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Fock type spaces with Riesz bases of reproducing kernels and de Branges spaces

by

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Abstract. We describe the radial Fock type spaces which have Riesz bases of normalized reproducing kernels and which are (or are not) isomorphic to de Branges spaces in terms of weight functions.

1. Introduction and main results. Let $\mathcal{F}$ be a Hilbert space of entire functions. Assume that

(i) $\mathcal{F}$ has the *division property*, that is, if $f \in \mathcal{F}$ and $f(\lambda) = 0$, then $f/(\cdot - \lambda) \in \mathcal{F}$, and
(ii) $\mathcal{F}$ has the *bounded point evaluation property*, that is, for each $\lambda \in \mathbb{C}$, the mapping $L_\lambda : f \mapsto f(\lambda)$ is a bounded linear functional on $\mathcal{F}$.

For every $\lambda \in \mathbb{C}$ there exists $\mathbf{k}_\lambda \in \mathcal{F}$, the *reproducing kernel* at λ in $\mathcal{F}$:

$$f(\lambda) = \langle f, \mathbf{k}_\lambda \rangle_{\mathcal{F}}, \qquad f \in \mathcal{F}.$$

Let $\mathbb{k}_\lambda = \mathbf{k}_\lambda / \|\mathbf{k}_\lambda\|$ be the normalized reproducing kernel at λ. Given a sequence $\Lambda \subset \mathbb{C}$, we say that $\{\mathbb{k}_\lambda\}_{\lambda \in \Lambda}$ is a *Riesz basis* (of normalized reproducing kernels) in $\mathcal{F}$ if it is complete and for some $c, C > 0$ we have

$$c \sum_{\lambda \in \Lambda} |a_\lambda|^2 \le \Big\| \sum_{\lambda \in \Lambda} a_\lambda \mathbb{k}_\lambda \Big\|^2 \le C \sum_{\lambda \in \Lambda} |a_\lambda|^2$$

for all finite sequences $\{a_\lambda\}$ of complex numbers. Equivalently, $\{\mathbb{k}_\lambda\}_{\lambda \in \Lambda}$ is a linear isomorphic image of an orthonormal basis in a separable Hilbert space.

The de Branges spaces $\mathcal{H}(\mathcal{E})$ are Hilbert spaces of entire functions determined by Hermite–Biehler class entire functions $\mathcal{E}$ (see [DB]). The norm in

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these spaces is given by $\|F\|^2_{\mathcal{H}(\mathcal{E})} = \int_{\mathbb{R}} |F(x)|^2/|\mathcal{E}(x)|^2\,dx$. They have Riesz bases of normalized reproducing kernels at real points $\{t_n\}_{n\in N}$. Correspondingly, every space $\mathcal{H}(\mathcal{E})$ can be identified with the space $\mathcal{H}$ of all entire functions of the form

$$F(z) = A(z)\sum_{n\in N} \frac{a_n\mu_n^{1/2}}{z-t_n}, \qquad \{a_n\} \in \ell^2,$$

where $\{t_n\}_{n\in N}$ is an increasing sequence such that $|t_n| \to \infty$ as $|n| \to \infty$, $N = \mathbb{Z}$, $\mathbb{Z}_+$ or $\mathbb{Z}_-$, $\sum_{n\in N} \mu_n \delta_{t_n}$ is a positive measure on $\mathbb{R}$ satisfying $\sum_{n\in N} \mu_n/(t_n^2+1) < \infty$, A is an entire function with zero set $\{t_n\}_{n\in N}$ which is real on the real line, and the norm of F is defined as $\|F\|_{\mathcal{H}} = \|\{a_n\}\|_{\ell^2}$. On the other hand, every $\mathcal{F}$ as above having a Riesz basis of normalized reproducing kernels at real points is (up to norm equivalence) a de Branges space (see, for instance, [BMS]).

Given a continuous function (a weight) h defined on $[0,\infty)$, we extend it to the whole complex plane $\mathbb{C}$ by $h(z) = h(|z|)$, and consider the Fock type space

$$\mathcal{F}_h = \Big\{ f \in \mathrm{Hol}(\mathbb{C}) : \|f\|^2 = \|f\|_h^2 = \int_{\mathbb{C}} |f(z)|^2 e^{-h(z)}\,dm(z) < \infty \Big\},$$

where dm is area Lebesgue measure. This is a Hilbert space of entire functions satisfying properties (i) and (ii).

Seip [S] proved that the standard Fock space $\mathcal{F}_h$, with $h(z) = |z|^2$, has no Riesz basis of normalized reproducing kernels. Later on, it was proved that $\mathcal{F}_h$ has no Riesz basis of normalized reproducing kernels for any sufficiently regular h such that $h(t) \gg (\log t)^2$ [BDK, BL]. On the other hand, the spaces $\mathcal{F}_h$ with $h(t) = (\log t)^\alpha$, $1 < \alpha \le 2$, do have Riesz bases of normalized reproducing kernels at real points [BL], and hence are de Branges spaces.

In the opposite direction, Theorem 1.2 of [BBB] states that the de Branges space $\mathcal{H}$ determined by the spectral data $(\{t_n\}_{n\in N}, \{\mu_n\}_{n\in N})$ coincides (up to norm equivalence) with a Fock type space $\mathcal{F}_h$ if and only if the sequence $\{t_n\}_{n\in N}$ is *lacunary*: $\liminf_{t_n\to\infty} t_{n+1}/t_n > 1$, $\liminf_{t_n\to-\infty} |t_n|/|t_{n+1}| > 1$, and for some $C > 0$ and any n,

$$\sum_{|t_k|\le|t_n|} \mu_k + t_n^2 \sum_{|t_k|>|t_n|} \frac{\mu_k}{t_k^2} \le C\mu_n.$$

(These de Branges spaces $\mathcal{H}$ are characterized by the property that every complete and minimal system of reproducing kernels is a strong Markushevich basis (strong M-basis)—see [BBB].)

The main question we deal with in this paper is to find conditions on h determining whether the Fock type space $\mathcal{F}_h$ has a Riesz basis of (normalized) reproducing kernels and whether $\mathcal{F}_h$ is a de Branges space, or equivalently whether the norm in $\mathcal{F}_h$ is equivalent to a weighted L^2 norm along the real line. Our work can be considered as completing a series of works [BL], [BMS], and [BBB]. The main novelty here is Theorem 1.3 below which states that there are Fock spaces which (a) have a Riesz basis of (normalized) reproducing kernels and (b) are not de Branges spaces.

