

## Indestructible guessing models and the continuum

by

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**Abstract.** We introduce a stronger version of an  $\omega_1$ -guessing model, which we call an indestructibly  $\omega_1$ -guessing model. The principle IGMP states that there are stationarily many indestructibly  $\omega_1$ -guessing models. This principle, which follows from PFA, captures many of the consequences of PFA, including the Suslin hypothesis and the singular cardinal hypothesis. We prove that IGMP is consistent with the continuum being arbitrarily large.

The idea of an  $\omega_1$ -guessing model was introduced by Viale–Weiss [10], who showed that the combinatorial principle  $\text{ISP}(\omega_2)$  of Weiss [11] is equivalent to the existence of stationarily many  $\omega_1$ -guessing models in  $P_{\omega_2}(H(\theta))$  for all cardinals  $\theta \geq \omega_2$ . They showed that PFA implies  $\text{ISP}(\omega_2)$ , and in turn  $\text{ISP}(\omega_2)$  implies many of the consequences of PFA, including the failure of square principles. In [10] it was asked whether  $\text{ISP}(\omega_2)$  determines the value of the continuum. We answered this question negatively in [5], by showing that  $\text{ISP}(\omega_2)$  is consistent with  $2^\omega$  having any value of uncountable cofinality greater than  $\omega_1$ .

In this paper we introduce a stronger kind of guessing model, which we call an *indestructibly  $\omega_1$ -guessing model*. We also introduce a new principle, denoted by IGMP, which asserts the existence of stationarily many indestructibly  $\omega_1$ -guessing models in  $P_{\omega_2}(H(\theta))$  for any cardinal  $\theta \geq \omega_2$ . As before, PFA implies IGMP, but IGMP captures more of the consequences of PFA than  $\text{ISP}(\omega_2)$  does, including the Suslin hypothesis. As with the principle  $\text{ISP}(\omega_2)$ , a natural question is whether the stronger principle IGMP determines the value of the continuum. The main result of this paper is that IGMP is consistent with  $2^\omega$  being equal to any  $\lambda \geq \omega_2$  with cofinality at least  $\omega_2$ .

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In Section 1, we review material which will be necessary for reading the paper. In Section 2, we discuss  $\omega_1$ -guessing models, and give new proofs of several previously known consequences of  $\text{ISP}(\omega_2)$ . In Section 3, we introduce indestructibly  $\omega_1$ -guessing models and the principle IGMP, and derive some consequences of IGMP, including Suslin's hypothesis and SCH.

In Section 5, we review the ideas of strong genericity and the strongly proper collapse. We also prove a new theorem about the preservation of strong properness after proper forcing. In Sections 6 and 7, we carefully develop a finite support iteration of specializing forcings, and prove that a certain quotient of such an iteration has the  $\omega_1$ -approximation property. In Section 8, we prove our consistency result, that IGMP is consistent with the continuum being greater than  $\omega_2$ .

**1. Preliminaries.** We review some background material which will be necessary for understanding the paper. We assume that the reader is already familiar with forcing, iterated forcing, and proper forcing.

If  $\kappa$  is a regular cardinal and  $X$  is a set,  $P_\kappa(X)$  denotes the set  $\{a \subseteq X : |a| < \kappa\}$ . The reader should be familiar with the basic definitions and facts regarding club and stationary subsets of  $P_\kappa(X)$ .

A *tree* is a strict partial ordering  $(T, <_T)$  such that for any  $t \in T$ , the set  $\{u \in T : u <_T t\}$  is well-ordered by  $<_T$ . We write  $u \leq_T v$  to mean that either  $u <_T v$  or  $u = v$ . We sometimes say that  $T$  is a tree without explicitly mentioning its partial order, which is always denoted by  $<_T$ . For an ordinal  $\alpha$ ,  $T_\alpha$  is the set of  $t \in T$  such that the set  $\{u \in T : u <_T t\}$ , ordered by  $<_T$ , has order type  $\alpha$ . The set  $T_\alpha$  is called *level*  $\alpha$  of  $T$ . The *height* of  $T$  is the least  $\delta$  such that  $T_\delta$  is empty. Let  $T \upharpoonright \beta := \bigcup \{T_\alpha : \alpha < \beta\}$ . A set  $b$  is a *branch* of  $T$  if it is a maximal linearly ordered subset of  $T$ .

For a regular uncountable cardinal  $\kappa$ , a  $\kappa$ -Aronszajn tree is a tree of height  $\kappa$ , all of whose levels have size less than  $\kappa$ , which has no branch of length  $\kappa$ . A *weak  $\kappa$ -Kurepa tree* is a tree of height and size  $\kappa$  which has more than  $\kappa$  many branches of length  $\kappa$ .

Let  $T$  be a tree. A *specializing function* for  $T$  is a function  $f : T \rightarrow \omega$  such that for all  $x, y \in T$ , if  $x <_T y$  then  $f(x) \neq f(y)$ . Note that if  $T$  has a specializing function, then  $T$  has no branches of length  $\omega_1$ . On the other hand, suppose that  $T$  is a tree which has no branches of length  $\omega_1$ . Define  $P(T)$  as the forcing poset, ordered by reverse inclusion, whose conditions are finite functions  $p$  from a subset of  $T$  into  $\omega$ , such that for all  $x, y \in \text{dom}(p)$ , if  $x <_T y$  in  $\text{dom}(p)$ , then  $p(x) \neq p(y)$ . Then  $P(T)$  is  $\omega_1$ -c.c., and if  $G$  is a  $V$ -generic filter for  $P(T)$ , then  $\bigcup G$  is a specializing function for  $T$  (see [2]).

We will frequently use the product lemma. This result says that if  $\mathbb{P}$  and  $\mathbb{Q}$  are forcing posets, then the  $V$ -generic filters for  $\mathbb{P} \times \mathbb{Q}$  are exactly those filters of the form  $G \times H$ , where  $G$  is a  $V$ -generic filter for  $\mathbb{P}$ , and  $H$  is a

$V[G]$ -generic filter for  $\mathbb{Q}$ . Moreover, in that case  $H$  is a  $V$ -generic filter for  $\mathbb{Q}$ ,  $G$  is a  $V[H]$ -generic filter for  $\mathbb{P}$ , and  $V[G \times H] = V[G][H] = V[H][G]$ .

Let  $\mathbb{P}$  and  $\mathbb{Q}$  be forcing posets, where  $\mathbb{P}$  is a suborder of  $\mathbb{Q}$ . We say that  $\mathbb{P}$  is a *regular suborder* of  $\mathbb{Q}$  if (a) for all  $p$  and  $q$  in  $\mathbb{P}$ , if  $p$  and  $q$  are incompatible in  $\mathbb{P}$ , then  $p$  and  $q$  are incompatible in  $\mathbb{Q}$ , and (b) if  $A$  is a maximal antichain of  $\mathbb{P}$ , then  $A$  is predense in  $\mathbb{Q}$ .

Let  $\mathbb{P}$  be a regular suborder of  $\mathbb{Q}$ , and let  $G$  be a  $V$ -generic filter on  $\mathbb{P}$ . In  $V[G]$ , let  $\mathbb{Q}/G$  be the forcing poset consisting of conditions  $q \in \mathbb{Q}$  such that for all  $s \in G$ ,  $q$  and  $s$  are compatible in  $\mathbb{Q}$ , with the same ordering as  $\mathbb{Q}$ . Then  $\mathbb{P} * (\mathbb{Q}/\dot{G}_{\mathbb{P}})$  is forcing equivalent to  $\mathbb{Q}$ . Moreover:

LEMMA 1.1 ([5, Lemma 1.6]). *Let  $\mathbb{P}$  be a regular suborder of  $\mathbb{Q}$ . Suppose that  $G$  is a  $V$ -generic filter on  $\mathbb{P}$ , and  $H$  is a  $V[G]$ -generic filter on  $\mathbb{Q}/G$ . Then  $H$  is a  $V$ -generic filter on  $\mathbb{Q}$ ,  $H \cap \mathbb{P} = G$ , and  $V[G][H] = V[H]$ .*

*Conversely, if  $H$  is a  $V$ -generic filter on  $\mathbb{Q}$ , then  $H \cap \mathbb{P}$  is a  $V$ -generic filter on  $\mathbb{P}$ ,  $H$  is a  $V[H \cap \mathbb{P}]$ -generic filter on  $\mathbb{Q}/(H \cap \mathbb{P})$ , and  $V[H] = V[H \cap \mathbb{P}][H]$ .*

LEMMA 1.2 ([5, Lemmas 1.1, 1.3]). *Let  $\mathbb{P}$  be a regular suborder of  $\mathbb{Q}$ . Then for all  $q \in \mathbb{Q}$ , there is  $s \in \mathbb{P}$  such that for all  $t \leq s$  in  $\mathbb{P}$ ,  $q$  and  $t$  are compatible in  $\mathbb{Q}$ . Moreover, this property of  $s$  is equivalent to  $s$  forcing in  $\mathbb{P}$  that  $q$  is in  $\mathbb{Q}/\dot{G}_{\mathbb{P}}$ .*

LEMMA 1.3 ([5, Lemma 1.5]). *Let  $\mathbb{P}$  and  $\mathbb{Q}$  be forcing posets, and assume that  $\mathbb{P}$  is a regular suborder of  $\mathbb{Q}$ . If  $D$  is a dense subset of  $\mathbb{Q}$ , then  $\mathbb{P}$  forces that  $D \cap (\mathbb{Q}/\dot{G}_{\mathbb{P}})$  is a dense subset of  $\mathbb{Q}/\dot{G}_{\mathbb{P}}$ .*

LEMMA 1.4. *Let  $\mathbb{P}$  and  $\mathbb{Q}$  be forcing posets, and assume that  $\mathbb{P}$  is a regular suborder of  $\mathbb{Q}$ . Let  $G$  be a  $V$ -generic filter on  $\mathbb{P}$ . Suppose that  $s \in G$  and  $p \in \mathbb{Q}/G$ . Then  $s$  and  $p$  are compatible in  $\mathbb{Q}/G$ .*

*Proof.* If not, then there is  $t \in G$  such that  $t$  forces in  $\mathbb{P}$  that (a)  $p$  is in  $\mathbb{Q}/\dot{G}_{\mathbb{P}}$ , and (b)  $p$  and  $s$  are incompatible in  $\mathbb{Q}/\dot{G}_{\mathbb{P}}$ . Fix  $u \in G$  with  $u \leq s, t$ .

As  $p \in \mathbb{Q}/G$  and  $u \in G$ , fix  $v \leq u, p$  in  $\mathbb{Q}$ . By Lemma 1.2, fix  $w \in \mathbb{P}$  such that for all  $z \leq w$  in  $\mathbb{P}$ ,  $v$  and  $z$  are compatible in  $\mathbb{Q}$ . Then in particular,  $v$  and  $w$  are compatible in  $\mathbb{Q}$ , and since  $v \leq u$ ,  $u$  and  $w$  are compatible in  $\mathbb{Q}$ . As  $\mathbb{P}$  is a regular suborder of  $\mathbb{Q}$  and  $u$  and  $w$  are in  $\mathbb{P}$ , it follows that  $u$  and  $w$  are compatible in  $\mathbb{P}$ . Fix  $y \leq w, u$  in  $\mathbb{P}$ .

Since  $y \leq w$ , every extension of  $y$  in  $\mathbb{P}$  is compatible with  $v$  in  $\mathbb{Q}$ . By Lemma 1.2,  $y$  forces in  $\mathbb{P}$  that  $v \in \mathbb{Q}/\dot{G}_{\mathbb{P}}$ . Now  $v \leq u, p$  in  $\mathbb{Q}$ . Since  $u \leq s$ , we have  $v \leq s, p$  in  $\mathbb{Q}$ . But  $y$  forces that  $v \in \mathbb{Q}/\dot{G}_{\mathbb{P}}$ , so  $y$  forces that  $s$  and  $p$  are compatible in  $\mathbb{Q}/\dot{G}_{\mathbb{P}}$ . This contradicts the choice of  $t$  and the fact that  $y \leq t$ . ■

Let  $\mathbb{P}$  and  $\mathbb{Q}$  be forcing posets. A function  $f : \mathbb{P} \rightarrow \mathbb{Q}$  is a *regular embedding* if (a) for all  $p, q \in \mathbb{P}$ , if  $q \leq p$  in  $\mathbb{P}$ , then  $f(q) \leq f(p)$  in  $\mathbb{Q}$ ;

(b) for all  $p, q \in \mathbb{P}$ , if  $p$  and  $q$  are incompatible in  $\mathbb{P}$ , then  $f(p)$  and  $f(q)$  are incompatible in  $\mathbb{Q}$ ; (c) if  $A$  is a maximal antichain of  $\mathbb{P}$ , then  $f[A]$  is predense in  $\mathbb{Q}$ . Note that if  $f : \mathbb{P} \rightarrow \mathbb{Q}$  is a regular embedding, then  $f[\mathbb{P}]$  is a regular suborder of  $\mathbb{Q}$ . A function  $f : \mathbb{P} \rightarrow \mathbb{Q}$  is a *dense embedding* if it satisfies (a) and (b) above, and  $f[\mathbb{P}]$  is dense in  $\mathbb{Q}$ . Note that any dense embedding is a regular embedding.

Assume that  $j : V \rightarrow M$  is an elementary embedding with critical point  $\kappa$ , living in some outer model  $W$  of  $V$ . Let  $\mathbb{P}$  be a forcing poset in  $V$  which is  $\kappa$ -c.c. We claim that  $j \upharpoonright \mathbb{P}$  is a regular embedding of  $\mathbb{P}$  into  $j(\mathbb{P})$ . Namely, the preservation of the order and incompatibility of conditions from  $\mathbb{P}$  to  $j(\mathbb{P})$  follows from the elementarity of  $j$ . And if  $A$  is a maximal antichain of  $\mathbb{P}$ , then by elementarity  $M$  models that  $j(A)$  is a maximal antichain of  $j(\mathbb{P})$ . By upward absoluteness,  $j(A)$  is a maximal antichain of  $j(\mathbb{P})$ . But since  $\mathbb{P}$  is  $\kappa$ -c.c.,  $|A| < \kappa$ , and therefore  $j(A) = j[A]$ . So  $j[A]$  is a maximal antichain of  $j(\mathbb{P})$ .

A function  $\pi : \mathbb{Q} \rightarrow \mathbb{P}$ , where  $\mathbb{P}$  and  $\mathbb{Q}$  are forcing posets, is called a *projection mapping* if (a)  $\pi(1_{\mathbb{Q}}) = 1_{\mathbb{P}}$ , (b)  $q \leq p$  in  $\mathbb{Q}$  implies that  $\pi(q) \leq \pi(p)$  in  $\mathbb{P}$ , and (c) whenever  $p \leq \pi(q)$  in  $\mathbb{P}$ , then there is some  $r \leq q$  in  $\mathbb{Q}$  such that  $\pi(r) \leq p$  in  $\mathbb{P}$ .

If  $\pi : \mathbb{Q} \rightarrow \mathbb{P}$  is a projection mapping and  $G$  is a  $V$ -generic filter on  $\mathbb{P}$ , then in  $V[G]$  we can define the forcing poset  $\mathbb{Q}/G$  whose conditions are those  $q \in \mathbb{Q}$  such that  $\pi(q) \in G$ , with the same ordering as  $\mathbb{Q}$ . Then  $\mathbb{Q}$  is forcing equivalent to  $\mathbb{P} * (\mathbb{Q}/\dot{G}_{\mathbb{P}})$ .

LEMMA 1.5. *Let  $\mathbb{P}$  be a suborder of  $\mathbb{Q}$ . Suppose that there exists a projection mapping  $\pi : \mathbb{Q} \rightarrow \mathbb{P}$  satisfying (i)  $\pi(p) = p$  for all  $p \in \mathbb{P}$ , and (ii)  $q \leq \pi(q)$  for all  $q \in \mathbb{Q}$ . Then  $\mathbb{P}$  is a regular suborder of  $\mathbb{Q}$ . Moreover, if  $G$  is a  $V$ -generic filter on  $\mathbb{P}$ , then the two kinds of quotients  $\mathbb{Q}/G$  in  $V[G]$  are the same.*

*Proof.* Suppose that  $p$  and  $q$  are in  $\mathbb{P}$ , and  $p$  and  $q$  are compatible in  $\mathbb{Q}$ . Let  $s \leq p, q$  in  $\mathbb{Q}$ . Then  $\pi(s) \leq \pi(p) = p$  and  $\pi(s) \leq \pi(q) = q$ . So  $p$  and  $q$  are compatible in  $\mathbb{P}$ .

Let  $A$  be a maximal antichain of  $\mathbb{P}$ , and we will show that  $A$  is predense in  $\mathbb{Q}$ . Let  $q \in \mathbb{Q}$ , and we will show that  $q$  is compatible with some condition in  $A$ . We have  $\pi(q) \in \mathbb{P}$ , so as  $A$  is a maximal antichain of  $\mathbb{P}$ , we can fix  $s \in A$  and  $r \in \mathbb{P}$  such that  $r \leq \pi(q), s$ . Since  $\pi$  is a projection mapping, we can fix  $t \leq q$  in  $\mathbb{Q}$  such that  $\pi(t) \leq r$ . But then  $t \leq q$ , and  $t \leq \pi(t) \leq r \leq s$ , so  $q$  is compatible with  $s$ .

Let  $G$  be a  $V$ -generic filter on  $\mathbb{P}$ . We will prove that for all  $q \in \mathbb{Q}$ ,  $q$  is compatible in  $\mathbb{Q}$  with every condition in  $G$  iff  $\pi(q) \in G$ . Assume that  $q$  is compatible with every condition in  $G$ . Since  $G$  is a  $V$ -generic filter on  $\mathbb{P}$ , to show that  $\pi(q) \in G$  it suffices to show that  $\pi(q)$  is compatible in  $\mathbb{P}$  with

every condition in  $G$ . So let  $s \in G$ , and we will show that  $\pi(q)$  and  $s$  are compatible in  $\mathbb{P}$ . By assumption,  $s$  and  $q$  are compatible in  $\mathbb{Q}$ , so fix  $t \leq q, s$ . Then  $\pi(t) \leq \pi(q)$  and  $\pi(t) \leq \pi(s) = s$ . Hence  $\pi(q)$  and  $s$  are compatible in  $\mathbb{P}$ .

Conversely, assume that  $\pi(q) \in G$ , and we will show that  $q$  is compatible in  $\mathbb{Q}$  with every condition in  $G$ . Fix  $s \in G$ . Then there is  $t \in G$  with  $t \leq \pi(q), s$ . Since  $\pi$  is a projection mapping, there is  $u \leq q$  such that  $\pi(u) \leq t$ . Then  $u \leq q$  and  $u \leq \pi(u) \leq t \leq s$ , so  $q$  and  $s$  are compatible in  $\mathbb{Q}$ . ■

A pair  $(V, W)$  of transitive sets or classes with  $V \subseteq W$  is said to have the  $\omega_1$ -covering property if for every set  $a \in W$  which is a subset of  $V \cap \text{On}$  and which  $W$  models is countable, there is a set of ordinals  $b \in V$  which  $V$  models is countable such that  $a \subseteq b$ . A forcing poset  $\mathbb{P}$  has the  $\omega_1$ -covering property if  $\mathbb{P}$  forces that  $(V, V[\dot{G}_{\mathbb{P}}])$  has the  $\omega_1$ -covering property.

For a set or class  $N$ , a set  $d \subseteq N$  is said to be countably approximated by  $N$  if for any set  $a$  in  $N$  which  $N$  models is countable,  $d \cap a \in N$ . A pair  $(V, W)$  of transitive sets or classes with  $V \subseteq W$  is said to have the  $\omega_1$ -approximation property if whenever  $d \in W$  is a bounded subset of  $V \cap \text{On}$  which is countably approximated by  $V$ , we have  $d \in V$ . A forcing poset  $\mathbb{P}$  has the  $\omega_1$ -approximation property if  $\mathbb{P}$  forces that  $(V, V[\dot{G}_{\mathbb{P}}])$  has the  $\omega_1$ -approximation property.

Note that if  $\mathbb{P}$  has the  $\omega_1$ -approximation property, then  $\mathbb{P}$  preserves  $\omega_1$ .

