ACTA ARITHMETICA
177.1 (2017)

Diophantine exponents for standard linear actions of SL_2 over discrete rings in $\mathbb{C}$

by

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1. Introduction. The set $\mathbb{Q}$ of rational numbers is dense in $\mathbb{R}$. However, one of the first works which tried to quantify this density came only in the nineteenth century from Dirichlet who stated that for any real number θ and all $Q > 1$, there exist integers p and q, $1 \leq q < Q$, such that

$$|q\theta - p| < \frac{1}{Q} < \frac{1}{q}. \tag{1.1}$$

This is a consequence of the pigeonhole principle (also known as Dirichlet's box principle). The inhomogeneous version was given by Minkowski using geometry of numbers: For any $\theta \in \mathbb{R} \setminus \mathbb{Q}$, and $\alpha \notin \mathbb{Z}\theta + \mathbb{Z}$, there exist integers p and q for which

$$|q\theta - \alpha - p| < \frac{1}{4|q|}. \tag{1.2}$$

This second statement, besides being only true for irrational θ, also has a weaker error estimate than in Dirichlet's theorem where it is in terms of $Q^{-1} < q^{-1}$ ($\Leftrightarrow q < Q$). If we now take two such inequalities with different α's, we are in the realm of simultaneous inhomogeneous Diophantine approximation [Cas57]. In other words, we are looking for infinitely many integral solutions (p_1, q_1, p_2, q_2) to the system of inequalities

$$\max\{|q_1\theta - p_1 - \alpha_1|, |q_2\theta - p_2 - \alpha_2|\} < \varepsilon. \tag{1.3}$$

An extra demand that the pairs (q_1, p_1) and (q_2, p_2) be primitive can be fulfilled by requiring that

$$\begin{pmatrix} q_1 & p_1 \\ q_2 & p_2 \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}). \tag{1.4}$$

2010 *Mathematics Subject Classification*: Primary 11J20, 11J13, 11A55; Secondary 22Fxx.
Key words and phrases: Diophantine approximation, asymptotic and uniform exponents, lattice subgroups.
Received 1 December 2015.
Published online 23 December 2016.

DOI: 10.4064/aa8370-6-2016

In recent times, mathematicians have been interested in understanding more generally the nature of dense orbits for the action of a group G on a homogeneous space X. Laurent [Lau] surveys various quantitative results regarding dense lattice orbits. In this connection, see also the work of Ghosh, Gorodnik and Nevo [GGN, GGN15] who relate the rate of approximation by 'rational points' on a homogeneous space X of a semisimple group G to the automorphic representations of G and compute the exact exponents for a number of examples. We point out upfront that their exponent κ_Γ is exactly the inverse of the value $\hat{\mu}_\Gamma$ introduced in Definition 1.2 below.

Let K be any number field whose ring of integers $\mathcal{O}_K$ is a discrete subring of $\mathbb{C}$. In addition, we require that any complex number z should be within unit distance of some element of $\mathcal{O}_K$. The only such rings correspond to the rings of integers for the quadratic number fields $\mathbb{Q}(\sqrt{-d})$ where $d = 1, 2, 3, 7$ or 11 (see [Dan15, Remark 2.4]). By $\Gamma = \mathrm{SL}_2(\mathcal{O}_K)$, we denote the lattice of special linear 2×2 matrices with entries in $\mathcal{O}_K$. Consider its action on the punctured complex plane $\mathbb{C}^2 \setminus \{0\}$ via matrix multiplication on the left. Abusing notation, we use $|\cdot|$ with matrices as well as complex numbers: for any matrix A, we let $|A|$ be the maximum of the modulus of its entries, while for z a complex number, $|z|$ stands for the Euclidean distance to the origin. We use lowercase Greek and both upper and lowercase Roman letters for various operating matrices, and vectors will be set in boldface (e.g. $\mathbf{z}$). The following terminology is motivated by Bugeaud and Laurent [BL05].

DEFINITION 1.1. Let $\mathbf{z}, \mathbf{y} \in \mathbb{C}^2$. The *asymptotic Diophantine exponent* $\mu_\Gamma(\mathbf{z}, \mathbf{y})$ is the quantity

$$\sup\{\omega \mid |\gamma\mathbf{z} - \mathbf{y}| \le |\gamma|^{-\omega} \text{ has infinitely many solutions in } \gamma \in \Gamma\}.$$

DEFINITION 1.2. The *uniform Diophantine exponent* $\hat{\mu}_\Gamma(\mathbf{z}, \mathbf{y})$ is the supremum of all ω's for which the system of inequalities

$$|\gamma\mathbf{z} - \mathbf{y}| \le T^{-\omega}, \quad |\gamma| \le T$$

has solutions for *all* T sufficiently large.

In this language, the results of Dirichlet and Minkowski respectively say that the uniform exponent for approximating $(\theta, 0)$ and the asymptotic exponent for approximating (θ, α) using an integral pair (p, q) are both ≥ 1. Further, measure-theoretic considerations dictate that in both cases, equality $(= 1)$ holds except on some set of Lebesgue measure zero. Proposition 3.13 and the subsequent discussion give us an analogue of Minkowski's theorem for approximating a complex pair (ξ, z) with the help of $\mathcal{O}_K$-integers.

It follows from the definitions that $\mu_\Gamma(\mathbf{z}, \mathbf{y}) \ge \hat{\mu}_\Gamma(\mathbf{z}, \mathbf{y})$ for all $\mathbf{z}, \mathbf{y} \in \mathbb{C}^2$. For the analogous situation of $\mathrm{SL}_2(\mathbb{Z})$ acting on $\mathbb{R}^2$, Laurent and Nogueira [LN12a] came up with estimates for the exponents defined above, and in some cases proved the lower and upper bounds to be equal, which turn out to be functions of the irrationality measures of the slopes of the starting and target points. For approximation in our setting, we follow in their footsteps to a large extent. When $d = 1$, so that $\mathcal{O}_K$ is the ring of Gaussian integers, a continued fraction expansion algorithm for complex numbers with partial quotients from $\mathcal{O}_K$ was given by Hurwitz [Hur87]. It is used to construct certain *convergent matrices*. The case $d = 3$ has the ring of Eisenstein integers as its integral ring, and we have an analogous nearest (Eisenstein) integer algorithm [Dan15]. These help us to approach any fixed target point $\mathbf{y} \in \mathbb{C}^2$ starting from some 'irrational' vector $\mathbf{z} = (z_1, z_2)^t \in \mathbb{C}^2$ both of whose coordinates are non-zero and the slope $\xi = z_1/z_2$ is in $\mathbb{C}' := \mathbb{C} \setminus K$. Note that $\mathbb{C}'$ is a full measure subset of $\mathbb{C}$, as K is only countable.

We give the following general definition inspired from that of irrationality measure for real irrational numbers.

Definition 1.3. The *K-irrationality measure* $\omega_K(z)$ for any $z \in \mathbb{C}'$ is the supremum of all real ω such that the inequality

$$|qz - p| \leq \frac{1}{|q|^{\omega}}$$

has infinitely many solutions in $p \in \mathcal{O}_K,\ q \in \mathcal{O}_K \setminus \{0\}$.

Sullivan [Sul82, Theorem 1] amongst others has formulated and proved the Khintchine theorem for Diophantine approximation of complex numbers by rationals from some fixed imaginary quadratic extension of $\mathbb{Q}$. In particular, his result implies that for all fields K considered here, the irrationality measure $\omega_K(z)$ is an almost everywhere constant function on $\mathbb{C}'$ with respect to the induced Lebesgue measure. Using the convergence case of the Borel–Cantelli lemma along with Dirichlet's box principle (see [Cas57, p. 1] and also Lemma 3.3 below), one can independently verify that its generic value is 1, and greater than 1 everywhere else. At this point, we remind the reader that the exponents μ_Γ and $\hat{\mu}_\Gamma$ defined above are invariant under the Γ-action, and therefore constant almost everywhere owing to the ergodicity of the action.

For non-negative functions f and g, the Vinogradov notation $f \ll g$ (similarly $f \gg g$) means that there exists some $C > 0$ for which $f(x) \leq Cg(x)$ for all x in the domain. When both $f \ll g$ and $g \ll f$, we write $f \asymp g$. The dependence of the implicit constant on some ambient parameters $a, b, c, \ldots$ will often be indicated in the subscript as $\ll_{a,b,c,\ldots}$.

The main result of this paper is given below.

