=1}^N$ and $Q = (Q_j(D))_{j=1}^M$ are two systems of order $m$ and $r$ respectively, satisfying condition $(\mathscr{H})$ and 1-equally strong, then $m = r$ and $\gamma_P = \gamma_Q$.

We are now ready to prove the following result:

Theorem 3.8. *Let $P = (P_j(D))_{j=1}^N$ and $Q = (Q_j(D))_{j=1}^M$ be systems of linear partial differential operators with constant coefficients, of order $m$ and $r$ respectively. Assume that $P$ and $Q$ satisfy condition $(\mathscr{H})$ of Definition* 3.1 *and that $Q$ is $h$-weaker than $P$. Let $\Omega$ be an open subset of $\mathbb{R}^n$. Let $\omega$ be a non-quasianalytic weight function and set $\omega'(t) = \omega(t^{1/s})$ for $s \geq \gamma_P/m$. Then*

$$\mathcal{E}_{(\omega')}^P(\Omega) \subseteq \mathcal{E}_{(\sigma')}^Q(\Omega), \tag{3.6}$$

$$\mathcal{E}_{\{\omega'\}}^P(\Omega) \subseteq \mathcal{E}_{\{\sigma'\}}^Q(\Omega), \tag{3.7}$$

*for $\sigma'(t) = \omega'(t^{r/(mh)}) = \omega(t^{r/(smh)})$.*

*Proof. Beurling case.* Let $u \in \mathcal{E}^P_{(\omega')}(\Omega)$. For every compact $K \subset \Omega$ there exist an open set $F$ relatively compact in $\Omega$ and $\delta > 0$ such that

$$K \subset F_{(M+1)\delta} \subset F \subset \Omega,$$

where

$$F_\sigma := \{x \in F : d(x, \partial F) > \sigma\}.$$

Note that $F_\sigma$ is always well defined, since $F$ is bounded in $\mathbb{R}^n$.

Moreover, for every $q \in \mathbb{N}$ there exists $C_q > 0$ such that

$$\sum_{|\beta|=\ell} \|P^\beta u\|_{L^2(F)} \le C_q e^{q\varphi^*_{\omega'}(\ell m/q)} = C_q e^{q\varphi^*_\omega(\ell ms/q)}, \quad \forall \ell \in \mathbb{N}, \tag{3.8}$$

by the definition of $\mathcal{E}^P_{(\omega')}(\Omega)$ and by (2.1).

By assumption $Q \prec_h P$ and, by Remark 3.5, there exists $\mu, \nu \in \mathbb{N}$ such that $h = \mu/\nu$.

Arguing as in [BC1, Thm. 2.4], we fix $\alpha \in \mathbb{N}_0^M$, choose $k_j, \ell_j \in \mathbb{N}_0$ such that $\alpha_j = k_j\nu + \ell_j$, with $l_j \le \nu - 1$, for $1 \le j \le M$, and set $k = \sum_{j=1}^M k_j$. From [BC1, formula (2.12)] there exist $C_1, C_2 > 0$ such that for every $u \in C^\infty(F)$,

$$\|Q^\alpha u\|_{L^2(F_{(M+1)\delta})} \le C_1^M \Bigg[\sum_{i=0}^M \binom{M}{i} M^i C_2^{k+i} \sum_{|\beta| \le k+i} \binom{k+i}{|\beta|} \cdot \left(\frac{k+i}{\delta}\right)^{(k+i-|\beta|)\gamma_P \mu} \|P^{\beta\mu} u\|_{L^2(F)}\Bigg]. \tag{3.9}$$

If $\gamma_P \le sm$, then from (3.8) we obtain, for all $\ell \le k$,

$$\begin{aligned} k^{(k-\ell)\gamma_P\mu} \sum_{|\beta|=\ell} \|P^{\beta\mu} u\|_{L^2(F)} &\le C_q k^{(k-\ell)sm\mu} e^{q\varphi^*_\omega(m\ell\mu s/q)} \\ &\le C_q \left(1 + \frac{\ell}{k-\ell}\right)^{\frac{k-\ell}{\ell} sm\mu\ell} (k-\ell)^{(k-\ell)sm\mu} e^{q\varphi^*_\omega(m\ell\mu s/q)} \\ &\le C_q e^{sm\mu\ell} [(k-\ell)sm\mu]^{(k-\ell)sm\mu} e^{q\varphi^*_\omega(m\ell\mu s/q)}. \end{aligned} \tag{3.10}$$