LEMMA 1.6. *Suppose that  $\mathbb{P}$  has the  $\omega_1$ -approximation property. Let  $T$  be a tree. Suppose that  $G$  is a  $V$ -generic filter on  $\mathbb{P}$ . Then any branch of  $T$  in  $V[G]$  whose length has uncountable cofinality is in  $V$ .*

*Proof.* Without loss of generality, assume that the underlying set of  $T$  is an ordinal. Let  $b$  be a branch of  $T$  in  $V[G]$  whose length has uncountable cofinality. We claim that  $b$  is countably approximated by  $V$ . Then  $b \in V$  and we are done. Let  $a$  be a countable set in  $V$ . Since the length of  $b$  has uncountable cofinality, there is  $y \in b$  such that  $a \cap b \subseteq \{x \in T : x <_T y\}$ . Then  $a \cap b = a \cap \{x \in T : x <_T y\}$ , which is in  $V$ . ■

LEMMA 1.7. *Suppose that  $(V_0, V_1)$  and  $(V_1, V_2)$  are pairs of transitive sets or classes which model ZFC, have the same ordinals, and satisfy  $V_0 \subseteq V_1 \subseteq V_2$ . Assume that both pairs have the  $\omega_1$ -covering property and the  $\omega_1$ -approximation property. Then  $(V_0, V_2)$  has the  $\omega_1$ -covering property and the  $\omega_1$ -approximation property.*

*Proof.* It is easy to check that  $(V_0, V_2)$  has the  $\omega_1$ -covering property. Let  $d$  be a bounded subset of  $V_0 \cap \text{On}$  in  $V_2$  which is countably approximated by  $V_0$ . We will show that  $d \in V_0$ .

We claim that  $d$  is countably approximated by  $V_1$ . Let  $a$  be a countable set in  $V_1$ . Since  $(V_0, V_1)$  has the  $\omega_1$ -covering property, we can fix a countable set  $b$  in  $V_0$  such that  $a \subseteq b$ . Since  $d$  is countably approximated by  $V_0$ ,  $b \cap d$  is in  $V_0$  and hence in  $V_1$ . Therefore  $a \cap d = a \cap (b \cap d)$  is in  $V_1$ .

Since  $(V_1, V_2)$  has the  $\omega_1$ -approximation property,  $d \in V_1$ . As  $d$  is countably approximated by  $V_0$  and  $(V_0, V_1)$  has the  $\omega_1$ -approximation property, we conclude that  $d \in V_0$ . ■

It follows that if  $\mathbb{P}$  has the  $\omega_1$ -covering property and the  $\omega_1$ -approximation property, and  $\mathbb{P}$  forces that  $\dot{\mathbb{Q}}$  has the  $\omega_1$ -covering property and the  $\omega_1$ -approximation property, then the two-step iteration  $\mathbb{P} * \dot{\mathbb{Q}}$  has the  $\omega_1$ -covering property and the  $\omega_1$ -approximation property.

LEMMA 1.8. *Suppose that  $V \subseteq W_0 \subseteq W$  are transitive sets or classes, and  $(V, W)$  has the  $\omega_1$ -approximation property. Then  $(V, W_0)$  has the  $\omega_1$ -approximation property. It follows that if  $\mathbb{P}$  is a regular suborder of  $\mathbb{Q}$ , and  $\mathbb{Q}$  has the  $\omega_1$ -approximation property, then  $\mathbb{P}$  has the  $\omega_1$ -approximation property.*

*Proof.* Straightforward. ■

A set  $N$  of size  $\omega_1$  is said to be *internally unbounded* if for any countable set  $a \subseteq N$ , there is a countable set  $b \in N$  such that  $a \subseteq b$ . If  $N \prec H(\theta)$  for some cardinal  $\theta \geq \omega_2$ , then clearly  $N$  is internally unbounded iff there exists a  $\subseteq$ -increasing sequence  $\langle N_i : i < \omega_1 \rangle$  of countable sets in  $N$  with union equal to  $N$ .

Finally, we will need to know some facts about the  $Y$ -c.c. property of forcing posets, which is a property introduced recently in [4]. The actual definition of being  $Y$ -c.c. is beyond the scope of this paper. In [4] it is proven that any  $Y$ -c.c. forcing poset is  $\omega_1$ -c.c. and has the  $\omega_1$ -approximation property, and any finite support iteration of  $Y$ -c.c. forcing posets is itself  $Y$ -c.c. Also, the forcing poset  $P(T)$  defined earlier in this section, for adding a specializing function for a tree which has no branches of length  $\omega_1$ , is  $Y$ -c.c.

**2. Guessing models and GMP.** Guessing models were introduced by Viale–Weiss [10].

DEFINITION 2.1. Let  $N$  be a set. A set  $d \subseteq N$  is said to be  $N$ -*guessed* if there exists  $e \in N$  such that  $d = e \cap N$ . We say that  $N$  is  $\omega_1$ -*guessing* if for any set  $d \subseteq N \cap \text{On}$  with  $\text{sup}(d) < \text{sup}(N \cap \text{On})$ , if  $d$  is countably approximated by  $N$ , then  $d$  is  $N$ -guessed.

A typical situation which we will consider is that  $N$  is an elementary substructure of  $H(\chi)$  for some cardinal  $\chi \geq \omega_2$ , and  $|N| = \omega_1$ . In the next section, we will also consider the case that  $N$  is an elementary substructure in an inner model over which the universe is a generic extension.

There is a useful characterization of being  $\omega_1$ -guessing in terms of the approximation property.

LEMMA 2.2 ([5, Lemma 1.10]). *Let  $N$  be an elementary substructure of  $H(\chi)$  for some uncountable cardinal  $\chi$ . Then the following are equivalent:*

- (1)  $N$  is  $\omega_1$ -guessing;
- (2) the pair  $(\bar{N}, V)$  has the  $\omega_1$ -approximation property, where  $\bar{N}$  is the transitive collapse of  $N$ .

Next we prove two technical lemmas about  $\omega_1$ -guessing models.

LEMMA 2.3. *Let  $N$  be in  $P_{\omega_2}(H(\chi))$ , where  $\chi \geq \omega_2$  is a cardinal, and assume that  $N \prec H(\chi)$  and  $N$  is  $\omega_1$ -guessing. Then for any cardinal  $\theta \in N$  with uncountable cofinality,  $\text{cf}(\sup(N \cap \theta)) = \omega_1$ . In particular,  $\omega_1 \subseteq N$ , and hence  $N \cap \omega_2 \in \omega_2$ .*

*Proof.* Since  $N$  has size at most  $\omega_1$ ,  $\text{cf}(\sup(N \cap \theta)) \leq \omega_1$ . Suppose for a contradiction that  $\text{cf}(\sup(N \cap \theta)) = \omega$ . Fix a sequence  $\langle \alpha_n : n < \omega \rangle$  of ordinals in  $N \cap \theta$  which is increasing and cofinal in  $\sup(N \cap \theta)$ .

We claim that the set  $\{\alpha_n : n < \omega\}$  is countably approximated by  $N$ . Let  $a \in N$  be countable, and we will show that  $a \cap \{\alpha_n : n < \omega\}$  is in  $N$ . By elementarity,  $\sup(a \cap \theta) \in N \cap \theta$ , so we can fix  $k$  such that  $\sup(a \cap \theta) < \alpha_k$ . Then  $a \cap \{\alpha_n : n < \omega\} \subseteq a \cap \{\alpha_n : n < k\}$ . So  $a \cap \{\alpha_n : n < \omega\}$  is a finite subset of  $N$ , and hence is in  $N$ .

Since the set  $\{\alpha_n : n < \omega\}$  is countably approximated by  $N$ , and is a bounded subset of  $N \cap \text{On}$ , there exists  $e \in N$  such that  $\{\alpha_n : n < \omega\} = N \cap e$ . We claim that  $e = \{\alpha_n : n < \omega\}$ . This is a contradiction, for then  $\sup(e) = \sup(N \cap \theta)$  would be in  $N$  by elementarity, which is impossible.

As  $N \cap e \subseteq \theta$ , by elementarity it follows that  $e \subseteq \theta$ . Also, for all  $\beta \in N \cap \theta$ ,  $N \cap e \cap \beta$  is finite; therefore by elementarity,  $e \cap \beta$  must be finite. So  $N$  models that every proper initial segment of  $e$  is finite, and hence  $e$  is at most countable. Therefore  $e \subseteq N$ , so  $e = N \cap e = \{\alpha_n : n < \omega\}$ . ■

LEMMA 2.4. *Let  $N$  be in  $P_{\omega_2}(H(\chi))$ , where  $\chi \geq \omega_2$  is a cardinal, and assume that  $N \prec H(\chi)$ . Let  $T \in N$  be a tree with height and size  $\omega_1$ . Suppose that  $W$  is an outer model of  $V$  with  $\omega_1^V = \omega_1^W$ , and  $N$  is  $\omega_1$ -guessing in  $W$ . Then every branch of  $T$  in  $W$  with length  $\omega_1$  is in  $N$ .*

*Proof.* Since  $N$  is  $\omega_1$ -guessing in  $W$ , it is easy to check that  $N$  is  $\omega_1$ -guessing in  $V$ . So by Lemma 2.3,  $\omega_1 \subseteq N$ .

Without loss of generality, assume that the underlying set of  $T$  is  $\omega_1$ . Let  $b$  be a branch of  $T$  of length  $\omega_1$  in  $W$ . Then  $b$  is a bounded subset of  $N \cap \text{On}$ . We claim that  $b$  is countably approximated by  $N$  in  $W$ . Let  $a \in N$  be countable. Since  $b$  has length  $\omega_1$ , we can fix  $y \in b$  such that  $a \cap b \subseteq \{x \in T : x <_T y\}$ . Then  $a \cap b = a \cap \{x \in T : x <_T y\}$ . Now

$N \prec H(\chi)$  in  $V$ , so the set  $\{x \in T : x <_T y\}$  is in  $N$ , and hence the set  $a \cap b$  is in  $N$ .

As  $N$  is  $\omega_1$ -guessing in  $W$ , there is  $e \in N$  such that  $N \cap e = b$ . Since  $N \cap e \subseteq T$ , it follows by elementarity that  $e \subseteq T$ . But the underlying set of  $T$  is  $\omega_1$ , which is a subset of  $N$ . So  $e \subseteq N$ . Hence  $e = e \cap N = b$ . So  $e = b$ , and  $b \in N$ . ■

**DEFINITION 2.5.** For a cardinal  $\theta \geq \omega_2$ , let  $\text{GMP}(\theta)$  be the statement that there exist stationarily many sets  $N \in P_{\omega_2}(H(\theta))$  such that  $N$  is  $\omega_1$ -guessing. Let  $\text{GMP}$  be the statement that  $\text{GMP}(\theta)$  holds for all cardinals  $\theta \geq \omega_2$  <sup>(1)</sup>.

It is easy to see that if  $\omega_2 \leq \theta_0 < \theta_1$  are cardinals,  $N \in P_{\omega_2}(H(\theta_1))$  is  $\omega_1$ -guessing,  $N \prec H(\theta)$ , and  $\theta_0 \in N$ , then  $N \cap H(\theta_0)$  is  $\omega_1$ -guessing. In particular,  $\text{GMP}(\theta_1)$  implies  $\text{GMP}(\theta_0)$ .

Weiss [11] introduced the principle  $\text{ISP}(\omega_2)$ , and Viale–Weiss [10] showed that it is equivalent to what we are calling  $\text{GMP}$ . They proved that  $\text{ISP}(\omega_2)$  follows from  $\text{PFA}$ , and that  $\text{ISP}(\omega_2)$  implies some of the consequence of  $\text{PFA}$ , such as the failure of square principles.

The original formulation of  $\text{ISP}(\omega_2)$  involves completely different concepts than guessing models, namely ineffable branches in slender  $P_\kappa(\lambda)$ -lists. Some of the consequences of  $\text{ISP}(\omega_2)$  were derived in [10] and [11] using these different concepts. We offer new proofs of two of the most important consequences of  $\text{ISP}(\omega_2)$  using arguments involving guessing models, namely, the failure of the approachability property on  $\omega_1$ , and the nonexistence of  $\omega_2$ -Aronszajn trees. We also derive a new consequence, namely the nonexistence of weak  $\omega_1$ -Kurepa trees.

For the original proof of the next result, see [10, Corollary 4.9].

**PROPOSITION 2.6.**  $\text{GMP}(\omega_3)$  implies  $\neg \text{AP}_{\omega_1}$ .

*Proof.* Suppose for a contradiction that the approachability property  $\text{AP}_{\omega_1}$  holds. So there exists a sequence  $\vec{a} = \langle a_i : i < \omega_2 \rangle$  of countable subsets of  $\omega_2$ , a club  $C \subseteq \omega_2$ , and a sequence  $\vec{c} = \langle c_\alpha : \alpha \in C \cap \text{cof}(\omega_1) \rangle$  such that for all  $\alpha \in C \cap \text{cof}(\omega_1)$ ,  $c_\alpha$  is a club subset of  $\alpha$  with order type  $\omega_1$ , and for all  $\beta < \alpha$ , there is  $i < \alpha$  such that  $c_\alpha \cap \beta = a_i$ .

Fix  $N$  in  $P_{\omega_2}(H(\omega_3))$  such that  $N \prec H(\omega_3)$ ,  $\vec{a}$ ,  $C$ , and  $\vec{c}$  are in  $N$ , and  $N$  is  $\omega_1$ -guessing. Since  $C \in N$ , we have  $N \cap \omega_2 \in C$ . As  $\omega_2 \in N$ , it follows that  $\text{cf}(N \cap \omega_2) = \omega_1$  by Lemma 2.3. So  $N \cap \omega_2 \in C \cap \text{cof}(\omega_1)$ .

Let  $\alpha := N \cap \omega_2$ . We claim that  $c_\alpha$  is countably approximated by  $N$ . Let  $a \in N$  be countable. Then for some  $\beta < \alpha$ ,  $c_\alpha \cap a \subseteq \beta$ . Fix  $i < \alpha$  such that  $c_\alpha \cap \beta = a_i$ . Then  $i \in N$ , and hence  $c_\alpha \cap \beta = a_i \in N$ . So  $c_\alpha \cap a = c_\alpha \cap \beta \cap a = a_i \cap a$ , which is in  $N$  since  $a$  and  $a_i$  are in  $N$ .

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<sup>(1)</sup>  $\text{GMP}$  stands for *guessing model principle*.



As  $N$  is  $\omega_1$ -guessing, we can fix  $e \in N$  such that  $e \cap N = c_\alpha$ . Since  $N \cap e \subseteq \omega_2$ , it follows that  $e \subseteq \omega_2$  by elementarity. And as  $N \cap e = c_\alpha$  is cofinal in  $N \cap \omega_2$ ,  $e$  is cofinal in  $\omega_2$  by elementarity. Therefore  $e$  has order type  $\omega_2$ . By elementarity, fix  $\gamma \in N \cap e$  such that  $e \cap \gamma$  has order type  $\omega_1$ . Since  $\omega_1 \subseteq N$ , we have  $e \cap \gamma \subseteq N$ , so  $e \cap \gamma = e \cap N \cap \gamma = c_\alpha \cap \gamma$ . Hence  $c_\alpha \cap \gamma$  has order type  $\omega_1$ , which contradicts  $c_\alpha$  having order type  $\omega_1$  and  $\gamma < \alpha$ . ■

The next result was originally proven in [11, Section 2].

**THEOREM 2.7.** *GMP( $\omega_3$ ) implies that there does not exist an  $\omega_2$ -Aronszajn tree.*

*Proof.* Let  $T$  be a tree of height  $\omega_2$ , all of whose levels have cardinality less than  $\omega_2$ . We will prove that there is a branch of  $T$  with order type  $\omega_2$ . Without loss of generality, assume that  $T$  has underlying set  $\omega_2$ , so that  $T \in H(\omega_3)$ . Fix  $N$  in  $P_{\omega_2}(H(\omega_3))$  such that  $N \prec H(\omega_3)$ ,  $T \in N$ , and  $N$  is  $\omega_1$ -guessing.

Let  $\alpha := N \cap \omega_2$ . For all  $\beta < \alpha$ ,  $T_\beta$  is in  $N$  by elementarity, and since  $T_\beta$  has size at most  $\omega_1$ , we have  $T_\beta \subseteq N$ . It follows that  $T \upharpoonright \alpha \subseteq N$ .

Fix a node  $y$  on level  $\alpha$  of  $T$ . Let  $d := \{x \in T : x <_T y\}$ . Then  $d$  is a bounded subset of  $N \cap \text{On}$ . We claim that  $d$  is countably approximated by  $N$ . Let  $a \in N$  be countable. Since  $\alpha$  has cofinality  $\omega_1$  by Lemma 2.3, there is  $y^* <_T y$  such that  $a \cap d = a \cap \{x \in T : x <_T y^*\}$ . As  $a$  and  $y^*$  are in  $N$ , so is  $a \cap d$ .

Since  $N$  is  $\omega_1$ -guessing, we can fix  $e \in N$  such that  $e \cap N = d$ . We claim that  $e$  is a branch of  $T$  with length  $\omega_2$ , which completes the proof. First, if  $x$  and  $y$  are in  $e \cap N$ , then  $x$  and  $y$  are in  $d$ , and since  $d$  is a branch, either  $x \leq_T y$  or  $y <_T x$ . By elementarity,  $e$  is a chain in  $T$ . Similarly, if  $x \in e \cap N$ ,  $x_0 \in N \cap T$ , and  $x_0 <_T x$ , then  $x_0 \in d$ , and hence  $x_0 \in d \subseteq e$ . By elementarity,  $e$  is closed downwards. If  $e$  does not have order type  $\omega_2$ , then by elementarity there is  $\beta < N \cap \omega_2 = \alpha$  such that  $e \subseteq T \upharpoonright \beta$ . But the node in  $d$  at level  $\beta$  is in  $e$ , which is a contradiction. ■

**THEOREM 2.8.** *GMP( $\omega_2$ ) implies that there does not exist a weak  $\omega_1$ -Kurepa tree.*

*Proof.* Suppose for a contradiction that  $T$  is a weak  $\omega_1$ -Kurepa tree. Without loss of generality, assume that the underlying set of  $T$  is  $\omega_1$ . Then  $T \in H(\omega_2)$ . Fix a set  $N \in P_{\omega_2}(H(\omega_2))$  such that  $N \prec H(\omega_2)$ ,  $T \in N$ , and  $N$  is  $\omega_1$ -guessing. By Lemma 2.4, every branch of  $T$  with length  $\omega_1$  is in  $N$ . But this is impossible, since  $N$  has size  $\omega_1$  and  $T$  has more than  $\omega_1$  many branches of length  $\omega_1$ . ■

Note that if CH holds, then there exists a weak  $\omega_1$ -Kurepa tree, namely the tree of functions in  $2^{<\omega_1}$ . Hence GMP( $\omega_2$ ) implies that CH fails. Viale–Weiss [10] asked whether GMP implies that  $2^\omega$  is equal to  $\omega_2$ . This question

was settled in [5], where we showed that GMP is consistent with  $2^\omega$  being equal to any given cardinal  $\lambda \geq \omega_2$  of uncountable cofinality.

Let us make an additional observation about the model constructed in [5]. Recall that the *pseudo-splitting number*  $\mathfrak{p}$  is the least size of a collection  $X$  of infinite subsets of  $\omega$ , closed under finite intersections, for which there is no set  $b$  such that  $b \setminus a$  is finite for all  $a \in X$ . Viale [9, Lemma 4.2] proved that under the assumption that  $\omega_1 < \mathfrak{p}$ , if  $\chi \geq \omega_2$  is a regular cardinal,  $N \in P_{\omega_2}(H(\chi))$ ,  $N \prec H(\chi)$ , and  $N$  is  $\omega_1$ -guessing, then  $N$  is internally unbounded.

Viale [9, Remark 4.3] asked whether it is consistent that there are stationarily many  $\omega_1$ -guessing models in a model where  $\mathfrak{p} = \omega_1$ . We point out that in the model constructed in [5], GMP holds and  $\mathfrak{p} = \omega_1$ , which settles this question. Namely, the model of [5] is obtained by forcing with a forcing poset of the form  $\mathbb{P} * \text{Add}(\omega, \lambda)$ , where  $\lambda > \omega_1$ . But by [3, Section 11.3], any model obtained by forcing with  $\text{Add}(\omega, \lambda)$ , where  $\lambda \geq \omega_1$ , satisfies  $\mathfrak{p} = \omega_1$ .

**3. Indestructibly guessing models and IGMP.** We introduce a stronger form of  $\omega_1$ -guessing, and with it a new principle.

**DEFINITION 3.1.** A set  $N$  is *indestructibly  $\omega_1$ -guessing* if for any forcing poset  $\mathbb{P}$  which preserves  $\omega_1$ ,  $\mathbb{P}$  forces that  $N$  is  $\omega_1$ -guessing.

**DEFINITION 3.2.** For a cardinal  $\theta \geq \omega_2$ , let  $\text{IGMP}(\theta)$  be the statement that there exist stationarily many sets  $N \in P_{\omega_2}(H(\theta))$  such that  $N$  is indestructibly  $\omega_1$ -guessing. Let **IGMP** be the statement that  $\text{IGMP}(\theta)$  holds for all cardinals  $\theta \geq \omega_2$  <sup>(2)</sup>.

