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Twisted Orlicz algebras, I

by

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Abstract. Let G be a locally compact group, let $\Omega : G \times G \to \mathbb{C}^*$ be a 2-cocycle, and let Φ be a Young function. In this paper, we consider the Orlicz space $L^\Phi(G)$ and investigate its algebraic properties under the twisted convolution $\circledast$ coming from Ω. We find sufficient conditions under which $(L^\Phi(G), \circledast)$ becomes a Banach algebra or a Banach $*$-algebra; we then call it a *twisted Orlicz algebra*. Furthermore, we study its harmonic analysis properties, such as symmetry, existence of functional calculus, regularity, and the Wiener property, mostly when G is a compactly generated group of polynomial growth. We apply our methods to several important classes of polynomial as well as subexponential weights, and demonstrate that our results could be applied to a variety of cases.

Introduction. In harmonic analysis, an important object related to a locally compact group G and its (left) Haar measure ds is the group algebra $L^1(G) := L^1(G, ds)$. Since the Haar measure is invariant under left translations, the group operations on G extend to $L^1(G)$ so that it becomes a Banach $*$-algebra with respect to the convolution

$$(f * g)(t) = \int_G f(s)g(s^{-1}t)\, ds$$

and the involution

$$f^*(t) = \overline{f(t^{-1})}\Delta(t^{-1}),$$

where Δ is the modular function of G. The properties of $L^1(G)$ have been well-studied over the last couple of decades, and one could deduce much information about G from $L^1(G)$ and vice versa. For instance, the unitary representations of G are in one-to-one correspondence with the non-degenerate bounded $*$-representations of $L^1(G)$. One could also consider the "twisted"

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convolution and involution on $L^1(G)$, i.e.

$$(f \circledast g)(t) = \int_G f(s)g(s^{-1}t)\Omega_{\mathbb{T}}(s, s^{-1}t)\, ds$$

and

$$f^*(t) = \overline{f(t^{-1})}\Delta(t^{-1})\overline{\Omega_{\mathbb{T}}(t, t^{-1})},$$

where $\Omega_{\mathbb{T}}$ is a 2-cocycle on G with values in the unit circle $\mathbb{T}$. These concepts appear naturally when one consider the "projective" unitary representations of G, as well as in other areas of mathematics such as non-commutative geometry or Gabor analysis (see, for example, [2], [6], [16]).

A generalization of $L^1(G)$ is the $L^p(G)$ space for $1 \leq p < \infty$ which is a Banach space, even a Banach module over $L^1(G)$ with respect to the convolution, but it is a Banach algebra only when G is compact [23]. On the other hand, a very natural phenomenon occurring in harmonic analysis is the appearance of a "weight" on a group or a "weighted norm" on an algebra. A *weight* ω on G is a locally bounded measurable function from G into the positive reals. For such a weight, one can extend the construction of $L^p(G)$ to the "weighted" spaces

$$L^p_\omega(G) := \{f : f\omega \in L^p(G) \text{ and } \|f\|_\omega = \|f\omega\|_p\}.$$

These spaces have various interesting properties and numerous applications in harmonic analysis. For instantce, by applying the Fourier transform, we find that Sobolev spaces $W^{k,2}(\mathbb{T})$ are nothing other than certain $l^2_\omega(\mathbb{Z})$ spaces.

A particular aspect of the behavior of weighted L^p spaces over locally compact groups is that they can form an algebra with respect to convolution. More precisely, when $p = 1$ and ω is submultiplicative, it follows routinely that $L^1_\omega(G)$ is a Banach algebra. Even though this may not hold in general if $p > 1$, there are sufficient conditions under which $L^p_\omega(G)$ is a Banach algebra with respect to convolution. This was first shown by J. Wermer [24] for $G = \mathbb{R}$; later Yu. N. Kuznetsova extended it to general locally compact groups. She has also studied some important Banach algebra properties of $L^p_\omega(G)$ such as the existence of an approximate identity and, for an abelian G, a description of their the maximal ideal space (see [9], [10] and the references therein). Moreover, Yu. N. Kuznetsova and C. Molitor-Braun [11] studied other properties such as symmetry, existence of functional calculus and the Wiener property.

Orlicz spaces are vast generalizations of L^p spaces. A variety of function spaces arise naturally in this way, like $L\log^+ L$ which is a Banach space related to Hardy–Littlewood maximal functions. Orlicz spaces can also contain certain Sobolev spaces as subspaces. Similar to L^p spaces, one can also consider weighted Orlicz spaces and study their properties. Very recently,

A. Osançlıol and S. Öztop [18] have looked at weighted Orlicz spaces as Banach algebras with respect to convolution. They found sufficient conditions for which the corresponding space becomes an algebra, and studied their properties such as existence of an approximate identity and the spectrum of the algebra when the underlying group is abelian. Their work, in part, extends some of the results of Kuznetsova to a wider classes of algebras.

Our goal in this paper is to continue studying convolution and possible algebraic structure on Orlicz spaces, but in a more general setting than in [18]. In fact, we would like to have a theory that encompasses everything we have discussed above. We start by considering twisted convolution coming from 2-cocycles with values in $\mathbb{C}^*$, the multiplicative group of complex numbers. We restrict ourselves to those 2-cocycles Ω for which $|\Omega|$ is a 2-coboundary determined by a submultiplicative weight ω. This approach has two advantages. First, it allows us to systematically and simultaneously study twisted convolution coming from 2-cocycles with values in $\mathbb{T}$, and the weighted spaces coming from ω. Secondly, the condition we find on Ω ensuring that the twisted Orlicz space becomes as algebra is that $|\Omega|$ satisfies a certain composition law (see (3.1)), and this is much better presented and understood if the twisted convolution is viewed as the one coming from a 2-cocycle (Section 3). We call the algebras we obtain *twisted Orlicz algebras*. When ω is a symmetric weight, we show that there is a natural involutive structure on twisted Orlicz algebras over unimodular locally compact groups. We study their symmetry as Banach $*$-algebras, and present a method to verify whether they are symmetric (Section 4).

We apply our methods to study twisted Orlicz algebras over compactly generated groups of polynomial growth. Following [11] we study various harmonic analysis properties of these algebras such as symmetry, existence of functional calculus and the Wiener property. We present three concrete and important classes of polynomial and subexponential weights on these groups and obtain a large family of symmetric twisted Orlicz algebras. This demonstrates that our methods can be applied to a vast variety of cases and extend the results even in the classical situation. For instance, it was left as open problems in [11] whether weighted L^p-algebras with subexponential weights over non-abelian groups are symmetric or have the Wiener property. Our methods yield affirmative answers to these questions in a much more general setting (Sections 5.2–5.4).

We finish by pointing out that throughout this paper, we concern ourselves with "bounded multiplications" for Banach algebras and Banach modules, as opposed to "contractive multiplications". Also weights for us are "weakly submultiplicative", as opposed to "submultiplicative". We have also investigated existence of approximate identities as well as cohomological properties of twisted Orlicz algebras; this will be presented in a subsequent paper.

1. Preliminaries. In this section, we give some definitions and state some technical results that will be crucial in the rest of this paper. Throughout, G denotes a locally compact group with a fixed left Haar measure ds.

1.1. Orlicz spaces. In this section, we recall some facts concerning Young functions and Orlicz spaces. Our main reference is [22].

A non-zero function $\Phi : [0,\infty) \to [0,\infty]$ is called a *Young function* if Φ is convex, $\Phi(0) = 0$, and $\lim_{x\to\infty} \Phi(x) = \infty$. For a Young function Φ, the *complementary function* Ψ of Φ is given by

$$\Psi(y) = \sup\{xy - \Phi(x) : x \geq 0\} \qquad (y \geq 0).$$

It is easy to check that Ψ is also a Young function. Also, if Ψ is the complementary function of Φ, then Φ is the complementary of Ψ and (Φ, Ψ) is called a *complementary pair*. We have the *Young inequality*

$$xy \leq \Phi(x) + \Psi(y) \qquad (x, y \geq 0)$$

for complementary functions Φ and Ψ. By our definition, a Young function can have the value ∞ at a point, and hence be discontinuous at that point. However, we always consider the pair of complementary Young functions (Φ, Ψ) with Φ being real-valued and continuous on $[0,\infty)$ and positive on $(0,\infty)$. Note that even though Φ is continuous, it may happen that Ψ is not.

Now suppose that G is a locally compact group with a fixed Haar measure ds, and (Φ, Ψ) is a complementary pair of Young functions. We define

$$\mathcal{L}^{\Phi}(G) = \Big\{ f : G \to \mathbb{C} : f \text{ is measurable and } \int_G \Phi(|f(s)|)\, ds < \infty \Big\}. \tag{1.1}$$

Since $\mathcal{L}^{\Phi}(G)$ is not always a linear space, we define the *Orlicz space* $L^{\Phi}(G)$ to be

$$L^{\Phi}(G) = \Big\{ f : G \to \mathbb{C} : \int_G \Phi(\alpha|f(s)|)\, ds < \infty \text{ for some } \alpha > 0 \Big\}, \tag{1.2}$$

where f indicates a member in the equivalence classes of measurable functions with respect to the Haar measure ds. Then the Orlicz space is a Banach space under the (Orlicz) norm $\|\cdot\|_{\Phi}$ defined for $f \in L^{\Phi}(G)$ by

$$\|f\|_{\Phi} = \sup\Big\{ \int_G |f(s)v(s)|\, ds : \int_G \Psi(|v(s)|)\, ds \leq 1 \Big\}, \tag{1.3}$$

where Ψ is the complementary function to Φ. One can also define the (Luxemburg) norm $N_{\Phi}(\cdot)$ on $L^{\Phi}(G)$ by

$$N_{\Phi}(f) = \inf\Big\{ k > 0 : \int_G \Phi(|f(s)|/k)\, ds \leq 1 \Big\}. \tag{1.4}$$

It is known that these two norms are equivalent:

$$N_{\Phi}(\cdot) \le \|\cdot\|_{\Phi} \le 2N_{\Phi}(\cdot), \tag{1.5}$$

and

$$N_{\Phi}(f) \le 1 \quad \text{if and only if} \quad \int_G \Phi(|f(s)|)\,ds \le 1. \tag{1.6}$$

Let $\mathcal{S}^{\Phi}(G)$ be the closure of the linear space of all step functions in $L^{\Phi}(G)$. Then $\mathcal{S}^{\Phi}(G)$ is a Banach space and contains $C_c(G)$, the space of all continuous functions on G with compact support, as a dense subspace [22, Proposition 3.4.3]. Moreover, $\mathcal{S}^{\Phi}(G)^*$, the dual of $\mathcal{S}^{\Phi}(G)$, can be identified with $L^{\Psi}(G)$ in a natural way [22, Theorem 4.1.6]. Another useful characterization of $\mathcal{S}^{\Phi}(G)$ is that $f \in \mathcal{S}^{\Phi}(G)$ if and only if $\alpha f \in \mathcal{L}^{\Phi}(G)$ for every $\alpha > 0$ [22, Definition 3.4.2 and Proposition 3.4.3].