Theorem 1.4. *Let K be $\mathbb{Q}(\sqrt{-d})$ where $d = 1, 2, 3, 7$ or 11. Also, suppose that a continued fraction algorithm for approximating an arbitrary com-*

plex number z with elements of K exists and has the following properties for all $n \gg 0$:

(1) *the denominators of the convergents rise monotonically, i.e., $|q_{n+1}| > |q_n|$, and*
(2) *there exist $r_0 \in \mathbb{N}$ and $\theta > 1$ for which $|q_{n+r_0}| \geq \theta |q_n|$.*

Then, for the full measure subset $\{\mathbf{z} = (z_1, z_2)^t \in \mathbb{C}^2 \mid z_1/z_2 \in \mathbb{C} \setminus K, \omega_K(z_1/z_2) = 1\}$ and $\Gamma = \mathrm{SL}_2(\mathcal{O}_K)$ acting linearly on the complex plane:

(i) *the exponent of approximation to the origin is $\mu_\Gamma(\mathbf{z}, \mathbf{0}) = \hat{\mu}_\Gamma(\mathbf{z}, \mathbf{0}) = 1$,*
(ii) *for almost all target points $\mathbf{y}$ with slope $y = y_1/y_2 \in \mathbb{C} \setminus K$ and $\omega_K(y) = 1$,*

$$1/3 \leq \hat{\mu}_\Gamma(\mathbf{z}, \mathbf{y}) \leq \mu_\Gamma(\mathbf{z}, \mathbf{y}) \leq 1/2, \tag{1.5}$$

(iii) *for target points $\mathbf{y}$ with slope $y \in K$,*

$$\mu_\Gamma(\mathbf{z}, \mathbf{y}) = \hat{\mu}_\Gamma(\mathbf{z}, \mathbf{y}) = 1/2. \tag{1.6}$$

While the discreteness of $\mathcal{O}_K$ (which is ensured by taking $K = \mathbb{Q}(\sqrt{-d})$, d as above) immediately implies that of $\mathrm{SL}_2(\mathcal{O}_K)$ and is also used in working out the generic K-irrationality measure ω_K, the exponential growth of denominators in the continued fraction algorithm helps in bounding the various intermediate matrices properly.

We emphasize again that the results of Laurent and Nogueira [LN12a] are valid for all starting points $\mathbf{x} \in \mathbb{R}^2 \setminus \{\mathbf{0}\}$ with irrational slopes and all target points, while we have nice answers only on a full measure subset of starting points where $\omega_K(z_1/z_2) = 1$. Furthermore, we get a reasonable lower bound for target points $\mathbf{y}$ with a K-irrational slope only when $\omega_K(y_1/y_2) = 1$ and the upper bound of $1/2$ in (1.5) is true for some full measure subset of $\mathbb{C}^2$ (coming from the Borel–Cantelli lemma and perhaps depending on $\mathbf{z}$).

A continued fraction theory as assumed in Theorem 1.4 is provided for $d = 1$ and 3 in [Hur87, Dan15, DN14]. We discuss and suitably modify some of their statements in the next section. The jugglery with approximating matrices comes in Section 3, which, in various parts, proves Theorem 1.4 (Propositions 3.2 for (i), 3.9 & 3.11 for (ii), and 3.15 for (iii), respectively). As a side result, we also record in Proposition 3.1 the value of the exponent for approximating the origin by dense orbits of triangle group actions on the real plane.

2. Continued fractions for complex numbers.

Dani and Nogueira [DN14] considered a family of continued fraction expansions for complex numbers where the partial quotients a_n are in $\mathbb{Z}[i]$, the ring of Gaussian integers. In [Dan15], Dani also dealt with continued fractions in terms of Eisenstein integers $a + b\zeta$ where $a, b \in \mathbb{Z}$ and $\zeta^2 + \zeta = -1$.

In particular, Dani and Nogueira gave the best known results for the rate of approximation by convergents coming from Hurwitz's algorithm. Hurwitz [Hur87] described a simple *nearest integer algorithm* which picks a Gaussian integer $a = a(z)$ nearest to any given complex number z (if there is more than one candidate satisfying the condition, choose any one of them). One then proceeds by induction, setting

$$z_0 = z, \quad a_n = a(z_n) \quad \text{and} \quad z_{n+1} = (z_n - a_n)^{-1}. \tag{2.1}$$

If $z = [a_0, a_1, \ldots]$, then $a_i \in \mathbb{Z}[i]$ for all i and $|a_i| > 1$ for $i \geq 1$. On defining the associated numerator and denominator (of the nth convergent) sequences of Gaussian integers in a recursive fashion as

$$\begin{aligned} p_{-2} &= 0, \quad p_{-1} = 1, \quad p_n = a_n p_{n-1} + p_{n-2} \quad \text{for } n \geq 0, \\ q_{-2} &= 1, \quad q_{-1} = 0, \quad q_n = a_n q_{n-1} + q_{n-2} \quad \text{for } n \geq 0, \end{aligned} \tag{2.2}$$

we ensure the exponential growth of the size of the denominators, namely $|q_n| > |q_{n-1}|$ and $|q_n| \geq \theta |q_{n-2}|$ for all $n \geq 1$, where $\theta = (\sqrt{5} + 1)/2$ [DN14, Corollary 5.3]. The latter guarantees that the distance between the complex number z and its nth convergent is small enough in terms of the size of the denominator q_{n+1} of the succeeding convergent.

We have a similar situation for the Eisenstein integers. Theorem 4.3 of [Dan15] tells us that for the continued fraction expansion with respect to the nearest integer algorithm, we have the monotonous rise of the denominator sizes as well as $|q_n| > 4|q_{n-2}|/3$ for $n \geq 1$. We now give a variant of [DN14, Proposition 2.1].

Lemma 2.1. *Let R be a discrete subring of $\mathbb{C}$, and $\mathrm{Frac}(R)$ its field of fractions. Further, let $\{a_n\} \subset R$ be a sequence which defines a continued fraction expansion of some $z \in \mathbb{C} \setminus \mathrm{Frac}(R)$, and p_n/q_n the corresponding sequence of convergents for which the hypothesis of Theorem* 1.4 *holds. Then there exist $C_1, n_0 > 0$ such that*

$$|q_n z - p_n| \leq C_1/|q_{n+1}| \quad \forall n > n_0. \tag{2.3}$$

Proof. We need to look more closely at the proof in [DN14] which goes through for any discrete ring. There, the authors have argued that

$$\left| z - \frac{p_n}{q_n} \right| \leq \sum_{k=0}^{\infty} \frac{1}{|q_{n+k} q_{n+k+1}|} \leq \frac{C_0}{|q_n|^2}, \quad \text{where} \quad C_0 = \frac{r_0 \theta^2}{\theta^2 - 1}. \tag{2.4}$$

We separate the first term from the series above:

$$\frac{1}{|q_n q_{n+1}|} + \sum_{k=1}^{\infty} \frac{1}{|q_{n+k} q_{n+k+1}|} \leq \frac{1}{|q_n q_{n+1}|} + \frac{C_0}{|q_{n+1}|^2}, \tag{2.5}$$

and the upper bound is arrived at by taking $n = n+1$ in the last step of their calculation. Multiplying (2.4) by $|q_n|$ ($\neq 0$) and recalling that $|q_n| < |q_{n+1}|$, we see that the scaled error $|\epsilon_n| = |q_n z - p_n|$ is $\leq C_1|q_{n+1}|^{-1}$, where $C_1 = C_0 + 1$ is an absolute constant. ■

Let us now try to obtain a lower bound for ϵ_n which is not a priori available. Unlike the simple continued fractions for real numbers, we get a very weak lower estimate for the nth error term. But first, consider two different convergents p_n/q_n and p_{n+r}/q_{n+r} for some $n \geq 0$, $r > 0$ arising from a continued fraction expansion of a fixed $z \in \mathbb{C}\backslash K$, where the associated partial quotients belong to the (discrete) ring of integers $\mathcal{O}_K$. Our claim is that the two convergents are not the same complex number. If they were, let $p_{n+r} = \kappa p_n$ and $q_{n+r} = \kappa q_n$ for some $\kappa \in \mathbb{C} \setminus \{0\}$. As $|q_{n+1}| > |q_n|$ for all n, we get $|\kappa| > 1$. Also,

$$|\kappa(p_n q_{n+r-1} - q_n p_{n+r-1})| = |p_{n+r}q_{n+r-1} - q_{n+r}p_{n+r-1}| = 1, \tag{2.6}$$

and hence the non-zero complex number $p_n q_{n+r-1} - q_n p_{n+r-1} \in \mathcal{O}_K$ has absolute value at least 1, $\mathcal{O}_K$ being a discrete ring. But this is a contradiction.