From now on let us fix

$$\psi(t) = h(e^t).$$

THEOREM 1.1. *If ψ'' increases to ∞, then $\mathcal{F}_h$ has no Riesz bases of* (*normalized*) *reproducing kernels.*

In this case, $(\log t)^2 = o(h(t))$ as $t \to \infty$. Under different (in some cases more restrictive) regularity conditions, a similar result follows from [S], [BDK], and [BL].

THEOREM 1.2. *If $\lim_{t\to\infty} \psi'(t) = \infty$ and ψ'' is a non-increasing positive function, and $|\psi'''(t)| = O(\psi''(t)^{5/3})$ as $t \to \infty$, then $\mathcal{F}_h$ has a Riesz basis of* (*normalized*) *reproducing kernels at real points, and hence is* (*up to norm equivalence*) *a de Branges space.*

In particular, the conclusions of Theorem 1.2 are valid for

$$h(t) = (\log t)^\alpha (\log\log t)^\beta, \qquad 1 < \alpha < 2,\ \beta \in \mathbb{R} \text{ or } \alpha = 2,\ \beta \le 0.$$

THEOREM 1.3. *Given any convex positive function φ with $x = o(\varphi(x))$ as $x \to \infty$, there exists a function ψ increasing to ∞, satisfying $\psi(t) = o(\varphi(t))$ as $t \to \infty$, and such that $\mathcal{F}_h$ has a Riesz basis of* (*normalized*) *reproducing kernels and is not* (*up to norm equivalence*) *a de Branges space.*

1.1. Families of reproducing kernels. Let $\{\mathbf{k}_\lambda\}_{\lambda\in\Lambda}$ be a complete minimal family of reproducing kernels in $\mathcal{F}_h$. Then there exists a so called generating function E with simple zeros at the points of Λ such that $E \notin \mathcal{F}_h$ and $E_\lambda = E/(\cdot - \lambda) \in \mathcal{F}_h$ for every $\lambda \in \Lambda$. The family $\{\|\mathbf{k}_\lambda\| E_\lambda/E'(\lambda)\}_{\lambda\in\Lambda}$ is biorthogonal to $\{\mathbb{k}_\lambda\}_{\lambda\in\Lambda}$. Consider the mappings $T_\Lambda : f \mapsto f|\Lambda$ and

$$R_\Lambda : \{a_\lambda\}_{\lambda\in\Lambda} \mapsto E(z) \sum_{\lambda\in\Lambda} \frac{a_\lambda}{E'(\lambda)} \cdot \frac{1}{z-\lambda}.$$

for finite sequences $\{a_\lambda\}_{\lambda\in\Lambda}$. Then $T_\Lambda R_\Lambda = \mathrm{Id}$. If T_Λ is bounded from $\mathcal{F}_h$ to $\ell^2(1/\|\mathbf{k}_\lambda\|^2)$ and R_Λ is bounded from $\ell^2(1/\|\mathbf{k}_\lambda\|^2)$ to $\mathcal{F}_h$, then the family $\{\|\mathbf{k}_\lambda\| E_\lambda/E'(\lambda)\}_{\lambda\in\Lambda}$ is a Riesz basis, and hence $\{\mathbb{k}_\lambda\}_{\lambda\in\Lambda}$ is a Riesz basis of normalized reproducing kernels in $\mathcal{F}_h$ (see, for instance, [N, Section C.3.1]).

1.2. Non-monotonic weights. In this paper, we deal mainly with non-decreasing weights h. This does restrict the class of Fock type spaces we consider.

PROPOSITION 1.4. *There exist continuous h_0 such that $\mathcal{F}_{h_0}$ does not coincide* (*up to norm equivalence*) *with $\mathcal{F}_h$ for any non-decreasing weight h.*

1.3. Notation and organization of the paper. If ψ is convex, then h is subharmonic and we set $\rho(z) = (\Delta h(z))^{-1/2}$ and $\tau(t) = (\psi'')^{-1/2}(t) = \rho(e^t)e^{-t}$. Later on, we will see that ρ will be a local metric coefficient and τ will be a local metric coefficient in the logarithmic scale. Given an analytic function f, we denote by $Z(f)$ its zero set.

The proofs of Theorems 1.1–1.3 are given in Sections 2–4, respectively. Proposition 1.4 is proved in Section 5.

2. Proof of Theorem 1.1. We follow the method proposed in [S] and later used in [BL]. It turns out that in some large annuli we can obtain precise estimates on the norm of the reproducing kernel. If $\{\Bbbk_\lambda\}_{\lambda\in\Lambda}$ is a Riesz basis, then Λ is ρ-separated and ρ-dense on these annuli (see Lemma 2.6 below), and this implies that the corresponding Hilbert transform is unbounded, which leads to a contradiction.

We assume in this section that ψ'' increases to ∞ or equivalently that τ decreases to 0.

LEMMA 2.1. *Given $A < \infty$ and $y_0 > 0$, there exists $y \ge y_0$ such that*

$$\left|\frac{\psi''(x)}{\psi''(y)} - 1\right| \le \frac{1}{A} \quad \textit{whenever} \quad |x-y| \le A\tau(y). \tag{2.1}$$

For similar statements see, for instance, [H].

Proof. Without loss of generality, we may assume that A is as large as we please. Set

$$y_n = y_{n-1} + 4A\tau(y_{n-1}), \qquad n \ge 1,$$

and suppose that

$$\psi''(y_n) \ge \frac{A+1}{A}\psi''(y_{n-1}), \qquad n \ge 1.$$

Then

$$\psi''(y_n) \ge \left(\frac{A+1}{A}\right)^n \psi''(y_0), \qquad n \ge 1,$$

and hence

$$\sum_{n\ge 1}(y_n - y_{n-1}) < \infty, \qquad \lim_{n\to\infty} y_n = y < \infty,$$

which is absurd.