A Young function Φ satisfies the Δ_2 *condition* if there exist a constant $K > 0$ and $x_0 \ge 0$ such that $\Phi(2x) \le K\Phi(x)$ for all $x \ge x_0$. In this case we write $\Phi \in \Delta_2$. If $\Phi \in \Delta_2$, then $L^{\Phi}(G) = \mathcal{S}^{\Phi}(G)$, so that $L^{\Phi}(G)^* = L^{\Psi}(G)$ [22, Corollary 3.4.5]. If in addition $\Psi \in \Delta_2$, then $L^{\Phi}(G)$ is a reflexive Banach space.

We will frequently use the (generalized) Hölder inequality for Orlicz spaces [22, Remark 3.3.1]. More precisely, for any complementary pair (Φ, Ψ) of Young functions and any $f \in L^{\Phi}(G)$ and $g \in L^{\Psi}(G)$, we have

$$\|fg\|_1 := \int_G |f(s)g(s)|\,ds \le \min\{N_{\Phi}(f)\|g\|_{\Psi}, \|f\|_{\Phi}N_{\Psi}(g)\}. \tag{1.7}$$

This in particular implies that $fg \in L^1(G)$.

For $1 \le p < \infty$ and the Young function $\Phi(x) = x^p/p$, the space $L^{\Phi}(G)$ becomes the Lebesgue space $L^p(G)$ and the norm $\|\cdot\|_{\Phi}$ is equivalent to the classical norm $\|\cdot\|_p$. If $p = 1$, then the complementary Young function of $\Phi(x) = x$ is

$$\Psi(y) = \begin{cases} 0 & \text{if } 0 \le y \le 1, \\ \infty & \text{otherwise}, \end{cases} \tag{1.8}$$

and $\|f\|_{\Phi} = \|f\|_1$ for all $f \in L^1(G)$ since $\int_G \Psi(|v(s)|)\,ds \le 1$ if and only if $|v(s)| \le 1$ locally almost everywhere on G. Note that Ψ defined in (1.8) is still a Young function as in the first definition of the Young function. If $1 < p < \infty$, then the complementary Young function of $\Phi(x) = x^p/p$ is $\Psi(y) = y^q/q$, where q is the conjugate of p, i.e. $1/p + 1/q = 1$.

1.2. 2-cocycles and 2-coboundaries. Throughout this article, $\mathbb{C}^*$ denotes the multiplicative group of complex numbers, i.e. $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$, $\mathbb{R}_+$ is the multiplicative group of positive reals, and $\mathbb{T}$ is the unit circle in $\mathbb{C}$.

Definition 1.1. Let G and H be locally compact groups such that H is abelian. A (*normalized*) 2-*cocycle on* G *with values in* H is a Borel measurable map $\Omega : G \times G \to H$ such that

$$\Omega(r,s)\Omega(rs,t) = \Omega(s,t)\Omega(r,st) \quad (r,s,t \in G) \tag{1.9}$$

and

$$\Omega(r,e_G) = \Omega(e_G,r) = e_H \quad (r \in G). \tag{1.10}$$

The set of all normalized 2-cocycles will be denoted by $\mathcal{Z}^2(G,H)$.

If $\omega : G \to H$ is measurable with $\omega(e_G) = e_H$, then it is easy to see that the mapping

$$(s,t) \mapsto \omega(st)\omega(s)^{-1}\omega(t)^{-1}$$

satisfies (1.9) and (1.10), hence it is a 2-cocycle; such maps are called 2-*coboundaries*. The set of 2-coboundaries will be denoted by $\mathcal{N}^2(G,H)$. It is easy to check that $\mathcal{Z}^2(G,H)$ is an abelian group under the product

$$\Omega_1\Omega_2(s,t) = \Omega_1(s,t)\Omega_2(s,t) \quad (s,t \in G),$$

and $\mathcal{N}^2(G,H)$ is a (normal) subgroup of $\mathcal{Z}^2(G,H)$. This in particular implies that

$$\mathcal{H}^2(G,H) := \mathcal{Z}^2(G,H)/\mathcal{N}^2(G,H)$$

turns into a group; it is called the second cohomology group of G into H with the trivial actions (i.e. $s \cdot \alpha = \alpha \cdot s = \alpha$ for all $s \in G$ and $\alpha \in H$).

We are mainly interested in the cases when H is $\mathbb{C}^*$, $\mathbb{R}_+$ or $\mathbb{T}$. One essential observation is that we can view $\mathbb{C}^* = \mathbb{R}_+\mathbb{T}$ as a (pointwise) direct product of groups. Hence, for any 2-cocycle Ω on G with values in $\mathbb{C}^*$ and $s,t \in G$, we can (uniquely) write $\Omega(s,t) = |\Omega(s,t)|e^{i\theta}$ for some $0 \le \theta < 2\pi$. Therefore, if we set

$$|\Omega|(s,t) := |\Omega(s,t)| \quad \text{and} \quad \Omega_{\mathbb{T}}(s,t) := e^{i\theta}, \tag{1.11}$$

then Ω decomposes (in a unique way) into the product $|\Omega|\Omega_{\mathbb{T}}$ of 2-cocycles, $|\Omega|$ and $\Omega_{\mathbb{T}}$ being 2-cocycles on G with values in $\mathbb{R}_+$ and $\mathbb{T}$, respectively.

2. Twisted group algebra. In this section, we gather what we need from the theory of twisted group algebras. To start, we must restrict ourselves to certain subgroups of 2-cocycles for which twisted group algebras can be defined. Throughout this section, G is a locally compact group with a fixed left Haar measure ds.

Definition 2.1. We denote by $\mathcal{Z}_b^2(G,\mathbb{C}^*)$ the group of *bounded* 2-*cocycles on* G *with values in* $\mathbb{C}^*$ which consists of all $\Omega \in \mathcal{Z}^2(G,\mathbb{C}^*)$ such that

(i) $\Omega \in L^\infty(G \times G)$;
(ii) $\Omega_{\mathbb{T}}$ is continuous.

We also define $\mathcal{Z}^2_{bw}(G,\mathbb{C}^*)$ to be the subgroup of $\mathcal{Z}^2_b(G,\mathbb{C}^*)$ consisting of all $\Omega \in \mathcal{Z}^2_b(G,\mathbb{C}^*)$ for which

$$|\Omega|(s,t) = \frac{\omega(st)}{\omega(s)\omega(t)} \quad (s,t \in G),$$

where $\omega : G \to \mathbb{R}_+$ is a locally integrable measurable function with $\omega(e) = 1$ and $1/\omega \in L^\infty(G)$. In this case, we call ω a *weight* on G and say that $|\Omega|$ is the 2-*coboundary determined* by ω, or alternatively, ω is the *weight associated to* $|\Omega|$.

Now suppose that $\Omega \in \mathcal{Z}^2_b(G,\mathbb{C}^*)$ and f and g are measurable functions on G. If there is a σ-finite measurable subset E of G such that $f = g = 0$ on $G \setminus E$, then we define the *twisted convolution of f and g under Ω* to be

$$(f \circledast g)(t) = \int_G f(s)g(s^{-1}t)\Omega(s,s^{-1}t)\,ds \quad (t \in G). \tag{2.1}$$

It follows from standard measure theory that since $E \times E$ is σ-finite and its complement in $G \times G$ is a null set, $f \circledast g$ is measurable on G. Moreover, if we let Δ be the modular function on G and define (for $s,t \in G$)

$$\begin{aligned} (\delta_s \circledast f)(t) &= f(s^{-1}t)\Omega(s,s^{-1}t), \\ (f \circledast \delta_s)(t) &= f(ts^{-1})\Delta(s^{-1})\Omega(ts^{-1},s), \end{aligned} \tag{2.2}$$

then both $\delta_s \circledast f$ and $f \circledast \delta_s$ are measurable functions on G and

$$(f \circledast g)(t) = \int_G f(s)(\delta_s \circledast g)(t)\,ds = \int_G g(s)(f \circledast \delta_s)(t)\,ds \quad (t \in G). \tag{2.3}$$

It now follows routinely (using Fubini's theorem) that for all $f,g \in L^1(G)$, we have $f \circledast g \in L^1(G)$ with $\|f \circledast g\|_1 \le \|\Omega\|_\infty \|f\|_1 \|g\|_1$. We conclude that $(L^1(G),\circledast)$ becomes a Banach algebra; it is called the *twisted group algebra.*

We wish to make some clarification with regard to our version of twisted group algebras. Even though in early stages in the works of Leptin and others, twisted group actions were defined in a much more general setting (see, for example, [2] and [14]), in recent years most authors have considered these concepts for $\Omega = \Omega_{\mathbb{T}}$, i.e. when the 2-cocycle has values in the unit circle. However, since we are mostly interested in when Orlicz spaces on a locally compact group G are algebras, and want to study their algebraic properties, the assumption $\Omega = \Omega_{\mathbb{T}}$ is not enough, as the corresponding Orlicz space is rarely an algebra with respect to the twisted convolution coming from $\Omega_{\mathbb{T}}$ (except in the trivial case when G is compact, see for example [7]). Hence we will consider a more general setting; we give more details in Section 3.