Lemma 2.2. *With the same notation and conventions as in Lemma* 2.1, *there exist* $C_2 > 0$ *and* $r_1 \in \mathbb{N}$ *such that*

$$|\epsilon_n| \geq C_2/|q_{n+r_1}|.$$

Proof. We apply the triangle inequality to the three numbers z, p_n/q_n and p_{n+r}/q_{n+r}, giving

$$\left|z - \frac{p_n}{q_n}\right| \geq \left|\frac{p_{n+r}}{q_{n+r}} - \frac{p_n}{q_n}\right| - \left|z - \frac{p_{n+r}}{q_{n+r}}\right| \geq \frac{1}{|q_n|\,|q_{n+r}|} - \frac{C_0}{|q_{n+r}|^2} \tag{2.7}$$

where we employ (2.4). This implies that

$$|\epsilon_n| = |q_n z - p_n| \geq \frac{1}{|q_{n+r}|} - \frac{|q_n|}{|q_{n+r}|}\,\frac{C_0}{|q_{n+r}|}. \tag{2.8}$$

Now, $|q_{n+r}| \geq \theta|q_{n+r-r_0}| \geq \cdots \geq \theta^{\lfloor r/r_0 \rfloor}|q_n|$ for all n, r by our assumption. Thus,

$$|\epsilon_n| \geq \frac{1}{|q_{n+r}|} - \frac{C_0}{\theta^{\lfloor r/r_0 \rfloor}|q_{n+r}|}. \tag{2.9}$$

As $\theta > 1$, the constant in the second term on the right side becomes less than 1 for some r_1 sufficiently large, and we get the required lower bound with some constant $C_2 > 0$ and $r_1 \in \mathbb{N}$ depending only on R and the continued fraction algorithm. ■

If $\omega_K(z)$ is finite and $\omega > \omega_K(z)$, then we must have $|q_{n+1}| \leq |q_n|^\omega$ for all $n \geq N_0(\omega)$. Combining Lemmata 2.1 and 2.2, we conclude that for all

$\omega > \omega_K(z)$,

$$\frac{C_2}{|q_{n+1}|^{\omega^{r_1}-1}} \le |\epsilon_n| \le \frac{C_1}{|q_{n+1}|} \tag{2.10}$$

for all large enough n. In addition, we get the usual identity

$$q_n p_{n-1} - p_n q_{n-1} = (-1)^n \tag{2.11}$$

as a bonus from the formal theory of continued fractions. It is this particular property which makes them of use in constructing the so-called *convergent matrices* in the next section.

3. Convergent matrices. Let $\xi \in \mathbb{C} \setminus K$ and let p_k/q_k for $p_k, q_k \in \mathcal{O}_K$ denote the convergent of order k to ξ, due to some continued fraction expansion algorithm which satisfies the hypothesis of Theorem 1.4. The construction of convergent matrices for the complex setting mimics the one for $\mathbb{R}^2$. As in [LN12a], we define the kth convergent matrix

$$M_k := \begin{pmatrix} q_k & -p_k \\ (-1)^{k-1} q_{k-1} & (-1)^k p_{k-1} \end{pmatrix} \in \mathrm{SL}_2(\mathcal{O}_K) \ = \Gamma. \tag{3.1}$$

The powers of -1 have been inserted so that the matrices are special linear once we have (2.11). The supremum norm of the above matrix is $\max(|q_k|, |p_k|)$ since the denominators increase monotonically, and the numerators p_k should increase accordingly in order to better and better approximate the fixed complex number ξ. If necessary, we pre-multiply the vector $\mathbf{z} \in \mathbb{C}^2$ by the $\mathrm{SL}_2(\mathcal{O}_K)$ matrix

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{3.2}$$

to have slope ξ with $|\xi| \le 1$ while the size $|J\mathbf{z}| = |\mathbf{z}|$ remains the same. Also, note that $|J\gamma| = |\gamma J| = |\gamma|$ for any 2×2 matrix γ. Thus, we can assume $|M_k| \asymp |q_k|$ and we will do so from here on without explicit mention. For the Γ-orbit of $\mathbf{z} = (z_1, z_2)^t$ with $z_1/z_2 = \xi$, we see that

$$M_k \mathbf{z} = z_2 \begin{pmatrix} \epsilon_k \\ (-1)^{k-1} \epsilon_{k-1} \end{pmatrix}, \tag{3.3}$$

implying that

$$|M_k \mathbf{z}| \le |\mathbf{z}| \frac{C_1}{|q_k|} \ll_K \frac{|\mathbf{z}|}{|M_k|} \text{ as } |\epsilon_k| \le \frac{C_1}{|q_{k+1}|} < \frac{C_1}{|q_k|} \text{ and } |\epsilon_{k-1}| \le \frac{C_1}{|q_k|}, \tag{3.4}$$

leveraging (2.3). Thus, there are infinitely many such matrices, and this immediately tells us that $\mu_\Gamma(\mathbf{z}, \mathbf{0}) \ge 1$. The proof of $\mu_\Gamma(\mathbf{z}, \mathbf{0}) \le 1$ goes along the same lines as in [LN12a, Lemma 1] with $|q_k|$ and $|q_{k+1}|$ replacing q_k and q_{k+1}

respectively and with appropriate constants. The proof so far uses no special feature of $\mathcal{O}_K$ other than the existence of a continued fraction algorithm whose denominator sizes increase monotonically. The triangle subgroup Γ_λ is the subgroup of $\mathrm{SL}_2(\mathbb{R})$ generated by the matrices J and

$$\begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}, \tag{3.5}$$

where either $\lambda = 2\cos(\pi/m)$, $m = 1, 2, \ldots$ or $\lambda > 2$. We have an associated continued fraction algorithm given by David Rosen [Ros54], which satisfies the hypothesis of Theorem 1.4 (see also [BHS13]). This observation leads to

Proposition 3.1. *For the linear action of the triangle group Γ_λ on the punctured real plane $\mathbb{R}^2 \setminus \{\mathbf{0}\}$, we have*

$$\mu_{\Gamma_\lambda}(\mathbf{x}, \mathbf{0}) = 1$$

for all $\mathbf{x} \in \mathbb{R}^2 \setminus \{\mathbf{0}\}$ whose slope does not have a finite Rosen λ-continued fraction.

The results on other exponents for the above triangle group actions have evaded us as yet because of the non-existence of a suitable height function on the rings of integers of $\mathbb{Q}(\lambda)$ which interacts properly with the Euclidean norm of real numbers. Returning to our main discussion on $\mathrm{SL}_2(\mathcal{O}_K)$-actions on $\mathbb{C}^2 \setminus \{\mathbf{0}\}$, we get an upper bound on $\widehat{\mu}_\Gamma(\mathbf{z}, \mathbf{0})$ from (3.4) trivially. For the reverse inequality, it suffices to consider the matrices M_k with

$$\begin{aligned} &|q_k| \asymp |M_k| \le T \le |M_{k+1}| \asymp |q_{k+1}|, \\ &|M_k \mathbf{z}| \ll_{\mathbf{z}} \frac{1}{|q_k|} \ll \frac{1}{|q_{k+1}|^{1/\omega}} \ll \frac{1}{T^{1/\omega}} \end{aligned} \tag{3.6}$$

for $\omega > \omega_K(\xi)$ and all $k > k_0 = k_0(\xi, \omega)$. Letting $\omega_K(\xi) \leftarrow \omega$ from the right, we obtain

Proposition 3.2. *For any $\mathbf{z} = (z_1, z_2)^t \in \mathbb{C}^2$ with slope $\xi = z_1/z_2 \in \mathbb{C}'$ such that a continued fraction expansion for ξ in terms of $\mathcal{O}_K$-integers exists and satisfies the conditions of Theorem 1.4, we have*

$$\mu_\Gamma(\mathbf{z}, \mathbf{0}) = 1 \quad \textit{and} \quad 1/\omega_K(\xi) \le \hat{\mu}_\Gamma(\mathbf{z}, \mathbf{0}) \le 1.$$

We now digress to prove a claim we made after Definition 1.3.