It is easy to see that if  $\omega_2 \leq \theta_0 < \theta_1$  are cardinals,  $N \in P_{\omega_2}(H(\theta_1))$  is indestructibly  $\omega_1$ -guessing,  $N \prec H(\theta_1)$ , and  $\theta_0 \in N$ , then  $N \cap H(\theta_0)$  is indestructibly  $\omega_1$ -guessing. In particular,  $\text{IGMP}(\theta_1)$  implies  $\text{IGMP}(\theta_0)$ .

We will prove in Section 4 that PFA implies **IGMP** <sup>(3)</sup>.

It turns out that indestructibly  $\omega_1$ -guessing models are internally unbounded. The proof is a variation of the proof of [9, Lemma 4.2] that if  $\mathfrak{p} > \omega_1$  and  $N$  is  $\omega_1$ -guessing, then  $N$  is internally unbounded.

**PROPOSITION 3.3.** *Let  $\theta \geq \omega_2$  be a cardinal. Suppose that  $N \in P_{\omega_2}(H(\theta))$ ,  $N \prec H(\theta)$ ,  $\text{cf}(\sup(N \cap \theta)) = \omega_1$ , and  $N$  is indestructibly  $\omega_1$ -guessing. Then  $N$  is internally unbounded.*

*Proof.* Suppose for a contradiction that  $x$  is a countable subset of  $N$  which is not covered by any set in  $P_{\omega_1}(N) \cap N$ . Without loss of generality,

<sup>(2)</sup> IGMP stands for *indestructibly guessing model principle*.

<sup>(3)</sup> The original proof in [10, Section 4] that PFA implies  $\text{ISP}(\omega_2)$  implicitly shows that PFA implies **IGMP**, although they did not formulate this principle.

$x$  is a set of ordinals. Since  $\text{cf}(N \cap \text{On}) = \text{cf}(N \cap \theta) = \omega_1$ ,  $x$  is bounded in  $\text{sup}(N \cap \text{On})$ . Then easily  $F = \{x \setminus y : y \in P_{\omega_1}(N) \cap N\}$  is a collection of infinite subsets of  $x$  which is closed under finite intersections.

Let  $\mathbb{P}$  be the  $\omega_1$ -c.c. Mathias forcing for adding a pseudointersection to  $F$ , that is, a subset of  $x$  which is almost contained modulo finite in every member of  $F$  (see, for example, [3, Theorem 7.7]). Let  $G$  be a  $V$ -generic filter on  $\mathbb{P}$ . Let  $b$  be the pseudointersection given by  $G$ . Then  $b \notin V$ .

We claim that  $b$  is countably approximated by  $N$ . Let  $a$  be a countable set in  $N$ . Then  $a \in P_{\omega_1}(N) \cap N$ , so  $x \setminus a \in F$ . Hence  $b \setminus (x \setminus a) = b \cap a$  is finite. As  $b \cap a$  is a finite subset of  $N$ , it is in  $N$ . Since  $N$  is indestructibly  $\omega_1$ -guessing and  $\mathbb{P}$  is  $\omega_1$ -c.c.,  $N$  is  $\omega_1$ -guessing in  $V[G]$ . As  $b$  is a bounded subset of  $N \cap \text{On}$  which is countably approximated by  $N$ , we can fix  $e \in N$  such that  $e \cap N = b$ . Note that  $e$  is countable, for otherwise by elementarity  $e \cap N$  would be uncountable, contradicting the fact that  $e \cap N = b$  and  $b$  is countable. Therefore  $e \subseteq N$ , so  $e = e \cap N = b$ . But this is impossible since  $e \in V$  and  $b \notin V$ . ■

**COROLLARY 3.4.** *Let  $\theta \geq \omega_2$  be a cardinal. Then  $\text{IGMP}(\theta^+)$  implies that there are stationarily many  $N \in P_{\omega_2}(H(\theta))$  such that  $N$  is internally unbounded and indestructibly  $\omega_1$ -guessing.*

*Proof.* If  $N \in P_{\omega_2}(H(\theta^+))$ ,  $N \prec H(\theta^+)$ , and  $N$  is  $\omega_1$ -guessing, then  $\text{sup}(N \cap \theta)$  has cofinality  $\omega_1$  by Lemma 2.3. So if  $N$  is indestructibly  $\omega_1$ -guessing, then by the comments after Definition 3.2 and by Proposition 3.3,  $N \cap H(\theta)$  is an elementary substructure of  $H(\theta)$ , is indestructibly  $\omega_1$ -guessing, and is internally unbounded. ■

**COROLLARY 3.5.** *IGMP implies SCH.*

*Proof.* By [9, Theorem 7.9], SCH holds provided that for all regular  $\theta \geq \omega_2$ , there are stationarily many  $N \in P_{\omega_2}(H(\theta))$  such that  $N$  is internally unbounded and  $\omega_1$ -guessing. This statement holds under IGMP by Corollary 3.4. ■

Now we move towards proving that IGMP implies the Suslin hypothesis.

**THEOREM 3.6.** *Assume  $\text{IGMP}(\omega_2)$ . Let  $T$  be a tree with height and size  $\omega_1$ . Assume that  $\mathbb{P}$  is a forcing poset which preserves  $\omega_1$ . Then  $\mathbb{P}$  does not add any new branches of length  $\omega_1$  to  $T$ .*

*Proof.* Without loss of generality, assume that the underlying set of  $T$  is  $\omega_1$ . Then  $T \in H(\omega_2)$ . By  $\text{IGMP}(\omega_2)$ , we can fix  $N \in P_{\omega_2}(H(\omega_2))$  such that  $N \prec H(\omega_2)$ ,  $T \in N$ , and  $N$  is indestructibly  $\omega_1$ -guessing. By Lemma 2.4, every branch of  $T$  with length  $\omega_1$  in a generic extension by  $\mathbb{P}$  is in  $N$ , and hence in  $V$ . ■

Let us say that a tree  $T$  is *nontrivial* if (1) for all  $t \in T$ , there are incomparable  $u, v$  in  $T$  with  $t \leq_T u, v$ , and (2) for all  $t \in T$  and for all  $\alpha$  less than the height of  $T$ , there is  $u \in T$  with height at least  $\alpha$  such that  $t \leq_T u$ .

Suppose that  $T$  is a nontrivial tree. Define  $\mathbb{P}_T$  as the forcing poset whose conditions are nodes in  $T$ , ordered by  $t \leq_{\mathbb{P}_T} s$  iff  $s \leq_T t$ . The assumption of  $T$  being nontrivial implies that the forcing poset  $\mathbb{P}_T$  adds a branch of  $T$  which is not in the ground model with length equal to the height of  $T$ .

**THEOREM 3.7.** *IGMP( $\omega_2$ ) implies that for any nontrivial tree  $T$  with height and size  $\omega_1$ ,  $\mathbb{P}_T$  collapses  $\omega_1$ .*

*Proof.* Let  $G$  be a  $V$ -generic filter on  $\mathbb{P}_T$ . Then in  $V[G]$ ,  $G$  is a branch of  $T$  which is not in  $V$ . By Theorem 3.6,  $\omega_1^V$  is not equal to  $\omega_1^{V[G]}$ . ■

We note that the conclusion of Theorem 3.7 was previously known to follow from PFA (see [1, Section 7]). Namely, under PFA, every tree with height and size  $\omega_1$  is special (see Definition 4.1 below). And adding a new branch of length  $\omega_1$  to a special tree by forcing will collapse  $\omega_1$  (see Proposition 4.3 below).

Recall that if  $T$  is an  $\omega_1$ -Suslin tree, then  $T$  is a tree with height and size  $\omega_1$ , and  $\mathbb{P}_T$  is  $\omega_1$ -c.c. In particular,  $\mathbb{P}_T$  preserves  $\omega_1$ . Moreover, if there exists an  $\omega_1$ -Suslin tree, then there exists a nontrivial  $\omega_1$ -Suslin tree.

**COROLLARY 3.8.** *IGMP( $\omega_2$ ) implies that there does not exist an  $\omega_1$ -Suslin tree, so the Suslin hypothesis holds.*

*Proof.* Immediate from Theorem 3.7. ■

On the other hand, the Suslin hypothesis is consistent with the principle GMP. For example, the model of GMP constructed in [5] is a generic extension by a forcing poset of the form  $\mathbb{P} * \text{Add}(\omega, \lambda)$ . But Shelah [7] proved that Cohen forcing  $\text{Add}(\omega)$  adds an  $\omega_1$ -Suslin tree, so there exists an  $\omega_1$ -Suslin tree in this model.

**THEOREM 3.9.** *Assume that  $2^\omega \leq \omega_2$ , and there exist cofinally many sets in  $P_{\omega_2}(H(\omega_2))$  which are indestructibly  $\omega_1$ -guessing. Suppose that  $W$  is a generic extension of  $V$ , and  $W$  contains a subset of  $\omega_1$  which is not in  $V$ . Then either  $W$  contains a real which is not in  $V$ , or  $\omega_2^V$  is not a cardinal in  $W$ .*

*Proof.* Since  $2^\omega \leq \omega_2$ ,  $H(\omega_1)$  has size at most  $\omega_2$ . So we can inductively define a  $\subseteq$ -increasing sequence  $\langle N_i : i < \omega_2 \rangle$ , whose union contains  $H(\omega_1)$ , such that for each  $i < \omega_2$ ,  $N_i$  is in  $P_{\omega_2}(H(\omega_2))$ ,  $\omega_1 + 1 \subseteq N_i$ , and  $N_i$  is indestructibly  $\omega_1$ -guessing.

Suppose that  $W \setminus V$  does not contain a real, and we will show that  $\omega_2^V$  is not a cardinal in  $W$ . Since every subset of  $\omega$  in  $W$  is in  $V$ , it follows that  $\omega_1^V = \omega_1^W$ , and  $W \setminus V$  contains no bounded subset of  $\omega_1$ . Fix  $b$  which is a

subset of  $\omega_1$  in  $W \setminus V$ . Then every proper initial segment of  $b$  is in  $V$ , and hence in  $H(\omega_1)^V$ . So we can fix, for each  $\alpha < \omega_1$ , the least ordinal  $i_\alpha < \omega_2^V$  such that  $b \cap \alpha \in N_{i_\alpha}$ . Note that the sequence  $\langle i_\alpha : \alpha < \omega_1 \rangle$  is in  $W$ .

We claim that this sequence is unbounded in  $\omega_2^V$ , which implies that  $\omega_2^V$  is not a cardinal in  $W$ . Otherwise there is  $\delta < \omega_2^V$  such that  $i_\alpha < \delta$  for all  $\alpha < \omega_1$ . Since the sequence  $\langle N_\alpha : \alpha < \omega_2 \rangle$  is  $\subseteq$ -increasing, it follows that for all  $\alpha < \omega_1$ ,  $b \cap \alpha \in N_\delta$ . As  $\omega_1 + 1 \subseteq N_\delta$ ,  $b$  is a bounded subset of  $N_\delta \cap \text{On}$ .

For any countable set  $a \in N_\delta$ ,  $a \cap b = a \cap (b \cap \alpha)$  for some  $\alpha < \omega_1$ , and hence  $a \cap b \in N_\delta$ . So  $b$  is countably approximated by  $N_\delta$  in  $W$ . Since  $N_\delta$  is indestructibly  $\omega_1$ -guessing in  $V$ ,  $N_\delta$  is  $\omega_1$ -guessing in  $W$ . So there is  $e \in N_\delta$  such that  $b = e \cap N_\delta$ . But then  $e$  and  $N_\delta$  are in  $V$ , and hence  $b \in V$ , which is a contradiction. ■

We note that the conclusion of Theorem 3.9 was previously shown to follow from PFA by Todorćević [8, Theorem 2].

**4. Trees and guessing models.** In this section we review some ideas of Baumgartner and Viale–Weiss concerning trees and guessing models. The main result here is Corollary 4.5, which gives a sufficient condition under which IGMP holds. We also observe that PFA implies IGMP.

The next definition is due to Baumgartner [1, Section 7].

**DEFINITION 4.1.** Let  $T$  be a tree. We say that  $T$  is *special* if there exists a function  $f : T \rightarrow \omega$  such that whenever  $s, t$ , and  $u$  are in  $T$  and  $f(s) = f(t) = f(u)$  if  $s <_T t$  and  $s <_T u$ , then  $t$  and  $u$  are comparable in  $T$ .

Recall that for a tree  $T$ , a function  $g : T \rightarrow \omega$  is a *specializing function* if for all  $s, t \in T$ , if  $s <_T t$  then  $g(s) \neq g(t)$ . Clearly if  $T$  has a specializing function, then  $T$  has no branches of length  $\omega_1$ . Baumgartner [1, Theorem 7.3] proved that for a tree  $T$  of height  $\omega_1$  which has no branches of length  $\omega_1$ ,  $T$  is special in the sense of Definition 4.1 iff  $T$  has a specializing function.

The next theorem appears as Theorem 7.5 of [1].

**THEOREM 4.2.** *Assume that every tree of height and size  $\omega_1$  which has no branches of length  $\omega_1$  is special. Then every tree of height and size  $\omega_1$  which has at most  $\omega_1$  many branches of length  $\omega_1$  is special.*

**PROPOSITION 4.3.** *Suppose that  $T$  is a tree with height  $\omega_1$  which is special. Then whenever  $W$  is an outer model of  $V$  with the same  $\omega_1$ , any branch of  $T$  in  $W$  of length  $\omega_1$  is in  $V$ .*

*Proof.* The proof follows easily from ideas of Baumgartner [1, Section 7]. Fix a function  $f : T \rightarrow \omega$  as in Definition 4.1. Let  $W$  be an outer model with  $\omega_1^V = \omega_1^W$ , and suppose that  $b$  is a branch of  $T$  with length  $\omega_1$  in  $W$ . We will show that  $b \in V$ .

Since  $f \upharpoonright b$  is a function from a set of size  $\omega_1$  into  $\omega$ , we can fix  $n < \omega$  such that the set  $\{t \in b : f(t) = n\}$  has size  $\omega_1$ . Fix  $s$  in  $b$  such that  $f(s) = n$ . Then the set

$$X := \{t \in b : s <_T t, f(t) = n\}$$

is uncountable. But  $X$  is a subset of the set

$$Y := \{t \in T : s <_T t, f(t) = n\},$$

and hence  $Y$  is uncountable. Note that  $Y$  is in  $V$ .

The set  $Y$  is an uncountable chain. For if  $t$  and  $u$  are in  $Y$ , then  $s <_T t$ ,  $s <_T u$ , and  $f(s) = f(t) = f(u) = n$ . Since  $T$  is special, it follows that  $t$  and  $u$  are comparable. Let  $c$  be the downward closure of  $Y$ . Then  $c$  is a branch of  $T$  in  $V$  with length  $\omega_1$ , and  $c \in V$ . There are cofinally many nodes above  $s$  in  $b$  which take the value  $n$  under  $f$ , and any such node is in  $c$ . So  $b = c$ , and therefore  $b \in V$ . ■

We now establish a connection between special trees and indestructibly  $\omega_1$ -guessing models. This connection involves constructing a tree from a guessing model; a similar construction was done previously [10, Lemma 4.6].

**PROPOSITION 4.4.** *Let  $\theta \geq \omega_2$  be a cardinal. Suppose that  $N$  is in  $P_{\omega_2}(H(\theta))$ ,  $N \prec H(\theta)$ , and  $N$  is internally unbounded and  $\omega_1$ -guessing. Then there exists a tree  $T$  of height and size  $\omega_1$  which has  $\omega_1$  many branches of length  $\omega_1$  such that  $T$  being special implies that  $N$  is indestructibly  $\omega_1$ -guessing.*

*Proof.* Since  $N$  is internally unbounded, we can fix a  $\subseteq$ -increasing sequence  $\langle N_i : i < \omega_1 \rangle$  with union equal to  $N$  such that for all  $i < \omega_1$ ,  $N_i \in P_{\omega_1}(N) \cap N$ .

Fix an uncountable ordinal  $\delta$  in  $N$ , and we will define a tree  $T_\delta$ . The desired tree  $T$  will then be the direct sum over all such trees  $T_\delta$ . The underlying set of  $T_\delta$  consists of all pairs in  $N$  of the form  $(i, f)$ , where  $i < \omega_1$  and  $f : N_i \cap \delta \rightarrow 2$ . For  $(i, f)$  and  $(j, g)$  in  $T_\delta$ , let  $(i, f) <_{T_\delta} (j, g)$  if  $i < j$  and  $f \subseteq g$ . Note that  $T_\delta$  is a tree with height and size  $\omega_1$ .

We claim that  $T_\delta$  has at most  $\omega_1$  many branches of length  $\omega_1$ . Consider a branch  $b$  of length  $\omega_1$ , and let  $F_b := \bigcup \{f : \exists i (i, f) \in b\}$ . Note that  $F_b$  is a function with domain equal to  $N \cap \delta$ , and  $b = \{(i, F_b \upharpoonright N_i) : i < \omega_1\}$ . Since  $b \subseteq T_\delta \subseteq N$ , we see that  $F_b \upharpoonright N_i \in N$  for all  $i < \omega_1$ .

Let  $A_b := \{\alpha \in N \cap \delta : F_b(\alpha) = 1\}$ . We claim that  $A_b$  is  $N$ -guessed. As  $N$  is  $\omega_1$ -guessing, it is enough to show that  $A_b$  is countably approximated by  $N$ . So let  $a$  be a countable set in  $N$ . Then  $a \subseteq N_i$  for some  $i < \omega_1$ . Therefore  $a \cap A_b = a \cap A_b \cap N_i$ . Now  $A_b \cap N_i = \{\alpha \in N_i \cap \delta : F_b(\alpha) = 1\}$ , which is definable in  $N$  from the parameters  $N_i$ ,  $\delta$ , and  $F_b \upharpoonright N_i$ . It follows that  $A_b \cap N_i$  is in  $N$ . Since  $a$  is also in  $N$ ,  $a \cap A_b = a \cap A_b \cap N_i$  is in  $N$ . Since  $N$  is  $\omega_1$ -guessing, we can fix  $e_b$  in  $N$  such that  $A_b = e_b \cap N$ .

Suppose that  $b$  and  $c$  are distinct branches of  $T$  with length  $\omega_1$ . Then clearly the functions  $F_b$  and  $F_c$  are different, and therefore the sets  $A_b$  and  $A_c$  are distinct. Since  $A_b = e_b \cap N$  and  $A_c = e_c \cap N$ , it follows that  $e_b$  and  $e_c$  are distinct. Thus the map  $b \mapsto e_b$  from the set of branches of  $T_\delta$  with length  $\omega_1$  into  $N$  is injective. Since  $N$  has size  $\omega_1$ , it follows that  $T_\delta$  has no more than  $\omega_1$  many branches of length  $\omega_1$ . This completes the analysis of  $T_\delta$ .

Let  $T$  be the disjoint sum of the trees  $T_\delta$  for all uncountable ordinals  $\delta$  in  $N$ . In other words, the underlying set of  $T$  consists of pairs of the form  $(\delta, t)$ , where  $\delta$  is an uncountable ordinal in  $N$  and  $t \in T_\delta$ . And the order on  $T$  is given by letting  $(\delta_1, t_1) <_T (\delta_2, t_2)$  iff  $\delta_1 = \delta_2$  and  $t_1 <_{T_\delta} t_2$ . Then  $T$  is a tree of height and size  $\omega_1$  which has at most  $\omega_1$  many branches of length  $\omega_1$ .

Suppose that  $T$  is special, and we will show that  $N$  is indestructibly  $\omega_1$ -guessing. Let  $W$  be an outer model of  $V$  with  $\omega_1^V = \omega_1^W$ . Assume that  $d$  is a bounded subset of  $N \cap \text{On}$  in  $W$  which is countably approximated by  $N$ . We will show that  $d$  is  $N$ -guessed. Fix an uncountable ordinal  $\delta$  in  $N$  such that  $d \subseteq \delta$ .

Let  $h : N \cap \delta \rightarrow 2$  be the characteristic function of  $d$ , in other words,  $h(\alpha) = 1$  if  $\alpha \in d$ , and  $h(\alpha) = 0$  otherwise. We claim that for all  $i < \omega_1$ ,  $(i, h \upharpoonright N_i)$  is in  $T_\delta$ . It suffices to show that  $h \upharpoonright N_i$  is in  $N$ . Since  $N_i \in N$ ,  $N_i$  is countable, and  $d$  is countably approximated by  $N$ , it follows that  $d \cap N_i \in N$ . But  $h \upharpoonright N_i$  is the characteristic function of  $d \cap N_i$ , so  $h \upharpoonright N_i \in N$ . Hence  $(i, h \upharpoonright N_i)$  is in  $T_\delta$ .