If $\Omega \in \mathcal{Z}^2_{bw}(G,\mathbb{C}^*)$, we can have an alternative representation of the twisted group algebra associated to Ω. More precisely, suppose that $\omega : G \to \mathbb{R}_+$ is a weight associated to $|\Omega|$ as in Definition 2.1 and consider the *weighted L^1-space*

$$L^1_\omega(G) := \{f : G \to \mathbb{C} : f\omega \in L^1(G)\}. \tag{2.4}$$

Then $L^1_\omega(G)$ with the norm $\|f\|_\omega = \|f\omega\|_1$ is a Banach space. Moreover, if $\circledast$ and $\circledast_{\mathbb{T}}$ are twisted convolutions with respect to Ω and $\Omega_{\mathbb{T}}$, respectively, then it is straightforward to verify that $(L^1_\omega(G), \circledast_{\mathbb{T}}, \|\cdot\|_\omega)$ becomes a Banach algebra, and the mapping

$$\Lambda_\omega : (L^1(G), \circledast, \|\cdot\|_1) \to (L^1_\omega(G), \circledast_{\mathbb{T}}, \|\cdot\|_\omega)$$

defined by

$$\Lambda_\omega(f) = f/\omega \quad (f \in L^1(G))$$

is an isometric algebra isomorphism. Furthermore, it is easy to see that the inclusion

$$\iota : (L^1_\omega(G), \circledast_{\mathbb{T}}, \|\cdot\|_\omega) \hookrightarrow (L^1(G), \circledast_{\mathbb{T}}, \|\cdot\|_1)$$

is a continuous injective algebra homomorphism with dense image. These relations will be used throughout this article.

We finish this section by pointing out that even though the assumption $\Omega \in \mathcal{Z}^2_{bw}(G, \mathbb{C}^*)$ might seem too restrictive, it does hold in most interesting cases. One of those cases (which includes all cases we discuss in this article) is when G is amenable, as we will see in the following lemma. This is well-known but we present a short proof for completeness.

LEMMA 2.2. *Suppose that G is amenable and $\Omega \in \mathcal{Z}^2_b(G, \mathbb{C}^*)$. Then $|\Omega|$ is the 2-coboundary determined by a weight $\omega : G \to \mathbb{R}_+$ with $\omega(e) = 1$ and $1/\omega \in L^\infty(G)$. In other words,*

$$\mathcal{Z}^2_b(G, \mathbb{C}^*) = \mathcal{Z}^2_{bw}(G, \mathbb{C}^*).$$

Proof. It is well-known that for an amenable group G we have $\mathcal{H}^2(G, \mathbb{R}) = \{0\}$, where we regard $\mathbb{R}$ as the additive group of real numbers. Since $(\mathbb{R}, +) \cong (\mathbb{R}_+, \cdot)$ as locally compact abelian groups, we conclude that $\mathcal{H}^2(G, \mathbb{R}_+) = \{0\}$, i.e. $|\Omega|$ is a 2-coboundary. Let $\omega : G \to \mathbb{R}_+$ be a weight function determining $|\Omega|$. Then $\omega(e) = 1$ and $\omega(st) \le C\omega(s)\omega(t)$ for all $s, t \in G$, where $C = \|\Omega\|_\infty$. In particular, $C\omega$ is a positive submultiplicative measurable function on G, and so, by the amenability of G, there is a continuous positive valued character χ on G such that $C\omega \ge \chi$ a.e. [25]. Thus if we replace ω with ω/χ, we get the desired result. ∎

3. Twisted Orlicz algebras. Throughout the rest of the paper, we assume that (Φ, Ψ) is a pair of complementary Young functions with Φ being continuous on $[0, \infty)$ and positive on $(0, \infty)$.

In this section, we would like to find sufficient conditions under which the twisted convolution (2.1) turns an Orlicz space into an algebra as formulated in the following definition.

DEFINITION 3.1. Let G be a locally compact group, let $\Omega \in \mathcal{Z}_b^2(G, \mathbb{C}^*)$, and let $\circledast$ be the twisted convolution coming from Ω. We say that $(L^\Phi(G), \circledast)$ is a *twisted Orlicz algebra* if $(L^\Phi(G), \circledast, \|\cdot\|_\Phi)$ is a Banach algebra, i.e. there is $C > 0$ such that for all $f, g \in L^\Phi(G)$ we have $f \circledast g \in L^\Phi(G)$ with

$$\|f \circledast g\|_\Phi \leq C\|f\|_\Phi \|g\|_\Phi.$$

The following lemma which is a generalization of [21, Proposition 1, p. 384] shows that one always has a natural $L^1(G)$-bimodule structure on $(L^\Phi(G), \circledast)$.

LEMMA 3.2. *Let G be a locally compact group, and let $\Omega \in \mathcal{Z}_b^2(G, \mathbb{C}^*)$. Then:*

(i) *There is $C > 0$ such that for all $f \in L^\Phi(G)$ and $s \in G$, both $\delta_s \circledast f$ and $f \circledast \delta_s$, defined in* (2.2), *belong to $L^\Phi(G)$ with*

$$\|\delta_s \circledast f\|_\Phi \leq C\|f\|_\Phi \quad \text{and} \quad \|f \circledast \delta_s\|_\Phi \leq C\|f\|_\Phi.$$

(ii) *$L^\Phi(G)$ is a Banach $L^1(G)$-bimodule with respect to the twisted convolution* (2.1).

(iii) *$\mathcal{S}^\Phi(G)$ becomes an essential Banach $L^1(G)$-submodule of $L^\Phi(G)$ with respect to the twisted convolution* (2.1).

Proof. (i) For every $f \in L^\Phi(G)$ and $t \in G$, we set

$$L_s f(t) = f(s^{-1}t) \quad \text{and} \quad R_s f(t) = f(ts)\Delta(s),$$

where Δ is the modular function of G. It follows from (1.2), (1.3), and the standard properties of the Haar measure that both $L_s f$ and $R_s f$ belong to $L^\Phi(G)$ with $\|L_s f\|_\Phi = \|R_s f\|_\Phi = \|f\|_\Phi$. On the other hand, by our hypothesis,

$$|\delta_s \circledast f| \leq C|L_s f| \quad \text{and} \quad |f \circledast \delta_s| \leq C|R_{s^{-1}} f|,$$

where $C = \|\Omega\|_\infty$. Therefore $\delta_s \circledast f$ and $f \circledast \delta_s$ belong to $L^\Phi(G)$ with

$$\|\delta_s \circledast f\|_\Phi \leq C\|L_s f\|_\Phi = C\|f\|_\Phi \quad \text{and} \quad \|f \circledast \delta_s\|_\Phi \leq C\|R_{s^{-1}} f\|_\Phi = C\|f\|_\Phi.$$

(ii) For all $f \in L^1(G)$, $g \in L^\Phi(G)$, and $h \in L^\Psi(G)$ with $\int_G \Psi(|h(s)|)\, ds \leq 1$, we have

$$\begin{aligned}
\int_G |(f \circledast g)(t)h(t)|\, dt &\leq \int_G \int_G |f(s)g(s^{-1}t)\Omega(s, s^{-1}t)h(t)|\, ds\, dt \\
&= \int_G |f(s)| \int_G |(\delta_s \circledast g)(t)h(t)|\, dt\, ds \qquad \text{(by (1.3))} \\
&\leq \int_G |f(s)|\, \|\delta_s \circledast g\|_\Phi\, ds \qquad \text{(by (i))} \\
&\leq C \int_G |f(s)|\, \|g\|_\Phi\, ds = C\|f\|_1 \|g\|_\Phi.
\end{aligned}$$

Therefore, again by (1.3), it follows that $\|f \circledast g\|_\Phi \le C\|f\|_1\|g\|_\Phi$, so that $L^\Phi(G)$ is a Banach left $L^1(G)$-module. The other case (Banach right module) follows similarly considering that for every $t \in G$,

$$(g \circledast f)(t) := \int_G f(s)g(ts^{-1})\Omega(ts^{-1}, s)\Delta(s^{-1})\,ds = \int_G f(s)(g \circledast \delta_s)(t)\,ds.$$

(iii) Suppose that $f, g \in C_c(G)$ and $\alpha > 0$. Since Φ is a positive continuous convex function on $\mathbb{R}^+$, it is increasing. Hence, for every $t \in G$,

$$\Phi(|\alpha(f \circledast g)(t)|) \le \Phi\big(\alpha\lambda(\operatorname{supp} f)\|f\|_\infty\|g\|_\infty\|\Omega\|_\infty\big).$$

Therefore

$$\begin{aligned}\int_G \Phi(|\alpha(f \circledast g)(t)|)\,dt &= \int_K \Phi(|\alpha(f \circledast g)(t)|)\,dt \\ &\le \lambda(K)\Phi\big(\alpha\lambda(\operatorname{supp} f)\|f\|_\infty\|g\|_\infty\|\Omega\|_\infty\big),\end{aligned}$$

where K is a compact set containing $\operatorname{supp} f \operatorname{supp} g$. Thus, by [22, Corollary 3.4.4], $f \circledast g \in \mathcal{S}^\Phi(G)$. The rest follows from part (ii) and the fact that $C_c(G)$ is norm dense in $(L^1(G), \|\cdot\|_1)$ and $(\mathcal{S}^\Phi(G), \|\cdot\|_\Phi)$. ∎

The following theorem provides a key step to obtain twisted Orlicz algebras. Roughly speaking, it states that one gets a twisted Orlicz algebra if the 2-cocycle $|\Omega|$ is dominated by the sum of two suitable positive functions.

THEOREM 3.3. *Let G be a locally compact group, and let $\Omega \in \mathcal{Z}^2_b(G, \mathbb{C}^*)$. Suppose that there exist non-negative measurable functions u and v in $L^\Psi(G)$ such that*

$$|\Omega(s,t)| \le u(s) + v(t) \qquad (s, t \in G). \tag{3.1}$$

Then for all $f, g \in L^\Phi(G)$, the twisted convolution (2.1) *is well-defined on $L^\Phi(G)$ and*

$$\|f \circledast g\|_\Phi \le \|fu\|_1\|g\|_\Phi + \|f\|_\Phi\|gv\|_1. \tag{3.2}$$

In particular, $(L^\Phi(G), \circledast)$ becomes a twisted Orlicz algebra having $\mathcal{S}^\Phi(G)$ as a closed subalgebra.