Lemma 3.3. *For $K = \mathbb{Q}(\sqrt{-d})$ where $d \in \{1, 2, 3, 7, 11\}$, the K-irrationality measure $\omega_K(z)$ is 1 for Lebesgue almost all $z \in \mathbb{C}$ and ≥ 1 for all $z \in \mathbb{C}'$.*

Proof. Given $Q > 1$, the number of $\mathcal{O}_K$ integers q with $|q| \le Q/2$ is $\ge c_K Q^2$ for some $c_K > 0$. For each of these q's, there exists a unique $p = p(q) \in \mathcal{O}_K$ such that the complex number $qz - p$ belongs to a fixed fundamental polygon $\mathcal{F}_K$ for $\mathcal{O}_K$ in $\mathbb{C}$. Therefore, we have at least $c_K Q^2$ distinct numbers in $\mathcal{F}_K$ as $z \notin K$. If we now divide this polygon into $\approx c_K Q^2$ subpolygons each of diameter $\le c_K^{-1/2} Q^{-1}$, then by Dirichlet's pigeonhole principle, one of them contains $q_1 z - p_1$ and $q_2 z - p_2$ for some $|q_1|, |q_2| \le Q/2$ and $q_1 \ne q_2$. In conclusion,

$$|(q_1 - q_2)z - (p_1 - p_2)| \le \frac{1}{\sqrt{c_K} Q} \ll_K \frac{1}{|q_1 - q_2|}, \tag{3.7}$$

giving $\omega_K(z) \ge 1$. To see that $\omega_K(z) = 1$ for almost all $z \in \mathbb{C}'$, notice that the number of $p \in \mathcal{O}_K$ such that $p/q \in \mathcal{F}_K$ is $\le b_K |q|^2$ for any fixed q, and the number of $q \in \mathcal{O}_K$ for which $|q| \sim Q$ is $\le b'_K Q$. Therefore, the series in the Borel–Cantelli lemma for the family of discs of radius $1/|q|^{1+s}$ around the K-rational point p/q is dominated by

$$\sum_{Q>1} b_K b'_K \frac{Q^3}{(Q^{1+s})^2}. \tag{3.8}$$

The latter converges for all $s > 1$, implying that for s in this range, the limsup set

$$\limsup_{p/q \in K} D(p/q, 1/|q|^{1+s}) \tag{3.9}$$

has Lebesgue measure zero. In other words, $\omega_K(z) = 1$ for almost all $z \in \mathbb{C}'$. ■

Hence, the generic value (in $\mathbf{z}$) of both $\mu_\Gamma(\mathbf{z}, \mathbf{0})$ and $\hat{\mu}_\Gamma(\mathbf{z}, \mathbf{0})$ is 1. We thus have the first claim of Theorem 1.4.

A function $h : X \to [0, \infty)$ on a countable space X is said to be a *height function* if for each $Q \ge 0$, the set $h^{-1}[0, Q]$ is finite. If (φ, G) is an action of a countable group G with height function h on a metric space X, then $\mu_\varphi(x, y)$ is defined to be

$$\sup\{\mu \mid \operatorname{dist}(gx, y) < h(g)^{-\mu} \text{ has infinitely many solutions in } g\}. \tag{3.10}$$

The uniform variant $\hat{\mu}_\varphi(x, y)$ is given in the same fashion for all $x, y \in X$. Next, we make a simple and more general observation whose proof is immediate from the definitions.

PROPOSITION 3.4. *Let (G_1, h_1) and (G_2, h_2) be countable groups with h_i being a height function on G_i, and let $\rho : G_1 \to G_2$ be a group homomorphism which respects h_1 in the sense that there exists $c > 1$ such that*

$$\frac{1}{c} h_1(g) \le h_2(\rho(g)) \le c h_1(g) \quad \forall g \in G_1. \tag{3.11}$$

Further, let (φ_i, G_i), $i = 1, 2$, be group actions on a metric space X and

$\varphi_2 \circ \rho = \varphi_1$. *Then, for all* $x, y \in X$,

$$\mu_{\varphi_2}(x,y) \geq \mu_{\varphi_1}(x,y) \quad \text{and} \quad \hat{\mu}_{\varphi_2}(x,y) \geq \hat{\mu}_{\varphi_1}(x,y). \tag{3.12}$$

Now, the group $\mathrm{SL}_2(\mathbb{Z}[i])$ sits inside $\mathrm{SL}_4(\mathbb{Z})$ owing to

$$a + ib \leftrightarrow \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \tag{3.13}$$

with the height function on $\mathrm{SL}_2(\mathbb{Z}[i])$ preserved and the standard linear actions on $\mathbb{C}^2 \cong \mathbb{R}^4$ coinciding under the resulting embedding. We therefore have the following corollary for simultaneous approximation by primitive integral vectors in dimension 4 by combining Propositions 3.2 and 3.4.

Corollary 3.5. *For almost all* $\mathbf{v} \in \mathbb{R}^4$, *the exponents* $\mu_{\mathrm{SL}_4(\mathbb{Z})}(\mathbf{v}, \mathbf{0}_4)$ *and* $\hat{\mu}_{\mathrm{SL}_4(\mathbb{Z})}(\mathbf{v}, \mathbf{0}_4)$ *for approaching the origin via the* $\mathrm{SL}_4(\mathbb{Z})$*-orbit are greater than or equal to* 1.

We will not write down the corresponding statements for other target points in $\mathbb{R}^4$. The next lemma bounds the size of a convergent matrix $\gamma \in \mathrm{SL}_2(\mathcal{O}_K)$ in terms of the entries in its decomposition. The idea here is to bring the starting point $\mathbf{z}$ sufficiently close to the origin using matrices M_k as above, spread it around as a lattice with the help of the subgroup

$$\mathcal{U} = \left\{ U^\ell := \begin{pmatrix} 1 & \ell \\ 0 & 1 \end{pmatrix} \;\middle|\; \ell \in \mathcal{O}_K \right\}, \tag{3.14}$$

and finally rotate the lattice so obtained by applying matrices N_j which attempt to take the 'complex line' $\langle (z_1, 0) \rangle$ closer to $\langle (z_1, \xi z_1) \rangle$.

Lemma 3.6 (Laurent and Nogueira [LN12a]). *Let* $k \in \mathbb{N}$ *and* $\ell \in \mathcal{O}_K$. *For any*

$$N = \begin{pmatrix} t & t' \\ s & s' \end{pmatrix} \in \Gamma, \tag{3.15}$$

the matrix $\gamma = NU^\ell M_k$ *satisfies*

$$|(\ell q_{k-1} + (-1)^{k-1} q_k)s| - |s' q_{k-1}| \leq |\gamma| \ll |\ell q_{k-1}|\,|N| + |N|\,|q_k|. \tag{3.16}$$

Proof. After two matrix multiplications we have

$$\begin{aligned} \gamma = NU^\ell M_k &= \begin{pmatrix} t & t' \\ s & s' \end{pmatrix} \begin{pmatrix} 1 & \ell \\ 0 & 1 \end{pmatrix} \begin{pmatrix} q_k & -p_k \\ (-)^{k-1} q_{k-1} & (-1)^k p_{k-1} \end{pmatrix} \\ &= \begin{pmatrix} tq_k + (-1)^{k-1} q_{k-1}(t\ell + t') & -tp_k + (-1)^k p_{k-1}(t\ell + t') \\ sq_k + (-1)^{k-1} q_{k-1}(s\ell + s') & -sp_k + (-1)^k p_{k-1}(s\ell + s') \end{pmatrix}. \end{aligned} \tag{3.17}$$

The bottom left entry of the matrix determines the lower bound in the lemma as soon as we employ the triangle inequality. Because we have already reduced to the case $|\xi| \le 1$, for all large enough n we have $|p_n| \ll |q_n|$, and then the upper bound is easy enough. ∎

We now take steps towards obtaining bounds for the vector $\gamma(\xi,1)^t$. The lemma below is again due to Laurent and Nogueira. We sketch its proof here to merely point out the minor difference(s) with the real case.

LEMMA 3.7. *Let k,ℓ,N and $\gamma = NU^\ell M_k = \left(\begin{smallmatrix} v_1 & u_1 \\ v_2 & u_2\end{smallmatrix}\right)$ be as in Lemma 3.6 and $y \in \mathbb{C}$. If $\delta = sy - t$ and $\delta' = s'y - t'$, then*

$$|v_1\xi + u_1 - y(v_2\xi + u_2)| \ll_K \frac{|\delta\ell + \delta'|}{|q_k|} + \frac{|\delta|}{|q_{k+1}|}.$$

Proof. To begin with,

$$\begin{aligned}(3.18)\qquad & y(v_2\xi + u_2) - (v_1\xi + u_1) \\ &= (-1 \quad y)\gamma\begin{pmatrix}\xi \\ 1\end{pmatrix} = (-1 \quad y)\begin{pmatrix} t & t' \\ s & s'\end{pmatrix} U^\ell M_k \begin{pmatrix}\xi \\ 1\end{pmatrix} \\ &= (\delta \quad \delta')U^\ell \begin{pmatrix}\epsilon_k \\ (-1)^{k-1}\epsilon_{k-1}\end{pmatrix} \\ &= \delta\epsilon_k + (-1)^{k-1}(\delta\ell + \delta')\epsilon_{k-1}.\end{aligned}$$

Since for the continued fraction expansions studied here, we have $|\epsilon_n| \ll_K |q_{n+1}|^{-1}$ for all $n \gg 1$, the claim follows. ∎