Hence, there exists $z \ge y_0$ such that

$$\psi''(z) \le \psi''(x) < \frac{A+1}{A}\psi''(z), \qquad 0 \le x - z \le 4A\tau(z).$$

It remains to set $y = z + 2A\tau(z)$. For large A we obtain (2.1). ∎

LEMMA 2.2. *Let $B>0$ and $N<\infty$. Given $e^y=R_1<R_2<\dots<R_N$, $R_{N+1}=\infty$, and integers $k_1,\dots,k_N$ such that*

$$\begin{aligned}&\frac{1}{B}\le\frac{R_{j+1}-R_j}{e^y\tau(y)}\le B, \quad && 1\le j<N,\\ &\frac{1}{B}\le k_j\tau(y)\le B, && 1\le j\le N,\end{aligned}$$

set

$$P(z)=\prod_{1\le j\le N}\Big(1-\Big(\frac{z}{R_j}\Big)^{k_j}\Big).$$

Next, set

$$v(t)=\begin{cases}0, & t<R_1,\\ \sum_{1\le s\le j}k_s(\log t-\log R_s), & R_j\le t<R_{j+1},\ 1\le j\le N.\end{cases}$$

Then for every $\delta>0$ there exists $D=D(\delta,B)$ independent of N such that for all $y>y(\delta,\psi,B)$ we have

$$\big|\log|P(z)|-v(|z|)\big|\le D(\delta,B)\quad\textit{whenever}\quad \operatorname{dist}(z,Z(P))>\delta e^y\tau(y).$$

Proof. It suffices to use the estimates

$$\begin{aligned}&\big|\log|1-z^k|\big|\le Ce^{-t}, \quad && |z|=1-t/k,\ t\ge1,\\ &\big|\log|1-z^k|\big|\le D_0(\delta), && 1-1/k<|z|<1+1/k,\ \operatorname{dist}(z,\Lambda_k)>\delta/k,\\ &\big|\log|1-z^k|-k\log|z|\big|\le Ce^{-c\min(t,k)}, && |z|=1+t/k,\ t\ge1,\end{aligned}$$

where $\Lambda_k=\{e^{2\pi ij/k}\}_{0\le j<k}$. ∎

Given $A,R>0$, set

$$\Omega_{R,A}=\big\{z\in\mathbb{C}:\big||z|-R\big|\le A\rho(R)\big\}.$$

LEMMA 2.3. *Given $A<\infty$, there exist $R>0$ and a polynomial Q such that $Z(Q)\setminus\{0\}\subset\Omega_{R,2A}$ and*

$$\begin{aligned}&\frac{1}{M}\le\frac{|Q(z)|\rho(R)e^{-h(z)}}{\operatorname{dist}(z,Z(Q))}\le M, \quad && z\in\Omega_{R,A},\\ &|Q(z)|\le Me^{h(z)}, && z\in\mathbb{C},\\ &\operatorname{dist}(w,Z(Q)\setminus\{w\})\ge\frac{\rho(R)}{M}, && w\in Z(Q)\setminus\{0\},\\ &\operatorname{dist}(z,Z(Q)\setminus\{w\})\le M\rho(R), && z\in\Omega_{R,A},\end{aligned}$$

for some absolute constant M.

Proof. The lemma follows immediately from Lemmas 2.1 and 2.2. Indeed, by Lemma 2.1 we can find R such that ψ'' is almost constant on a suitable interval around $\log R$. Then we find k_j, R_j in Lemma 2.2 and $a\in\mathbb{R}$,

$k \in \mathbb{N}$ such that $a + kt + v(\exp t)$ is close to $\psi(t)$ for $\exp t \in \Omega_{R,2A}$. Finally, we set $Q(z) = e^a z^k P(z)$, where P is given by Lemma 2.2. ∎

For a similar construction see [BDK]; for an alternative way to produce analytic functions Q satisfying such conditions see the atomization procedure in [LM].

Now we can proceed as in [BL].

LEMMA 2.4. *Given A and R such that the conclusions of Lemma 2.3 are valid, and given $w \in \Omega_{R,A/2}$, there exists a function Φ_w analytic in the disc $D_w = \{z \in \mathbb{C} : |z - w| < \rho(R)\}$ and such that*

$$1/M \le |\Phi_w(z)| e^{-h(z)} \le M, \qquad z \in D_w,$$

for some absolute constant M.

Proof. The proof of this folklore result is identical to that of [BL, Lemma 2.1]. ∎

LEMMA 2.5. *Given A and R such that the conclusions of Lemma 2.3 are valid, we have*

$$1/M \le \|\mathbf{k}_z\| e^{-h(z)/2} \rho(R) \le M, \qquad z \in \Omega_{R,A/3},$$

for some absolute constant M.

Proof. Analogous to that of [BL, Lemma 2.3] with F there replaced by our polynomial Q. ∎

LEMMA 2.6. *In the notation of Lemma 2.3, let a sequence $\Lambda \subset \mathbb{C}$ be such that $\{\Bbbk_\lambda\}_{\lambda\in\Lambda}$ is a Riesz basis in $\mathcal{F}_h$. Then*

(a) $\operatorname{dist}(\lambda, \Lambda \setminus \{\lambda\}) \ge \beta\rho(R)$ *for all* $\lambda \in \Lambda \cap \Omega_{R,A/4}$,
(b) $\operatorname{dist}(z, \Lambda) \le \rho(R)/\beta$ *for all* $z \in \Omega_{R,A/4}$,

for some $\beta = \beta(\Lambda)$ and for $A > A(\Lambda)$.