Proof. Fix $f, g \in L^\Phi(G)$. Then, for every $t \in G$,

$$\begin{aligned}\int_G |f(s)g(s^{-1}t)\Omega(s, s^{-1}t)|\,ds &\le \int_G |f(s)g(s^{-1}t)|\,|u(s)|\,ds \\ &\quad + \int_G |f(s)g(s^{-1}t)|\,|v(s^{-1}t)|\,ds \\ &= |fu| * |g|(t) + |f| * |gv|(t).\end{aligned}$$

Since from Hölder's inequality (1.7) both fu and gv belong to $L^1(G)$ and, by Lemma 3.2, $L^\Phi(G)$ is a Banach $L^1(G)$-bimodule under convolution, it follows that the measurable function

$$t \mapsto (f \circledast g)(t) = \int_G f(s)g(s^{-1}t)\Omega(s, s^{-1}t)\, ds$$

belongs to $L^\Phi(G)$. Moreover, for every $h \in L^\Psi(G)$ with $\int_G \Psi(|h(s)|)\, ds \le 1$, we have

$$\begin{aligned}\int_G |(f \circledast g)(t)h(t)|\, dt &\le \int_G\int_G |f(s)g(s^{-1}t)\Omega(s, s^{-1}t)h(t)|\, ds\, dt \\ &\le \int_G |fu| * |g|(t)|h(t)|\, dt + \int_G |f| * |gv|(t)|\,|h(t)|\, dt \\ &\le \|\,|fu| * |g|\,\|_\Phi + \|\,|f| * |gv|\,\|_\Phi \quad \text{(by Lemma 3.2)} \\ &\le \|fu\|_1 \|g\|_\Phi + \|f\|_\Phi \|gv\|_1.\end{aligned}$$

Therefore, by (1.3), we obtain (3.2). This, together with the repeated use of Hölder's inequality (1.7), implies that

$$\|f \circledast g\|_\Phi \le C\|f\|_\Phi \|g\|_\Phi,$$

where $C = N(u)_\Psi + N(v)_\Psi$. Hence $(L^\Phi(G), \circledast)$ becomes a twisted Orlicz algebra. ■

We have the following immediate corollary. We recall that a function $\mathcal{L} : G \to \mathbb{R}^+$ is called *weakly subadditive* if there is $C > 0$ such that

$$\mathcal{L}(st) \le C(\mathcal{L}(s) + \mathcal{L}(t)) \quad (s, t \in G). \tag{3.3}$$

COROLLARY 3.4. *Let G be a locally compact group, let $\Omega \in \mathcal{Z}^2_b(G, \mathbb{C}^*)$, and let $\circledast$ be the twisted convolution coming from Ω. Suppose that*

$$|\Omega(s, t)| \le \frac{\mathcal{L}(st)}{\mathcal{L}(s)\mathcal{L}(t)} \quad (s, t \in G), \tag{3.4}$$

where $\mathcal{L} : G \to \mathbb{R}^+$ is a weakly subadditive function with $1/\mathcal{L} \in L^\Psi(G)$. Then $(L^\Phi(G), \circledast)$ is a twisted Orlicz algebra.

Proof. Since $\mathcal{L}$ is weakly subadditive, it satisfies (3.3). Combining this with (3.4), we get

$$|\Omega(s, t)| \le \frac{C}{\mathcal{L}(s)} + \frac{C}{\mathcal{L}(t)} \quad (s, t \in G).$$

Hence if we set

$$u = v = C/\mathcal{L},$$

then Ω satisfies (3.1) with $u, v \in L^\Psi(G)$. It now follows from Theorem 3.3 that $(L^\Phi(G), \circledast)$ is a twisted Orlicz algebra. ■

The preceding corollary gives a useful tool to determine when we have twisted Orlicz algebras. We will apply it mostly for compactly generated groups of polynomial growth (see Section 5). However, as demonstrated by the following example, it can be applied to other classes of groups as well. The example is taken from [20, Example 1 and Remark 2].

Example 3.5. Let G be a locally compact group for which there is an increasing sequence $\{G_i\}_{i\in\mathbb{N}}$ of compact subgroups of G such that $G := \bigcup_{i\in\mathbb{N}} G_i$. Take an increasing sequence $\{n_i\}_{i\in\mathbb{N}} \in [1,\infty)$. Define $\omega : G \to [1,\infty)$ by

$$\omega = 1 + \sum_{i=1} n_i 1_{G_{i+1}\setminus G_i}.$$

It is easy to see that

$$\omega(st) = \max\{\omega(s), \omega(t)\} \quad (s,t \in G).$$

This, in particular, implies that ω is a weakly additive weight on G. Moreover, we can pick $\{n_i\}$ in such a way that $1/\omega \in L^1(G) \cap L^\infty(G) \subseteq L^\Psi(G)$, where the inclusion is an easy consequence of (1.2). Therefore, in this case, by Corollary 3.4, $(L^\Phi(G), \circledast)$ is a twisted Orlicz algebra, where $\Omega \in \mathcal{Z}_b^2(G, \mathbb{C}^*)$ with $|\Omega|$ determined by ω.

4. Symmetry. In this section, we investigate symmetry for twisted Orlicz algebras. The notion of symmetry plays an important role in the theory of Banach $*$-algebras. Let A be a Banach $*$-algebra. We say that A is *symmetric* if $\sigma_A(a^*a) \subseteq [0,\infty)$ for every $a \in A$, where $\sigma_A(b)$ is the spectrum of $b \in A$. It is well-known that C*-algebras are symmetric, and a commutative Banach $*$-algebra is symmetric if and only if every multiplicative linear functional on A is a $*$-homomorphism. Also the group algebra of a compactly generated group with polynomial growth is symmetric [15], whereas the group algebra of the free group on n generators is not symmetric for $n \geq 2$.

To consider symmetry of twisted Orlicz algebras, we first restrict ourselves to those weights for which a natural involution can be defined on the twisted group algebra.

Definition 4.1. Let G be a locally compact group. We denote by $\mathcal{Z}_{bs}^2(G, \mathbb{C}^*)$ the group of *bounded symmetric* 2-*cocycles on* G *with values in* $\mathbb{C}^*$ which consists of all $\Omega \in \mathcal{Z}_{bw}^2(G, \mathbb{C}^*)$ for which there is a weight ω associated to $|\Omega|$ such that

$$\omega(s) = \omega(s^{-1}) \quad (s \in G).$$

Such weights on G are called *symmetric*.

Now suppose that $\Omega \in \mathcal{Z}_{bs}^2(G, \mathbb{C}^*)$, ω is a symmetric weight associated to $|\Omega|$, and $\circledast$ and $\circledast_{\mathbb{T}}$ are twisted convolutions with respect to Ω and $\Omega_{\mathbb{T}}$, respectively. It is well-known and easily seen that $(L^1(G), \circledast_{\mathbb{T}}, \|\cdot\|_1)$ is a

Banach $*$-algebra with the involution defined by

$$f^*(s) = \overline{f(s^{-1})}\Delta(s^{-1})\overline{\Omega_{\mathbb{T}}(s, s^{-1})} \quad (f \in L^1(G),\, s \in G). \tag{4.1}$$

On the other hand, by what was discussed in Section 2, the two Banach algebras $(L^1(G), \circledast, \|\cdot\|_1)$ and $(L^1_\omega(G), \circledast_{\mathbb{T}}, \|\cdot\|_\omega)$ are the same, and the latter can be viewed as a subalgebra of $(L^1(G), \circledast_{\mathbb{T}}, \|\cdot\|_1)$. This, in particular, allows us to define an involution on $L^1_\omega(G)$ as the restriction of that from $L^1(G)$, and it is routine to verify that $(L^1_\omega(G), \circledast_{\mathbb{T}}, \|\cdot\|_\omega)$ becomes a Banach $*$-algebra. Hence, when it comes to symmetry, it is more useful to consider the "weighted" representation for twisted group algebras.

Our next step is to generalize the preceding discussion to twisted Orlicz spaces. We do this in the following lemma whose proof is straightforward, so we omit it.

Lemma 4.2. *Let G be a locally compact group, let $\Omega \in \mathcal{Z}^2_{bw}(G, \mathbb{C}^*)$, and let ω be a weight associated to $|\Omega|$. Define the weighted L^Φ-space*

$$L^\Phi_\omega(G) := \{f : G \to \mathbb{C} : f\omega \in L^\Phi(G)\}. \tag{4.2}$$

Then $L^\Phi_\omega(G)$ with the norm $\|f\|_{\Phi,\omega} = \|f\omega\|_\Phi$ is a Banach space. Moreover, if $\circledast$ and $\circledast_{\mathbb{T}}$ are twisted convolutions with respect to Ω and $\Omega_{\mathbb{T}}$, respectively, then $(L^\Phi_\omega(G), \circledast_{\mathbb{T}}, \|\cdot\|_{\Phi,\omega})$ becomes a Banach $(L^1_\omega(G), \circledast_{\mathbb{T}}, \|\cdot\|_\omega)$-bimodule so that the mapping

$$\Lambda_\omega : L^\Phi(G) \to L^\Phi_\omega(G), \qquad \Lambda_\omega(f) = f/\omega, \tag{4.3}$$

is a linear isometric isomorphism satisfying $(f \in L^1(G),\, g \in L^\Phi(G))$

$$\Lambda_\omega(f \circledast g) = \Lambda_\omega(f) \circledast_{\mathbb{T}} \Lambda_\omega(g) \quad \text{and} \quad \Lambda_\omega(g \circledast f) = \Lambda_\omega(g) \circledast_{\mathbb{T}} \Lambda_\omega(f). \tag{4.4}$$

If, in addition, $(L^\Phi(G), \circledast, \|\cdot\|_\Phi)$ is a twisted Orlicz algebra, then (4.4) *holds for all $f, g \in L^\Phi(G)$, so that $(L^\Phi_\omega(G), \circledast_{\mathbb{T}}, \|\cdot\|_{\Phi,\omega})$ is a Banach algebra. Furthermore, if ω is symmetric and G is unimodular, then, with the involution* (4.1), *$(L^\Phi_\omega(G), \circledast_{\mathbb{T}})$ becomes a Banach $*$-algebra with either of the norms* (1.3) *or* (1.4).