If $(\Lambda_1, \Lambda_2)^t$ is the difference $\gamma\mathbf{z} - \mathbf{y}$, then

$$(3.19)\qquad \Lambda_1 = z_2(v_1\xi + u_1) - y_1, \qquad \Lambda_2 = z_2(v_2\xi + u_2) - y_2,$$

and on further choosing $y = y_1/y_2$, we get

$$\begin{aligned}(3.20)\qquad |\Lambda_1 - y\Lambda_2| &= |z_2((v_1\xi + u_1) - y(v_2\xi + u_2))| \\ &\ll_K |z_2|\left(\frac{|\delta\ell + \delta'|}{|q_k|} + \frac{|\delta|}{|q_{k+1}|}\right).\end{aligned}$$

Once we bound one of the components (say Λ_2) and the difference $|\Lambda_1 - y\Lambda_2|$, the vector $(\Lambda_1, \Lambda_2)^t$ is bounded automatically. We proceed to do just that. After a slight adjustment in the proof of Lemma 3.7, we deduce

$$\begin{aligned}(3.21)\qquad \Lambda_2 &= z_2(v_2\xi + u_2) - y_2 \\ &= z_2(s\epsilon_k + (-1)^{k-1}(s\ell + s')\epsilon_{k-1}) - y_2 = (-1)^{k-1}z_2 s\epsilon_{k-1}(\ell - \rho),\end{aligned}$$

where

$$(3.22)\qquad \rho = \frac{(-1)^{k-1}y_2}{z_2 s\epsilon_{k-1}} - \frac{(-1)^{k-1}\epsilon_k}{\epsilon_{k-1}} - \frac{s'}{s}$$

helps us to decide the value of ℓ such that $|\ell - \rho| \le C_3$ for some constant C_3 depending only on $\mathcal{O}_K$ (or K) and $|\ell| \le |\rho|$, having fixed M_k and N first.

3.1. Generic target points. In the next few pages, we discuss the situation where the target point $\mathbf{y} = (y_1, y_2)^t \in \mathbb{C}^2$ has slope $y = y_1/y_2 \in \mathbb{C}' = \mathbb{C} \setminus K$. As such points constitute a set of full measure in $\mathbb{C}^2$, we shall be inferring properties of almost all points in the complex plane.

Let t_j/s_j and t_{j-1}/s_{j-1} be consecutive convergents in an $\mathcal{O}_K$-continued fraction expansion of y for our fixed target point $\mathbf{y}$. As argued for ξ, we may also suppose $|y| \leq 1$ thanks to the J of (3.2). When

$$t = t_j, \quad s = s_j, \quad t' = (-1)^{j-1} t_{j-1}, \quad s' = (-1)^{j-1} s_{j-1}, \tag{3.23}$$

the matrix $N_j := \left(\begin{smallmatrix} t & t' \\ s & s' \end{smallmatrix}\right)$ belongs to $\Gamma = \mathrm{SL}_2(\mathcal{O}_K)$ and for any $\omega > \omega_K(\xi)$, the auxiliary term ρ in (3.22) is confined within the range

$$\frac{1}{C_1}\left|\frac{y_2 q_k}{z_2 s_j}\right| - 2 \leq |\rho| \leq \frac{|y_2|\,|q_k|^{\omega^{r_1-1}}}{|C_2 z_2 s_j|} + 2 \tag{3.24}$$

for all k large enough using Lemmata 2.1 and 2.2, the fact that the error term is

$$\epsilon_n = \frac{(-1)^n}{z_1 \cdots z_{n+1}} \tag{3.25}$$

with $|z_i| \geq 1$ for $i > 0$ [Dan15, Prop. 2.1(i)] and a monotonous increase of the denominators s_j. This in turn tells us that the optimal choice of ℓ obeys

$$\frac{1}{C_1}\left|\frac{y_2 q_k}{z_2 s_j}\right| - (C_3 + 2) \leq |\ell| \ll_{\mathbf{y},\mathbf{z},K} \frac{|q_k|^{\omega^{r_1-1}}}{|s_j|} + 1. \tag{3.26}$$

Substituting this in Lemmata 3.6 and 3.7, we have an $\mathcal{O}_K$-analogue of [LN12a, Lemma 4].

Lemma 3.8. *Let $j \in \mathbb{N}$, $k \gg 0$ and $\omega > \omega_K(\xi)$. There exists $\gamma = N_j U^\ell M_k \in \Gamma$ for some $\ell \in \mathcal{O}_K$ such that*

$$\left|\frac{|y_2|}{|C_1 z_2|}|q_k q_{k-1}| - |s_j q_k|\right| - (C_3+3)|s_j q_{k-1}| \leq |\gamma| \ll_{\mathbf{y},\mathbf{z},K} |q_{k-1}|\,|q_k|^{\omega^{r_1-1}} + |s_j q_k|$$

as well as

$$|\gamma \mathbf{z} - \mathbf{y}| \ll_{\mathbf{y},\mathbf{z},K} \frac{|q_k|^{\omega^{r_1-1}-1}}{|s_j s_{j+1}|} + \left|\frac{s_j}{q_k}\right|. \tag{3.27}$$

Proof. The bounds for γ are straightforward. As for $\gamma\mathbf{z} - \mathbf{y}$, we have

$$|\Lambda_2| \leq C_1 C_3 \left|\frac{z_2 s_j}{q_k}\right|, \tag{3.28}$$

using (3.21) and an optimal choice of ℓ as explained immediately after (3.22). Moreover, $|\Lambda_1| \leq |y\Lambda_2| + |\Lambda_1 - y\Lambda_2| \leq |\Lambda_2| + |\Lambda_1 - y\Lambda_2|$ as $|y| \leq 1$. The quantity $|\Lambda_1 - y\Lambda_2|$ is bounded using (3.20), while we recall that $|\delta| \leq C_1/|s_{j+1}|$ and $|\delta'| \leq C_1/|s_j|$. ∎

With the above lemma at hand, we now make an appropriate choice of the indices j and k so that

$$|q_{k-1}|^{1/3} < |s_j| \le |q_k|^{1/3} < |s_{j+1}|. \tag{3.29}$$

The existence of arbitrarily large pairs (j, k) satisfying (3.29) is quaranteed, as $|q_k|$'s and $|s_j|$'s are strictly increasing sequences of real numbers. Such a pair is then fed into the statement of Lemma 3.8 to give

$$|\gamma\mathbf{z} - \mathbf{y}| \ll_{\mathbf{y},\mathbf{z},K} \frac{1}{|q_{k-1}|^{1/3}|q_k|^{4/3-\omega^{r_1-1}}} + \frac{1}{|q_k|^{2/3}}. \tag{3.30}$$

In this work, we are mostly concerned with the Γ-orbits of generic points in $\mathbb{C}^2$ whose slope has K-irrationality measure 1. Thus, it is fair to assume that $1 \le \omega_K(\xi) < 3$, where $\xi = z_1/z_2 \in \mathbb{C}'$ is the slope of the starting point $\mathbf{z}$. For $\omega > \omega_K(\xi) \ge 1$, the first summand on the right side of (3.30) dominates the second, for all k's large enough. Also, for $\omega > \omega_K(\xi)$, $|q_k| \le |q_{k-1}|^{\omega}$ for $k \gg 0$, where we remind the reader that p_k/q_k's are convergents to ξ coming from a continued fraction expansion algorithm. Furthermore, under the condition (3.29) and for $\omega < 3$, the second term in the upper bound for $|\gamma|$ is much smaller than the first, implying the existence of a $\gamma \in \Gamma$ which satisfies

$$|\gamma| \ll_{\mathbf{y},\mathbf{z},K} |q_k|^{\omega^{r_1-1}+1}, \qquad |\gamma\mathbf{z} - \mathbf{y}| \ll_{\mathbf{y},\mathbf{z},K} \frac{1}{|q_k|^{1/(3\omega)+4/3-\omega^{r_1-1}}}. \tag{3.31}$$

The preceding lemma also tells us that $|\gamma| \gg |q_k q_{k-1}|$ for the choice of j and k according to (3.29). This ensures the existence of infinitely many matrices $\gamma \in \mathrm{SL}_2(\mathcal{O}_K)$ satisfying the above system of inequalities.