Consequently,  $b := \{(i, h \upharpoonright N_i) : i < \omega_1\}$  is a branch of  $T_\delta$ , and hence of  $T$ , with length  $\omega_1$ . Since  $T$  is special, any branch of  $T$  in  $W$  with length  $\omega_1$  is in  $V$  by Proposition 4.3. It follows that  $b \in V$ . Therefore  $h \in V$ , and hence  $d \in V$ . Since  $d$  is countably approximated by  $N$  in  $W$ , it is also countably approximated by  $N$  in  $V$ . As  $N$  is  $\omega_1$ -guessing in  $V$ ,  $d$  is  $N$ -guessed. ■

**COROLLARY 4.5.** *Suppose that every tree of height and size  $\omega_1$  which has no branches of length  $\omega_1$  is special. Assume that for all sufficiently large regular cardinals  $\theta \geq \omega_2$ , there are stationarily many sets  $N$  in  $P_{\omega_2}(H(\theta))$  such that  $N$  is internally unbounded and  $\omega_1$ -guessing. Then IGMP holds.*

*Proof.* By Theorem 4.2, every tree of height and size  $\omega_1$  which has at most  $\omega_1$  many branches of length  $\omega_1$  is special. By the comments after Definition 3.2, it suffices to show that for all sufficiently large regular cardinals  $\theta \geq \omega_2$ , IGMP( $\theta$ ) holds. By assumption, for all sufficiently large regular cardinals  $\theta \geq \omega_2$ , there are stationarily many  $N \in P_{\omega_2}(H(\theta))$  such that  $N \prec H(\theta)$ ,  $N$  is internally unbounded, and  $N$  is  $\omega_1$ -guessing. By Proposition 4.4, for any such  $N$  there exists a tree  $T$  with height and size  $\omega_1$  which has at most  $\omega_1$  many branches of length  $\omega_1$  such that if  $T$  is special then  $N$  is indestructibly  $\omega_1$ -guessing. By our assumption about trees,  $T$  is indeed special, so  $N$  is indestructibly  $\omega_1$ -guessing. ■

COROLLARY 4.6. PFA *implies* IGMP.

*Proof.* By [10, Section 4], PFA implies that for all regular cardinals  $\theta \geq \omega_2$ , there are stationarily many  $N$  in  $P_{\omega_2}(H(\lambda))$  which are internally unbounded and  $\omega_1$ -guessing. By [2], MA, and hence PFA, implies that every tree of height and size  $\omega_1$  which has no branches of length  $\omega_1$  is special. The result now follows from Corollary 4.5. ■

Corollary 3.4 and Proposition 4.4 suggest an alternative definition of indestructibly  $\omega_1$ -guessing. Let us call an internally unbounded  $\omega_1$ -guessing model a *special  $\omega_1$ -guessing model* if some tree as described in the proof of Proposition 4.4 is special. The argument of Proposition 4.4 shows that in that case,  $N$  is  $\omega_1$ -guessing in any outer model  $W$  with the same  $\omega_1$ . This conclusion about  $N$  is apparently stronger than being indestructibly  $\omega_1$ -guessing, since the latter property is restricted to outer models  $W$  which are generic extensions of  $V$ .

Thus we could formulate another principle which asserts that there exist stationarily many special  $\omega_1$ -guessing models, and this principle clearly implies IGMP. Note that by the proof of Corollary 4.6, PFA implies this principle. We do not know whether the two principles are equivalent, so we leave this as an open question. They are equivalent if IGMP implies that every tree of height and size  $\omega_1$  with at most  $\omega_1$  many branches is special, but that is not known.

**5. Strong genericity and the strongly proper collapse.** We now turn to developing the forcing posets which will be used in the consistency result of Section 8. In the present section we review the ideas of strong genericity and strong properness, prove a theorem about the preservation of strong properness by proper forcing, and discuss the strongly proper collapse. More details on these topics can be found in [5].

DEFINITION 5.1. Let  $\mathbb{Q}$  be a forcing poset,  $q \in \mathbb{Q}$ , and  $N$  a set. Then  $q$  is a *strongly  $(N, \mathbb{Q})$ -generic condition* if for any set  $D$  which is a dense subset of the forcing poset  $N \cap \mathbb{Q}$ ,  $D$  is predense in  $\mathbb{Q}$  below  $q$ .

If  $\mathbb{Q}$  is understood from context, we say that  $q$  is a *strongly  $N$ -generic condition*.

LEMMA 5.2 ([5, Lemma 2.2]). *Let  $\mathbb{Q}$  be a forcing poset,  $q \in \mathbb{Q}$ , and  $N$  a set. Then  $q$  is strongly  $N$ -generic iff there exists a function  $r \mapsto r \upharpoonright N$ , defined on conditions  $r \leq q$ , such that  $r \upharpoonright N \in N \cap \mathbb{Q}$ , and for all  $v \leq r \upharpoonright N$  in  $N \cap \mathbb{Q}$ ,  $r$  and  $v$  are compatible.*

For a forcing poset  $\mathbb{Q}$ , let  $\lambda_{\mathbb{Q}}$  denote the smallest cardinal  $\lambda$  such that  $\mathbb{Q} \subseteq H(\lambda)$ .



DEFINITION 5.3. A forcing poset  $\mathbb{Q}$  is *strongly proper on a stationary set* if there are stationarily many  $N$  in  $P_{\omega_1}(H(\lambda_{\mathbb{Q}}))$  such that whenever  $p \in N \cap \mathbb{Q}$ , there is  $q \leq p$  which is a strongly  $N$ -generic condition.

Standard arguments show that being strongly proper on a stationary set is equivalent to the property above, where we replace  $\lambda_{\mathbb{Q}}$  with any cardinal  $\theta \geq \lambda_{\mathbb{Q}}$ .

PROPOSITION 5.4. *If  $\mathbb{Q}$  is strongly proper on a stationary set, then  $\mathbb{Q}$  has the  $\omega_1$ -covering property and the  $\omega_1$ -approximation property.*

*Proof.* A condition which is strongly  $N$ -generic is also  $N$ -generic in the sense of proper forcing. By standard proper forcing arguments,  $\mathbb{Q}$  has the  $\omega_1$ -covering property. For a proof that  $\mathbb{Q}$  has the  $\omega_1$ -approximation property, see [5, comments after Proposition 2.13]. ■

THEOREM 5.5. *Suppose that  $\mathbb{Q}$  is strongly proper on a stationary set, and  $\mathbb{P}$  is proper. Then  $\mathbb{P}$  forces that  $\mathbb{Q}$  is strongly proper on a stationary set.*

*Proof.* Fix  $\theta$  such that  $\mathbb{P}$  forces that  $\theta$  is a cardinal and  $\theta \geq \lambda_{\mathbb{Q}}$ . Fix a  $\mathbb{P}$ -name  $\dot{F}$  for a function from  $(H(\theta)^{V[\dot{G}_{\mathbb{P}}]})^{<\omega}$  to  $H(\theta)^{V[\dot{G}_{\mathbb{P}}]}$ , and let  $p \in \mathbb{P}$ . We will find  $u \leq p$ , and a name  $\dot{M}$  for a countable subset of  $H(\theta)^{V[\dot{G}_{\mathbb{P}}]}$  which is closed under  $\dot{F}$ , such that  $u$  forces that for all  $s \in \dot{M} \cap \mathbb{Q}$ , there is  $t \leq s$  which is strongly  $(\dot{M}, \mathbb{Q})$ -generic.

Let  $\chi$  be a regular cardinal larger than  $2^{|\mathbb{P}|}$  such that  $\mathbb{P}$ ,  $\mathbb{Q}$ ,  $\theta$ , and  $\dot{F}$  are in  $H(\chi)$ . Since  $\mathbb{P}$  is proper and  $\mathbb{Q}$  is strongly proper on a stationary set, we can fix  $N$  in  $P_{\omega_1}(H(\chi))$  satisfying:

- (1)  $N \prec (H(\chi), \in, \mathbb{P}, p, \mathbb{Q}, \theta, \dot{F})$ ;
- (2) for all  $p_0 \in N \cap \mathbb{P}$ , there is  $q_0 \leq p_0$  which is  $(N, \mathbb{P})$ -generic;
- (3) for all  $s \in N \cap \mathbb{Q}$ , there is  $t \leq s$  which is strongly  $(N, \mathbb{Q})$ -generic.

Since  $p \in N \cap \mathbb{P}$ , by (2) we can fix  $q \leq p$  which is  $(N, \mathbb{P})$ -generic. We claim that  $q$  forces that

$$\dot{M} := N[\dot{G}_{\mathbb{P}}] \cap H(\theta)^{V[\dot{G}_{\mathbb{P}}]}$$

is as required.

Since  $q$  is  $(N, \mathbb{P})$ -generic,  $q$  forces that  $N[\dot{G}_{\mathbb{P}}] \cap V = N$ . By (1),  $\mathbb{P}$  forces that

$$N[\dot{G}_{\mathbb{P}}] \prec (H(\chi)^{V[\dot{G}_{\mathbb{P}}]}, \in, \dot{F}),$$

and therefore that  $N[\dot{G}_{\mathbb{P}}]$  is closed under  $\dot{F}$ . Hence  $\mathbb{P}$  forces that  $\dot{M}$  is closed under  $\dot{F}$ .

Let  $r \leq q$  and  $\dot{s}$  be given such that  $r$  forces in  $\mathbb{P}$  that  $\dot{s} \in \dot{M} \cap \mathbb{Q}$ . Then  $r$  forces that  $\dot{s} \in N[\dot{G}_{\mathbb{P}}] \cap \mathbb{Q} \subseteq N[\dot{G}_{\mathbb{P}}] \cap V = N$ . So we can fix  $u \leq r$  and  $s \in N$  such that  $u$  forces that  $\dot{s} = \check{s}$ .

By (3), let  $t \leq s$  be strongly  $(N, \mathbb{Q})$ -generic. Then there exists a function  $g : \{z \in \mathbb{Q} : z \leq t\} \rightarrow N \cap \mathbb{Q}$  such that for all  $z \leq t$  in  $\mathbb{Q}$ , if  $w \leq g(z)$  is in  $N \cap \mathbb{Q}$ , then  $w$  and  $z$  are compatible in  $\mathbb{Q}$ . Note that by upward absoluteness,  $g$  is forced to satisfy the same property in  $V[\dot{G}_{\mathbb{P}}]$ . But  $u$  forces that  $N \cap \mathbb{Q} = N[\dot{G}_{\mathbb{P}}] \cap \mathbb{Q} = \dot{M} \cap \mathbb{Q}$ . Therefore  $u$  forces that  $g : \{z \in \mathbb{Q} : z \leq t\} \rightarrow \dot{M} \cap \mathbb{Q}$  and for all  $z \leq t$  in  $\mathbb{Q}$ , whenever  $w \leq g(z)$  is in  $\dot{M} \cap \mathbb{Q}$ , then  $w$  and  $z$  are compatible in  $\mathbb{Q}$ . In other words,  $u$  forces that  $t$  is strongly  $(\dot{M}, \mathbb{Q})$ -generic. ■

Assume that  $\kappa$  is a strongly inaccessible cardinal. In the proof of the consistency result in Section 8, we will use the forcing poset  $\mathbb{P}$  from [5, Section 6], which is called a *strongly proper collapse*. The forcing poset  $\mathbb{P}$  is strongly proper,  $\kappa$ -c.c., has size  $\kappa$ , and collapses  $\kappa$  to become  $\omega_2$ . Roughly speaking, this forcing poset consists of finite adequate sets of countable elementary substructures, ordered by reverse inclusion. A more detailed description of this forcing poset is beyond the scope of this paper; see [5, Section 6] for more details.

We will need one more technical fact about the strongly proper collapse  $\mathbb{P}$ .

**PROPOSITION 5.6.** *Let  $\lambda \geq \kappa$  be a cardinal. Then  $\mathbb{P} \times \text{Add}(\omega, \lambda)$  is strongly proper. Moreover, if  $\mathbb{P}_0$  is any regular suborder of  $\mathbb{P} \times \text{Add}(\omega, \lambda)$ , then  $\mathbb{P}_0$  forces that  $(\mathbb{P} \times \text{Add}(\omega, \lambda))/\dot{G}_{\mathbb{P}_0}$  is strongly proper on a stationary set.*

*Proof.* This follows from [5, Theorem 2.11 and Proposition 7.3]. ■

**6. Special iterations.** To obtain a model in which IGMP holds, we will use a finite support iteration  $\mathbb{P}$  of forcings which specialize trees of height and size  $\omega_1$  which have no branches of length  $\omega_1$ . It was proven recently in [4] that such an iteration has the  $\omega_1$ -approximation property.

For our purposes, we will need to know that a certain quotient of such an iteration has the  $\omega_1$ -approximation property. Specifically, we will have an elementary substructure  $N$  of size  $\omega_1$ , and we will need to know that the regular suborder  $N \cap \mathbb{P}$  forces that  $\mathbb{P}/\dot{G}_{N \cap \mathbb{P}}$  has the  $\omega_1$ -approximation property. Unlike the situation in [5], we do not have a general result which implies that such a quotient has the  $\omega_1$ -approximation property. Instead, we will prove directly that  $\mathbb{P}/\dot{G}_{N \cap \mathbb{P}}$  is forced by  $N \cap \mathbb{P}$  to be forcing equivalent to a finite support iteration of specializing forcings.

Let  $\text{Fn}(\omega_1, \omega)$  denote the set of all finite functions whose domain is a subset of  $\omega_1$  and whose range is a subset of  $\omega$ . Recall that if  $T$  is a tree with no branches of length  $\omega_1$ , then  $P(T)$  is the forcing poset described in Section 1 for adding a specializing function to  $T$ .

DEFINITION 6.1. For an ordinal  $\lambda$ , let  $S(\lambda)$  denote the set of all functions  $p$ , whose domain is a finite subset of  $\lambda$ , such that  $p(\alpha) \in \text{Fn}(\omega_1, \omega)$  for all  $\alpha \in \text{dom}(p)$ . Define a partial order on  $S(\lambda)$  by letting  $q \leq p$  if  $\text{dom}(p) \subseteq \text{dom}(q)$ , and  $p(\alpha) \subseteq q(\alpha)$  for all  $\alpha \in \text{dom}(p)$ .

Note that if  $W$  is an outer model of  $V$  with  $\omega_1^V = \omega_1^W$ , then  $S(\lambda)^V = S(\lambda)^W$ , and the order on  $S(\lambda)$  is the same in  $V$  and  $W$ .

Instead of working directly with forcing iterations, we will use a simpler dense suborder.

DEFINITION 6.2. For an ordinal  $\lambda$  and a set  $A \subseteq \lambda$ , a sequence  $\langle \mathbb{P}_i : i \leq \lambda \rangle$  is said to be a *special  $A$ -iteration* if there exists a sequence  $\langle \dot{T}_i : i \in A \rangle$  such that the following statements are satisfied:

- (1) for all  $i \leq \lambda$ ,  $\mathbb{P}_i$  is a suborder of  $S(i)$ ;
- (2) for all  $i \in A$ ,  $\dot{T}_i$  is a  $\mathbb{P}_i$ -name for a tree with underlying set  $\omega_1$  which has no branches of length  $\omega_1$ ;
- (3)  $\mathbb{P}_0 = \{\emptyset\}$ ;
- (4) for all  $i \in A$ , a set  $p \in S(i+1)$  is in  $\mathbb{P}_{i+1}$  iff  $p \upharpoonright i \in \mathbb{P}_i$ , and if  $i \in \text{dom}(p)$  then  $p \upharpoonright i \Vdash_{\mathbb{P}_i} p(i) \in P(\dot{T}_i)$ ;
- (5)  $\mathbb{P}_{i+1} = \mathbb{P}_i$  for all  $i \in \lambda \setminus A$ ;
- (6) for all  $\beta \leq \lambda$  limit, a set  $p \in S(\beta)$  is in  $\mathbb{P}_\lambda$  iff  $p \upharpoonright i \in \mathbb{P}_i$  for all  $i < \beta$ .

Note that if  $p \in \mathbb{P}_\lambda$ , then  $\text{dom}(p) \subseteq A$ . If  $A = \lambda$ , then we say that the sequence is a *special iteration*. The partial ordering  $\mathbb{P}_\lambda$  itself is said to be a *special  $A$ -iteration*.

The next two lemmas provide some basic facts about special  $A$ -iterations.

LEMMA 6.3. *Let  $\langle \mathbb{P}_i : i \leq \lambda \rangle$  be a special  $A$ -iteration. Let  $i < j \leq \lambda$ . Then:*

- (1)  $\mathbb{P}_i \subseteq \mathbb{P}_j$ ;
- (2) if  $p \in \mathbb{P}_j$ , then  $p \upharpoonright i \in \mathbb{P}_i$ ;
- (3) if  $p \in \mathbb{P}_j$  and  $q \leq p \upharpoonright i$  in  $\mathbb{P}_i$ , then  $q \cup p \upharpoonright [i, j]$  is in  $\mathbb{P}_j$  and is below  $p$ ;
- (4) the function  $p \mapsto p \upharpoonright i$  is a projection mapping of  $\mathbb{P}_j$  onto  $\mathbb{P}_i$ , and this map satisfies  $p \upharpoonright i = p$  for all  $p \in \mathbb{P}_i$ , and  $q \leq q \upharpoonright i$  for all  $q \in \mathbb{P}_j$ ;
- (5)  $\mathbb{P}_i$  is a regular suborder of  $\mathbb{P}_j$ .

*Proof.* (1)–(3) can be easily proven by induction. (4) follows from (3), and (4) implies (5) by Lemma 1.5. ■

LEMMA 6.4. *Let  $\langle \mathbb{P}_i : i \leq \lambda \rangle$  be a special  $A$ -iteration, with sequence of names  $\langle \dot{T}_i : i \in A \rangle$ . Let  $p$  and  $q$  be conditions in  $\mathbb{P}_\lambda$  such that  $p(i) \subseteq q(i)$  for all  $i \in \text{dom}(p) \cap \text{dom}(q)$ . Define  $p + q$  as the function with domain  $\text{dom}(p) \cup \text{dom}(q)$  such that for all  $i \in \text{dom}(p + q)$ , if  $i \in \text{dom}(q)$ , then  $(p + q)(i) = q(i)$ , and if  $i \in \text{dom}(p) \setminus \text{dom}(q)$ , then  $(p + q)(i) = p(i)$ . Then  $p + q$  is in  $\mathbb{P}_\lambda$ , and  $p + q \leq p, q$ .*

*Proof.* Let  $r := p + q$ . It is clear that  $r$  is in  $S(\lambda)$ , and  $r \leq p, q$  in  $S(\lambda)$ . In fact, for all  $i \leq \lambda$ , we have  $r \restriction i \in S(i)$ , and  $r \restriction i \leq p \restriction i, q \restriction i$  in  $S(i)$ . To show that  $r \in \mathbb{P}_\lambda$ , we will prove by induction that  $r \restriction i \in \mathbb{P}_i$  for all  $i \leq \lambda$ .

For  $i = 0$ ,  $r \restriction 0 = \emptyset$  is in  $\mathbb{P}_0$  by Definition 6.2(3). If  $\beta \leq \lambda$  is a limit ordinal and  $r \restriction \gamma \in \mathbb{P}_\gamma$  for all  $\gamma < \beta$ , then  $r \in \mathbb{P}_\beta$  by Definition 6.2(6).

Suppose that  $i = i_0 + 1$  and  $r \restriction i_0 \in \mathbb{P}_{i_0}$ . Then  $(r \restriction i) \restriction i_0 = r \restriction i_0 \in \mathbb{P}_{i_0}$ . If  $i_0 \notin \text{dom}(r)$ , then  $r \restriction i = r \restriction i_0 \in \mathbb{P}_{i_0} \subseteq \mathbb{P}_i$ , and we are done. Suppose that  $i_0 \in \text{dom}(r)$ . Then  $i_0 \in A$ , and  $r(i_0) = s(i_0)$ , where  $s$  is either  $p$  or  $q$ . But  $r \restriction i_0 \leq s \restriction i_0$ , and  $s \restriction i_0 \Vdash_{\mathbb{P}_{i_0}} s(i_0) \in P(\dot{T}_{i_0})$ . Hence  $r \restriction i_0 \Vdash_{\mathbb{P}_{i_0}} r(i_0) = s(i_0) \in P(\dot{T}_{i_0})$ . So  $r \in \mathbb{P}_i$ . ■

The next lemma says that a special  $A$ -iteration is forcing equivalent to a finite support iteration of specializing forcings.