Proof. Analogous to that of [BL, Lemma 2.4]. ∎

Proof of Theorem 1.1. We follow the proof of [BL, Theorem 2.5]. Suppose that $\{\Bbbk_\lambda\}_{\lambda\in\Lambda}$ is a Riesz basis in $\mathcal{F}_h$. Fix a large A, choose R and Q as in Lemma 2.3, and define E as in Subsection 1.1. Lemma 2.5 implies that

$$\left\|\frac{E}{\cdot - \lambda}\right\|^2 \asymp \frac{|E'(\lambda)|^2}{\|\mathbf{k}_z\|^2} \asymp |E'(\lambda)|^2 (\rho(\lambda))^2 e^{-h(\lambda)}, \qquad \lambda \in \Lambda \cap \Omega_{R,A/4}.$$

Consider the function $E/[(\cdot - \lambda)\Phi_\lambda]$. Applying the mean value property we obtain

$$\int_{\mathbb{C}} \frac{|E(z)|^2}{|z-\lambda|^2} e^{-h(z)}\, dm(z) \le \frac{C}{(\rho(R))^2} \int_{|z-\lambda|<\rho(R)} |E(z)|^2 e^{-h(z)}\, dm(z).$$

By Lemma 2.6(a), we have

$$\int\limits_{\mathbb{C}}\Big(\sum_{|\lambda-R|<N\rho(R),\,\lambda\in\Lambda}\frac{1}{|z-\lambda|^2}\Big)|E(z)|^2e^{-h(z)}\,dm(z)\le\frac{C}{(\rho(R))^2}\int\limits_{|R-z|<(N+1)\rho(R)}|E(z)|^2e^{-h(z)}\,dm(z),$$

with $A/4-1<N\le A/4$, and, as a result,

$$\inf_{z:\,|z-R|<(N+1)\rho(R)}\Big[(\rho(R))^2\sum_{|\lambda-R|<N\rho(R),\,\lambda\in\Lambda}\frac{1}{|z-\lambda|^2}\Big]\le C. \tag{2.2}$$

Finally, by Lemma 2.6(b),

$$(\rho(R))^2\sum_{|\lambda-R|<N\rho(R),\,\lambda\in\Lambda}\frac{1}{|z-\lambda|^2}\ge C\int\limits_{\rho(R)<|\zeta|<N\rho(R)}\frac{dm(\zeta)}{|z-\zeta|^2}.$$

For large A (and hence large N) we get a contradiction to (2.2). ∎

3. Proof of Theorem 1.2. The situation for h of slow growth is quite different from that considered in Section 2. In particular, along some special (lacunary) sequence of points $\{\lambda_n\}_{n\ge 0}$ we show that $\|\mathbf{k}_{\lambda_n}\|^2\asymp e^{h(\lambda_n)}/(\lambda_n\rho(\lambda_n))\gtrsim e^{h(\lambda_n)}/(\rho(\lambda_n))^2$ instead of $\|\mathbf{k}_{\lambda_n}\|^2\asymp e^{h(\lambda_n)}/(\rho(\lambda_n))^2$ in Lemma 2.5. To prove the boundedness of the operator R_Λ introduced in Subsection 1.1 we use the results of [BMS].

For $n\ge\psi'(0)/2-1$ we choose y_n such that $\psi'(y_n)=2n+2$ and set $\lambda_n=\exp y_n$. Define

$$g(t)=\psi(y_n+t)-(y_n+t)\psi'(y_n),\qquad -y_n\le t<\infty.$$

Set $\alpha_n=\psi''(y_n)=g''(0)$. Then $g'(0)=0$, $0<g''(t)\le\psi''(0)$, $|g'''(t)|=O((g''(t))^{5/3})$, and hence

$$g(t)-g(0)\asymp\alpha_nt^2,\qquad |t|\le C\alpha_n^{-2/3}, \tag{3.1}$$

$$g'(t)\asymp\alpha_nt,\qquad |t|\le C\alpha_n^{-2/3}, \tag{3.2}$$

$$g(t)-g(0)\ge C\alpha_n^{-1/3}+C\alpha_n^{1/3}|t|,\qquad |t|\ge C\alpha_n^{-2/3}. \tag{3.3}$$

To the finite number of y_n not yet defined, we assign arbitrary values in such a way that the sequence $\{y_n\}_{n\ge0}$ of positive numbers is strictly increasing.

Lemma 3.1.

$$\|z^n\|^2\asymp e^{(2n+2)y_n-\psi(y_n)}(\psi''(y_n))^{-1/2},\qquad n\ge 0.$$

Proof. We have

$$\|z^n\|^2=O(1)+2\pi\int_1^\infty t^{2n+1}e^{-\psi(\log t)}\,dt=O(1)+2\pi\int_0^\infty e^{(2n+2)s-\psi(s)}\,ds.$$

By (3.1) and (3.3) we have

$$\frac{C_1}{\sqrt{\alpha_n}} \le \int_{-y_n}^{\infty} e^{g(0)-g(t)}\,dt \le \frac{C_2}{\sqrt{\alpha_n}},$$

and hence

$$\frac{C_1}{\sqrt{\alpha_n}} \le \|z^n\|^2 e^{\psi(y_n)-(2n+2)y_n} \le \frac{C_2}{\sqrt{\alpha_n}}. \ \blacksquare$$

Next, we estimate the norm of the reproducing kernel at the points λ_n (cf. also Lemma 2.5).

LEMMA 3.2.

$$\|\mathbf{k}_{\lambda_n}\|^2 \asymp e^{\psi(y_n)-2y_n}(\psi''(y_n))^{1/2} = e^{h(\lambda_n)}/(\lambda_n\rho(\lambda_n)), \qquad n \ge 0.$$

Proof. First of all, by Lemma 3.1, we have

$$\|\mathbf{k}_{\lambda_n}\|^2 \ge \frac{\lambda_n^{2n}}{\|z^n\|^2} \ge Ce^{\psi(y_n)-2y_n}(\psi''(y_n))^{1/2}.$$

Let $f \in \mathcal{F}_h$ and

$$F(t) = \Big(\int_0^{2\pi} |f(te^{i\theta}|^2\,d\theta\Big)^{1/2}, \qquad \omega(s) = \log F(\exp s).$$

Hardy's convexity theorem [D, Chapter 1] states that ω is convex.