As hinted before, we could have started with an ω close enough to 1 so that the exponent $1/(3\omega) + 4/3 - \omega^{r_1-1}$ in (3.31) is positive. For such an $\omega > \omega_K(\xi)$, we therefore have

$$\mu_\Gamma(\mathbf{z}, \mathbf{y}) \ge \frac{1 + 4\omega - 3\omega^{r_1}}{3\omega(\omega^{r_1-1} + 1)}. \tag{3.32}$$

In the limit $\omega_K(\xi) \leftarrow \omega$ from the right, and the generic value of the former being 1, we get

Proposition 3.9. *For all $\mathbf{z} \in \mathbb{C}^2 \setminus \{\mathbf{0}\}$ having slope ξ with $\omega_K(\xi) = 1$ and for all $\mathbf{y} \in \mathbb{C}^2$ with slope $y \in \mathbb{C} \setminus K$,*

$$\mu_\Gamma(\mathbf{z}, \mathbf{y}) \ge 1/3.$$

We now calculate lower bounds for $\hat{\mu}_\Gamma$. For almost all target points $\mathbf{y} \in \mathbb{C}^2 \setminus \{\mathbf{0}\}$ with slope y belonging to $\mathbb{C}'$, we prove a result mildly stronger than Proposition 3.9:

$$\hat{\mu}_\Gamma(\mathbf{z}, \mathbf{y}) \ge 1/3 \tag{3.33}$$

for almost all $(\mathbf{z},\mathbf{y}) \in \mathbb{C}^2 \times \mathbb{C}^2$. In proving this, the K-irrationality measure $\omega_K(y)$ associated with $\mathbf{y}$ is used as an auxiliary tool. The result below is [LN12a, Lemma 6] mutatis mutandis, and the proof is omitted.

LEMMA 3.10. *Let $\omega > \omega_K(\xi)$ (≥ 1) and define*

$$\tau := \frac{\omega_K(y)}{2\omega_K(y)+1}\omega^{r_1-1}. \tag{3.34}$$

Given any $\varepsilon > 0$ and $k_0 = k_0(\varepsilon) \in \mathbb{N}$, there exists $\gamma \in \Gamma$ such that

$$|\gamma| \ll_{\mathbf{y},\mathbf{z},K} |q_k|^{1+\omega^{r_1-1}} \quad \text{and} \quad |\gamma\mathbf{z}-\mathbf{y}| \leq |q_k|^{\tau-1+\varepsilon} \quad \forall k > k_0. \tag{3.35}$$

In addition to our assumption that $1/(3\omega) + 4/3 - \omega^{r_1-1} > 0$, we now also suppose that $\omega < 2^{1/(r_1-1)}$. This is to ensure that the quantity τ of Lemma 3.10 remains less than 1. Next, we restrict to ε small enough so that for given τ, the exponent $1-\tau-\varepsilon$ which shall meet soon is greater than 0. After this, we recourse to the old but helpful idea of sandwiching (also used in [LN12a]), which given any sufficiently large real positive number T, picks a k large enough in terms of T so that

$$C|q_k|^{1+\omega^{r_1-1}} \leq T < C|q_{k+1}|^{1+\omega^{r_1-1}}, \tag{3.36}$$

where C is the hidden absolute constant in the upper bound for $|\gamma|$ given by Lemma 3.10. Such a choice of k will mean that both

$$|\gamma| \leq T \quad \text{and} \quad |\gamma\mathbf{z}-\mathbf{y}| \leq \frac{1}{|q_k|^{1-\tau-\varepsilon}} \leq \frac{1}{T^{(1-\tau-\varepsilon)/(\omega+\omega^{r_1})}}, \tag{3.37}$$

giving

$$\hat{\mu}_\Gamma(\mathbf{z},\mathbf{y}) \geq \frac{1-\tau-\varepsilon}{\omega+\omega^{r_1}}. \tag{3.38}$$

As the bound obtained is valid for all sufficiently small $\varepsilon > 0$ and $\omega > \omega_K(\xi)$, in the limit $\varepsilon \to 0_+$ and $\omega \to \omega_K(\xi)_+$ we get

$$\hat{\mu}_\Gamma(\mathbf{z},\mathbf{y}) \geq \frac{\big(2-(\omega_K(\xi))^{r_1-1}\big)\omega_K(y)+1}{(2\omega_K(y)+1)\big(\omega_K(\xi)+(\omega_K(\xi))^{r_1}\big)} \tag{3.39}$$

when the starting point $\mathbf{z} \in \mathbb{C}^2$ has slope whose K-irrationality measure $\omega_K(\xi)$ is very close to that of any generic point in the complex plane. Since we are only concerned with generic pairs $(\mathbf{z},\mathbf{y}) \in \mathbb{C}^2 \times \mathbb{C}^2$, we may as well take both $\omega_K(\xi)$ and $\omega_K(y)$ to be 1, whereby $\hat{\mu}_\Gamma$ comes out to be at least $1/3$.

From the literature, we mention the work of Pollicott [Pol11] about calculating the error term in the equidistribution sum associated with the linear action of cocompact lattices $\Gamma \subset \mathrm{SL}_2(\mathbb{C})$ on $\mathbb{C}^2$. The bounds for the generic value of Diophantine exponents then fall out as a corollary. A work in the same spirit for the $\mathrm{SL}_2(\mathbb{Z})$-action on $\mathbb{R}^2 \setminus \{\mathbf{0}\}$ was carried out by Maucourant and Weiss [MW12] with much weaker estimates than those of [LN12a]. Applicable in a broader framework, the machinery of Ghosh, Gorodnik and

Nevo [GGN, GGN15] is vastly superior and gives the generic values of exponents for an array of lattice actions on homogeneous varieties of connected almost simple, semisimple algebraic groups. In [GGN], they consider many examples where the upper and lower bounds for the generic exponents turn out to be equal. The case of $\mathrm{SL}_2(\mathcal{O}_K)$-actions on the punctured complex plane does not feature on this list.

Let us next turn our attention to Theorem 3.1 of [GGN] in our search for upper bounds on μ_Γ and $\hat{\mu}_\Gamma$. The punctured plane $\mathbb{C}^2 \setminus \{\mathbf{0}\}$ can be realized as the special linear group $\mathrm{SL}_2(\mathbb{C})$ quotiented by the closed upper unipotent subgroup H. The non-uniform lattice $\Gamma = \mathrm{SL}_2(\mathcal{O}_K)$ acts ergodically on G/H, and we verify that the hypothesis of the theorem is valid in this scenario. In the terminology of [GGN], the coarse volume growth exponent a for the upper unipotent group $H \subset \mathrm{SL}_2(\mathbb{C}) = G$ and the lower local dimension d' of the homogeneous space $G/H \approx \mathbb{C}^2 \setminus \{\mathbf{0}\}$ equal 2 and 4, respectively. We then find that for any $\mathbf{z} \in \mathbb{C}^2$ with a dense Γ-orbit and almost all $\mathbf{y} \in \mathbb{C}^2$,

$$\kappa(\mathbf{z}, \mathbf{y}) := 1/\hat{\mu}_\Gamma(\mathbf{z}, \mathbf{y}) \geq 2, \tag{3.40}$$

which is the same as saying that $\hat{\mu}_\Gamma(\mathbf{z}, \mathbf{y}) \leq 1/2$ for all $\mathbf{z} \in \mathbb{C}^2$ with slope $\xi \in \mathbb{C}'$ and $\mathbf{y}$ belonging to a full Lebesgue measure subset (depending on $\mathbf{z}$) of the complex plane. The same proof can be modified to replace $\hat{\mu}_\Gamma$ with μ_Γ. To see this, one should apply Borel–Cantelli as soon as we have estimates for the number of lattice elements $\gamma \in \Gamma \cap G_t$ of bounded size e^t and such that $\gamma\mathbf{z}$ lies within a unit distance of the target point $\mathbf{y}$. This is given by $e^{2t+\varepsilon}$ up to a constant multiple depending on ε alone. In summary,

PROPOSITION 3.11. *For all pairs* $(\mathbf{z}, \mathbf{y}) \in \mathbb{C}^2 \times \mathbb{C}^2$ *with the slopes* ξ *of* $\mathbf{z}$ *and* y *of* $\mathbf{y}$ *both having* K*-irrationality measure* 1, *we have*

$$1/3 \leq \hat{\mu}_\Gamma(\mathbf{z}, \mathbf{y}) \leq \mu_\Gamma(\mathbf{z}, \mathbf{y}), \tag{3.41}$$

and for all $\mathbf{z} \in \mathbb{C}^2$ *with slope* $\xi \in \mathbb{C}'$ *and almost all* (*depending on* $\mathbf{z}$) *target points* $\mathbf{y}$,

$$\hat{\mu}_\Gamma(\mathbf{z}, \mathbf{y}) \leq \mu_\Gamma(\mathbf{z}, \mathbf{y}) \leq 1/2. \tag{3.42}$$

3.2. Target point with K-rational slope. The task of computing exponents is much easier when $\mathbf{y}$ has slope $y = y_1/y_2 = a/b \in K$, where $a, b \in \mathcal{O}_K$ and $|\gcd_{\mathcal{O}_K}(a, b)| = 1$. Without any loss of generality, assume as before that $\max\{1, |a|\} \leq |b|$. The column vector $(a, b)^t$ is taken to be the first column of our matrix N, as the fraction a/b is the best approximation to y by K-rational points. After this step, the second column can be chosen to be some $(a', b')^t \in \mathcal{O}_K^2$ such that $ab' - a'b = 1$ and $|b'| \leq |b|$. This is possible because $\mathcal{O}_{\mathbb{Q}(\sqrt{-d})}$ is a Euclidean domain for $d = 1, 2, 3, 7$ and 11 [Mot49], and then it is clear that $|N| \asymp |b|$. As in Section 3.1, pick $\omega > \omega_K(\xi)$ (≥ 1) where $\xi = z_1/z_2 \in \mathbb{C}'$. If necessary, we will take ω very close to $\omega_K(\xi)$.