LEMMA 6.5. *Let  $\langle \mathbb{P}_i : i \leq \lambda \rangle$  be a special  $A$ -iteration, with sequence of names  $\langle \dot{T}_i : i \in A \rangle$ . Then there exists a finite support iteration*

$$\langle \mathbb{P}_i^*, \dot{Q}_j^* : i \leq \lambda, j < \lambda \rangle$$

*satisfying the following properties:*

- (1) *for all  $i < \lambda$ , the function which sends  $p \in \mathbb{P}_i$  to  $p^* \in \mathbb{P}_i^*$ , where  $\text{dom}(p^*) = \text{dom}(p)$  and for all  $\alpha \in \text{dom}(p^*)$ ,  $p^*(\alpha)$  is the canonical  $\mathbb{P}_\alpha^*$ -name for  $p(\alpha)$ , is an isomorphism from  $\mathbb{P}_i$  into a dense suborder of the separative quotient of  $\mathbb{P}_i^*$ ;*
- (2) *for all  $i \in A$ ,  $\mathbb{P}_i^*$  forces that  $\dot{Q}_i^* = P(\dot{T}_i^*)$ , where  $\dot{T}_i^*$  is the canonical translation of the  $\mathbb{P}_i$ -name  $\dot{T}_i$  to a  $\mathbb{P}_i^*$ -name using the isomorphism described in (1);*
- (3) *for all  $j \in \lambda \setminus A$ ,  $\mathbb{P}_j^*$  forces that  $\dot{Q}_j^* = \{\emptyset\}$  is the trivial forcing poset.*

*Proof.* This proof follows by standard arguments. ■

COROLLARY 6.6. *Let  $\mathbb{P}$  be a special  $A$ -iteration. Then  $\mathbb{P}$  is  $\omega_1$ -c.c. and has the  $\omega_1$ -approximation property.*

*Proof.* By Lemma 6.5,  $\mathbb{P}$  is forcing equivalent to a finite support iteration of forcings which specialize trees of height and size  $\omega_1$  which have no branches of length  $\omega_1$ . So  $\mathbb{P}$  is forcing equivalent to a finite support iteration of  $Y$ -c.c. forcing posets, and hence is  $Y$ -c.c. But any  $Y$ -c.c. forcing poset is  $\omega_1$ -c.c. and has the  $\omega_1$ -approximation property. ■

PROPOSITION 6.7. *Let  $\lambda$  be a cardinal of cofinality at least  $2^{\omega_1}$ . Then there exists a special iteration  $\langle \mathbb{P}_i : i \leq \lambda \rangle$  which forces that every tree with underlying set  $\omega_1$  which has no branches of length  $\omega_1$  is special.*

*Proof.* We just give a sketch of the proof. The special iteration is constructed by induction. We only need to specify the names  $\dot{T}_j$  for  $j < \lambda$ ,

since the rest of the definition is determined by conditions (3)–(6) of Definition 6.2.

For  $j < \lambda$ ,  $\mathbb{P}_j$  will be a subset of  $S(j)$ , and hence will have size at most  $\omega_1 \cdot |j|$ , which is less than  $\lambda$ . Also,  $\mathbb{P}_j$  is  $\omega_1$ -c.c. So it is possible to enumerate all nice  $\mathbb{P}_j$ -names for trees with underlying set  $\omega_1$  with no branches of length  $\omega_1$  in order type less than or equal to  $\lambda$ . Using a bookkeeping function, we choose  $\dot{T}_j$  to be such a name which was enumerated at some stage less than or equal to  $j$ . The bookkeeping function will ensure that any name that is enumerated will eventually be chosen as  $\dot{T}_j$  for some  $j < \lambda$ .

Given a nice  $\mathbb{P}_\lambda$ -name for a tree with underlying set  $\omega_1$  with no branches of length  $\omega_1$ , since the cofinality of  $\lambda$  is greater than  $\omega_1$  and  $\mathbb{P}_\lambda$  is  $\omega_1$ -c.c., it is easy to see that the name is actually a  $\mathbb{P}_j$ -name for some  $j < \lambda$ . So at some stage earlier than  $\lambda$ , we forced with  $P(\dot{T}_j)$ , and specialized the tree  $\dot{T}_j$ . ■

**COROLLARY 6.8.** *Let  $\lambda$  be a cardinal with cofinality at least  $2^{\omega_1}$ . Then there exists a special iteration  $\langle \mathbb{P}_i : i \leq \lambda \rangle$  which forces that every tree with height and size  $\omega_1$  which has no branches of length  $\omega_1$  is special.*

*Proof.* If  $T$  is a tree with height and size  $\omega_1$ , then clearly  $T$  is isomorphic to a tree with underlying set  $\omega_1$ . Now apply Proposition 6.7. ■

We conclude this section by proving that a tail of a special iteration is forcing equivalent to a special iteration in an intermediate extension. The proof is elementary, but somewhat tedious; the reader should not feel guilty in just accepting the statement of Proposition 6.9 and skipping the proof.

Fix a special  $A$ -iteration  $\langle \mathbb{P}_i : i \leq \lambda \rangle$  with sequence of names  $\langle \dot{T}_i : i \in A \rangle$ . Let  $i < j \leq \lambda$ , and suppose that  $G_i$  is a  $V$ -generic filter on  $\mathbb{P}_i$ . By Lemma 1.1, if  $G_{i,j}$  is a  $V[G_i]$ -generic filter on  $\mathbb{P}_j/G_i$ , then  $G_{i,j}$  is a  $V$ -generic filter on  $\mathbb{P}_j$ ,  $G_{i,j} \cap \mathbb{P}_i = G_i$ , and  $V[G_i][G_{i,j}] = V[G_{i,j}]$ . In particular, if  $j \in A$ , then in  $V[G_i][G_{i,j}] = V[G_{i,j}]$ ,  $\dot{T}_j^{G_{i,j}}$  is a tree with underlying set  $\omega_1$  which has no branches of length  $\omega_1$ . In this situation, we will write  $\dot{T}_{i,j}$  for a  $\mathbb{P}_j/G_i$ -name in  $V[G_i]$  for this tree.

**PROPOSITION 6.9.** *Let  $i < \lambda$ , and let  $G_i$  be a  $V$ -generic filter on  $\mathbb{P}_i$ . Then in  $V[G_i]$ , there exists a special  $(A \setminus i)$ -iteration  $\langle \mathbb{P}'_j : j \leq \lambda \rangle$ , with sequence of names  $\langle \dot{T}'_j : j \in A \setminus i \rangle$ , satisfying the following properties:*

- (1) *for each  $j \leq \lambda$ , the map  $p \mapsto p \upharpoonright (A \setminus i)$  is a dense embedding of  $\mathbb{P}_j/G_i$  into  $\mathbb{P}'_j$ ;*
- (2) *for each  $j \in A \setminus i$ ,  $\dot{T}'_j$  is the canonical translation of the name  $\dot{T}_{i,j}$  to a  $\mathbb{P}'_j$ -name using the dense embedding from (1).*

*Proof.* First consider  $j \leq i$ . Then  $\mathbb{P}'_j$  is the trivial poset  $\{\emptyset\}$ , and  $\mathbb{P}_j/G_i$  is equal to  $G_i \cap \mathbb{P}_j$ . Using the fact that  $G_i \cap \mathbb{P}_j$  is a filter to show preservation

of incompatibility, we easily see the map  $p \mapsto p \upharpoonright (A \setminus i) = \emptyset$  is a dense embedding of  $\mathbb{P}_j/G_i = G_j \cap \mathbb{P}_j$  into  $\mathbb{P}'_j = \{\emptyset\}$ .

Suppose that  $i \leq \beta \leq \lambda$  is a limit ordinal, and for all  $\gamma < \beta$ , the map  $p \mapsto p \upharpoonright (A \setminus i)$  is a dense embedding of  $\mathbb{P}_\gamma/G_i$  into  $\mathbb{P}'_\gamma$ . Using the fact that  $\mathbb{P}_\beta/G_i = \bigcup\{\mathbb{P}_\gamma/G_i : \gamma < \beta\}$  and  $\mathbb{P}'_\beta = \bigcup\{\mathbb{P}'_\gamma : \gamma < \beta\}$ , it is easy to check that the same is true of  $\mathbb{P}_\beta/G_i$  and  $\mathbb{P}'_\beta$ .

Finally, assume that  $i < j < \lambda$ , and the map  $p \mapsto p \upharpoonright (A \setminus i)$  is a dense embedding of  $\mathbb{P}_j/G_i$  into  $\mathbb{P}'_j$ . We will prove that the same is true for  $\mathbb{P}_{j+1}/G_i$  and  $\mathbb{P}'_{j+1}$ . First assume that  $j \notin A$ . Then  $\mathbb{P}_{j+1} = \mathbb{P}_j$ , so  $\mathbb{P}_{j+1}/G_i = \mathbb{P}_j/G_i$ . Also,  $\mathbb{P}'_{j+1} = \mathbb{P}'_j$ . So we are done by the inductive hypothesis.

Suppose that  $j \in A$ . In  $V[G_i]$ ,  $\dot{T}_{i,j}$  is a  $\mathbb{P}_j/G_i$ -name for a tree with underlying set  $\omega_1$  which has no branches of length  $\omega_1$ . Since  $p \mapsto p \upharpoonright (A \setminus i)$  is a dense embedding of  $\mathbb{P}_j/G_i$  into  $\mathbb{P}'_j$ , the name  $\dot{T}_{i,j}$  translates under this dense embedding to a  $\mathbb{P}'_j$ -name  $\dot{T}'_j$  for the same tree.

Assume that  $q \in \mathbb{P}_{j+1}/G_i$ , and we will show that  $q \upharpoonright (A \setminus i)$  is in  $\mathbb{P}'_{j+1}$ . This follows from the inductive hypothesis if  $j \notin \text{dom}(q)$ , so suppose that  $j \in \text{dom}(q)$ . Since  $q \upharpoonright j \in \mathbb{P}_j/G_i$ , by the inductive hypothesis  $(q \upharpoonright j) \upharpoonright (A \setminus i) = (q \upharpoonright (A \setminus i)) \upharpoonright j$  is in  $\mathbb{P}'_j$ . As  $q \in \mathbb{P}_{j+1}$ , we have  $q \upharpoonright j \Vdash_{\mathbb{P}_j}^V q(j) \in P(\dot{T}_j)$ . By the choice of  $\dot{T}_{i,j}$ ,  $q \upharpoonright j \Vdash_{\mathbb{P}_j/G_i}^{V[G_i]} q(j) \in P(\dot{T}_{i,j})$ . Hence  $(q \upharpoonright (A \setminus i)) \upharpoonright j \Vdash_{\mathbb{P}'_j}^{V[G_i]} q(j) \in P(\dot{T}'_j)$ . So  $q \upharpoonright (A \setminus i) \in \mathbb{P}'_{j+1}$ .

Let  $p \in \mathbb{P}'_{j+1}$ . We will find a condition  $u$  in  $\mathbb{P}_{j+1}/G_i$  such that  $u \upharpoonright (A \setminus i) \leq p$ . If  $j \notin \text{dom}(p)$ , then  $p \in \mathbb{P}'_j$ . By the inductive hypothesis, there is  $q \in \mathbb{P}_j/G_i$  such that  $q \upharpoonright (A \setminus i) \leq p$ . Since  $\mathbb{P}_j \subseteq \mathbb{P}_{j+1}$  and  $q \upharpoonright i \in G_i$ , it follows that  $q \in \mathbb{P}_{j+1}/G_i$ , and we are done.

Suppose that  $j \in \text{dom}(p)$ . Let  $x := p(j)$ . By the inductive hypothesis, fix  $q \in \mathbb{P}_j/G_i$  such that  $q \upharpoonright (A \setminus i) \leq p \upharpoonright i$ . The proof would be finished if  $q \cup \{(j, x)\}$  was in  $\mathbb{P}_{j+1}/G_i$ , that is, if  $q \cup \{(j, x)\}$  was in  $\mathbb{P}_{j+1}$ . Unfortunately, we do not know that  $q$  forces over  $V$  that  $x$  is in  $P(\dot{T}_j)$ , so it could be the case that  $q \cup \{(j, x)\}$  is not in  $\mathbb{P}_{j+1}$ , and we have to work a bit harder.

Since  $q \upharpoonright (A \setminus i) \leq p \upharpoonright i$ , it follows that

$$q \upharpoonright (A \setminus i) \Vdash_{\mathbb{P}'_j}^{V[G_i]} x \in P(\dot{T}'_j).$$

We claim that

$$q \Vdash_{\mathbb{P}_j/G_i}^{V[G_i]} x \in P(\dot{T}_{i,j}).$$

Let  $G_{i,j}$  be a  $V[G_i]$ -generic filter on  $\mathbb{P}_j/G_i$  with  $q \in G_{i,j}$ . Let  $T := \dot{T}_{i,j}^{G_{i,j}}$ . We will prove that  $x \in P(T)$ . The image of  $G_{i,j}$  under the dense embedding  $s \mapsto s \upharpoonright (A \setminus i)$  generates a  $V[G_i]$ -generic filter  $G'_j$  on  $\mathbb{P}'_j$ . Since  $q \in G_{i,j}$ , we have  $q \upharpoonright (A \setminus i) \in G'_j$ . And since  $q \upharpoonright (A \setminus i) \leq p \upharpoonright j$ , it follows that  $p \upharpoonright j \in G'_j$ . As

$p \in \mathbb{P}'_{j+1}$ , by definition we have

$$p \upharpoonright j \Vdash_{\mathbb{P}'_j}^{V[G_i]} p(j) = x \in P(\dot{T}'_j).$$

Since  $p \in G'_j$ , it follows that  $x \in P((\dot{T}'_j)^{G'_j})$ . But by the choice of the name  $\dot{T}'_j$  we have  $(\dot{T}'_j)^{G'_j} = (\dot{T}_{i,j})^{G_{i,j}} = T$ . So  $x \in P(T)$ , proving the claim.

Fix a condition  $s \in G_i$  such that

$$s \Vdash_{\mathbb{P}_i}^V q \Vdash_{\mathbb{P}_j/\dot{G}_i}^{V[\dot{G}_i]} x \in P(\dot{T}_{i,j}).$$

Since  $q \upharpoonright i \in G_i$ , without loss of generality we may assume that  $s \leq q \upharpoonright i$ . It follows that  $t := s \cup (q \upharpoonright [i, \lambda))$  is in  $\mathbb{P}_j$ ,  $t \leq q$ , and since  $t \upharpoonright i = s \in G_i$ , we have  $t \in \mathbb{P}_j/G_i$ . Now the choice of  $s$  implies that

$$t \Vdash_{\mathbb{P}_j}^V p(j) \in \dot{T}_j,$$

as can be easily checked. So  $u := t \cup \{(j, x)\}$  is in  $\mathbb{P}_{j+1}/G_i$ , and  $u \upharpoonright (A \setminus i) \leq p$ .

It is obvious that the map  $p \mapsto p \upharpoonright (A \setminus i)$  preserves order. For the preservation of incompatibility, suppose that  $p$  and  $q$  are in  $\mathbb{P}_{j+1}/G_i$ , and  $r \leq p \upharpoonright (A \setminus i), q \upharpoonright (A \setminus i)$  in  $\mathbb{P}'_{j+1}$ . By what we just proved, there is  $r_0 \in \mathbb{P}_{j+1}/G_i$  such that  $r_0 \upharpoonright (A \setminus i) \leq r$  in  $\mathbb{P}'_{j+1}$ . Since  $G_i$  is a filter, without loss of generality we may assume that  $r_0 \upharpoonright i \leq p \upharpoonright i, q \upharpoonright i$ . Then clearly  $r_0 \leq p, q$  in  $\mathbb{P}_{j+1}/G_i$ . ■

**COROLLARY 6.10.** *Let  $\langle \mathbb{P}_i : i \leq \lambda \rangle$  be a special  $A$ -iteration. Fix  $i < \lambda$ , and let  $G_i$  be a  $V$ -generic filter on  $\mathbb{P}_i$ . Then in  $V[G_i]$ , for all  $j \leq \lambda$ ,  $\mathbb{P}_j/G_i$  is  $\omega_1$ -c.c. and has the  $\omega_1$ -approximation property.*

*Proof.* By Proposition 6.9,  $\mathbb{P}_j/G_i$  is forcing equivalent to a special  $(A \setminus i)$ -iteration. So we are done by Corollary 6.6. ■

**7. Factoring a special iteration over an elementary substructure.** In this section we will prove the following result.

**THEOREM 7.1.** *Let  $\vec{P} = \langle \mathbb{P}_i : i \leq \lambda \rangle$  be a special iteration, with sequence of names  $\vec{T} = \langle \dot{T}_i : i < \lambda \rangle$ . Let  $\theta \geq \omega_2$  be a regular cardinal such that  $\vec{P}$  and  $\vec{T}$  are in  $H(\theta)$ . Let  $N$  be an elementary substructure of  $H(\theta)$  such that  $N$  has size  $\omega_1$ ,  $\omega_1 \subseteq N$ , and  $\vec{P}$  and  $\vec{T}$  are in  $N$ . Then:*

- (1)  $N \cap \mathbb{P}_\lambda$  is a regular suborder of  $\mathbb{P}_\lambda$ ;
- (2)  $N \cap \mathbb{P}_\lambda$  forces that  $\mathbb{P}_\lambda/\dot{G}_{N \cap \mathbb{P}_\lambda}$  is forcing equivalent to a special  $(\lambda \setminus N)$ -iteration;
- (3)  $N \cap \mathbb{P}_\lambda$  forces that  $\mathbb{P}_\lambda/\dot{G}_{N \cap \mathbb{P}_\lambda}$  is  $\omega_1$ -c.c. and has the  $\omega_1$ -approximation property.

Note that (3) follows from (2) by Corollary 6.6.

The proof of Theorem 7.1(1) is given in Lemma 7.5(2). The rest of the section after that is devoted to proving Theorem 7.1(2). The latter is tedious; we suggest that the reader skip it on a first reading.

For the remainder of the section fix  $\vec{P}$ ,  $\vec{T}$ ,  $\theta$ , and  $N$  as described in Theorem 7.1.

LEMMA 7.2. *Let  $i < j \leq \lambda$ , where  $j \in N$ . Then:*

- (1) *the map  $p \mapsto p \upharpoonright i$  is a projection mapping from  $\mathbb{P}_j \cap N$  into  $\mathbb{P}_i \cap N$ ;*
- (2)  *$p \upharpoonright i = p$  for all  $p \in \mathbb{P}_i \cap N$ , and  $q \leq q \upharpoonright i$  for all  $q \in \mathbb{P}_j \cap N$ ;*
- (3)  *$\mathbb{P}_i \cap N$  is a regular suborder of  $\mathbb{P}_j \cap N$ .*

*Proof.* (2) follows from Lemma 6.3(4), and (3) follows from (1), (2), and Lemma 1.5. It remains to prove (1). Let  $i_N := \min((N \cap (\lambda + 1)) \setminus i)$ . By Lemma 6.3(4), the map  $p \mapsto p \upharpoonright i_N$  is a projection mapping of  $\mathbb{P}_j$  into  $\mathbb{P}_{i_N}$ . Using the elementarity of  $N$ , it easily follows that the same map restricted to  $\mathbb{P}_j \cap N$  is a projection mapping of  $\mathbb{P}_j \cap N$  into  $\mathbb{P}_{i_N} \cap N$ . But if  $q \in \mathbb{P}_{i_N}$ , then by elementarity,  $\text{dom}(q) \subseteq i_N \cap N \subseteq i$ , and hence  $q \in \mathbb{P}_i$ . It follows that  $\mathbb{P}_{i_N} \cap N = \mathbb{P}_i \cap N$ , and  $p \upharpoonright i = p \upharpoonright i_N$  for all  $p \in \mathbb{P}_j \cap N$ . So this map is a projection mapping of  $\mathbb{P}_j \cap N$  into  $\mathbb{P}_i \cap N$ . ■

DEFINITION 7.3. For each  $i \leq \lambda$ , let  $E_i$  denote the set of  $p \in \mathbb{P}_i$  such that  $p \upharpoonright N$ , the restriction of the function  $p$  to  $N \cap i$ , is in  $\mathbb{P}_i$ .