We remark that for $\Phi(x) = x^p/p$ $(1 < p < \infty)$ and $\Omega_{\mathbb{T}} = 1$, the space $L^\Phi_\omega(G)$ in Lemma 4.2 is precisely $L^p_\omega(G)$ with the convolution considered in [11].

We are now ready to investigate symmetry of twisted Orlicz algebras. As we have seen so far, our approach is to first determine this property for twisted group algebras. This is a generalization of the approach in [11] applying to symmetry of $L^p_\omega(G)$ with the (untwisted) convolution.

4.1. Symmetry of twisted group algebras. In order to investigate symmetry for twisted group algebras, we rely on a natural relation between the twisted convolution on a locally compact group G and the standard (untwisted) convolution on a certain central extension of G [2, Section 3].

We can then apply what is known for symmetry of weighted group algebras to obtain our results. This is similar to the approach in [6, Section 2.1]. Below, we present it in a more general setting:

Let G be a locally compact group, let $\Omega \in \mathcal{Z}^2_{bw}(G, \mathbb{C}^*)$, and let $\widetilde{G} := G \times \mathbb{T}$. Then $\widetilde{G}$ becomes a group with the multiplication

$$(s, \alpha) \cdot (t, \beta) := (st, \alpha\beta\Omega_{\mathbb{T}}(s, t)) \quad (s, t \in G,\ \alpha, \beta \in \mathbb{T}).$$

Moreover, there is a locally compact group topology on $\widetilde{G}$, coinciding with the usual product topology (the separable case is due to G. W. Mackey [16] and the non-separable one follows from the work of M. Leinert [13]). In particular, the Haar measure on $\widetilde{G}$ is just the product Haar measure $\int_G \int_{\mathbb{T}} \cdots \, dx \, d\alpha$. Now consider the mapping

$$\Gamma : C_c(G) \to L^1(\widetilde{G}), \quad \Gamma(f)(s, \alpha) = \overline{\alpha} f(s).$$

If $\circledast_{\mathbb{T}}$ is the twisted convolution on $L^1(G)$ coming from $\Omega_{\mathbb{T}}$, and $*$ is the (ordinary) convolution on $\widetilde{G}$, then it is straightforward to check the following (see [2, Lemma 3.2] or [6, Lemma 2.3 and Corollary 2.5]):

(1) Γ extends to an isometric $*$-algebra isomorphism from $(L^1(G), \circledast_{\mathbb{T}})$ into $(L^1(\widetilde{G}), *)$;
(2) $\operatorname{Im} \Gamma$ is an ideal in $(L^1(\widetilde{G}), *)$.

Moreover, if ω is a symmetric weight associated to $|\Omega|$ and we consider $\tilde{\omega} : \widetilde{G} \to \mathbb{R}^+$ defined by

$$\tilde{\omega}(s, \alpha) = \omega(s) \quad (s \in G,\ \alpha \in \mathbb{C}), \tag{4.5}$$

then it is straightforward to verify $\tilde{\omega}$ is a symmetric weight on $\widetilde{G}$ so that the preceding statements (1) and (2) remain valid if we replace $(L^1(G), \circledast_{\mathbb{T}})$ and $(L^1(\widetilde{G}), *)$ with their weighted analogs $(L^1_\omega(G), \circledast_{\mathbb{T}})$ and $(L^1_{\tilde{\omega}}(\widetilde{G}), *)$, respectively. This observation allows us to recover symmetry of the twisted group algebra from that of the weighted group algebra. We can summarize all this discussion in the following.

Proposition 4.3. *Let G be a locally compact group, let $\Omega \in \mathcal{Z}^2_{bs}(G, \mathbb{C}^*)$, and let ω be a symmetric weight associated to $|\Omega|$. The twisted group algebra $(L^1_\omega(G), \circledast_{\mathbb{T}})$ can be viewed as a $*$-closed two-sided ideal with a bounded approximate identity of the* (*untwisted*) *weighted group algebra $(L^1_{\tilde{\omega}}(\widetilde{G}), *)$. In particular, if $(L^1_{\tilde{\omega}}(\widetilde{G}), *)$ is symmetric, then so is $(L^1_\omega(G), \circledast_{\mathbb{T}})$.*

4.2. Symmetry of twisted Orlicz algebras. A very useful technique to investigate symmetry of a Banach algebra as well as being inverse-closed in the concept of differential norm. Assume that $A \subseteq B$ are Banach algebras with a common unit element. A *differential norm* is a norm on A that satisfies

$$\|ab\|_A \le C(\|a\|_A\|b\|_B + \|a\|_B\|b\|_A) \tag{4.6}$$

for all $a, b \in A$. In this case, we call A a *differential subalgebra of* B. This concept has appeared in various articles; the above formulation is given in [5, Section 3.1]. The following lemma demonstrates the main property of differential subalgebras which we need. The proof is well-known, has been presented in several articles and is rather straightforward, so we omit it (see, for example, [5, Lemma 3.2] and the references therein).

LEMMA 4.4. *Let A be a differential subalgebra of a Banach algebra B, and let r_A and r_B be the spectral radius functions of A and B, respectively. Suppose that either A and B are simultaneously unital with the same unit, or they are both non-unital. Then:*

(i) *$r_A(a) = r_B(a)$ for every $a \in A$.*
(ii) *A is inverse-closed in B.*
(iii) *Suppose that B is a Banach $*$-algebra having A as a $*$-subalgebra. If B is symmetric, then so is A.*

The following theorem which is the main result of this section demonstrates that certain twisted Orlicz algebras can be viewed as differential subalgebras of twisted group algebras. Thus we can determine their symmetry by applying the preceding lemma together with the classical results concerning symmetry of twisted group algebras discussed in Section 4.1.

THEOREM 4.5. *Let G be a locally compact unimodular group, let $\Omega \in \mathcal{Z}^2_{bs}(G, \mathbb{C}^*)$, and let σ be a symmetric weight associated to $|\Omega|$. Suppose that there exists a symmetric weakly subadditive weight ω on G with $1/\omega \in L^\Psi(G)$ and $M > 0$ such that*

$$\frac{\sigma(st)}{\sigma(s)\sigma(t)} \le \frac{M\omega(st)}{\omega(s)\omega(t)} \quad (s, t \in G). \tag{4.7}$$

Then $\rho := \sigma/\omega$ is a symmetric weight on G such that $(L^\Phi_\sigma(G), \circledast_{\mathbb{T}}, \|\cdot\|_{\Phi,\sigma})$ becomes a differential $*$-subalgebra of $(L^1_\rho(G), \circledast_{\mathbb{T}}, \|\cdot\|_{1,\rho})$, where $\circledast_{\mathbb{T}}$ is the twisted convolution coming from $\Omega_{\mathbb{T}}$. In particular, $(L^\Phi_\sigma(G), \circledast_{\mathbb{T}}, \|\cdot\|_{\Phi,\sigma})$ is symmetric whenever $(L^1_\rho(G), \circledast_{\mathbb{T}}, \|\cdot\|_{1,\rho})$ is symmetric.

Proof. We first note that (4.7) is nothing but (3.4) for $\mathcal{L} = \omega/M$. Thus, by our hypothesis and Corollary 3.4, $(L^\Phi(G), \circledast, \|\cdot\|_\Phi)$ is a twisted Orlicz algebra, where $\circledast$ is the twisted convolution coming from Ω. Therefore, by Lemma 4.2, $(L^\Phi_\sigma(G), \circledast_{\mathbb{T}}, \|\cdot\|_{\Phi,\sigma})$ is a Banach $*$-algebra. Now consider the function $\rho : G \to \mathbb{R}^+$ given by

$$\rho(s) = \frac{\sigma(s)}{\omega(s)} \quad (s \in G). \tag{4.8}$$

We first show that ρ is a symmetric weight on G. It is clear that ρ is measurable and symmetric with $\rho(e) = 1$. Also, since ω is a symmetric weight, it is bounded away from 0, i.e. there is $K > 0$ such that

$$\omega(s) \geq K \quad (s \in G).$$

Therefore, by (4.8), $\rho \leq \sigma/K$, and so ρ is locally integrable since σ is. Finally, (4.7) is clearly equivalent to

$$\rho(st) \leq M\rho(s)\rho(t) \quad (s,t \in G).$$

That is, ρ is a weight on G.

Our next step is to show that $(L^{\Phi}_{\sigma}(G), \circledast_{\mathbb{T}}, \|\cdot\|_{\Phi,\sigma})$ is a differential $*$-subalgebra of $(L^1_{\rho}(G), \circledast_{\mathbb{T}}, \|\cdot\|_{1,\rho})$. We first note that

$$L^{\Phi}_{\sigma}(G) \subseteq L^1_{\rho}(G). \tag{4.9}$$

To see this, suppose that $f \in L^{\Phi}_{\sigma}(G)$. Since, by hypothesis, $1/\omega \in L^{\Psi}(G)$, we have

$$\begin{aligned}\|f\|_{1,\rho} &= \int_G |f(s)|\rho(s)\,ds = \int_G |f(s)|\sigma(s)\frac{1}{\omega(s)}\,ds \\ &\leq \|f\sigma\|_{\Phi} N_{\Psi}(1/\omega) \quad \text{(by Hölder's inequality (1.7))} \\ &= \|f\|_{\Phi,\sigma} N_{\Psi}(1/\omega).\end{aligned}$$

Hence (4.9) holds. This in particular implies that $(L^{\Phi}_{\sigma}(G), \circledast_{\mathbb{T}})$ is a $*$-subalgebra of $(L^1_{\rho}(G), \circledast_{\mathbb{T}})$. It now remains to show that the corresponding relation (4.6) holds. Since ω is weakly subadditive, it satisfies (3.3) for a fixed constant $C > 0$. Thus, if we apply (4.7), for all $s,t \in G$ we have

$$|\Omega(s,t)| = \frac{\sigma(st)}{\sigma(s)\sigma(t)} \leq \frac{M\omega(st)}{\omega(s)\omega(t)} \leq \frac{CM}{\omega(s)} + \frac{CM}{\omega(t)} = u(s) + u(t),$$

where $u := CM/\omega$. Since, by our hypothesis, $u \in L^{\Psi}(G)$, it follows from (3.2) that

$$\|f \circledast g\|_{\Phi} \leq \|fu\|_1 \|g\|_{\Phi} + \|f\|_{\Phi}\|gu\|_1 \quad (f,g \in L^{\Phi}(G)).$$

Alternatively, if we apply Lemma 4.2 and use the equivalent weighted reformulation, we get

$$\begin{aligned}\|f \circledast_{\mathbb{T}} g\|_{\Phi,\sigma} &\leq \|f\sigma u\|_1 \|g\sigma\|_{\Phi} + \|f\sigma\|_{\Phi}\|g\sigma u\|_1 \\ &= CM(\|f\|_{1,\rho}\|g\|_{\Phi,\sigma} + \|f\|_{\Phi,\sigma}\|g\|_{1,\rho})\end{aligned}$$

for all $f,g \in L^{\Phi}_{\sigma}(G)$. Hence $(L^{\Phi}_{\sigma}(G), \circledast_{\mathbb{T}}, \|\cdot\|_{\Phi,\sigma})$ is a differential $*$-subalgebra of $(L^1_{\rho}(G), \circledast_{\mathbb{T}}, \|\cdot\|_{1,\rho})$. The final statement follows from Lemma 4.4. ■

We finish this section by pointing out that the preceding theorem is particularly useful when investigating symmetry of twisted Orlicz algebras of compactly generated groups with polynomial growth, as demonstrated in the following section. However, we can also apply it to other cases.