Lemma 3.12 (cf. [LN12a, Lemma 5]). *Let $k \in \mathbb{N}$ be large enough. Given $\mathbf{y} \in \mathbb{C}^2$ with slope $y \in K$, there exist $\ell \in \mathcal{O}_K$ and $\gamma = NU^\ell M_k \in \Gamma$ satisfying*

$$|q_k q_{k-1}| \ll_{\mathbf{y},\mathbf{z},K} |\gamma| \ll_{\mathbf{y},\mathbf{z},K} |q_{k-1}|\,|q_k|^{\omega^{r_1-1}}$$

and

$$|\gamma\mathbf{z} - \mathbf{y}| \ll_K |bz_2/q_k|. \tag{3.43}$$

Proof. Here, the quantities δ and δ' defined in Lemma 3.7 equal $by - a = 0$ and $b'y - a' = 1/b$, respectively, and thereby

$$|\Lambda_1 - y\Lambda_2| \ll_K \left|\frac{z_2}{bq_k}\right|. \tag{3.44}$$

For the same reason, b replaces s_j in (3.28), and then the triangle inequality gives the required bound on $|\gamma\mathbf{z} - \mathbf{y}|$. The same change made to (3.26) gives, in conjunction with Lemma 3.6,

$$\begin{aligned} |\gamma| &\ll |\ell b q_{k-1}| + |bq_k| \ll_{\mathbf{y},\mathbf{z},K} |q_{k-1}|\,|q_k|^{\omega^{r_1-1}} + |q_{k-1}| + |q_k| \\ &\ll |q_{k-1}|\,|q_k|^{\omega^{r_1-1}}. \end{aligned} \tag{3.45}$$

On the other hand,

$$|\gamma| \gg_{\mathbf{z}} |\ell q_{k-1}| - |q_k| \gg_{\mathbf{y},\mathbf{z},K} |q_k q_{k-1}| \tag{3.46}$$

on looking at the modified (3.26) again, as $|q_{k-1}| \gg |b|$ for all large enough k's. ∎

From the bounds on the size of γ in the above lemma, we get

$$|q_k| \ll_{\mathbf{y},\mathbf{z},K} |\gamma| \ll_{\mathbf{y},\mathbf{z},K} |q_k|^{\omega^{r_1-1}+1}. \tag{3.47}$$

The right inequality gives

$$|1/q_k| \ll_{\mathbf{y},\mathbf{z},K} |\gamma|^{-1/(\omega^{r_1-1}+1)}, \tag{3.48}$$

which in turn implies that

$$|\gamma\mathbf{z} - \mathbf{y}| \ll_{\mathbf{y},\mathbf{z},K} |\gamma|^{-1/(\omega^{r_1-1}+1)} \tag{3.49}$$

for infinitely many $\gamma \in \mathrm{SL}_2(\mathcal{O}_K)$. This means that $\mu_\Gamma(\mathbf{z},\mathbf{y}) \geq (\omega^{r_1-1}+1)^{-1}$ for all $\mathbf{y}$ with slope $y \in K$ and $\omega > \omega_K(\xi)$. Taking the limit $\omega_K(\xi) \leftarrow \omega$ from the right, we conclude that for any starting point $\mathbf{z} \in \mathbb{C}^2 \setminus \{\mathbf{0}\}$ whose slope ξ has K-irrationality measure $\omega_K(\xi)$ and any target point $\mathbf{y}$ with 'K-rational slope',

$$\mu_\Gamma(\mathbf{z},\mathbf{y}) \geq \frac{1}{(\omega_K(\xi))^{r_1-1}+1}. \tag{3.50}$$

Next on our agenda is a lower bound for $\hat{\mu}_\Gamma$ when $\mathbf{y}$ has a K-rational slope. This value is in general lower than that for μ_Γ above, but equals the same for almost every $\mathbf{z}$. Given $T \gg 0$ and $\omega > \omega_K(\xi)$, we choose k with

$$|q_{k-1}|\,|q_k|^{\omega^{r_1-1}} \le T < |q_k|\,|q_{k+1}|^{\omega^{r_1-1}},$$

whereby

$$(3.51)\qquad |\gamma| \ll_{\mathbf{y},\mathbf{z},K} |q_{k-1}|\,|q_k|^{\omega^{r_1-1}} \le T \quad \text{and} \quad T \le |q_k|^{1+\omega^{r_1}}$$

for γ given by Lemma 3.12. No more input, apart from repeating the same set of arguments, is required to now deduce that

$$(3.52)\qquad \hat{\mu}_\Gamma(\mathbf{z},\mathbf{y}) \ge \frac{1}{(\omega_K(\xi))^{r_1}+1}.$$

Borrowing an idea from [LN12b], we get the following transference result:

PROPOSITION 3.13. *Let $\xi, y \in \mathbb{C}$. Then*

$$\hat{\omega}_K(\xi,y) \ge \frac{1}{(\omega_K(\xi))^{r_1}+1} \quad \textit{and} \quad \omega_K(\xi,y) \ge \frac{1}{(\omega_K(\xi))^{r_1-1}+1}.$$

Proof. In the above observations, let $\mathbf{z} = (\xi,1)^t$ and $\mathbf{y} = (y,y)^t$. Either of the two rows of the various matrix solutions $\{\gamma_i\} \subset \mathrm{SL}_2(\mathcal{O}_K)$ thus obtained will do the job. ■

As a special case, when $\omega_K(\xi) = 1$, we deduce that for all $\varepsilon > 0$, there exists a $T_0 > 0$ such that for all $T > T_0$, we have a pair $(q,p) \in \mathcal{O}_K^2$ for which

$$(3.53)\qquad |q\xi + p - y| < 1/T^{1/2-\varepsilon} \quad \text{and} \quad \max\{|p|,|q|\} \le T.$$

The lemma below helps us to obtain an upper bound for $\mu_\Gamma(\mathbf{z},\mathbf{y})$ when the starting point has dense $\mathrm{SL}_2(\mathcal{O}_K)$-orbit in $\mathbb{C}^2$ and the target point has a K-rational slope. The method used is Laurent and Nogueira's factorization technique [LN12a, Theorem 4] to break down any candidate matrix $\gamma \in \Gamma$ in terms of well-known entities like N and M_k in order to be able to say something about the size of $\gamma\mathbf{z} - \mathbf{y}$ and of the various components appearing in between. As $\hat{\mu}_\Gamma \le \mu_\Gamma$, this will trivially give us an upper bound for $\hat{\mu}_\Gamma(\mathbf{z},\mathbf{y})$.

LEMMA 3.14. *Let $\mathbf{z} \in \mathbb{C}^2 \setminus \{\mathbf{0}\}$ have a slope $\xi \in \mathbb{C}'$ and let $\mathbf{y}$ be a fixed target point with slope $y = a/b \in K$ as introduced at the beginning of this section. For all k large enough and $\gamma \in \Gamma$ such that*

$$|\gamma| \le \frac{1}{3C_1}\left|\frac{y_2}{z_2}\right| |q_k q_{k+1}|, \quad \textit{we must have} \quad |\gamma\mathbf{z} - \mathbf{y}| \ge \left|\frac{z_2}{3b}\right| \frac{1}{|q_k|}.$$

Here, C_1 is the constant in (2.3), distilled from Dani's continued fraction theory for complex numbers in terms of $\mathcal{O}_K$-integers.