LEMMA 7.4. *Let  $i < j \leq \lambda$ . Then:*

- (1)  *$E_i \subseteq E_j$ ;*
- (2) *if  $p \in E_j$ , then  $p \upharpoonright i \in E_i$ .*

*Proof.* Let  $p \in E_i$ . Then  $p \in \mathbb{P}_i$  and  $p \upharpoonright N \in \mathbb{P}_i$ . Since  $\mathbb{P}_i \subseteq \mathbb{P}_j$  by Lemma 6.3(1), it follows that  $p \in \mathbb{P}_j$  and  $p \upharpoonright N \in \mathbb{P}_j$ . So  $p \in E_j$ , which proves (1). For (2), let  $p \in E_j$ . Then  $p \in \mathbb{P}_j$  and  $p \upharpoonright N \in \mathbb{P}_j$ . By Lemma 6.3(2),  $p \upharpoonright i \in \mathbb{P}_i$  and  $(p \upharpoonright i) \upharpoonright N = (p \upharpoonright N) \upharpoonright i \in \mathbb{P}_i$ . So  $p \upharpoonright i \in E_i$ . ■

We now prove Theorem 7.1(1). We also prove that  $E_i$  is dense in  $\mathbb{P}_i$ , since that fact follows by the same argument.

LEMMA 7.5. *Let  $i \leq \lambda$ . Then:*

- (1)  *$E_i$  is dense in  $\mathbb{P}_i$ ;*
- (2)  *$N \cap \mathbb{P}_i$  is a regular suborder of  $\mathbb{P}_i$ .*

*Proof.* Suppose  $p$  and  $q$  are in  $N \cap \mathbb{P}_i$ , and are compatible in  $\mathbb{P}_i$ . We will show that  $p$  and  $q$  are compatible in  $\mathbb{P}_i \cap N$ . Let  $i_N := \min((N \cap (\lambda + 1)) \setminus i)$ . Since  $\mathbb{P}_i$  is a regular suborder of  $\mathbb{P}_{i_N}$  by Lemma 6.3(5),  $p$  and  $q$  are compatible in  $\mathbb{P}_{i_N}$ . By elementarity, there is  $r \in N \cap \mathbb{P}_{i_N}$  such that  $r \leq p, q$ . Then  $\text{dom}(r) \subseteq N \cap i_N \subseteq i$ , so  $r \in \mathbb{P}_i$ . Hence  $r \in \mathbb{P}_i \cap N$  and  $r \leq p, q$ .



Let  $A$  be a maximal antichain in  $N \cap \mathbb{P}_i$ , and we will show that  $A$  is predense in  $\mathbb{P}_i$ . Let  $p \in \mathbb{P}_i$ . We will find  $r \in \mathbb{P}_i$  and  $s \in A$  such that  $r \leq s, p$ . Moreover, we will have  $r \upharpoonright N \in \mathbb{P}_i$ , and hence  $r \in E_i$ . This proves both that  $A$  is predense in  $\mathbb{P}_i$ , and that  $E_i$  is dense in  $\mathbb{P}_i$ .

Since  $\omega_1 \subseteq N$ ,  $\text{Fn}(\omega_1, \omega)$  is a subset of  $N$ . It follows that  $p \upharpoonright N$  is equal to  $p \cap N$ , which is in  $N$  since  $p \cap N$  is finite. However, we do not know that  $p \cap N$  is in  $\mathbb{P}_i$ . Since  $p \in \mathbb{P}_i \subseteq \mathbb{P}_{i_N}$ , by the elementarity of  $N$  we can fix  $p' \in \mathbb{P}_{i_N} \cap N$  such that  $p \upharpoonright N \subseteq p'$ .

Since  $A$  is a maximal antichain in  $\mathbb{P}_i \cap N$ , it is also a maximal antichain in  $\mathbb{P}_{i_N} \cap N$ . So we can fix  $s \in A$  and  $q \in \mathbb{P}_{i_N} \cap N$  such that  $q \leq p', s$ . Then by elementarity,  $\text{dom}(q) \subseteq i_N \cap N \subseteq i$ , so  $q \in \mathbb{P}_i \cap N$ . As  $\text{dom}(q) \subseteq N$  and  $\text{dom}(p \upharpoonright N) \subseteq \text{dom}(p') \subseteq \text{dom}(q)$ , we have  $\text{dom}(p) \cap \text{dom}(q) = \text{dom}(p \upharpoonright N)$ . And since  $p \upharpoonright N \subseteq p'$  and  $q \leq p'$ , it follows that  $p(i) \subseteq q(i)$  for all  $i \in \text{dom}(p \upharpoonright N)$ . By Lemma 6.4,  $r := p + q$  exists and  $r \leq p, q$ . Hence  $r \leq p, s$ . Moreover, by the definition of  $p + q$ , we have  $r \upharpoonright N = q \in \mathbb{P}_i$ , so  $r \in E_i$ . ■

NOTATION 7.6. Fix  $i \leq \lambda$ , and let  $G_{i,N}$  be a  $V$ -generic filter on  $\mathbb{P}_i \cap N$ . Let  $E_i^N$  denote the set  $E_i \cap (\mathbb{P}_i/G_{i,N})$ .

Since  $E_i$  is dense in  $\mathbb{P}_i$ ,  $E_i^N$  is dense in  $\mathbb{P}_i/G_{i,N}$  by Lemma 1.3.

LEMMA 7.7. Fix  $i \leq \lambda$ , and let  $G_{i,N}$  be a  $V$ -generic filter on  $\mathbb{P}_i \cap N$ . Then for all  $p \in E_i$ ,  $p \in E_i^N$  iff  $p \upharpoonright N \in G_{i,N}$ .

*Proof.* Using Lemma 6.4, it is easy to check that the map  $p \mapsto p \upharpoonright N$  is a projection mapping from  $E_i$  into  $E_i \cap N = \mathbb{P}_i \cap N$ , which satisfies  $p \upharpoonright N = p$  for all  $p \in E_i \cap N$ , and  $q \leq q \upharpoonright N$  for all  $q \in E_i$ . By Lemma 1.5, it follows that the set of  $p \in E_i$  such that  $p$  is compatible in  $E_i$  with every condition in  $G_{i,N}$  is equal to the set of  $p \in E_i$  such that  $p \upharpoonright N \in G_{i,N}$ . But a condition in  $E_i$  is compatible in  $E_i$  with every condition in  $G_{i,N}$  iff it is in  $E_i \cap (\mathbb{P}_i/G_{i,N}) = E_i^N$ . ■

Given a  $V$ -generic filter  $G_N$  on  $\mathbb{P}_\lambda \cap N$ , for each  $i \leq \lambda$ , let  $G_{i,N} := G_N \cap \mathbb{P}_i$ . Since  $\mathbb{P}_i \cap N$  is a regular suborder of  $\mathbb{P}_\lambda \cap N$ ,  $G_{i,N}$  is a  $V$ -generic filter on  $\mathbb{P}_i \cap N$ . Also,  $G_{i,N} \subseteq G_{j,N}$  for all  $i < j \leq \lambda$ , and  $s \upharpoonright i \in G_{i,N}$  for all  $s \in G_{j,N}$ .

LEMMA 7.8. Let  $i < j \leq \lambda$ . Then:

- (1)  $E_i^N \subseteq E_j^N$ ;
- (2) if  $p \in E_j^N$ , then  $p \upharpoonright i \in E_i^N$ .

*Proof.* (1) Let  $p \in E_i^N$ . Then  $p \in E_i$ , and by Lemma 7.7,  $p \upharpoonright N \in G_{i,N}$ . But  $E_i \subseteq E_j$  by Lemma 7.4(1), and  $G_{i,N} \subseteq G_{j,N}$ . So  $p \in E_j$  and  $p \upharpoonright N \in G_{j,N}$ . By Lemma 7.7,  $p \in E_j^N$ .

(2) Let  $p \in E_j^N$ . Then by Lemma 7.7,  $p \in E_j$  and  $p \upharpoonright N \in G_{j,N}$ . By Lemma 7.4(2),  $p \upharpoonright i \in E_i$ , and by the comments preceding the present lemma,  $(p \upharpoonright i) \upharpoonright N = (p \upharpoonright N) \upharpoonright i$  is in  $G_{i,N}$ . By Lemma 7.7,  $p \upharpoonright i \in E_i^N$ . ■

NOTATION 7.9. Let  $i \leq \lambda$ . In  $V[G_{i,N}]$ , define  $\mathbb{P}_i^N$  as the suborder of  $S(i)$  consisting of all functions of the form  $p \upharpoonright (i \setminus N)$ , where  $p \in E_i^N$ .

So  $\mathbb{P}_i^N$  consists of the functions in  $E_i^N$  with their fragment belonging to  $N$  removed.

LEMMA 7.10. *Let  $i < j \leq \lambda$ . Then:*

- (1)  $\mathbb{P}_i^N \subseteq \mathbb{P}_j^N$ ;
- (2) for all  $p \in \mathbb{P}_j^N$ ,  $p \upharpoonright i \in \mathbb{P}_i^N$ .

*Proof.* (1) Let  $p \in \mathbb{P}_i^N$ . Then there is  $p_0 \in E_i^N$  such that  $p = p_0 \upharpoonright (i \setminus N)$ . Since  $E_i^N \subseteq E_j^N$  by Lemma 7.8(1), we have  $p_0 \in E_j^N$ . Clearly  $p = p_0 \upharpoonright (j \setminus N)$ , so  $p \in \mathbb{P}_j^N$ .

(2) Let  $p \in \mathbb{P}_j^N$ . Then there is  $p_0 \in E_j^N$  such that  $p = p_0 \upharpoonright (j \setminus N)$ . By Lemma 7.8(2),  $p_0 \upharpoonright i \in E_i^N$ , and  $p \upharpoonright i = (p_0 \upharpoonright (j \setminus N)) \upharpoonright i = (p_0 \upharpoonright i) \upharpoonright (i \setminus N)$ . So  $p \upharpoonright i \in \mathbb{P}_i^N$ . ■

DEFINITION 7.11. Define  $\pi_i : E_i^N \rightarrow \mathbb{P}_i^N$  in  $V[G_{i,N}]$  by letting

$$\pi_i(p) = p \upharpoonright (i \setminus N)$$

for all  $p \in E_i^N$ .

LEMMA 7.12. *The function  $\pi_i$  is a dense embedding of  $E_i^N$  onto  $\mathbb{P}_i^N$ . Hence  $\mathbb{P}_i^N$  is forcing equivalent to  $\mathbb{P}_i/G_{i,N}$ .*

*Proof.* The second statement follows from the first together with the fact that  $E_i^N$  is dense in  $\mathbb{P}_i/G_{i,N}$ . The function  $\pi_i$  is surjective by the definition of  $\mathbb{P}_i^N$ , and clearly  $q \leq p$  in  $S(i)$  implies that  $\pi_i(q) \leq \pi_i(p)$  in  $S(i)$ . To see that  $\pi_i$  preserves incompatibility, suppose that  $p$  and  $q$  are in  $E_i^N$ , and  $r \leq \pi_i(p), \pi_i(q)$  in  $\mathbb{P}_i^N$ . We will show that  $p$  and  $q$  are compatible in  $E_i^N$ . Fix  $r_0 \in E_i^N$  such that  $r = r_0 \upharpoonright (i \setminus N)$ .

Since  $r_0, p$ , and  $q$  are in  $E_i^N$ , it follows that  $r_0 \upharpoonright N, p \upharpoonright N$ , and  $q \upharpoonright N$  are in  $G_{i,N}$  by Lemma 7.7. As  $G_{i,N}$  is a filter, we can fix  $s \in G_{i,N}$  with  $s \leq r_0 \upharpoonright N, p \upharpoonright N, q \upharpoonright N$ . Then  $\text{dom}(s) \cap \text{dom}(r_0) = \text{dom}(s)$ , and  $r_0(\gamma) \subseteq s(\gamma)$  for all  $\gamma \in \text{dom}(s)$ . By Lemma 6.4,  $r_0 + s$  is a condition in  $\mathbb{P}_i$  which is below  $r_0$  and  $s$ . Moreover,  $(r_0 + s) \upharpoonright N = s \in G_{i,N}$ , so  $r_0 + s \in E_i^N$ . Now  $(r_0 + s) \upharpoonright N = s \leq p \upharpoonright N, q \upharpoonright N$ , and  $(r_0 + s) \upharpoonright (i \setminus N) = r \leq \pi_i(p) = p \upharpoonright (i \setminus N), \pi_i(q) = q \upharpoonright (i \setminus N)$ . So  $r_0 + s \leq p, q$ . ■

Recall that for  $i \leq \lambda$ ,  $\mathbb{P}_i \cap N$  is a regular suborder of both  $\mathbb{P}_i$  and  $\mathbb{P}_\lambda \cap N$ , by Lemmas 7.2 and 7.5(2). So if  $G_i$  is a  $V$ -generic filter on  $\mathbb{P}_i$ , then  $G_i \cap N$  is a  $V$ -generic filter on  $\mathbb{P}_i \cap N$ . Therefore we can form the forcing poset  $(\mathbb{P}_\lambda \cap N)/(G_i \cap N)$ .

LEMMA 7.13. *Let  $i \in N \cap (\lambda + 1)$ . Then:*

- (1)  $\mathbb{P}_i$  forces that  $(\mathbb{P}_\lambda \cap N)/(\dot{G}_i \cap N)$  is equal to  $(\mathbb{P}_\lambda/\dot{G}_i) \cap N[G_i]$ ;
- (2)  $\mathbb{P}_i$  forces that  $(\mathbb{P}_\lambda \cap N)/(G_i \cap N)$  is  $\omega_1$ -c.c. and has the  $\omega_1$ -approximation property.

*Proof.* (1) Let  $G_i$  be a  $V$ -generic filter on  $\mathbb{P}_i$ . We will show that

$$(\mathbb{P}_\lambda \cap N)/(G_i \cap N) = (\mathbb{P}_\lambda/G_i) \cap N[G_i].$$

Let  $p \in (\mathbb{P}_\lambda \cap N)/(G_i \cap N)$ . Then by the definition of the quotient forcing,  $p \in \mathbb{P}_\lambda \cap N$  and  $p \upharpoonright i \in G_i \cap N$ . In particular,  $p \upharpoonright i \in G_i$ . So  $p \in \mathbb{P}_\lambda/G_i$ . Also, since  $p \in N$  and  $N \subseteq N[G_i]$ , it follows that  $p \in N[G_i]$ . Hence  $p \in (\mathbb{P}_\lambda/G_i) \cap N[G_i]$ .

Conversely, let  $p \in (\mathbb{P}_\lambda/G_i) \cap N[G_i]$ . By the definition of the quotient forcing,  $p \in \mathbb{P}_\lambda$  and  $p \upharpoonright i \in G_i$ . Since  $\mathbb{P}_i$  is  $\omega_1$ -c.c.,  $N[G_i] \cap V = N$  by standard arguments. Therefore  $p \in N[G_i] \cap \mathbb{P}_\lambda = N \cap \mathbb{P}_\lambda$ . So  $p \in \mathbb{P}_\lambda \cap N$ . Since  $p$  and  $i$  are in  $N$ , we have  $p \upharpoonright i \in N$ . So  $p \upharpoonright i \in G_i \cap N$ . Hence  $p \in (\mathbb{P}_\lambda \cap N)/(G_i \cap N)$ .

(2) By Proposition 6.9, there exists in  $V[G_i]$  a special  $(\lambda \setminus i)$ -iteration  $\mathbb{P}'_\lambda$  such that the map  $f(p) = p \upharpoonright (\lambda \setminus i)$  is a dense embedding of  $\mathbb{P}_\lambda/G_i$  into  $\mathbb{P}'_\lambda$ . By the elementarity of  $N[G_i]$ , we can choose  $\mathbb{P}'_\lambda$  to be in  $N[G_i]$ . Again by elementarity, it is easy to check that  $f \upharpoonright N[G_i]$  is a dense embedding of  $(\mathbb{P}_\lambda/G_i) \cap N[G_i]$  into  $\mathbb{P}'_\lambda \cap N[G_i]$ . Hence  $(\mathbb{P}_\lambda/G_i) \cap N[G_i]$  is forcing equivalent to  $\mathbb{P}'_\lambda \cap N[G_i]$ . By (1), it follows that  $(\mathbb{P}_\lambda \cap N)/(G_i \cap N)$  is forcing equivalent to  $\mathbb{P}'_\lambda \cap N[G_i]$ .

By Lemma 7.5(2) applied in  $V[G_i]$  to  $\mathbb{P}'_\lambda$  and  $N[G_i]$ ,  $\mathbb{P}'_\lambda \cap N[G_i]$  is a regular suborder of  $\mathbb{P}'_\lambda$ . But  $\mathbb{P}'_\lambda$  is a special  $(\lambda \setminus i)$ -iteration, and hence is  $\omega_1$ -c.c. and has the  $\omega_1$ -approximation property. Since  $\mathbb{P}'_\lambda \cap N[G_i]$  is a regular suborder of  $\mathbb{P}'_\lambda$ , it is also  $\omega_1$ -c.c., and by Lemma 1.8 it has the  $\omega_1$ -approximation property. Hence  $(\mathbb{P}_\lambda \cap N)/(G_i \cap N)$  is  $\omega_1$ -c.c. and has the  $\omega_1$ -approximation property. ■

LEMMA 7.14. *Let  $i \leq \lambda$ . Then  $\mathbb{P}_i$  forces that  $(\mathbb{P}_\lambda \cap N)/(\dot{G}_i \cap N)$  has the  $\omega_1$ -approximation property.*

*Proof.* Let  $G_i$  be a  $V$ -generic filter on  $\mathbb{P}_i$ , and let  $H$  be a  $V[G_i]$ -generic filter on  $(\mathbb{P}_\lambda \cap N)/(G_i \cap N)$ . We will show that the pair  $(V[G_i], V[G_i][H])$  has the  $\omega_1$ -approximation property.

Let  $j := \min((N \cap (\lambda + 1)) \setminus i)$ . Let  $G_{i,j}$  be a  $V[G_i][H]$ -generic filter on  $\mathbb{P}_j/G_i$ . By the product lemma and Lemma 1.1,

$$V[G_i][H][G_{i,j}] = V[G_i][G_{i,j}][H] = V[G_{i,j}][H],$$

and  $G_{i,j}$  is a  $V$ -generic filter on  $\mathbb{P}_j$  such that  $G_{i,j} \cap \mathbb{P}_i = G_i$ .

We claim that  $G_i \cap N = G_{i,j} \cap N$ . Since  $G_i \subseteq G_{i,j}$ , the forward inclusion is immediate. Conversely, let  $p \in G_{i,j} \cap N$ , and we will show that  $p \in G_i$ . Then  $p \in \mathbb{P}_j \cap N$ . By elementarity,  $\text{dom}(p) \subseteq N \cap j \subseteq i$ , so  $p \in G_{i,j} \cap \mathbb{P}_i = G_i$ . It follows from the claim that  $(\mathbb{P}_\lambda \cap N)/(G_i \cap N) = (\mathbb{P}_\lambda \cap N)/(G_{i,j} \cap N)$ .

Since  $j \in N$ , by Lemma 7.13 it follows that  $(\mathbb{P}_\lambda \cap N)/(G_{i,j} \cap N)$  has the  $\omega_1$ -c.c. and the  $\omega_1$ -approximation property in  $V[G_{i,j}]$ . Therefore the pair

$$(V[G_{i,j}], V[G_{i,j}][H])$$

has the  $\omega_1$ -covering property and the  $\omega_1$ -approximation property. That is, the pair

$$(V[G_i][G_{i,j}], V[G_i][G_{i,j}][H])$$

has the  $\omega_1$ -covering property and the  $\omega_1$ -approximation property. By Corollary 6.10, the forcing poset  $\mathbb{P}_j/G_i$  is  $\omega_1$ -c.c. and has the  $\omega_1$ -approximation property in  $V[G_i]$ . So the pair

$$(V[G_i], V[G_i][G_{i,j}])$$

has the  $\omega_1$ -covering property and the  $\omega_1$ -approximation property. By Lemma 1.7, it follows that the pair

$$(V[G_i], V[G_i][G_{i,j}][H])$$

has the  $\omega_1$ -approximation property. Since  $V[G_i] \subseteq V[G_i][H] \subseteq V[G_i][G_{i,j}][H]$ , by Lemma 1.8 the pair

$$(V[G_i], V[G_i][H])$$

has the  $\omega_1$ -approximation property. ■

For the remainder of the section, fix a  $V$ -generic filter  $G_N$  on  $\mathbb{P}_\lambda \cap N$ . Our goal is to prove that in  $V[G_N]$ ,  $\mathbb{P}_\lambda/G_N$  is forcing equivalent to a special  $(\lambda \setminus N)$ -iteration. For  $i \leq \lambda$ , let  $G_{i,N} := G_N \cap \mathbb{P}_i$ . Recall that  $E_i^N$  is dense in  $\mathbb{P}_i/G_{i,N}$ , and  $\pi_i : E_i^N \rightarrow \mathbb{P}_i^N$  is a dense embedding, where  $\pi_i(p) := p \upharpoonright (i \setminus N)$  for all  $p \in E_i^N$ . So it suffices to show that in  $V[G_N]$ ,  $\mathbb{P}_i^N$  is forcing equivalent to a special  $(i \setminus N)$ -iteration for all  $i \leq \lambda$ .