Since $\omega(t)$ is a non-quasianalytic weight function, condition $(\beta)$ implies $\omega(t) = o(t)$, and hence for every $q' \in \mathbb{N}$ there exists $C_{q'} > 0$ such that from [AJO, Rem. 2.4],

$$y \log y \le y + q' \varphi^*_\omega\left(\frac{y}{q'}\right) + C_{q'}, \quad \forall y > 0. \tag{3.11}$$

Applying the above inequality to (3.10) we find that

$$k^{(k-\ell)\gamma_P\mu}\sum_{|\beta|=\ell}\|P^{\beta\mu}u\|_{L^2(F)} \leq C_q e^{sm\mu\ell}e^{(k-\ell)sm\mu}e^{q'\varphi^*_\omega((k-\ell)m\mu s/q')}e^{C_{q'}}e^{q\varphi^*_\omega(m\ell\mu s/q)}. \tag{3.12}$$

By condition $(\alpha)$ of Definition 2.1 there exists $\tilde{L}>0$ such that

$$\omega(et)\leq \tilde{L}(1+\omega(t)), \qquad \forall t\geq 0.$$

Then from [BJ3, Prop. 21(e) and Rem. 22] we find that for every $\rho,\lambda>0$ there exist $\lambda', D_{\rho,\lambda}>0$ such that

$$\rho^j e^{\lambda\varphi^*_\omega(j/\lambda)} \leq D_{\rho,\lambda}e^{\lambda'\varphi^*_\omega(j/\lambda')}, \qquad \forall j\in\mathbb{N}_0, \tag{3.13}$$

with $\lambda'=\lambda/\tilde{L}^{[\log\rho+1]}$ and $D_{\rho,\lambda}=\exp\{\lambda[\log\rho+1]\}$, where $[x]$ is the integer part of $x$.

Applying (3.13) in (3.12) we find that for every $\lambda>0$ there exists $C_\lambda>0$ such that

$$k^{(k-\ell)\gamma_P\mu}\sum_{|\beta|=\ell}\|P^{\beta\mu}u\|_{L^2(F)} \leq C_\lambda e^{\lambda\varphi^*_\omega(m\ell\mu s/\lambda)}e^{\lambda\varphi^*_\omega((k-\ell)m\mu s/\lambda)}. \tag{3.14}$$

From condition $(\alpha)$ of Definition 2.1, by [BMT, Lemma 1.2] we see that there exists $L'>0$ such that

$$\omega(u+v)\leq L'(\omega(u)+\omega(v)+1), \qquad \forall u,v\geq 0,$$

and hence for all $j,k\in\mathbb{N}_0$ and $\lambda>0$,

$$\begin{aligned} e^{\lambda\varphi^*(j/\lambda)+\lambda\varphi^*(k/\lambda)} &= \sup_{s\geq 0}e^{js-\lambda\varphi_\omega(s)}\cdot\sup_{t\geq 0}e^{kt-\lambda\varphi_\omega(t)} \\ &= \sup_{u,v\geq 1}e^{j\log u+k\log v-\lambda(\omega(u)+\omega(v))} \\ &\leq \sup_{u,v\geq 1}u^jv^ke^{-\frac{\lambda}{L'}\omega(u+v)}e^\lambda \leq e^\lambda\sup_{u,v\geq 1}(u+v)^{j+k}e^{-\frac{\lambda}{L'}\omega(u+v)} \\ &\leq e^\lambda\sup_{\sigma\geq 0}e^{(j+k)\sigma-\frac{\lambda}{L'}\varphi_\omega(\sigma)} = e^\lambda e^{\frac{\lambda}{L'}\varphi^*_\omega(\frac{j+k}{\lambda/L'})}. \end{aligned} \tag{3.15}$$

Applying this to (3.14) we find that for every $\tilde{q}\in\mathbb{N}$ there exists $C_{\tilde{q}}>0$ such that for all $\ell\leq k$,

$$k^{(k-\ell)\gamma_P\mu}\sum_{|\beta|=\ell}\|P^{\beta\mu}u\|_{L^2(F)} \leq C_{\tilde{q}}e^{\tilde{q}\varphi^*_\omega(k\mu ms/\tilde{q})}. \tag{3.16}$$