EXAMPLE 4.6. Let G, ω be as in Example 3.5 with $1/\omega \in L^1(G) \cap L^\infty(G) \subseteq L^\Psi(G)$. Suppose that $\widetilde{G}$ is the central extension of G considered in Section 4.1. Take $\Omega_{\mathbb{T}} \in \mathcal{Z}_b^2(G, \mathbb{T})$, $p > 0$ and set $\rho = \omega^p$. It is clear that $\tilde{\rho}$, defined by (4.5), is a weakly additive symmetric weight on $\widetilde{G}$ with $\tilde{\rho}^{-p} \in L^1(\widetilde{G})$. Therefore, by [20, Theorem 1], $(L^1_{\tilde{\rho}}(\widetilde{G}), *)$ is symmetric, and so, by Theorem 5.1, $(L^1_\rho(G), \circledast_{\mathbb{T}})$ is symmetric, where $\circledast_{\mathbb{T}}$ is the twisted convolution coming from $\Omega_{\mathbb{T}}$. Hence if we apply Theorem 4.5 for $\sigma := \omega^{p+1}$ and $\rho := \omega^p$, it follows that $(L^\Phi_{\omega^{p+1}}(G), \circledast_{\mathbb{T}})$ is a symmetric twisted Orlicz algebra.

5. Groups with polynomial growth

5.1. General theory. Let G be a compactly generated group with a fixed compact symmetric generating neighborhood U of the identity. The group G is said to have *polynomial growth* if there exist $C > 0$ and $d \in \mathbb{N}$ such that for every $n \in \mathbb{N}$,

$$\lambda(U^n) \leq Cn^d \quad (n \in \mathbb{N}).$$

Here $\lambda(S)$ is the Haar measure of any measurable $S \subseteq G$ and

$$U^n = \{u_1 \cdots u_n : u_i \in U,\ i = 1, \dots, n\}.$$

The smallest such d is called the *order of growth* of G and it is denoted by $d(G)$. It can be shown that the order of growth of G does not depend on the symmetric generating set U, i.e. it is a universal constant for G. Also, by [3, Lemma 2.3], U can be chosen to have strictly polynomial growth, i.e. there are positive numbers C_1 and C_2 such that

$$C_1 n^d \leq \lambda(U^n) \leq C_2 n^d \quad (n \in \mathbb{N}). \tag{5.1}$$

It is immediate that compact groups are of polynomial growth. More generally, every group with the property that the conjugacy class of every element is relatively compact has polynomial growth [19, Theorem 12.5.17]. Also every (compactly generated) nilpotent group (hence every abelian group) has polynomial growth [19, Theorem 12.5.17].

Using the generating set U of G we can define a *length function* $\tau_U : G \to [0, \infty)$ by

$$\tau_U(x) = \inf\{n \in \mathbb{N} : x \in U^n\} \quad \text{for } x \neq e, \qquad \tau_F(e) = 0. \tag{5.2}$$

When there is no risk of ambiguity, we write τ instead of τ_U. It is straightforward to verify that τ is a symmetric subadditive function on G, i.e.

$$\tau(xy) \leq \tau(x) + \tau(y) \quad \text{and} \quad \tau(x) = \tau(x^{-1}) \quad (x, y \in G). \tag{5.3}$$

We can use τ to define various weights on G. More precisely, for all $0 < \alpha \leq 1$, $\beta \geq 0$, $\gamma > 0$, and $C > 0$, we can define the *polynomial weight* ω_β on G of order β by

$$\omega_\beta(s) = (1+\tau(s))^\beta \quad (s \in G), \tag{5.4}$$

and the *subexponential weights* $\sigma_{\alpha,C}$ and $\rho_{\beta,C}$ on G by

$$\sigma_{\alpha,C}(s) = e^{C\tau(x)^\alpha} \quad (s \in G), \tag{5.5}$$

$$\rho_{\gamma,C}(s) = e^{C\tau(s)/(\ln(1+\tau(s)))^\gamma} \quad (s \in G). \tag{5.6}$$

5.2. Symmetric twisted Orlicz algebras over groups with polynomial growth. Throughout the rest of this section, we assume that G is a compactly generated group of polynomial growth.

Let ω be a symmetric weight on G. We say that ω satisfies the *GRS-condition* if for every $s \in G$,

$$\lim_{n\to\infty} \omega(s^n)^{1/n} = 1.$$

In [3], it is shown that the GRS-condition (among several others) characterizes precisely the symmetry of $L^1_\omega(G)$ (see also [4]). Their work heavily depends on a method developped by Hulanicki [8] as well as on symmetry of $L^1(G)$ [3, Section 3]. This result was applied in [11] to determine symmetry of $L^p_\omega(G)$ for several cases including polynomial weights defined in (5.4).

Our goal is to modify and extend the results in [11] to twisted Orlicz algebras over G. We start by extending the main result of [3] to twisted group algebras over G by applying Proposition 4.3.

THEOREM 5.1. *Let $\Omega \in \mathcal{Z}^2_{bw}(G, \mathbb{C}^*)$, and let ω be a symmetric weight associated to $|\Omega|$. If ω satisfies the GRS-condition, then $(L^1_\omega(G), \circledast_{\mathbb{T}})$ is symmetric, where $\circledast_{\mathbb{T}}$ is the twisted convolution coming from $\Omega_{\mathbb{T}}$.*

Proof. Let $\widetilde{G} := G \times \mathbb{T}$ with the locally compact group structure coming from $\Omega_{\mathbb{T}}$ as explained in Section 4.1. Since G has polynomial growth, so does its compact extension $\widetilde{G}$. Also it is easy to see that $\tilde{\omega} := \omega \times 1$ is a symmetric weight on $\widetilde{G}$ satisfying the GRS-condition. Hence, by [3, Theorem 1.3], $(L^1_{\tilde{\omega}}(\widetilde{G}), *)$ is symmetric (with the ordinary convolution). Thus, by Proposition 4.3, $(L^1_\omega(G), \circledast_{\mathbb{T}})$ is symmetric. ∎

Due to the more complicated nature of $L^\Phi_\omega(G)$, it is not clear how one can have a result as good as Theorem 5.1 for twisted Orlicz algebras. However, we can still show that for lots of important classes of weights, including the polynomial and subexponential weights defined in Section 5.1, we can determine when a twisted Orlicz algebra is symmetric.

We start with the following theorem which deals with the case of weakly additive weights.

THEOREM 5.2. *Let $\Omega_{\mathbb{T}} \in \mathcal{Z}^2_b(G, \mathbb{T})$, let $\circledast_{\mathbb{T}}$ be the twisted convolution coming from $\Omega_{\mathbb{T}}$, and let ω be a symmetric weakly subadditive weight on G*

such that $1/\omega \in L^{\Psi}(G)$. *Then* $(L^{\Phi}_{\omega}(G), \circledast_{\mathbb{T}})$ *is a symmetric twisted Orlicz algebra.*

Proof. By Theorem 5.1, $(L^1(G), \circledast_{\mathbb{T}})$ is symmetric. Hence, the result is an immediate consequence of Theorem 4.5 by setting $\sigma = \omega$. ∎

COROLLARY 5.3. *Let* $\Omega_{\mathbb{T}} \in \mathcal{Z}^2_b(G, \mathbb{T})$, *let* $\circledast_{\mathbb{T}}$ *be the twisted convolution coming from* $\Omega_{\mathbb{T}}$, *and let* ω_β *be the polynomial weight defined in* (5.4). *Then* $(L^{\Phi}_{\omega_\beta}(G), \circledast_{\mathbb{T}})$ *is a symmetric twisted Orlicz algebra if* $\beta > d/l$. *Here* $d := d(G)$ *is the degree of the growth of* G *and* $l \geq 1$ *is such that* $\lim_{x\to 0^+} \Psi(x)/x^l$ *exists.*

Proof. Suppose that $\beta > d/l$. It is easy to check that ω_β is weakly subadditive. In fact,

$$\omega_\beta(st) \leq 2^\beta(\omega_\beta(s) + \omega_\beta(t)) \quad (s, t \in G).$$

Hence, by Theorem 5.2, it suffices to show that $1/\omega_\beta \in L^{\Psi}(G)$. Since $\lim_{x\to 0^+} \Psi(x)/x^l$ exists, $\Psi(x)/x^l$ is bounded when x approaches 0^+. Thus there exist $M, N > 0$ such that

$$\Psi(x) \leq Mx^l \quad (0 \leq x \leq N).$$

In particular,

$$\Psi\left(\frac{N}{\omega_\beta(s)}\right) \leq \frac{MN^l}{\omega_\beta(s)^l} \quad (s \in G).$$