Next, by (3.2), for large n we have $y_n - \alpha_n^{-1/2} > y_{n-1}$ and

$$\begin{aligned}\|f\|^2 &= \int_0^{\infty} (F(t))^2 e^{-h(t)} t\,dt \ge \int_0^{\infty} e^{2\omega(s)-\psi(s)+2s}\,ds \\ &\ge \int_{y_n-\alpha_n^{-1/2}}^{y_n+\alpha_n^{-1/2}} e^{2\omega(s)-\psi(s)+2s}\,ds \\ &\ge Ce^{-g(0)} \int_{-\alpha_n^{-1/2}}^{\alpha_n^{-1/2}} e^{2\omega(y_n+s)-2n(y_n+s)}\,ds =: I.\end{aligned}$$

If $\delta = \alpha_0^{-1/2}$, then by convexity of ω we have

$$\begin{aligned}I &\ge Ce^{-g(0)}\alpha_n^{-1/2} \int_{-\delta}^{\delta} e^{2\omega(y_n+s)-2n(y_n+s)}\,ds \\ &= Ce^{-g(0)}\alpha_n^{-1/2} \int_{y_n-\delta}^{y_n+\delta} e^{2\omega(s)-2ns}\,ds \\ &= Ce^{-g(0)}\alpha_n^{-1/2} \int_{\exp(y_n-\delta)}^{\exp(y_n+\delta)} \frac{|f(z)|^2}{|z|^{2n+2}}\,dm(z).\end{aligned}$$

Applying the mean value theorem to the function $f(z)z^{-n-1}$, we conclude that

$$|f(e^{y_n})|^2 \le C\|f\|^2 e^{g(0)}\alpha_n^{1/2} e^{2ny_n} = C\|f\|^2 e^{\psi(y_n)-2y_n}\alpha_n^{1/2}. \blacksquare$$

Next, we consider a continuous piecewise linear function ℓ such that

$$\ell'(t) = 2n+2, \qquad y_n < t < y_{n+1},\ n \ge 0.$$

Then $\ell'(t) \le \psi'(t)$ for large t, and hence $\ell(t) \le \psi(t) + O(1)$ as $t \to \infty$. On the other hand, $\ell'(t)+2 \ge \psi'(t)$ for large t, and hence $\ell(t)+2t \ge \psi(t)+O(1)$ as $t \to \infty$.

Lemma 3.3. *We have*

(a) $$\int_0^\infty e^{\ell(t)-\psi(t)}\,dt < \infty,$$

(b) $$\int_{y_n-\delta}^\infty e^{\ell(t)-\psi(t)}\,dt \le Ce^{\ell(y_n)-\psi(y_n)}(\psi''(y_n))^{-1/2},$$

(c) $$\int_0^{y_n-\delta} e^{\ell(t)-\psi(t)+2t}\,dt \le Ce^{\ell(y_n)-\psi(y_n)+2y_n}(\psi''(y_n))^{-1/2},$$

(d) $$\sum_{s=0}^n e^{\psi(y_s)-\ell(y_s)}(\psi''(y_s))^{1/2} \asymp e^{\psi(y_n)-\ell(y_n)}(\psi''(y_n))^{1/2},$$

(e) $$\sum_{s=n}^\infty e^{\psi(y_s)-\ell(y_s)-2y_s}(\psi''(y_s))^{1/2} \asymp e^{\psi(y_n)-\ell(y_n)-2y_n}(\psi''(y_n))^{1/2},$$

for any fixed $\delta \in (0, y_1)$.

Proof. Set $u = \psi - \ell$. Then

$$\begin{aligned} u'(y_n+t) &\ge C\alpha_n t, && 0 \le t \le C\alpha_n^{-2/3},\\ u'(y_n+t) &\ge C\alpha_n^{1/3}, && C\alpha_n^{-2/3} < t < y_{n+1}-y_n,\\ u'(y_n-t) &\ge 1, && 0 \le t \le C\alpha_n^{-2/3}. \end{aligned}$$

Therefore,

$$u(y_n) \ge u(y_{n-1}) + C\alpha_n^{-2/3}.$$

Since ψ'' does not increase, (a), (b), and (d) follow immediately.

Next, set $w(t) = \ell(t) + 2t - \psi(t)$. Then

$$\begin{aligned} w'(y_n+t) &\ge 1, && 0 \le t \le C\alpha_n^{-2/3},\\ w'(y_n-t) &\ge C\alpha_n t, && 0 \le t \le C\alpha_n^{-2/3},\\ w'(y_n-t) &\ge C\alpha_n^{1/3}, && C\alpha_n^{-2/3} < t < y_n-y_{n-1}. \end{aligned}$$

Therefore,

$$w(y_{n+1}) \ge w(y_n) + C\alpha_n^{-2/3}.$$

Since ψ'' does not increase, (c) and (e) follow immediately. ■

Set

$$E(z)=\prod_{n\ge 0}\Big(1-\frac{z}{\lambda_n}\Big).$$

Arguing as in the proof of Lemma 2.2, for small $\delta>0$ we obtain

$$|E(e^t)|^2\le Ce^{\ell(t)},\qquad t\ge 0,$$
$$|E(e^t)|^2\asymp e^{\ell(t)},\qquad \operatorname{dist}(t,\{y_n\}_{n\ge 0})\ge\delta.$$

Therefore, for every entire function $F\not\equiv 0$ we have

$$FE\notin\mathcal{F}_h. \tag{3.4}$$

Indeed,

$$\int_{\mathbb{C}}|F(z)|^2|E(z)|^2e^{-h(z)}\,dm(z)\ge C\int_0^\infty e^{\ell(t)-\psi(t)+2t}\,dt=\infty.$$

In a similar way, by Lemma 3.3(a), $E/(\cdot-\lambda_0)\in\mathcal{F}_h$. Furthermore,

$$|E'(\lambda_n)|^2\asymp e^{\ell(y_n)-2y_n}. \tag{3.5}$$

Set $d\mu(z)=|E(z)|^2e^{-h(z)}\,dm(z)$, $v_n=e^{\psi(y_n)-\ell(y_n)}(\psi''(y_n))^{1/2}$, $\Omega_n=\{z: e^{y_n-\delta}<|z|<e^{y_{n+1}-\delta}\}$, for $n\ge 0$. By Lemma 3.3(b), we have

$$\begin{aligned}\int_{\Omega_n}\frac{d\mu(z)}{|z-\lambda_n|^2}&=\int_{e^{y_n-\delta}<|z|<e^{y_{n+1}-\delta}}\Big|\frac{E(z)}{z-e^{y_n}}\Big|^2e^{-h(z)}\,dm(z)\\&\asymp\int_{y_n-\delta}^{y_{n+1}-\delta}e^{\ell(t)-2(t-y_n)^+-\psi(t)+2(t-y_n)}\,dt\\&\le\int_{y_n-\delta}^{\infty}e^{\ell(t)-\psi(t)}\,dt\\&\le C\frac{e^{\ell(y_n)-\psi(y_n)}}{(\psi''(y_n))^{1/2}}=\frac{C}{v_n},\qquad n\ge 0.\end{aligned}\tag{3.6}$$