Substituting in (3.9) we obtain, for some constant $A > 0$,

$$\begin{aligned}(3.17)\qquad \|Q^\alpha u\|_{L^2(F_{(M+1)\delta})} &\le A^k C_{\tilde q} e^{\tilde q \varphi^*_\omega((k+M)\mu ms/\tilde q)}\\ &\le A^k C_{\tilde q} e^{\frac{\tilde q}{2}\varphi^*_\omega(2k\mu ms/\tilde q)} e^{\frac{\tilde q}{2}\varphi^*_\omega(2M\mu ms/\tilde q)}\\ &\le C'_{\tilde q} A^{\mu msk} e^{\frac{\tilde q}{2}\varphi^*_\omega(\frac{k\mu ms}{\tilde q/2})}\\ &\le C_{\tilde q'} e^{\tilde q'\varphi^*_\omega(k\mu ms/\tilde q')}\end{aligned}$$

by the convexity of $\varphi^*_\omega$ and by (3.13), for $\tilde q' = \tilde q/(2\tilde L^{[\log A+1]})$.

Since $k \le |\alpha|/\nu$ by construction, from (3.17) we thus conclude that for every $q \in \mathbb{N}$ there exists $D_q > 0$ such that

$$\begin{aligned}\|Q^\alpha u\|_{L^2(K)} &\le \|Q^\alpha u\|_{L^2(F_{(M+1)\delta})}\\ &\le D_q e^{q\varphi^*_\omega(|\alpha|\mu ms/(\nu q))} = D_q e^{q\varphi^*_{\sigma'}(|\alpha| r/q)}, \qquad \forall \alpha \in \mathbb{N}_0^M,\end{aligned}$$

by (2.1), since $\sigma'(t) = \omega(t^{r/(smh)})$. This proves that $u \in \mathcal{E}^Q_{(\sigma')}(\Omega)$.

*Roumieu case.* It is similar to the Beurling case: in (3.8) we take $(1/q)\varphi^*_{\omega'}(\ell mq)$ instead of $q\varphi^*_{\omega'}(\ell m/q)$ and a fixed constant $C$ instead of $C_q$, and similarly later on for $q', q'', \ldots$.

The proof is complete. ■

COROLLARY 3.9. *Let $P = (P_j(D))_{j=1}^N$ and $Q = (Q_j(D))_{j=1}^M$ be systems of order $m$ satisfying condition $(\mathscr{H})$ and 1-equally strong. Let $\Omega$ be an open subset of $\mathbb{R}^n$. Let $\omega$ be a non-quasianalytic weight function and set $\omega'(t) = \omega(t^{1/s})$ for $s \ge \gamma_P/m = \gamma_Q/m$. Then*

$$\mathcal{E}^P_{(\omega')}(\Omega) = \mathcal{E}^Q_{(\omega')}(\Omega) \quad \text{and} \quad \mathcal{E}^P_{\{\omega'\}}(\Omega) = \mathcal{E}^Q_{\{\omega'\}}(\Omega).$$

From Remark 3.3 we obtain the Theorem of the Iterates as a corollary of Theorem 3.8:

COROLLARY 3.10. *Let $P = (P_j(D))_{j=1}^N$ be an elliptic system of order $m$. Let $\Omega$ be an open subset of $\mathbb{R}^n$ and $\omega$ a non-quasianalytic weight function. Then*

$$(3.18)\qquad \mathcal{E}^P_{(\omega)}(\Omega) = \mathcal{E}_{(\omega)}(\Omega) \quad \text{and} \quad \mathcal{E}^P_{\{\omega\}}(\Omega) = \mathcal{E}_{\{\omega\}}(\Omega).$$

*Proof. Beurling case.* Let us first prove the inclusion

$$(3.19)\qquad \mathcal{E}^P_{(\omega)}(\Omega) \subseteq \mathcal{E}_{(\omega)}(\Omega).$$

To this end we consider the system $Q = (D_j)_{j=1}^n$ for $D_j = -i\partial_{x_j}$. The operators $Q_j(D) = D_j$ are not hypoelliptic, but the system $Q$ satisfies condition $(\mathscr{H})$. The system $P$ satisfies (3.4) and hence condition $(\mathscr{H})$ for $\gamma_P = m$, by Remark 3.3.