Let U be the symmetric neighborhood of the identity in G for which (5.1) holds. In particular,

$$\lambda(U^n) \sim n^d \quad (n \in \mathbb{N}). \tag{5.7}$$

For $C := MN^l$, we have

$$\begin{aligned}
\int_G \Psi\left(\frac{N}{\omega_\beta(s)}\right) ds &\leq \int_G \frac{C}{(1+\tau(s))^{l\beta}}\, ds \leq C + C\sum_{n=1}^{\infty} \int_{U^n\setminus U^{n-1}} \frac{1}{(1+\tau(s))^{l\beta}}\, ds \\
&= C + C\sum_{n=1}^{\infty} \frac{\lambda(U^n \setminus U^{n-1})}{(1+n)^{l\beta}} \\
&= C + C\sum_{n=1}^{\infty} \frac{\lambda(U^n) - \lambda(U^{n-1})}{(1+n)^{l\beta}} + \lim_{n\to\infty} \frac{\lambda(U^n)}{(1+n)^{l\beta}} \\
&\leq C + \frac{C\lambda(\{e\})}{2^{l\beta}} + C\sum_{n=1}^{\infty} \lambda(U^n)\left[\frac{1}{(1+n)^{l\beta}} - \frac{1}{(2+n)^{l\beta}}\right],
\end{aligned}$$

as, by (5.7), $\lim_{n\to\infty} \lambda(U^n)/(1+n)^{l\beta} = 0$. Furthermore, again by (5.7), the series in the last line of the preceding expression converges since

$$\lambda(U^n)\left[\frac{1}{(1+n)^{l\beta}} - \frac{1}{(2+n)^{l\beta}}\right] \sim \frac{n^d}{(1+n)^{l\beta+1}} \sim n^{d-l\beta-1}$$

and $d - l\beta < 0$. This completes the proof (see (1.2)). ∎

Remark 5.4. (i) If Ψ is a Young function, it is convex and positive. In this case, it is easy to see that the function $\Psi(x)/x$ is positive and decreasing on $\mathbb{R}^+$, and so $\lim_{x\to 0^+} \Psi(x)/x$ exists. Therefore, in Corollary 5.3, we can always assume that $l = 1$. Of course, we would like to pick the largest possible l to get the most optimal estimation.

(ii) We recall from [22, p. 20] that two Young functions Ψ_1 and Ψ_2 are *strongly equivalent* and write $\Psi_1 \approx \Psi_2$ if there exist $0 < a \le b < \infty$ such that

$$\Psi_1(ax) \le \Psi_2(x) \le \Psi_1(bx) \quad (x \ge 0).$$

It is clear from the definition of the Orlicz space (1.2) that strongly equivalent Young functions generate the same Orlicz space. In particular, this allows us to consider different strongly equivalent Young functions in our computation, for example to verify the conditions of Corollary 5.3.

Example 5.5. Let $\mathbb{Z}^d$ be the group of d-dimensional integers. The usual choice of generating set for $\mathbb{Z}^d$ is

$$F = \{(x_1, \ldots, x_d) \mid x_i \in \{-1, 0, 1\}\}.$$

It is straightforward to see that

$$|F^n| = (2n+1)^d \quad (n = 0, 1, 2, \ldots),$$

so that the order of growth of $\mathbb{Z}^d$ is d.

Now suppose that $\Omega_{\mathbb{T}} : \mathbb{Z}^d \times \mathbb{Z}^d \to \mathbb{T}$ is a 2-cocycle, $\circledast_{\mathbb{T}}$ is the twisted convolution coming from $\Omega_{\mathbb{T}}$, and ω_β is the polynomial weight on $\mathbb{Z}^d$ defined in (5.4).

(i) If $\Phi(x) = x^p/p$ $(1 < p < \infty)$, then $\Psi(x) = x^q/q$ with $1/p + 1/q = 1$, so that $(L^{\Phi}_{\omega_\beta}(\mathbb{Z}^d), \circledast_{\mathbb{T}})$ is a symmetric Banach $*$-algebra provided that $\beta > d/q$.

(ii) Suppose that there is $\delta > 0$ such that Ψ'' exists and is continuous on $[0, \delta]$ with $\Psi'_+(0) = 0$. Then, by applying the l'Hospital rule repeatedly,

$$\lim_{x\to 0^+} \frac{\Psi(x)}{x^2} = \lim_{x\to 0^+} \frac{\Psi'(x)}{2x} = \frac{\Psi''(0)}{2}.$$

Therefore $(L^{\Phi}_{\omega_\beta}(\mathbb{Z}^d), \circledast)$ is a symmetric Banach $*$-algebra if $\beta > d/2$. This can be applied to various Young functions, a few of which we list below (see [17, Proposition 2.11] and [22, p. 15]).

(1) If $\Phi(x) = x\ln(1+x)$, then $\Psi(x) \approx \cosh x - 1$.

(2) If $\Phi(x) = \cosh x - 1$, then $\Psi(x) \approx x\ln(1+x)$.

(3) If $\Phi(x) = e^x - x - 1$, then $\Psi(x) = (1+x)\ln(1+x) - x$.
(4) If $\Phi(x) = (1+x)\ln(1+x) - x$, then $\Psi(x) = e^x - x - 1$.

5.3. The case of subexponential weights. Throughout this article, in order to obtain the (symmetric) twisted Orlicz algebra $(L^\Phi(G), \circledast)$, we rely heavily on the existence of two suitable functions in $L^\Psi_+(G)$ whose sum dominates the 2-cocycle $|\Omega|$. However, it is not immediately clear whether this can be done if the weight associated to $|\Omega|$ is not weakly additive. A similar obstacle occurred in [12] while investigating the possibility of the isomorphism of a weighted group algebra of a finitely generated group of polynomial growth with an operator algebra. There, one needed $|\Omega|$ to be dominated by the sum of two absolutely square summable functions.

In [12], a method was developed to bypass the obstacle pointed out in the preceding paragraph. More precisely, it is shown in [12, Theorem 3.3] that if ω_β and $\sigma_{\alpha,C}$ are the weights defined in (5.4) and (5.5), respectively, then $\omega := \sigma_{\alpha,C}/\omega_\beta$ is also a weight on G provided that $0 < \alpha < 1$ and $\beta > 0$ is large enough. This in particular allows us to use the weak subadditivity of ω_β to show that $l^1_{\sigma_{\alpha,C}}(G)$ is isomorphic to an operator algebra.

In this section, we will apply the above approach of [12] to show that in most cases, we get symmetric twisted Orlicz algebras if $|\Omega|$ is determined by one of the weights $\sigma_{\alpha,C}$ or ρ_γ (the latter defined in (5.6)). We start with the following technical lemma which we will use to show that, similar to the case of $\sigma_{\alpha,C}$, the function $\omega := \rho_\gamma/\omega_\beta$ is also a weight on G provided that $\gamma > 0$.

LEMMA 5.6. *Let $\beta, \gamma, C > 0$ and define $p : [0,\infty) \to \mathbb{R}$ by*

$$p(x) = \frac{Cx}{(\ln(e+x))^\beta} - \gamma \ln(1+x). \tag{5.8}$$

Then there are $x_0, M > 0$ such that

(i) *p is positive and increasing on $[x_0,\infty)$;*
(ii) *p' is positive and decreasing on $[x_0,\infty)$;*
(iii) *for all $x \ge x_0$ and $y \ge 0$,*

$$0 < p(x+y) \le p(x) + p(y) + M. \tag{5.9}$$

Proof. It is clear that $p(x) \to \infty$ as $x \to \infty$. Also we have

$$p'(x) = C(\ln(e+x))^{-\beta} - \frac{C\beta x}{e+x}(\ln(e+x))^{-1-\beta} - \frac{\gamma}{1+x}.$$

Hence, by rewriting $p'(x)$ as

$$p'(x) = C(\ln(e+x))^{-\beta}\left(1 - \frac{\beta x}{(e+x)\ln(e+x)} - \frac{\gamma(\ln(e+x))^\beta}{C(1+x)}\right)$$

and using the relations

$$\lim_{x\to\infty}\frac{x}{(e+x)\ln(e+x)}=\lim_{x\to\infty}\frac{(\ln(e+x))^{\beta}}{1+x}=0,$$

it follows that $p'(x)>0$ for $x>0$ large. On the other hand, by computing the second derivative of p, we get

$$p''(x)=\frac{C\beta(\ln(e+x))^{-2-\beta}}{(e+x)^2}q(x),$$

where

$$q(x)=(1+\beta)x-2e\ln(e+x)-x\ln(e+x)+\frac{\gamma(\ln(e+x))^{2+\beta}(e+x)^2}{C\beta(1+x)^2}.$$

In particular, $p''(x)<0$ if and only if $q(x)<0$. Now if we write $q(x)$ as

$$q(x)=x\left[1+\beta-\frac{\ln(e+x)}{2}\right]+x\ln(e+x)\left[\frac{\gamma(\ln(e+x))^{1+\beta}(e+x)^2}{C\beta x(1+x)^2}-\frac{2e}{x}-\frac{1}{2}\right],$$

then one easily sees that $q(x)<0$ for $x>0$ large. Thus there is $x_0>0$ such that $p,p'>0$ and $p''<0$ on $[x_0,\infty)$. In particular, p is increasing and p' is decreasing on $[x_0,\infty)$. This proves (i) and (ii).