NOTATION 7.15. In  $V[G_{i,N}]$ , let  $\dot{G}_i$  be a  $\mathbb{P}_i^N$ -name for the  $V[G_{i,N}]$ -generic filter on  $\mathbb{P}_i/G_{i,N}$  generated by  $\pi_i^{-1}(\dot{G}_{\mathbb{P}_i^N})$ . Also in  $V[G_{i,N}]$ , let  $\dot{T}_i^N$  be a  $\mathbb{P}_i^N$ -name for  $(\dot{T}_i)^{\dot{G}_i}$ .

LEMMA 7.16. *The forcing poset  $\mathbb{P}_i^N$  forces over  $V[G_N]$  that  $\dot{T}_i^N$  is a tree with underlying set  $\omega_1$  which has no branches of length  $\omega_1$ .*

*Proof.* Let  $G_i^N$  be a  $V[G_N]$ -generic filter on  $\mathbb{P}_i^N$ . We will show that in  $V[G_N][G_i^N]$ ,  $(\dot{T}_i^N)^{G_i^N}$  is a tree with underlying set  $\omega_1$  which has no branches of length  $\omega_1$ . Let  $G_i$  denote the filter on  $\mathbb{P}_i/G_{i,N}$  generated by  $\pi_i^{-1}(G_i^N)$ . Since  $\pi_i$  is a dense embedding and  $E_i^N$  is dense in  $\mathbb{P}_i/G_{i,N}$ ,  $G_i$  is a  $V[G_N]$ -generic filter on  $\mathbb{P}_i/G_{i,N}$ . Moreover,

$$V[G_N][G_i^N] = V[G_N][G_i].$$

We claim that  $G_i \cap N = G_{i,N}$  and  $V[G_{i,N}][G_i] = V[G_i]$ . Since  $G_i$  is a  $V[G_N]$ -generic filter on  $\mathbb{P}_i/G_{i,N}$ , it is also a  $V[G_{i,N}]$ -generic filter on  $\mathbb{P}_i/G_{i,N}$ .

By Lemma 1.1,  $G_i$  is a  $V$ -generic filter on  $\mathbb{P}_i$ ,  $G_i \cap (\mathbb{P}_i \cap N) = G_{i,N}$ , and  $V[G_{i,N}][G_i] = V[G_i]$ . But  $G_i \cap (\mathbb{P}_i \cap N) = G_i \cap N$ , proving the claim.

Note that both of the forcing posets  $\mathbb{P}_i/G_{i,N}$  and  $(\mathbb{P}_\lambda \cap N)/G_{i,N}$  are in  $V[G_{i,N}]$ . By Lemma 1.1,

$$V[G_N] = V[G_{i,N}][G_N],$$

and  $G_N$  is a  $V[G_{i,N}]$ -generic filter on  $(\mathbb{P}_\lambda \cap N)/G_{i,N}$ . So  $G_i$  is a  $V[G_{i,N}][G_N]$ -generic filter on  $\mathbb{P}_i/G_{i,N}$ . By the product lemma,

$$V[G_N][G_i] = V[G_{i,N}][G_N][G_i] = V[G_{i,N}][G_i][G_N],$$

and  $G_N$  is a  $V[G_{i,N}][G_i]$ -generic filter on  $(\mathbb{P}_\lambda \cap N)/G_{i,N}$ .

By the above claim,  $G_{i,N} = G_i \cap N$  and  $V[G_{i,N}][G_i] = V[G_i]$ . So

$$V[G_N][G_i] = V[G_{i,N}][G_i][G_N] = V[G_i][G_N],$$

and  $G_N$  is a  $V[G_i]$ -generic filter on  $(\mathbb{P}_\lambda \cap N)/(G_i \cap N)$ . By Lemma 7.14, it follows that in  $V[G_i]$ ,  $(\mathbb{P}_\lambda \cap N)/(G_i \cap N)$  has the  $\omega_1$ -approximation property. Hence the pair  $(V[G_i], V[G_i][G_N])$  has the  $\omega_1$ -approximation property.

Recall that  $\dot{T}_i$  is a  $\mathbb{P}_i$ -name for a tree with underlying set  $\omega_1$  which has no branches of length  $\omega_1$ . Let  $T_i := \dot{T}_i^{G_i}$ . Then in  $V[G_i]$ ,  $T_i$  is a tree with underlying set  $\omega_1$  which has no branches of length  $\omega_1$ . By upward absoluteness, in  $V[G_N][G_i^N] = V[G_N][G_i] = V[G_i][G_N]$ ,  $T_i$  is a tree with underlying set  $\omega_1$ . Since the pair  $(V[G_i], V[G_i][G_N])$  has the  $\omega_1$ -approximation property,  $T_i$  has no branches of length  $\omega_1$  in  $V[G_i][G_N]$ . By Notation 7.15,  $(\dot{T}_i^N)^{G_i^N}$  is equal to  $\dot{T}_i^{G_i} = T_i$ , and this is a tree with underlying set  $\omega_1$  with no branches of length  $\omega_1$  in  $V[G_i][G_N] = V[G_N][G_i] = V[G_N][G_i^N]$ . ■

Let us return to proving that in  $V[G_N]$ ,  $\mathbb{P}_i^N$  is forcing equivalent to a special  $(i \setminus N)$ -iteration for all  $i \leq \lambda$ . In the model  $V[G_N]$ , consider the sequence

$$\langle \mathbb{P}'_i : i \leq \lambda \rangle$$

which is the special  $(\lambda \setminus N)$ -iteration defined from  $\langle \dot{T}_i^N : i \in \lambda \setminus N \rangle$ . In other words, the sequence is defined inductively using (3)–(6) of Definition 6.2. Of course, this only makes sense if for each  $i \in \lambda \setminus N$ , the name  $\dot{T}_i^N$  from Notation 7.15 is a  $\mathbb{P}'_i$ -name for a tree with underlying set  $\omega_1$  which has no branches of length  $\omega_1$ .

We will prove inductively that for each  $i \leq \lambda$ ,  $\mathbb{P}_i^N$  is a dense subset of  $\mathbb{P}'_i$ . Then for all  $i \in \lambda \setminus N$ ,  $\dot{T}_i^N$  literally is a  $\mathbb{P}'_i$ -name. And since  $\mathbb{P}_i^N$  forces over  $V[G_N]$  that  $\dot{T}_i^N$  is a tree with underlying set  $\omega_1$  which has no branches of length  $\omega_1$  by Lemma 7.16, so does  $\mathbb{P}'_i$ . So in the end,  $\mathbb{P}_\lambda^N$  is forcing equivalent to  $\mathbb{P}'_\lambda$ , which is a special  $(\lambda \setminus N)$ -iteration, completing the proof of Theorem 7.1.

It remains to prove the following lemma.

LEMMA 7.17. *For all  $i \leq \lambda$ ,  $\mathbb{P}_i^N$  is a dense subset of  $\mathbb{P}'_i$ .*

*Proof.* This is trivial for  $i = 0$  by Definition 6.2(3), and it follows easily from the inductive hypothesis when  $i$  is a limit ordinal by Definition 6.2(6).

Let  $i < \lambda$ , and suppose that  $\mathbb{P}_i^N$  is a dense subset of  $\mathbb{P}'_i$ . We will prove that  $\mathbb{P}_{i+1}^N$  is a dense subset of  $\mathbb{P}'_{i+1}$ . First, assume the easier case that  $i \in N$ . Then  $\mathbb{P}'_{i+1} = \mathbb{P}'_i$  by Definition 6.2(5). So it suffices to show that  $\mathbb{P}_{i+1}^N = \mathbb{P}_i^N$ . But  $\mathbb{P}_i^N \subseteq \mathbb{P}_{i+1}^N$  by Lemma 7.10(1). Conversely, let  $p \in \mathbb{P}_{i+1}^N$ , and we will show that  $p \in \mathbb{P}_i^N$ . Then  $p = p_0 \upharpoonright ((i+1) \setminus N)$  for some  $p_0 \in E_{i+1}^N$ . But  $i \in N$ , so  $(i+1) \setminus N = i \setminus N$ . Hence  $p = p_0 \upharpoonright ((i+1) \setminus N) = p_0 \upharpoonright (i \setminus N) = (p_0 \upharpoonright i) \upharpoonright (i \setminus N)$ . But  $p_0 \upharpoonright i \in E_i^N$  by Lemma 7.8(2), so  $p \in \mathbb{P}_i^N$ .

Secondly, assume that  $i \notin N$ . We will show that  $\mathbb{P}_{i+1}^N$  is dense in  $\mathbb{P}'_{i+1}$ . Let  $p \in \mathbb{P}'_{i+1}$ , and we will find a condition in  $\mathbb{P}_{i+1}^N$  which is below  $p$ . If  $i \notin \text{dom}(p)$ , then  $p = p \upharpoonright i \in \mathbb{P}'_i$ . By the inductive hypothesis, there is  $q \leq p$  in  $\mathbb{P}_i^N$ . Then  $q \in \mathbb{P}_{i+1}^N$  by Lemma 7.10(1), and  $q \leq p$ , so we are done.

Suppose that  $i \in \text{dom}(p)$ . Then by the definition of  $\mathbb{P}'_{i+1}$ ,  $p \upharpoonright i \in \mathbb{P}'_i$ , and

$$p \upharpoonright i \Vdash_{\mathbb{P}'_i}^{V[G_N]} p(i) \in P(\dot{T}_i^N).$$

Let  $x = p(i)$ . By the inductive hypothesis, we can fix  $q \leq p \upharpoonright i$  in  $\mathbb{P}_i^N$ . Then

$$q \Vdash_{\mathbb{P}'_i}^{V[G_N]} x \in P(\dot{T}_i^N).$$

Since  $\mathbb{P}'_i$  is a dense suborder of  $\mathbb{P}_i^N$  and  $\dot{T}_i^N$  is a  $\mathbb{P}_i^N$ -name, we have

$$q \Vdash_{\mathbb{P}_i^N}^{V[G_N]} x \in P(\dot{T}_i^N).$$

CLAIM 7.18.  $q \Vdash_{\mathbb{P}_i^N}^{V[G_{i,N}]} x \in P(\dot{T}_i^N)$ .

*Proof.* If not, then there is  $r \leq q$  in  $\mathbb{P}_i^N$  such that

$$r \Vdash_{\mathbb{P}_i^N}^{V[G_{i,N}]} x \notin P(\dot{T}_i^N).$$

Let  $K$  be a  $V[G_N]$ -generic filter on  $\mathbb{P}'_i$  with  $r \in K$ , and let  $T := (\dot{T}_i^N)^K$ . Since  $r \leq p \upharpoonright i$ , we have  $p \upharpoonright i \in K$ . As  $p \upharpoonright i \in K$ , in  $V[G_N][K]$  we know that  $p(i) = x \in P(T)$ . On the other hand,  $K \cap \mathbb{P}_i^N$  is a  $V[G_{i,N}]$ -generic filter on  $\mathbb{P}_i^N$ , and  $T = (\dot{T}_i^N)^{K \cap \mathbb{P}_i^N}$ . Since  $r \in K \cap \mathbb{P}_i^N$ , in  $V[G_{i,N}][K \cap \mathbb{P}_i^N]$  we have  $x \notin P(T)$ . But  $P(T)$  is the same in  $V[G_{i,N}][K \cap \mathbb{P}_i^N]$  and  $V[G_N][K]$ , which is a contradiction. ■

Since  $V[G_{i,N}]$  models that  $q$  forces in  $\mathbb{P}_i^N$  that  $x \in P(\dot{T}_i^N)$ , we can fix  $s \in G_{i,N}$  such that

$$s \Vdash_{\mathbb{P}_i \cap N}^V q \Vdash_{\mathbb{P}_i^N}^{V[\dot{G}_{i,N}]} x \in P(\dot{T}_i^N).$$

As  $q \in \mathbb{P}_i^N$ , fix  $q_0 \in E_i^N$  such that  $q = q_0 \upharpoonright (i \setminus N)$ . Then  $q_0 \in \mathbb{P}_i/G_{i,N}$ . By Lemma 1.4,  $q_0$  and  $s$  are compatible in  $\mathbb{P}_i/G_{i,N}$ , so fix  $r_0 \leq q_0, s$  in  $\mathbb{P}_i/G_{i,N}$ .

By extending  $r_0$  if necessary, we may assume that  $r_0 \in E_i^N$ . By Lemma 7.7,  $r_0 \upharpoonright N \in G_{i,N}$ , and since  $s \in N$  and  $r_0 \leq s$ , we have  $r_0 \upharpoonright N \leq s$ .

CLAIM 7.19.  $r_0 \Vdash_{\mathbb{P}_i}^V x \in P(\dot{T}_i)$ .

*Proof.* To prove this, let  $H_i$  be a  $V$ -generic filter on  $\mathbb{P}_i$  with  $r_0 \in H_i$ . We will show that  $x \in P((\dot{T}_i)^{H_i})$ . Let  $H_{i,N} := H_i \cap N$ , which is a  $V$ -generic filter on  $\mathbb{P}_i \cap N$  by Lemma 7.5(2). Since  $r_0 \leq s$  and  $s \in \mathbb{P}_i \cap N$ , it follows that  $s \in H_i \cap N = H_{i,N}$ . By Lemma 1.1,  $V[H_i] = V[H_{i,N}][H_i]$ , and  $H_i$  is a  $V[H_{i,N}]$ -generic filter on  $\mathbb{P}_i/H_{i,N}$ . Therefore  $\pi_i[H_i \cap E_i^N]$  generates a  $V[H_{i,N}]$ -generic filter  $H_i^N$  on  $\mathbb{P}_i^N$ , where  $E_i^N$  and  $\mathbb{P}_i^N$  are defined in  $V[H_{i,N}]$  from  $H_{i,N}$ , instead of from  $G_{i,N}$  as above. Moreover,  $V[H_i] = V[H_{i,N}][H_i] = V[H_{i,N}][H_i^N]$ . Since  $r_0 \in H_i$ , we have  $r_0 \upharpoonright (i \setminus N) \in H_i^N$ . Now  $r_0 \leq q_0$ , so  $r_0 \upharpoonright (i \setminus N) \leq q_0 \upharpoonright (i \setminus N) = q$ . Hence  $q \in H_i^N$ . Let  $T := (\dot{T}_i^N)^{H_i^N}$ .

Recall that

$$s \Vdash_{\mathbb{P}_i \cap N}^V q \Vdash_{\mathbb{P}_i^N}^{V[G_{i,N}]} x \in P(\dot{T}_i^N).$$

Since  $s \in H_{i,N}$  and  $q \in H_i^N$ , it follows that  $x \in P(T)$ . By Notation 7.15,  $T$  is equal to  $\dot{T}_i^L$ , where  $L$  is the filter on  $\mathbb{P}_i/H_{i,N}$  generated by  $\pi_i^{-1}(H_i^N)$ . By the definition of  $H_i^N$ , we know that  $\pi_i[H_i \cap E_i^N] \subseteq H_i^N$ , and therefore  $H_i \cap E_i^N \subseteq \pi_i^{-1}(H_i^N) \subseteq L$ . Since clearly  $H_i \cap E_i^N$  generates the filter  $H_i$ , we have  $H_i \subseteq L$ . As  $H_i$  and  $L$  are both  $V[H_{i,N}]$ -generic filters on  $\mathbb{P}_i/H_{i,N}$ , it follows that  $H_i = L$ . So  $T = (\dot{T}_i)^{H_i}$ . Consequently, in  $V[H_i]$ ,  $x$  is in  $P((\dot{T}_i)^{H_i})$ , which proves the claim. ■

Since  $r_0 \in \mathbb{P}_i$  and  $r_0 \Vdash_{\mathbb{P}_i}^V x \in P(\dot{T}_i)$ , by the definition of  $\mathbb{P}_{i+1}$  it follows that  $r_1 := r_0 \cup \{(i, x)\}$  is in  $\mathbb{P}_{i+1}$ . Moreover, since  $i \notin N$  and  $r_0 \in E_i^N$ ,

$$r_1 \upharpoonright N = r_0 \upharpoonright N \in G_{i,N} \subseteq G_{i+1,N}.$$

So  $r_1 \in E_{i+1}^N$ . Hence  $r := r_1 \upharpoonright ((i+1) \setminus N)$  is in  $\mathbb{P}_{i+1}^N$ . Since  $r_0 \leq q_0$ ,

$$r \upharpoonright i = r_0 \upharpoonright (i \setminus N) \leq q_0 \upharpoonright (i \setminus N) = q,$$

and so  $r \upharpoonright i \leq q \leq p \upharpoonright i$ . Since  $p(i) = x$ , we have  $r = (r \upharpoonright i) \cup \{(i, x)\} \leq p$ . ■

**8. IGMP and the continuum.** We now construct a model in which IGMP holds and  $2^\omega > \omega_2$ . We begin with a ground model  $V$  which satisfies GCH, in which  $\kappa$  is a supercompact cardinal, and  $\lambda \geq \kappa$  is a cardinal with cofinality at least  $\omega_2$ . We will define a forcing poset of the form

$$(\mathbb{P} \times \text{Add}(\omega, \lambda)) * \dot{\mathbb{Q}}$$

which forces  $\kappa = \omega_2$ ,  $2^\omega = \lambda$ , and IGMP.

Let  $\mathbb{P}$  be the strongly proper collapse of  $\kappa$  to become  $\omega_2$  which was discussed in Section 5. Then  $\mathbb{P}$  is strongly proper,  $\kappa$ -c.c., has size  $\kappa$ , and collapses  $\kappa$  to become  $\omega_2$ . Since GCH holds in the ground model, standard

arguments show that  $\mathbb{P}$  forces that  $2^\omega$  and  $2^{\omega_1}$  are bounded by  $\kappa$  (in fact, they are both equal to  $\kappa$ ).

Consider the product forcing  $\mathbb{P} \times \text{Add}(\omega, \lambda)$ . Note that the forcing poset  $\text{Add}(\omega, \lambda)$  is the same in  $V$  and  $V^\mathbb{P}$ . Since  $\mathbb{P}$  forces that  $\text{Add}(\omega, \lambda)$  is  $\omega_1$ -c.c., it follows that  $\mathbb{P} \times \text{Add}(\omega, \lambda)$  is  $\kappa$ -c.c., and has size  $\lambda$ . So by standard arguments,  $\mathbb{P} \times \text{Add}(\omega, \lambda)$  forces that  $2^\omega = 2^{\omega_1} = \lambda$ .

Applying Corollary 6.8, let  $\dot{Q}$  be a  $\mathbb{P} \times \text{Add}(\omega, \lambda)$ -name for a special iteration of length  $\lambda$  which forces that every tree of height and size  $\omega_1$  which has no branches of length  $\omega_1$  is special. Then  $\mathbb{P} \times \text{Add}(\omega, \lambda)$  forces that  $\dot{Q}$  is  $\omega_1$ -c.c., and has size  $\lambda$ . So the forcing poset

$$(\mathbb{P} \times \text{Add}(\omega, \lambda)) * \dot{Q}$$

is  $\kappa$ -c.c., and forces that  $2^\omega = 2^{\omega_1} = \lambda$ .

By Corollary 4.5, in order to prove that  $(\mathbb{P} \times \text{Add}(\omega, \lambda)) * \dot{Q}$  forces IGMP, it suffices to show that  $(\mathbb{P} \times \text{Add}(\omega, \lambda)) * \dot{Q}$  forces that for all sufficiently large regular cardinals  $\theta \geq \omega_2$ , there are stationarily many sets  $N$  in  $P_{\omega_2}(H(\theta))$  such that  $N$  is internally unbounded and  $\omega_1$ -guessing.

Fix a regular cardinal  $\theta \geq \kappa$  such that the forcing poset  $(\mathbb{P} \times \text{Add}(\omega, \lambda)) * \dot{Q}$  is a member of  $H(\theta)$ . Let  $\dot{F}$  be a  $(\mathbb{P} \times \text{Add}(\omega, \lambda)) * \dot{Q}$ -name for a function  $\dot{F} : (H(\theta))^{<\omega} \rightarrow H(\theta)$ . We will prove that  $(\mathbb{P} \times \text{Add}(\omega, \lambda)) * \dot{Q}$  forces that there exists a set  $N$  satisfying:

- (1)  $N$  is in  $P_\kappa(H(\theta))$ ;
- (2)  $N \prec H(\theta)$ ;
- (3)  $N$  is closed under  $\dot{F}$ ;
- (4)  $N$  is internally unbounded and  $\omega_1$ -guessing.

This will complete the proof <sup>(4)</sup>.

Let us abbreviate  $\text{Add}(\omega, \lambda)$  to  $\text{Add}$ .