Proof. Assume, if possible, that for some $\gamma = \left(\begin{smallmatrix} v_1 & u_1 \\ v_2 & u_2 \end{smallmatrix}\right)$ as above, the vector $\gamma\mathbf{z} - \mathbf{y} = \left(\begin{smallmatrix} \Lambda_1 \\ \Lambda_2 \end{smallmatrix}\right)$ has supremum norm strictly less than $|z_2|\,|3bq_k|^{-1}$. Without loss of generality, we may suppose that $|a| \le |b|$ because of the matrix J from (3.2). Given the complex number a/b with $a, b \in \mathcal{O}_K$ and $|\gcd_{\mathcal{O}_K}(a,b)| = 1$, we take $N = \left(\begin{smallmatrix} a & a' \\ b & b' \end{smallmatrix}\right) \in \Gamma$ with $|b| \le |N| < 2|b|$. As in [LN12a], let

$$\gamma' := N^{-1}\gamma = \begin{pmatrix} v_1' & u_1' \\ v_2' & u_2' \end{pmatrix}. \tag{3.54}$$

Since $b'y_1 - a'y_2 = y_2/b$, here too we get

$$\gamma' = \begin{pmatrix} b'(v_1y_2 - v_2y_1)/y_2 + v_2/b & b'(u_1y_2 - u_2y_1)/y_2 + u_2/b \\ -b(v_1y_2 - v_2y_1)/y_2 & -b(u_1y_2 - u_2y_1)/y_2 \end{pmatrix} \tag{3.55}$$

after adding and subtracting equal quantities to both the entries in the first row. Also,

$$\begin{pmatrix} z_2(v_1'\xi + u_1') \\ z_2(v_2'\xi + u_2') \end{pmatrix} = \gamma'\mathbf{z} = N^{-1}\left(\mathbf{y} + \begin{pmatrix} \Lambda_1 \\ \Lambda_2 \end{pmatrix}\right) = \begin{pmatrix} y_2/b + b'\Lambda_1 - a'\Lambda_2 \\ -b\Lambda_2 + a\Lambda_2 \end{pmatrix}. \tag{3.56}$$

Next,

$$v_1y_2 - v_2y_1 = \begin{vmatrix} v_1 & y_1 \\ v_2 & y_2 \end{vmatrix} = \begin{vmatrix} v_1 & \gamma\mathbf{z} - \begin{pmatrix} \Lambda_1 \\ \Lambda_2 \end{pmatrix} \\ v_2 & \end{vmatrix} = z_2 - \begin{vmatrix} v_1 & \Lambda_1 \\ v_2 & \Lambda_2 \end{vmatrix},$$

so that

$$\begin{aligned} |v_1y_2 - v_2y_1| &\le |z_2| + 2|\gamma| \max\{|\Lambda_1|, |\Lambda_2|\} \\ &\le |z_2| + \left|\frac{2y_2}{9C_1b} q_{k+1}\right| \le \left|\frac{y_2}{4C_1b}\right| |q_{k+1}| \end{aligned} \tag{3.57}$$

for all k such that $|q_k| > 36C_1|bz_2|/|y_2|$. Combining (3.55)–(3.57), we conclude that

$$|v_2'| = \left|\frac{b}{y_2}(v_1y_2 - v_2y_1)\right| < \frac{|q_{k+1}|}{4C_1},$$

and

$$|v_2'\xi + u_2'| = \frac{1}{|z_2|}|-b\Lambda_1 + a\Lambda_2| \le 2\left|\frac{b}{z_2}\right| \max\{|\Lambda_1|, |\Lambda_2|\} < \frac{2}{3} \cdot |q_k|^{-1} \tag{3.58}$$

as $|a| \le |b|$. Now, consider the $\mathrm{SL}_2(\mathcal{O}_K)$ matrix $g := N^{-1}\gamma M_k^{-1}$. Then $g = \left(\begin{smallmatrix} * & * \\ * & v_2'p_k + q_ku_2' \end{smallmatrix}\right)$, and the bottom right entry has size

$$\begin{aligned} |v_2'p_k + q_ku_2'| &= |-v_2'(q_k\xi - p_k) + q_k(v_2'\xi + u_2')| \\ &\le \left|\frac{C_1v_2'}{q_{k+1}}\right| + |q_k|\,|v_2'\xi + u_2'| < \frac{1}{4} + \frac{2}{3} < 1. \end{aligned} \tag{3.59}$$

As the ring $\mathcal{O}_K$ was taken to be discrete, it has no non-zero element whose Euclidean norm is less than one. This means g has to be $\left(\begin{smallmatrix} m & \zeta \\ -\zeta^{-1} & 0 \end{smallmatrix}\right)$ for some

$m \in \mathbb{Z}$ and $\zeta \in \mathcal{O}_K^*$ with $|\zeta| = 1$. Then (3.56) tells us that

$$\gamma'\mathbf{z} = \begin{pmatrix} y_2/b + b'\Lambda_1 - a'\Lambda_2 \\ -b\Lambda_2 + a\Lambda_2 \end{pmatrix} = gM_k\mathbf{z} = z_2\begin{pmatrix} m\epsilon_k + (-1)^{k-1}\zeta\epsilon_{k-1} \\ -\zeta^{-1}\epsilon_k \end{pmatrix}. \tag{3.60}$$

We concentrate on the entry in the first coordinate to get

$$\begin{aligned} \left|\frac{y_2}{b}\right| - \frac{4}{3}\left|\frac{z_2}{q_k}\right| \le \left|\frac{y_2}{b} + b'\Lambda_1 - a'\Lambda_2\right| &= |z_2|\,|m\epsilon_k + (-1)^{k-1}\zeta\epsilon_{k-1}| \\ &\le C_1|z_2|\left(\left|\frac{m}{q_{k+1}}\right| + \left|\frac{1}{q_k}\right|\right), \end{aligned} \tag{3.61}$$

which gives a lower bound on $|m|$,

$$|m| \ge \left|\frac{101y_2}{108C_1bz_2}\right||q_{k+1}| > 33, \tag{3.62}$$

as $C_1 > 1$ and we recall that $|q_{k+1}| > |q_k| > 36C_1|bz_2|/|y_2|$. We now have a decomposition $\gamma = NgM_k$ which helps us to get a handle on the size of γ. To be precise,

$$\gamma = \pm\begin{pmatrix} a & a' \\ b & b' \end{pmatrix}\begin{pmatrix} m & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} q_k & -p_k \\ (-1)^{k-1}q_{k-1} & (-1)^k p_{k-1} \end{pmatrix}, \tag{3.63}$$

so that $|\gamma| \ge |b|(|m| - 2)|q_k|$ by the triangle inequality. As $|m| > 33$ for all suitably large k's, the quantity $|m| - 2$ should be strictly bigger than $31|m|/33$. We deduce that

$$|\gamma| > \frac{31}{33}|bmq_k| \ge \frac{4}{5C_1}\left|\frac{y_2}{z_2}\right||q_kq_{k+1}|, \tag{3.64}$$

but the hypothesis was $|\gamma| \le \frac{|y_2|}{3C_1|z_2|}|q_kq_{k+1}|$, a contradiction. ■

For any $\gamma \in \Gamma$ with $|\gamma|$ sufficiently large, pick k with

$$\frac{1}{3C_1}\left|\frac{y_2}{z_2}\right||q_{k-1}q_k| < |\gamma| \le \frac{1}{3C_1}\left|\frac{y_2}{z_2}\right||q_kq_{k+1}|, \tag{3.65}$$

and for $\omega_K(\xi)$ finite, choose any $\omega > \omega_K(\xi)$ so that eventually $|q_{k-1}| \ge |q_k|^{1/\omega}$. Consequently,

$$|\gamma\mathbf{z} - \mathbf{y}| \ge \left|\frac{z_2}{3b}\right|\frac{1}{|q_k|} \gg_{\mathbf{y},\mathbf{z}} \frac{1}{|\gamma|^{\omega/(\omega+1)}}, \tag{3.66}$$

and letting ω approach $\omega_K(\xi)$ from the right, we get

$$\mu_\Gamma(\mathbf{z},\mathbf{y}) \le \frac{\omega_K(\xi)}{\omega_K(\xi)+1}. \tag{3.67}$$

The statement is also true for $\omega_K(\xi) = \infty$, as can be easily checked. However, as discussed in Section 1, $\omega_K(\xi) = 1$ for Lebesgue almost all $\xi \in \mathbb{C}$.

Therefore, for a full measure subset of $\mathbb{C}^2 \setminus \{\mathbf{0}\}$, (3.52) and (3.67) together give

PROPOSITION 3.15. *For all* $\mathbf{z} \in \mathbb{C}^2 \setminus \{\mathbf{0}\}$ *whose slope* ξ *has* K*-irrationality measure* 1, *and for all* $\mathbf{y} \in \mathbb{C}^2$ *with slope* $y \in K$,

$$\mu_\Gamma(\mathbf{z}, \mathbf{y}) = \hat{\mu}_\Gamma(\mathbf{z}, \mathbf{y}) = 1/2.$$

This completes the proof of Theorem 1.4.

Acknowledgments. The author thanks his supervisor Anish Ghosh for suggesting the problem, reading the various preliminary versions of the manuscript and for being a constant source of encouragement. I am indebted to Professors S. G. Dani and Yann Bugeaud for their many careful observations and generously pointing out possible extensions as a future work. Financial support from CSIR, Govt. of India under SPM-07/858(0199)/2014-EMR-I is duly acknowledged.

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