For (iii), fix $y\ge 0$. Define $r:[x_0,\infty)\to\mathbb{R}$ by

$$r(x)=p(x+y)-p(x)\qquad(x\ge x_0).$$

Since p' is decreasing on $[x_0,\infty)$, we find that $r'\le 0$ on $[x_0,\infty)$. Thus $r(x)\le r(x_0)$, or equivalently, $p(x+y)-p(x)\le p(x_0+y)-p(x_0)$ for every $x\in[x_0,\infty)$. Hence, by the Mean Value Theorem,

$$p(x+y)-p(x)-p(y)\le p(x_0+y)-p(y)-p(x_0)=x_0p'(z)-p(x_0)$$

for some $y<z<x_0+y$. The conclusion follows since p' is continuous on $[0,\infty)$ and $\lim_{x\to\infty}p'(x)=0$ so that p' is bounded on $[0,\infty)$. ∎

PROPOSITION 5.7. *Let G be a compactly generated group of polynomial growth, $\beta,\gamma>0$, and ω_β and ρ_γ the weights* (5.4) *and* (5.6), *respectively. If $\omega:=\rho_\gamma/\omega_\beta$, then ω is a symmetric weight on G satisfying the GRS-condition.*

Proof. It is clear that ω is symmetric and satisfies the GRS-condition. It remains to show that ω is a weight, i.e. there is a constant $K>0$ such that

$$\omega(st)\le K\omega(s)\omega(t)\qquad(s,t\in G).\tag{5.10}$$

Let p be the function defined in Lemma 5.6. Then clearly

$$\omega(s)=e^{p(\tau(s))}\qquad(s\in G).$$

Let $x_0,M>0$ be the constants obtained in Lemma 5.6. Take $s,t\in G$. We consider three cases:

Case I: $\max\{\tau(s), \tau(t)\} < x_0$. Then $\tau(st) \le \tau(s) + \tau(t) < 2x_0$. Hence we can set $K_1 = \max\{e^{p(x)-p(y)-p(z)} : x, y, z \in [0, 2x_0]\}$ to obtain (5.10).

Case II: $\max\{\tau(s), \tau(t)\} \ge x_0$ and $\tau(st) < x_0$. In this case, by (5.9),

$$0 < p(\tau(s) + \tau(t)) \le p(\tau(s)) + p(\tau(t)) + M.$$

Hence

$$e^{p(\tau(st))} \le K_2 e^{p(\tau(s))} e^{p(\tau(t))},$$

where $K_2 = e^M \max\{e^{p(x)} : x \in [0, x_0]\}$.

Case III: $\max\{\tau(s), \tau(t)\} \ge x_0$ and $\tau(st) \ge x_0$. Then, since p is increasing on $[x_0, \infty)$ (Lemma 5.6(i)), we can again apply (5.9) to get

$$e^{p(\tau(st))} \le e^{p(\tau(s))+p(\tau(t))} \le K_3 e^{p(\tau(s))} e^{p(\tau(t))},$$

where $K_3 = e^M$.

The conclusion follows with $K = \max\{K_1, K_2, K_3\}$. ∎

We are now ready to state the main result of this section, whose proof relies on Theorem 4.5.

Theorem 5.8. *Let $\Omega_{\mathbb{T}} \in \mathcal{Z}_b^2(G, \mathbb{T})$ and let $\circledast_{\mathbb{T}}$ be the twisted convolution coming from $\Omega_{\mathbb{T}}$. Then:*

(i) *If $\sigma_{\alpha,C}$ is the subexponential weight defined in* (5.5) *with $0 < \alpha < 1$, then $(L^{\Phi}_{\sigma_{\alpha,C}}(G), \circledast_{\mathbb{T}})$ is a symmetric twisted Orlicz algebra.*

(ii) *If $\rho_{\gamma,C}$ is the subexponential weight defined in* (5.6) *with $\gamma > 0$, then $(L^{\Phi}_{\rho_{\gamma,C}}(G), \circledast_{\mathbb{T}})$ is a symmetric twisted Orlicz algebra.*

Proof. (i) Fix $0 < \alpha < 1$ and $C > 0$. Choose

$$\beta > \max\left\{1, \frac{6}{C\alpha(1-\alpha)}, d(G)\right\},$$

where $d(G)$ is the order of growth of G. As shown in the proof of Corollary 5.3, $1/\omega_\beta \in L^{\Psi}(G)$. Moreover, by [12, Theorem 3.3], there is $M > 0$ (depending only on α, β, and C) such that

$$\frac{\sigma_{\alpha,C}(st)}{\sigma_{\alpha,C}(s)\sigma_{\alpha,C}(t)} \le \frac{M\omega_\beta(st)}{\omega_\beta(s)\omega_\beta(t)} \quad (s, t \in G). \tag{5.11}$$

But ω_β is weakly subadditive: in fact,

$$\omega_\beta(st) \le 2^\beta(\omega_\beta(s) + \omega_\beta(t)) \quad (s, t \in G).$$

Therefore

$$\frac{\sigma_{\alpha,C}(st)}{\sigma_{\alpha,C}(s)\sigma_{\alpha,C}(t)} \le \frac{2^\beta M}{\omega_\beta(s)} + \frac{2^\beta M}{\omega_\beta(t)} \quad (s, t \in G).$$

Hence if we set

$$u = 2^\beta M/\omega_\beta,$$

then $u \in L^{\Psi}(G)$ with

$$\frac{\sigma_{\alpha,C}(st)}{\sigma_{\alpha,C}(s)\sigma_{\alpha,C}(t)} \le u(s) + u(t) \quad (s,t \in G).$$

Now suppose that $\Omega \in \mathcal{Z}^2_{bw}(G,\mathbb{C}^*)$ is a 2-cocycle for which $\sigma_{\alpha,C}$ is a weight associated to $|\Omega|$. Then the preceding inequality is

$$|\Omega(s,t)| \le u(s) + u(t) \quad (s,t \in G).$$

This is a particular case of (3.1), and so, by Theorem 3.3, $(L^{\Phi}(G), \circledast)$ is an algebra. In fact (see (3.2)), for all $f, g \in L^{\Phi}(G)$ we have

$$\|f \circledast g\|_{\Phi} \le \|fu\|_1 \|g\|_{\Phi} + \|f\|_{\Phi} \|gu\|_1 \le C\|f\|_{\Phi}\|g\|_{\Phi},$$

where $C = 2N_{\Psi}(u)$ (the last inequality follows from Hölder's inequality (1.7)). Alternatively, if we apply Lemma 4.2 and use the equivalent weighted reformulation, we get

$$\|f \circledast_{\mathbb{T}} g\|_{\Phi,\sigma_{\alpha,C}} \le C\|f\sigma_{\alpha,C}\|_{\Phi}\|g\sigma_{\alpha,C}\|_{\Phi} = C\|f\|_{\Phi,\sigma_{\alpha,C}}\|g\|_{\Phi,\sigma_{\alpha,C}}$$

for all $f, g \in L^{\Phi}_{\sigma,\alpha}(G)$. That is, $(L^{\Phi}_{\sigma_{\alpha,C}}(G), \circledast_{\mathbb{T}})$ is an algebra. Moreover, by setting $\sigma := \sigma_{\alpha,C}$, $\omega := \omega_\beta$, and $\rho := \sigma_{\alpha,C}/\omega_\beta$ and applying Theorem 4.5, we find that ρ is a symmetric weight on G (see the proof of Theorem 4.5, in particular compare (4.7) with (5.11)) and $(L^{\Phi}_{\sigma_{\alpha,C}}(G), \circledast_{\mathbb{T}})$ is a differential $*$-subalgebra of $(L^1_\rho(G), \circledast_{\mathbb{T}})$. On the other hand, since both $\sigma_{\alpha,C}$ and ω_β satisfy the GRS-condition, so does ρ. Therefore, by Theorem 5.1, $(L^1_\rho(G), \circledast_{\mathbb{T}})$ is symmetric, and so $(L^{\Phi}_{\sigma_{\alpha,C}}(G), \circledast_{\mathbb{T}})$ is symmetric, once again from Theorem 4.5.

(ii) In view of Proposition 5.7, the proof is similar to that of (i). ∎

5.4. Functional calculus, Wiener property, and minimal ideals. Let G be a compactly generated group of polynomial growth, and let ω be a symmetric weight on G. In [11, Sections 4–8], the authors investigated other important and related properties of the Banach $*$-algebra $(L^p_\omega(G), *)$ such as existence of C^∞-functional calculus on compactly supported self-adjoint elements of $(L^p_\omega(G), *)$, regularity, (weak) Wiener property and existence of the minimal ideal associated to a given closed subset of the dual of $(L^p_\omega(G), *)$. Their approach is to apply in their setting what is known about these properties and techniques in the case of $(L^1_\omega(G), *)$. In fact, when one examines carefully the arguments in [11, Sections 4–8], one realizes that to obtain their results, it suffices that the following hold for the Banach $*$-algebra $(L^p_\omega(G), *)$:

(i) $(L^p_\omega(G), *)$ is a symmetric subalgebra of $(L^1(G), *)$;
(ii) $(L^p_\omega(G), *)$ is an essential Banach $(L^1_\omega(G), *)$-bimodule;
(iii) $(L^1_\omega(G), *)$ has a bounded approximate identity consisting of compactly supported self-adjoint elements;

(iv) there is an appropriate C^∞-functional calculus on compactly supported self-adjoint elements of $(L^1_\omega(G), *)$.

Now suppose that $\Omega \in \mathcal{Z}^2_{bw}(G, \mathbb{C}^*)$ and ω is a symmetric weight associated to $|\Omega|$. If ω satisfies the GRS-condition, then, by Theorem 5.1, both $(L^1(G), \circledast_{\mathbb{T}})$ and $(L^1_\omega(G), \circledast_{\mathbb{T}})$ are symmetric, where $\circledast_{\mathbb{T}}$ is the twisted convolution coming from $\Omega_{\mathbb{T}}$. If, in addition, $(L^\Phi(G), \circledast)$ is a symmetric twisted Orlicz algebra, then by Lemma 3.2 and Theorem 3.3, $(\mathcal{S}^\Phi_\omega(G), \circledast_{\mathbb{T}})$ satisfies assumptions (i) and (ii) above with $(L^p_\omega(G), *)$ replaced by $(\mathcal{S}^\Phi_\omega(G), \circledast_{\mathbb{T}})$ and the (untwisted) convolution $*$ by the twisted convolution $\circledast_{\mathbb{T}}$. Furthermore, by Lemma 4.2, Proposition 4.3 and [1], $(L^1_\omega(G), \circledast_{\mathbb{T}})$ satisfies assumptions (iii) and (iv) above. Hence all the results presented in [11, Sections 4–8] remain valid in our setting if we replace $(L^p_\omega(G), *)$ with $(\mathcal{S}^\Phi_\omega(G), \circledast_{\mathbb{T}})$. In particular, this applies to all the cases considered in Sections 5.2 and 5.3. Since one does not need any further major argument to achieve this (beyond what we explained above), we do not present anything further and just refer the interested reader to [11, Sections 4–8] and [1] (see also [4]).

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