Furthermore, again by Lemma 3.3(b), we have

$$\sum_{m=n+1}^\infty\int_{\Omega_m}\frac{d\mu(z)}{|z|^2}=\int_{y_{n+1}-\delta}^\infty e^{\ell(t)-\psi(t)}\,dt\le\frac{C}{v_{n+1}},\quad n\ge 0.$$

By Lemma 3.3(d), we find that

$$\sum_{s=0}^n v_s\asymp v_n,\quad n\ge 0.$$

Therefore,

$$\sum_{s=0}^n v_s\sum_{m=n+1}^\infty\int_{\Omega_m}\frac{d\mu(z)}{|z|^2}\le C,\quad n\ge 0. \tag{3.7}$$

Next, by Lemma 3.3(c), we obtain

$$\sum_{m=0}^{n} \int_{\Omega_m} d\mu(z) = \int_{0}^{y_{n+1}-\delta} e^{\ell(t)-\psi(t)+2t}\,dt \le C\frac{e^{2y_{n+1}}}{v_{n+1}}, \quad n \ge 0.$$

By Lemma 3.3(e), we have

$$\sum_{s=n+1}^{\infty} \frac{v_s}{|\lambda_s|^2} \asymp \frac{v_{n+1}}{|\lambda_{n+1}|^2}, \quad n \ge 0.$$

Therefore,

$$\sum_{s=n+1}^{\infty} \frac{v_s}{|\lambda_s|^2} \sum_{m=0}^{n} \int_{\Omega_m} d\mu(z) \le C, \quad n \ge 0. \tag{3.8}$$

Set $\Lambda = \{\lambda_n\}_{n\ge0}$ and consider the mappings T_Λ and R_Λ defined in Subsection 1.1. The argument in the proof of Lemma 3.2 implies that the mapping T_Λ is bounded from $\mathcal{F}_h$ to $\ell^2(1/\|\mathbf{k}_\lambda\|^2)$. By (3.4), T_Λ is injective. By Lemma 3.2 and by (3.5), $v_n \asymp \|\mathbf{k}_{\lambda_n}\|^2/|E'(\lambda_n)|^2$. Estimates (3.6)–(3.8) permit us to apply [BMS, Theorem 1.1] to show that $d\mu$ is a Carleson measure for the space of the corresponding discrete Hilbert transforms. Thus, R_Λ is bounded from $\ell^2(1/\|\mathbf{k}_\lambda\|^2)$ to $\mathcal{F}_h$. By the argument in Subsection 1.1 we conclude that $\{\Bbbk_{\lambda_n}\}_{n\ge0}$ is a Riesz basis in $\mathcal{F}_h$.

4. Proof of Theorem 1.3. In this section we construct an increasing weight h and a corresponding set Λ consisting of (increasing numbers of) equidistributed points on rapidly growing concentric circles such that $\{\Bbbk_\lambda\}_{\lambda\in\Lambda}$ is a Riesz basis in $\mathcal{F}_h$.

Given an increasing sequence $\{R_n\}_{n\ge1}$ we set

$$\psi(t) = t + 2\sum_{s\le n} s(t-\log R_s) + n\min(t-\log R_n, \log R_{n+1}-t),$$
$$\log R_n \le t \le \log R_{n+1},\ n \ge 1.$$

Clearly, ψ is an increasing function.

An elementary geometric argument permits us to choose a sequence $\{R_n\}_{n\ge1}$ satisfying the following two properties:

(I) $R_1 \ge 2, \quad R_{n+1} \ge R_n^2, \quad n \ge 1,$
(II) $\psi(t) = o(\varphi(t)), \quad t \to \infty.$

In particular, $\log R_n \ge 2^n \log 2$. As above, we fix $h(t) = \psi(\log t)$. Next, we define

$$E(z) = \prod_{n\ge1}\Bigl(1-\Bigl(\frac{z}{R_n}\Bigr)^n\Bigr), \quad \Lambda = Z(E).$$

By an argument similar to that in the proof of Lemma 2.2, we have

$$
\begin{aligned}
&|E(z)| \asymp e^{h(z)/2}\frac{n\,\mathrm{dist}(z,\Lambda)}{R_n^{3/2}}, && \big||z|-R_n\big| \le \frac{R_n}{n},\\
(4.1)\quad &|E'(\lambda)| \asymp e^{h(R_n)/2}\frac{n}{R_n^{3/2}}, && \lambda\in\Lambda,\ |\lambda| = R_n,\\
&|E(z)| \asymp e^{h(z)/2}|z|^{-1/2}\left|\frac{R_n}{z}\right|^{n/2}, && \frac{n+1}{n}R_n \le |z| \le \sqrt{R_nR_{n+1}},\\
&|E(z)| \asymp e^{h(z)/2}|z|^{-1/2}\left|\frac{z}{R_{n+1}}\right|^{n/2}, && \sqrt{R_nR_{n+1}} \le |z| \le \frac{nR_{n+1}}{n+1}.
\end{aligned}
$$

Let $F \ne 0$ be an entire function. Then

$$
\begin{aligned}
(4.2)\quad &\int_{\mathbb{C}} |F(z)|^2|E(z)|^2 e^{-h(z)}\,dm(z)\\
&\ge C\sum_{n\ge1}\int_{\frac{n-1}{n}\le\frac{|z|}{R_n}\le\frac{n+1}{n}} |F(z)|^2\frac{n^2\,\mathrm{dist}^2(z,\Lambda)}{R_n^3}\,dm(z) \ge \sum_{n\ge1}\frac{R_n}{n} = \infty.
\end{aligned}
$$

Thus, $E\notin\mathcal{F}_h$, and more generally, $FE\notin\mathcal{F}_h$ for any entire function $F\ne0$.