Since (3.4) implies that $Q$ is $1/m$-weaker than $P$, from Theorem 3.8, with $s = 1 = \gamma_P/m$ and hence $\omega'(t) = \omega(t)$, we find that

$$\mathcal{E}^P_{(\omega)}(\Omega) \subseteq \mathcal{E}^Q_{(\sigma)}(\Omega) = \mathcal{E}_{(\sigma)}(\Omega), \tag{3.20}$$

for $\sigma(t) = \omega(t^{1/(m\cdot\frac{1}{m})}) = \omega(t)$, and hence (3.19) is proved.

Conversely, since every $P_j(\xi)$ is a polynomial of degree $m$, we clearly see that $P$ is $m$-weaker than $Q$ and, from Theorem 3.8,

$$\mathcal{E}_{(\omega)}(\Omega) = \mathcal{E}^Q_{(\omega)}(\Omega) \subseteq \mathcal{E}^P_{(\sigma)}(\Omega)$$

for $\sigma(t) = \omega(t^{\frac{m}{1\cdot m}}) = \omega(t)$, so that also the opposite inclusion

$$\mathcal{E}_{(\omega)}(\Omega) \subseteq \mathcal{E}^P_{(\omega)}(\Omega)$$

is valid, and hence the equality (3.18) is proved in the Beurling case.

*Roumieu case.* The proof is the same as in the Beurling case, using (3.7) instead of (3.6). ∎

EXAMPLE 3.11. Let us consider in $\mathbb{R}^2$ the system $P = (P_j(D))_{j=1}^2$ defined by

$$P_1(D_1, D_2) = D_1^2, \qquad P_2(D_1, D_2) = D_2^2.$$

These operators are not hypoelliptic but the system $P$ satisfies conditon $(\mathscr{H})$ for $\gamma_P = 2$.

Let us next consider $Q = Q(D) = \Delta = -D_1^2 - D_2^2$. This is an elliptic operator of order 2, and hence satisfies condition $(\mathscr{H})$ for $\gamma_Q = 2$ (see Remark 3.3).

Moreover, $P$ and $Q$ are 1-equally strong and $\gamma_P/m = 1$. We can then apply Corollaries 3.9 and 3.10 with $\omega'(t) = \omega(t)$ and deduce that, for any open subset $\Omega$ of $\mathbb{R}^2$ and for every non-quasianalytic weight function $\omega$,

$$\mathcal{E}^P_{(\omega)}(\Omega) = \mathcal{E}^Q_{(\omega)}(\Omega) = \mathcal{E}_{(\omega)}(\Omega).$$

This means that the elements $u \in \mathcal{E}_{(\omega)}(\Omega)$ can be equivalently determined by estimating their derivatives $D^\alpha u(x) = D_1^{\alpha_1} D_2^{\alpha_2} u(x)$, or the iterates of $Q(D)$, i.e. $\Delta^\beta u(x)$, or the iterates of the system $P = (P_j(D))_{j=1}^2$, i.e. $P^\gamma u(x) = D_1^{2\gamma_1} D_2^{2\gamma_2} u(x)$ for $\alpha, \gamma \in \mathbb{N}_0^2$, $\beta \in \mathbb{N}_0$.

The same holds also in the Roumieu case.

**4. A necessary condition.** In order to obtain a necessary condition for the inclusions (3.6) or (3.7), we first need to introduce the following:

DEFINITION 4.1. We say that a non-quasianalytic weight function $\omega$ satisfies the growth *condition B-M-M* if there exists a constant $H \geq 1$ such that

$$2\omega(t) \leq \omega(Ht) + H, \qquad \forall t \geq 0. \tag{4.1}$$

REMARK 4.2. Condition B-M-M was introduced in [BMM] in order to characterize those weight functions $\omega$ for which $\mathcal{E}_{(\omega)}(\Omega)$ (or $\mathcal{E}_{\{\omega\}}(\Omega)$) can also be considered as a Denjoy–Carleman class $\mathcal{E}_{(M_p)}(\Omega)$ (or $\mathcal{E}_{\{M_p\}}(\Omega)$, respectively) as defined in [K2], for some sequence $\{M_p\}$.