Given $\lambda\in\Lambda$, we set $E_\lambda = E/(\cdot-\lambda)$. Next, we are going to deal with the scalar products $\langle E_\lambda, E_\mu\rangle_{\mathcal{F}_h}$. For $\lambda\in\Lambda$ with $|\lambda| = R_n$, we have

$$
\begin{aligned}
(4.3)\quad \|E_\lambda\|^2 &= \int_{\mathbb{C}}\frac{|E(z)|^2}{|z-\lambda|^2}e^{-h(z)}\,dm(z)\\
&\le C\Bigg(\int_{\frac{n-1}{n}\le\frac{|z|}{R_n}\le\frac{n+1}{n}}\frac{n^2\,\mathrm{dist}^2(z,\Lambda)}{R_n^3\,|z-\lambda|^2}\,dm(z)\\
&\qquad + \sum_{s\ge1,\,s\ne n}\int_{\frac{s-1}{s}\le\frac{|z|}{R_s}\le\frac{s+1}{s}}\frac{s^2\,\mathrm{dist}^2(z,\Lambda)}{R_s^3\,|z-\lambda|^2}\,dm(z)\\
&\qquad + \sum_{s\ge1}\int_{\frac{s+1}{s}R_s\le|z|\le\sqrt{R_sR_{s+1}}}\left|\frac{R_s}{z}\right|^s\frac{dm(z)}{|z|\cdot|z-\lambda|^2}\\
&\qquad + \sum_{s\ge1}\int_{\sqrt{R_sR_{s+1}}\le|z|\le\frac{s}{s+1}R_{s+1}}\left|\frac{z}{R_{s+1}}\right|^s\frac{dm(z)}{|z|\cdot|z-\lambda|^2}\Bigg)\\
&\le \frac{C}{R_n} + C\sum_{s\ge1,\,s\ne n}\frac{R_s}{s\max(R_s,R_n)^2} \asymp \frac{1}{R_n}.
\end{aligned}
$$

Furthermore, if $\lambda,\mu\in\Lambda$ with $R_j = |\lambda| < |\mu| = R_k$, then in a similar way

$$
(4.4)\qquad \int_{\mathbb{C}}\frac{|E(z)|^2}{|z-\lambda|\cdot|z-\mu|}e^{-h(z)}\,dm(z) \le \frac{C\log(k+1)}{R_k}.
$$

Next we fix $n \ge 1$ and set $R = R_n$. Then

$$E(z) = \Big(1 + O\Big(\frac{1}{R}\Big)\Big)\Big(1 - \Big(\frac{z}{R}\Big)^n\Big) \prod_{1\le s<n} \Big(\frac{z}{R_s}\Big)^s, \qquad \big||z| - R\big| \le \frac{R}{2}.$$

Let $\alpha \ne \beta$ with $|\alpha| = |\beta| = 1$, and let $\lambda = \alpha R$, $\mu = \beta R$ belong to Λ. Then

$$\begin{aligned}
&\int_{\mathbb{C}} \frac{|E(z)|^2}{(z-\lambda)\,\overline{z-\mu}} e^{-h(z)}\, dm(z) \\
&\quad = O\Big(\frac{1}{nR}\Big) + \int_{R/2\le|z|\le R} \frac{\big(1-\big(\frac{z}{R}\big)^n\big)\big(1-\big(\frac{\bar z}{R}\big)^n\big)}{(z-\lambda)\,\overline{z-\mu}} \Big|\frac{z}{R}\Big|^{n-1} \frac{dm(z)}{|z|} \\
&\qquad + \int_{R\le|z|\le 3R/2} \frac{\big(1-\big(\frac{z}{R}\big)^n\big)\big(1-\big(\frac{\bar z}{R}\big)^n\big)}{(z-\lambda)\,\overline{z-\mu}} \Big|\frac{R}{z}\Big|^{3n} \frac{dm(z)}{|z|} \\
&\quad = O\Big(\frac{1}{nR}\Big) + \frac{1}{R}\int_{1/2}^{1} \Big(\sum_{s=0}^{n-1} \alpha^s w^{n-1-s}\Big) \overline{\sum_{s=0}^{n-1} \beta^s w^{n-1-s}}\, |w|^{n-2}\, dm(w) \\
&\qquad + \frac{1}{R}\int_{1}^{3/2} \Big(\sum_{s=0}^{n-1} \alpha^s w^{n-1-s}\Big) \overline{\sum_{s=0}^{n-1} \beta^s w^{n-1-s}}\, |w|^{-3n-1}\, dm(w) \\
&\quad = O\Big(\frac{1}{nR}\Big) + \frac{1}{R}\sum_{s=0}^{n-1} 2\pi(\alpha\overline{\beta})^s \Big(\frac{1}{2s+n+1} + \frac{1}{3n-2s-2}\Big).
\end{aligned}$$

Set

$$\begin{aligned}
a_s &= \frac{1}{2s+n+1} + \frac{1}{3n-2s-2}, \qquad 0 \le s < n, \\
b_{j,k} &= \sum_{s=0}^{n-1} e^{2\pi i(j-k)s/n} a_s, \qquad 0 \le j,k < n, \\
A &= (b_{j,k})_{0\le j,k<n}.
\end{aligned}$$

Then A is a circulant matrix. It is well known and easily verified that A has an orthonormal basis of eigenvectors $n^{-1/2}(e^{2\pi ikq/n})_{0\le k<n}$, $0 \le q < n$, and the corresponding eigenvalues are $\sum_{k=0}^{n-1} b_{0,k} e^{2\pi ikq/n}$, $0 \le q < n$. Therefore, the norm of the corresponding operator acting on ℓ^2_n is equal to

$$\max_{0\le q<n} \Big|\sum_{k=0}^{n-1} e^{2\pi ikq/n} \sum_{s=0}^{n-1} e^{-2\pi iks/n} a_s\Big| = n \max_{0\le q<n} |a_q| = O(1), \qquad n \to \infty.$$

Thus, if

$$B_n = (\langle E_\lambda, E_\mu\rangle_{\mathcal{F}_h})_{\lambda,\,\mu\in\Lambda,\,|\lambda|=|\mu|=R_n},$$

then

$$\|B_n\|_{\ell^2_n\to\ell^2_n} = O(1/R_n), \qquad n \to \infty. \tag{4.5}$$

Now, for $|\lambda| = R_n$ we estimate $\|\mathbf{k}_\lambda\|$. Applying the mean value theorem to the function

$$g(z) = f(z) \prod_{1\le s<R_n} \left(\frac{R_s}{z}\right)^s$$

in the disc $D = \{z : |z-\lambda| < R_n/n\}$, we obtain

$$(4.6) \qquad |f(\lambda)|^2 e^{-h(\lambda)} \le \frac{Cn^2}{R_n^2} \int_D |f(w)|^2 e^{-h(w)}\, dm(w),$$

and hence

$$\|\mathbf{k}_\lambda\| \le Ce^{h(R_n)/2}\frac{n}{R_n}, \qquad |\lambda| = R_n.$$

By (4.1) and (4.3), for $\lambda \in \Lambda$ with $|\lambda| = R_n$ we have

$$\|\mathbf{k}_\lambda\| \ge \frac{|E_\lambda(\lambda)|}{\|E_\lambda\|} = \frac{|E'(\lambda)|}{\|E_\lambda\|} \ge Ce^{h(R_n)/2}\frac{n}{R_n}.$$

Replacing $E(z)$ by $E(e^{i\theta}z)$ we get the same estimate for all λ with $|\lambda| = R_n$. Thus,

$$(4.7) \qquad \frac{1}{C}e^{h(R_n)/2}\frac{n}{R_n} \le \|\mathbf{k}_\lambda\| \le Ce^{h(R_n)/2}\frac{n}{R_n}, \qquad |\lambda| = R_n,$$

for some C independent of λ and n.

Now, we consider the mappings T_Λ and R_Λ defined in Subsection 1.1. Estimate (4.6) implies that T_Λ is bounded from $\mathcal{F}_h$ to $\ell^2(1/\|\mathbf{k}_\lambda\|^2)$. By (4.2), T_Λ is injective. Set $e_\lambda = \{\|\mathbf{k}_\lambda\|\delta_{\lambda\mu}\}_{\mu\in\Lambda}$. Then $\|e_\lambda\|_{\ell^2(1/\|\mathbf{k}_\lambda\|^2)} = 1$ and $R_\Lambda e_\lambda = E_\lambda\|\mathbf{k}_\lambda\|/E'(\lambda)$. Estimates (4.1), (4.4), (4.5), and (4.7) show that the matrix $(\langle R_\Lambda e_\lambda, R_\Lambda e_\mu\rangle)_{\lambda,\mu\in\Lambda}$ consists of (i) a sequence of squares along the diagonal with uniformly bounded norms and (ii) rapidly decaying off-diagonal terms. Thus, R_Λ is bounded from $\ell^2(1/\|\mathbf{k}_\lambda\|^2)$ to $\mathcal{F}_h$. By the argument in Subsection 1.1 we conclude that $\{\mathbb{k}_\lambda\}_{\lambda\in\Lambda}$ is a Riesz basis in $\mathcal{F}_h$.

Finally, for $n \ge 3$ we set

$$f_n(z) = \frac{E(z)}{z-R_n} \prod_{1\le s<n/2} \frac{z - e^{-2\pi is/n}R_n}{z - e^{2\pi is/n}R_n}.$$

Then $|E_{R_n}| = |f_n|$ on the real line, and

$$\begin{aligned} \|f_n\|^2 &= \int_{\mathbb{C}} |f_n(z)|^2 e^{-h(z)}\, dm(z) \\ &\ge \int_{||z|-R_n|\le R_n/n,\, |\arg z-\pi/2|\le\pi/4} |f_n(z)|^2 e^{-h(z)}\, dm(z) \\ &\ge \frac{C}{nR_n}2^{n/20} \ge n\|E_{R_n}\|^2, \qquad n\to\infty. \end{aligned}$$

This shows that $\mathcal{F}_h$ is not (up to norm equivalence) isomorphic to a de Branges space.

5. Proof of Proposition 1.4. Let $\mathcal{F}_h = \mathcal{F}_{h_0}$ (up to norm equivalence), and suppose that all polynomials belong to $\mathcal{F}_{h_0}$. We have

$$\|z^n\|^2_{\mathcal{F}_h} = \int_0^\infty 2\pi r^{2n+1} e^{-h(r)}\, dr = \int_0^\infty x^n u_h(x)\, dx,$$

where $u_h(x) = \pi e^{-h(\sqrt{x})}$.

If h is continuous and non-decreasing, we can find a non-increasing $\tilde{u} \in C^1(\mathbb{R})$ such that $u_h \le \tilde{u} \le 2u_h$ and $\tilde{u}' \le 0$. Since

$$\int_0^\infty x^n \tilde{u}(x)\, dx = -\frac{1}{n+1} \int_0^\infty x^{n+1} \tilde{u}'(x)\, dx,$$

by the Cauchy–Schwarz inequality we deduce that for some bounded sequence $\{c_n\}_{n\ge1}$,

(5.1) the sequence $\{\log[n\|z^n\|^2_{\mathcal{F}_{h_0}}] + c_n\}_{n\ge1}$ is convex.

On the other hand, let

$$p_n = \log \int_0^\infty x^n\, d\mu(x), \quad \text{where} \quad d\mu(x) = \sum_{k\ge1} e^{-s_k t_k} \delta_{\exp s_k}$$

with some $t_k \in \mathbb{N}$ and $s_k \ge 1$ for $k \ge 0$. If $t_k > t_{k-1} + 1$ and $s_k > 2t_k s_{k-1}$ for $k > 1$, then

$$p_n = \log \sum_{k\ge1} e^{(n-t_k)s_k} = \max_{k\ge1}[(n-t_k)s_k] + O(1), \qquad n \ge 1,$$

and

$$p_n = (n-t_k)s_k + O(1) \qquad \text{for } t_k + 1 \le n \le t_{k+1},\ k \ge 1.$$

Finally, suppose $t_k \ge t^2_{k-1}$ for $k > 1$, let u_{h_0} be a continuous positive function such that

$$\Big| p_n - \log \int_0^\infty x^n u_{h_0}(x)\, dx \Big| \le 1, \qquad n \ge 1,$$

and let $u_{h_0}(x) = \pi e^{-h_0(\sqrt{x})}$. Then h_0 is continuous and there exists a bounded sequence $\{d_n\}_{n\ge1}$ such that the sequence $\{\log[\|z^n\|^2_{\mathcal{F}_{h_0}}] + d_n\}_{n\ge1}$ is linear for $t_k + 1 \le n \le t_{k+1}$, $k \ge 1$, which contradicts (5.1).

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