Gevrey weights satisfy condition B-M-M.

Let us now prove that the condition $Q \prec_h P$ of Theorem 3.8 is also necessary for the inclusions (3.6) and (3.7).

To this end we first recall, from [JH, Lemma 4.7], the following:

LEMMA 4.3. *For all $h, \lambda > 0$ and $t \geq 1$,*

(i) $\sup_{j\in\mathbb{N}_0} t^j \exp\{-\lambda\varphi^*(hj/\lambda)\} \leq \exp\{\lambda\omega(t^{1/h})\}$,

(ii) $\sup_{j\in\mathbb{N}_0} t^j \exp\{-\lambda\varphi^*(hj/\lambda)\} \geq \frac{1}{t}\exp\{\lambda\omega(t^{1/h})\}$.

We can then prove:

THEOREM 4.4. *Let $\Omega$ be an open subset of $\mathbb{R}^n$ and $\omega$ a non-quasianalytic weight function satisfying condition B-M-M. Let $P = (P_j(D))_{j=1}^N$ be a system of linear partial differential operators of order $m$ with constant coefficients satisfying condition $(\mathscr{C})$ of Definition 2.4 and let $Q = (Q_j(D))_{j=1}^M$ be a generic system of linear partial differential operators of order $r$ with constant coefficients. If there exists $h > 0$ such that either*

$$\mathcal{E}^P_{(\omega)}(\Omega) \subseteq \mathcal{E}^Q_{(\sigma)}(\Omega), \tag{4.2}$$

*or*

$$\mathcal{E}^P_{\{\omega\}}(\Omega) \subseteq \mathcal{E}^Q_{\{\sigma\}}(\Omega), \tag{4.3}$$

*for $\sigma(t) = \omega(t^{r/(mh)})$, then $Q$ is $h$-weaker than $P$.*

*Proof. Roumieu case.* We follow the ideas of Juan-Huguet [JH], replacing the assumption, in [JH, Thm. 4.5], that the single operator $P(D)$ is hypoelliptic, with the weaker assumption that the system $P$ satisfies condition $(\mathscr{C})$, in the spirit of [BC1].

Let us now assume that (4.3) is satisfied and fix a compact set $K_0 \subset \Omega$ which coincides with the closure of its interior.

We have the following inclusions:

$$\begin{aligned}\mathcal{E}^P_{(\omega)}(\Omega) \subseteq \mathcal{E}^P_{\{\omega\}}(\Omega) &\subseteq \mathcal{E}^Q_{\{\sigma\}}(\Omega)\\ &= \operatorname*{proj}_{\overleftarrow{K\subset\subset\Omega}} \operatorname*{ind}_{\overrightarrow{\ell\in\mathbb{N}}} \mathcal{E}^Q_{\sigma,1/\ell}(K) \subseteq \operatorname*{ind}_{\overrightarrow{\ell\in\mathbb{N}}} \mathcal{E}^Q_{\sigma,1/\ell}(K_0).\end{aligned}$$

By assumption the system $P$ satisfies condition $(\mathscr{C})$, and hence, by Remark 2.8, $\mathcal{E}^P_{(\omega)}(\Omega)$ is a Fréchet space and $\operatorname{ind}_{\overrightarrow{\ell\in\mathbb{N}}} \mathcal{E}^Q_{\sigma,1/\ell}(K_0)$ is an (LF)-space.

We can therefore apply the Closed Graph Theorem and Grothendieck's Factorization Theorem (see [MV, Thms. 24.31 and 24.33]) to conclude that there exists $\ell_0 \in \mathbb{N}$ such that

$$\mathcal{E}^P_{(\omega)}(\Omega) \subseteq \mathcal{E}^Q_{\sigma,1/\ell_0}(K_0)$$

with continuous inclusion. Then there exist a constant $C > 0$, a compact $K \subset\subset \Omega$ and $\lambda > 0$ such that, for all $f \in \mathcal{E}^P_{(\omega)}(\Omega)$,

$$\sup_{\beta\in\mathbb{N}_0^M} \|Q^\beta(D)f\|_{L^2(K_0)} e^{-\frac{1}{\ell_0}\varphi^*_\sigma(|\beta|r\ell_0)} \leq C \sup_{\alpha\in\mathbb{N}_0^N} \|P^\alpha(D)f\|_{L^2(K)} e^{-\lambda\varphi^*_\omega(|\alpha|m/\lambda)}. \tag{4.4}$$

For $\xi \in \mathbb{R}^n$, we denote $f_\xi(x) := e^{i\langle x,\xi\rangle}$ and remark that $f_\xi \in \mathcal{E}^P_{(\omega)}(\Omega)$, because for every compact $K \subset\subset \Omega$ and $\lambda > 0$,

$$\begin{aligned}\|P^\alpha(D)f_\xi\|_{L^2(K)} &= \|P^\alpha(\xi)f_\xi\|_{L^2(K)} \leq m(K)|P^\alpha(\xi)| \\ &\leq C(1+|\xi|^{m|\alpha|}) \leq C_\xi e^{\lambda'\varphi^*_\omega(|\alpha|m/\lambda')}\end{aligned}$$

for some $C_\xi > 0$ and $\lambda' > 0$, by (3.13). Since $f_\xi \in \mathcal{E}^P_{(\omega)}(\Omega)$, we can apply (4.4) to $f_\xi$, obtaining

$$\sup_{\beta\in\mathbb{N}_0^M} |Q^\beta(\xi)| e^{-\frac{1}{\ell_0}\varphi^*_\sigma(|\beta|r\ell_0)} \leq C' \sup_{\alpha\in\mathbb{N}_0^N} |P^\alpha(\xi)| e^{-\lambda\varphi^*_\omega(|\alpha|m/\lambda)} \tag{4.5}$$

for some $C' > 0$. Therefore

$$\begin{aligned} &\sup_{\beta\in\mathbb{N}_0^M} \Big(\sum_{j=1}^M \Big|\frac{Q_j}{M}(\xi)\Big|\Big)^{|\beta|} e^{-\frac{1}{\ell_0}\varphi^*_\sigma(|\beta|r\ell_0)} \qquad (4.6)\\ &\leq \sup_{\beta\in\mathbb{N}_0^M}\Big(\sum_{\beta_1+\cdots+\beta_M=|\beta|} \frac{|\beta|!}{\beta_1!\cdots\beta_M!}|Q_1(\xi)|^{\beta_1}\cdots|Q_M(\xi)|^{\beta_M}\frac{1}{M^{|\beta|}}e^{-\frac{1}{\ell_0}\varphi^*_\sigma(|\beta|r\ell_0)}\Big)\\ &\leq \sup_{\beta\in\mathbb{N}_0^M} |Q^\beta(\xi)| e^{-\frac{1}{\ell_0}\varphi^*_\sigma(|\beta|r\ell_0)} \leq C' \sup_{\alpha\in\mathbb{N}_0^N} |P^\alpha(\xi)| e^{-\lambda\varphi^*_\omega(|\alpha|m/\lambda)}\\ &\leq C'' \sup_{\alpha\in\mathbb{N}_0^N}\Big(\sum_{j=1}^N |P_j(\xi)|\Big)^{|\alpha|} e^{-\lambda\varphi^*_\omega(|\alpha|m/\lambda)}.\end{aligned}$$

From Lemma 4.3 it follows that if

$$\sum_{j=1}^M \Big|\frac{Q_j}{M}(\xi)\Big| \geq 1 \quad \text{and} \quad \sum_{j=1}^N |P_j(\xi)| \geq 1, \tag{4.7}$$

then

$$(4.8)\qquad \Big(\sum_{j=1}^{M}\Big|\frac{Q_j}{M}(\xi)\Big|\Big)^{-1}\exp\Big\{\frac{1}{\ell_0}\sigma\Big(\Big(\sum_{j=1}^{M}\Big|\frac{Q_j}{M}(\xi)\Big|\Big)^{1/r}\Big)\Big\} \le \tilde{C}\exp\Big\{\lambda\omega\Big(\Big(\sum_{j=1}^{N}|P_j(\xi)|\Big)^{1/m}\Big)\Big\}$$

for some $\tilde{C} > 0$.

From property $(\gamma)$ of the weight function $\sigma(t)$ we see that (4.8) implies that if (4.7) holds, then for some $\lambda' > 0$,

$$\exp\Big\{\lambda'\sigma\Big(\Big(\sum_{j=1}^{M}\Big|\frac{Q_j}{M}(\xi)\Big|\Big)^{1/r}\Big)\Big\} \le \tilde{C}\exp\Big\{\lambda\omega\Big(\Big(\sum_{j=1}^{N}|P_j(\xi)|\Big)^{1/m}\Big)\Big\}.$$

Since $\sigma(t) = \omega(t^{r/(mh)})$ by assumption, we thus obtain

$$(4.9)\qquad \omega\Big(\Big(\sum_{j=1}^{M}\Big|\frac{Q_j}{M}(\xi)\Big|\Big)^{1/(mh)}\Big) \le A\Big(1+\omega\Big(\Big(\sum_{j=1}^{N}|P_j(\xi)|\Big)^{1/m}\Big)\Big) \le \omega\Big(A'\Big(\sum_{j=1}^{N}|P_j(\xi)|\Big)^{1/m}\Big)$$

for some $A' > 0$ if (4.7) holds, because condition B-M-M implies that for every $k \in \mathbb{N}$ there exists a constant $H_k \ge 1$ such that $2^{k-1}\omega(t) \le \omega(H_k t)$ for all $t \ge 1$.

Since $\omega(t)$ is increasing, (4.9) implies that there exists a constant $B > 1$ such that if (4.7) holds, then

$$(4.10)\qquad \sum_{j=1}^{M}|Q_j(\xi)| \le B\Big(1+\sum_{j=1}^{N}|P_j(\xi)|\Big)^{h}.$$

However, (4.10) is trivial if (4.7) does not hold, because of (4.5), so that (4.10) is satisfied for all $\xi \in \mathbb{R}^n$ and $Q$ is $h$-weaker than $P$.

*Beurling case.* The proof is similar, but easier, as in the Roumieu case, since $\mathcal{E}^P_{(\omega)}(\Omega)$ and $\mathcal{E}^Q_{(\sigma)}(\Omega)$ are metrizable, and hence the inclusion (4.2) implies (4.4). ∎

REMARK 4.5. By Remark 2.9, instead of condition ($\mathscr{C}$) we can consider, in Theorem 4.4, the weaker assumption that $\mathcal{E}^P_{(\omega)}(\Omega)$ is a Fréchet space and then take on $\mathcal{E}^Q_{\sigma,1/\ell}(K_0)$ the fundamental system of seminorms $\{\tau^Q_{K_0,\ell,m}\}_{m\in\mathbb{N}}$ defined by (2.7), to make $\varinjlim_{\ell\in\mathbb{N}}\mathcal{E}^Q_{\sigma,1/\ell}(K_0)$ an (LF)-space.

As a consequence of Theorem 4.4 we have the converse of Corollary 3.10:

COROLLARY 4.6. *Let $\Omega$ be an open subset of $\mathbb{R}^n$. Let $\omega$ be a non-quasi-analytic weight function satisfying condition B-M-M, and let $P=(P_j(D))_{j=1}^N$*

*be a system of order m satisfying condition* ($\mathscr{C}$)*. If either*

$$\mathcal{E}^P_{(\omega)}(\Omega) \subseteq \mathcal{E}_{(\omega)}(\Omega), \tag{4.11}$$

*or*

$$\mathcal{E}^P_{\{\omega\}}(\Omega) \subseteq \mathcal{E}_{\{\omega\}}(\Omega), \tag{4.12}$$

*then the system P is elliptic.*

*Proof. Beurling case.* Let us consider the system $Q = (D_j)_{j=1}^n$. Then $\mathcal{E}^Q_{(\omega)}(\Omega) = \mathcal{E}_{(\omega)}(\Omega)$ and (4.11) implies (4.2) with $\sigma(t) = \omega(t) = \omega(t^{r/(mh)})$ for $r = 1$ and $h = 1/m$.

By Theorem 4.4 we find that $Q$ is $1/m$-weaker than $P$, i.e.

$$\sum_{j=1}^n |\xi_j| \le C\Big(1 + \sum_{j=1}^N |P_j(\xi)|\Big)^{1/m}, \quad \forall \xi \in \mathbb{R}^n.$$

This proves that the system $P$ is elliptic, and hence the corollary is proved.

*Roumieu case.* The proof is similar to that in the Beurling case, using (4.12) and (4.3) instead of (4.11) and (4.2). ∎

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