NOTATION 8.1. Fix a  $V$ -generic filter  $G \times H$  on  $\mathbb{P} \times \text{Add}$ , and fix a  $V[G \times H]$ -generic filter  $I$  on  $\dot{Q} := (\dot{Q})^{G \times H}$ . Let  $F := (\dot{F})^{(G \times H) * I}$ .

We will prove that in  $V[(G \times H) * I]$ , there exists a set  $N$  in  $P_\kappa(H(\theta))$  such that  $N \prec H(\theta)$ ,  $N$  is closed under  $F$ , and  $N$  is internally unbounded and  $\omega_1$ -guessing.

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<sup>(4)</sup> The proof actually shows the existence of stationarily many  $N$  satisfying (1)–(4) which are internally stationary. The reason is that for all regular  $\omega_2 \leq \theta < \kappa$ ,  $\mathbb{P}$  forces that  $H(\theta)$  is internally stationary. In fact, we can strengthen this to internally club, by modifying the adequate set forcing  $\mathbb{P}$  by simultaneously adding  $\kappa$  many club subsets of  $\omega_1$  to ensure that for all regular  $\omega_2 \leq \theta < \kappa$ ,  $\mathbb{P}$  forces that  $H(\theta)$  is internally club. This modification of adequate set forcing is beyond the scope of the paper. Alternatively, this stronger result could also be obtained by replacing  $\mathbb{P}$  with the decorated sequence forcing of Neeman [6].



NOTATION 8.2. By the supercompactness of  $\kappa$ , fix in  $V$  an elementary embedding  $j : V \rightarrow M$  with critical point  $\kappa$  such that  $j(\kappa) > |H(\theta)|$  and  $M^{|H(\theta)|} \subseteq M$ .

Note that by the closure of  $M$ , we have  $H(\theta)^V = H(\theta)^M$  and  $j \upharpoonright H(\theta)^V \in M$ . Since  $H(\theta)^V = H(\theta)^M$  and  $\mathbb{P} \times \text{Add} \in H(\theta)^V$ , it follows that

$$H(\theta)^{V[G \times H]} = H(\theta)^V[G \times H] = H(\theta)^M[G \times H] = H(\theta)^{M[G \times H]},$$

and also

$$H(\theta)^{V[(G \times H) * I]} = H(\theta)^V[(G \times H) * I] = H(\theta)^M[(G \times H) * I] = H(\theta)^{M[(G \times H) * I]}.$$

Since the critical point of  $j$  is  $\kappa$  and  $\mathbb{P} \times \text{Add}$  is  $\kappa$ -c.c., it follows that  $j \upharpoonright (\mathbb{P} \times \text{Add})$  is a regular embedding of  $\mathbb{P} \times \text{Add}$  into  $j(\mathbb{P} \times \text{Add})$ , as proven in Section 1. Hence  $j \upharpoonright [\mathbb{P} \times \text{Add}]$  is a regular suborder of  $j(\mathbb{P} \times \text{Add})$ ,  $j \upharpoonright [G \times H]$  is a  $V$ -generic filter on  $j \upharpoonright [\mathbb{P} \times \text{Add}]$ , and  $V[G \times H] = V[j \upharpoonright [G \times H]]$ . So in  $V[G \times H]$ , we can form the forcing poset  $j(\mathbb{P} \times \text{Add})/j \upharpoonright [G \times H]$ .

NOTATION 8.3. Let  $J$  be a  $V[(G \times H) * I]$ -generic filter on the forcing poset  $j(\mathbb{P} \times \text{Add})/j \upharpoonright [G \times H]$ .

Then in particular,  $J$  is a  $V[G \times H]$ -generic filter on  $j(\mathbb{P} \times \text{Add})/j \upharpoonright [G \times H]$ . So by Lemma 1.1,  $J$  is a  $V$ -generic filter on  $j(\mathbb{P} \times \text{Add})$ ,  $J \cap j \upharpoonright [\mathbb{P} \times \text{Add}] = j \upharpoonright [G \times H]$ , and  $V[G \times H][J] = V[J]$ . It follows that  $J$  is an  $M$ -generic filter on  $j(\mathbb{P} \times \text{Add})$ , and  $j \upharpoonright [G \times H] \subseteq J$ . So in  $V[J]$ , standard arguments show that we can extend  $j$  to

$$j : V[G \times H] \rightarrow M[J]$$

by letting

$$j(\dot{a}^{G \times H}) := j(\dot{a})^J$$

for any  $\mathbb{P} \times \text{Add}$ -name  $\dot{a}$  in  $V$ .

As noted above,  $j \upharpoonright H(\theta)^V \in M$ . Since  $H(\theta)^{V[G \times H]} = H(\theta)^V[G \times H]$ , the definition of  $j$  above shows that  $j \upharpoonright H(\theta)^{V[G \times H]}$  is definable in  $M[J]$  from  $H(\theta)^V$ ,  $j \upharpoonright H(\theta)^V$ ,  $G \times H$ , and  $J$ . Hence  $j \upharpoonright H(\theta)^{V[G \times H]}$  is in  $M[J]$ .

Since  $J$  is a  $V[G \times H][I]$ -generic filter on  $j(\mathbb{P} \times \text{Add})/j \upharpoonright [G \times H]$ , by the product lemma,  $I$  is a  $V[G \times H][J]$ -generic filter on  $\mathbb{Q}$ . But  $V[G \times H][J] = V[J]$ , so  $I$  is a  $V[J]$ -generic filter on  $\mathbb{Q}$ . Hence  $I$  is an  $M[J]$ -generic filter on  $\mathbb{Q}$ . Also by the product lemma,

$$V[J][I] = V[G \times H][J][I] = V[G \times H][I][J].$$

Since  $j \upharpoonright \mathbb{Q}$  is an isomorphism in  $M[J]$  from  $\mathbb{Q}$  to  $j \upharpoonright [\mathbb{Q}]$ ,  $j \upharpoonright [I]$  is an  $M[J]$ -generic filter on  $j \upharpoonright [\mathbb{Q}]$ .

As  $J$  is a  $V[G \times H]$ -generic filter on  $j(\mathbb{P} \times \text{Add})/j \upharpoonright [G \times H]$ , it is also an  $M[G \times H]$ -generic filter on  $j(\mathbb{P} \times \text{Add})/j \upharpoonright [G \times H]$ . So by Lemma 1.1,  $M[G \times H][J] = M[J]$ . By the product lemma,

$$M[J][I] = M[G \times H][J][I] = M[G \times H][I][J].$$

Let

$$N_0 := j[H(\theta)^{V[G \times H]}].$$

Note that since  $j \upharpoonright H(\theta)^{V[G \times H]}$  is in  $M[J]$ ,  $N_0$  is in  $M[J]$ . We would like to apply Theorem 7.1 in  $M[J]$  to the special iteration  $j(\mathbb{Q})$  and the model  $N_0$ . Now  $N_0$  has the same cardinality as  $H(\theta)^{V[G \times H]}$ , which in turn has the same cardinality as  $H(\theta)^V$ . But  $|H(\theta)|^V < j(\kappa)$ , and in  $M[J]$ ,  $j(\kappa)$  is equal to  $\omega_2$  and  $\omega_1^M$  is preserved. It follows that in  $M[J]$ ,  $N_0$  has size  $\omega_1$ . Also clearly  $\omega_1 \subseteq N_0$ .

We claim that  $N_0$  is an elementary substructure of  $H(j(\theta))^{M[J]}$ . Let  $F^*$  be a Skolem function for the structure  $H(\theta)^{V[G \times H]}$  in  $V[G \times H]$ . Since  $H(\theta)^{V[G \times H]}$  is closed under  $F^*$ , it easily follows that  $N_0$  is closed under  $j(F^*)$ . By the elementarity of  $j$ ,  $j(F^*)$  is a Skolem function for  $H(j(\theta))^{M[J]}$  in  $M[J]$ , which proves the claim.

So all of the assumptions of Theorem 7.1 for  $j(\mathbb{Q})$  and  $N_0$  hold in  $M[J]$ . By Theorem 7.1, it follows that in  $M[J]$ ,  $N_0 \cap j(\mathbb{Q})$  is a regular suborder of  $j(\mathbb{Q})$ , and  $N_0 \cap j(\mathbb{Q})$  forces that  $j(\mathbb{Q})/\dot{G}_{N_0 \cap j(\mathbb{Q})}$  is  $\omega_1$ -c.c. and has the  $\omega_1$ -approximation property.

LEMMA 8.4. *In the model  $M[J]$ ,  $j[\mathbb{Q}]$  is a regular suborder of  $j(\mathbb{Q})$ , and  $j[\mathbb{Q}]$  forces that  $j(\mathbb{Q})/\dot{G}_{j[\mathbb{Q}]}$  is  $\omega_1$ -c.c. and has the  $\omega_1$ -approximation property.*

*Proof.* By the comments preceding this lemma, it suffices to show that  $j[\mathbb{Q}] = j(\mathbb{Q}) \cap N_0$ . Let  $q \in \mathbb{Q}$ , and we will show that  $j(q) \in j(\mathbb{Q}) \cap N_0$ . Since  $\dot{\mathbb{Q}} \in H(\theta)^V$ , we have  $H(\theta)^{V[G \times H]}$ . So  $q \in H(\theta)^{V[G \times H]}$ . Hence  $j(q) \in N_0$ . Also  $j(q) \in j(\mathbb{Q})$  by the elementarity of  $j$ . Conversely, let  $q^* \in j(\mathbb{Q}) \cap N_0$ , and we will show that  $q^* \in j[\mathbb{Q}]$ . Then by the definition of  $N_0$ , there is  $q \in H(\theta)^{V[G \times H]}$  such that  $j(q) = q^*$ . Since  $j(q) = q^* \in j(\mathbb{Q})$ , it follows that  $q \in \mathbb{Q}$  by the elementarity of  $j$ . Hence  $q^* \in j[\mathbb{Q}]$ . ■

Since  $j[I]$  is an  $M[J]$ -generic filter on  $j[\mathbb{Q}]$ , we can form the forcing poset  $j(\mathbb{Q})/j[I]$  in  $M[J][I]$ . In particular, this forcing poset is in  $V[G \times H][I][J]$ .

NOTATION 8.5. Let  $K$  be a  $V[G \times H][I][J]$ -generic filter on  $j(\mathbb{Q})/j[I]$ .

Since  $V[G \times H][I][J] = V[J][I]$ ,  $K$  is a  $V[J][I]$ -generic filter on  $j(\mathbb{Q})/j[I]$ . Hence  $K$  is an  $M[J][I]$ -generic filter on  $j(\mathbb{Q})/j[I]$ . By Lemma 1.1, it follows that  $K$  is an  $M[J]$ -generic filter on  $j(\mathbb{Q})$ ,  $K \cap j[\mathbb{Q}] = j[I]$ , and  $M[J][I][K] = M[J][K]$ . In particular,  $j[I] \subseteq K$ . By standard arguments, in  $V[J][I][K]$  we can extend the elementary embedding  $j : V[G \times H] \rightarrow M[J]$  to

$$j : V[G \times H][I] \rightarrow M[J][K]$$

by letting

$$j(\dot{a}^I) := j(\dot{a})^K$$

for any  $\dot{a}$  which is a  $\mathbb{Q}$ -name in  $V[G \times H]$ . Note that since  $j \upharpoonright H(\theta)^{V[G \times H]}$  is in  $M[J]$ , and  $H(\theta)^{V[G \times H][I]} = H(\theta)^{V[G \times H][I]}$ , it follows that  $j \upharpoonright H(\theta)^{V[G \times H][I]}$  is definable in  $M[J][K]$  from the parameters  $H(\theta)^{V[G \times H]}$ ,  $j \upharpoonright H(\theta)^{V[G \times H]}$ ,  $I$ , and  $K$ . Therefore  $j \upharpoonright H(\theta)^{V[G \times H][I]}$  is in  $M[J][K]$ .

LEMMA 8.6. *The pair*

$$(M[G \times H][I], M[J][K])$$

*has the  $\omega_1$ -covering property and the  $\omega_1$ -approximation property.*

*Proof.* We have

$$M[J][K] = M[J][I][K] = M[G \times H][J][I][K] = M[G \times H][I][J][K].$$

So it suffices to show that the pair

$$(M[G \times H][I], M[G \times H][I][J][K])$$

has the  $\omega_1$ -covering property and the  $\omega_1$ -approximation property.

By Proposition 5.6, for any regular suborder  $\mathbb{P}_0$  of  $\mathbb{P} \times \text{Add}$ ,  $\mathbb{P}_0$  forces that  $(\mathbb{P} \times \text{Add})/\dot{G}_{\mathbb{P}_0}$  is strongly proper on a stationary set. By the elementarity of  $j$ , the same is true of  $j(\mathbb{P} \times \text{Add})$  in  $M$ . In particular, this is true in  $M$  of the regular suborder  $j[\mathbb{P} \times \text{Add}]$  of  $j(\mathbb{P} \times \text{Add})$ . It follows that in  $M[G \times H]$ , the forcing poset  $j(\mathbb{P} \times \text{Add})/j[G \times H]$  is strongly proper on a stationary set.

In  $M[G \times H]$ ,  $\mathbb{Q}$  is a special iteration, and hence is  $\omega_1$ -c.c., and therefore proper. By Theorem 5.5, since  $I$  is an  $M[G \times H]$ -generic filter on  $\mathbb{Q}$ , in  $M[G \times H][I]$  the forcing poset  $j(\mathbb{P} \times \text{Add})/j[G \times H]$  is strongly proper on a stationary set. Therefore in  $M[G \times H][I]$ ,  $j(\mathbb{P} \times \text{Add})/j[G \times H]$  has the  $\omega_1$ -covering property and the  $\omega_1$ -approximation property. Since  $J$  is an  $M[G \times H][I]$ -generic filter on  $j(\mathbb{P} \times \text{Add})/j[G \times H]$ , it follows that the pair

$$(M[G \times H][I], M[G \times H][I][J])$$

has the  $\omega_1$ -covering property and the  $\omega_1$ -approximation property.

By Lemma 8.4, in the model  $M[J] = M[G \times H][J]$ ,  $j[\mathbb{Q}]$  is a regular suborder of  $j(\mathbb{Q})$ , and  $j[\mathbb{Q}]$  forces that  $j(\mathbb{Q})/\dot{G}_{j[\mathbb{Q}]}$  is  $\omega_1$ -c.c. and has the  $\omega_1$ -approximation property. Also,  $j[I]$  is an  $M[J]$ -generic filter on  $j[\mathbb{Q}]$ . So in the model  $M[J][I]$ ,  $j(\mathbb{Q})/j[I]$  is  $\omega_1$ -c.c. and has the  $\omega_1$ -approximation property. Since  $K$  is an  $M[J][I]$ -generic filter on  $j(\mathbb{Q})/j[I]$ , the pair

$$(M[J][I], M[J][I][K])$$

has the  $\omega_1$ -covering property and the  $\omega_1$ -approximation property. But  $M[J] = M[G \times H][J]$ , so the pair

$$(M[G \times H][J][I], M[G \times H][J][I][K])$$

has the  $\omega_1$ -covering property and the  $\omega_1$ -approximation property. Since

$M[G \times H][J][I] = M[G \times H][I][J]$ , the pair

$$(M[G \times H][I][J], M[G \times H][I][J][K])$$

has the  $\omega_1$ -covering property and the  $\omega_1$ -approximation property.

To summarize, we have shown that the pairs

$$(M[G \times H][I], M[G \times H][I][J])$$

and

$$(M[G \times H][I][J], M[G \times H][I][J][K])$$

both have the  $\omega_1$ -covering property and the  $\omega_1$ -approximation property. By Lemma 1.7, it follows that the pair

$$(M[G \times H][I], M[G \times H][I][J][K])$$

has the  $\omega_1$ -covering property and the  $\omega_1$ -approximation property. ■

Recall that we are trying to prove that in  $V[G \times H][I]$ , there is a set  $N$  in  $P_\kappa(H(\theta))$  such that  $N \prec H(\theta)$ ,  $N$  is closed under  $F$ , and  $N$  is internally unbounded and  $\omega_1$ -guessing. In the model  $V[J][I][K]$ , we have an elementary embedding  $j : V[G \times H][I] \rightarrow M[J][K]$ . So by the elementarity of  $j$ , it suffices to prove that in  $M[J][K]$ , there exists a set  $N$  satisfying:

- (a)  $N$  is in  $P_{j(\kappa)}(H(j(\theta)))^{M[J][K]}$ ;
- (b)  $N \prec H(j(\theta))^{M[J][K]}$ ;
- (c)  $N$  is closed under  $j(F)$ ;
- (d)  $N$  is internally unbounded and  $\omega_1$ -guessing.

Let

$$N := j[H(\theta)^{V[G \times H][I]}].$$

We will show that  $N$  is in  $M[J][K]$ , and  $M[J][K]$  models that  $N$  satisfies properties (a)–(d) above.

Since  $j \upharpoonright H(\theta)^{V[G \times H][I]} \in M[J][K]$ , as observed above,  $N \in M[J][K]$ . Also, by the elementarity of  $j$ ,  $N \subseteq j(H(\theta)^{V[G \times H][I]}) = H(j(\theta))^{M[J][K]}$ . Now  $N$  has the same cardinality as  $H(\theta)^{V[G \times H][I]}$ , which in turn has the same cardinality as  $H(\theta)^V$ . But  $|H(\theta)^V| < j(\kappa)$ , and in  $M[J][K]$ ,  $j(\kappa)$  is equal to  $\omega_2$  and  $\omega_1^M$  is preserved. It follows that in  $M[J][K]$ ,  $N$  has size  $\omega_1$ . Hence  $N$  is in  $P_{j(\kappa)}(H(j(\theta)))^{M[J][K]}$ .

We claim that  $N$  is an elementary substructure of  $H(j(\theta))^{M[J][K]}$ . Let  $F^*$  be a Skolem function for the structure  $H(\theta)^{V[G \times H][I]}$  in  $V[G \times H][I]$ . Since  $H(\theta)^{V[G \times H][I]}$  is closed under  $F^*$ , it easily follows that  $N$  is closed under  $j(F^*)$ . By elementarity of  $j$ ,  $j(F^*)$  is a Skolem function for  $H(j(\theta))^{M[J][K]}$  in  $M[J][K]$ . So  $N$  is an elementary substructure of  $H(j(\theta))^{M[J][K]}$ . The same argument shows that  $N$  is closed under  $j(F)$ .

It remains to show that  $N$  is internally unbounded and  $\omega_1$ -guessing in  $M[J][K]$ . To show that  $N$  is internally unbounded, let  $a$  be a countable sub-

set of  $N$  in  $M[J][K]$ . Then  $b := j^{-1}[a]$  is a countable subset of  $H(\theta)^{V[G \times H][I]}$  in  $M[J][K]$ . Since the pair  $(M[G \times H][I], M[J][K])$  has the  $\omega_1$ -covering property by Lemma 8.6, we can fix a countable set  $c \subseteq H(\theta)^{V[G \times H][I]} = H(\theta)^{M[G \times H][I]}$  in  $M[G \times H][I]$  such that  $b \subseteq c$ . But  $\theta$  is regular and uncountable in  $M[G \times H][I]$ , so  $c \in H(\theta)^{M[G \times H][I]}$ . Hence  $j(c) = j[c]$  is in  $N$ ,  $j(c)$  is countable, and  $a = j[b] \subseteq j[c] = j(c)$ .

To show that  $N$  is  $\omega_1$ -guessing in  $M[J][K]$ , by Lemma 2.2 it suffices to show that the pair

$$(\bar{N}, M[J][K])$$

satisfies the  $\omega_1$ -approximation property, where  $\bar{N}$  is the transitive collapse of  $N$ . Since  $N$  is isomorphic to  $H(\theta)^{V[G \times H][I]}$ , which is transitive, we have  $\bar{N} = H(\theta)^{V[G \times H][I]} = H(\theta)^{M[G \times H][I]}$ . Hence it suffices to show that the pair

$$(H(\theta)^{M[G \times H][I]}, M[J][K])$$

has the  $\omega_1$ -approximation property.

Let  $d$  be a bounded subset of  $H(\theta)^{M[G \times H][I]} \cap \text{On} = \theta$  in  $M[J][K]$  which is countably approximated by  $H(\theta)^{M[G \times H][I]}$ . We will show that  $d$  is in  $H(\theta)^{M[G \times H][I]}$ . We claim that  $d$  is countably approximated by  $M[G \times H][I]$ . Consider a countable set  $a$  in  $M[G \times H][I]$ . Then  $a \cap \theta$  is a countable subset of  $\theta$  in  $M[G \times H][I]$ , and hence is in  $H(\theta)^{M[G \times H][I]}$ . Since  $d \subseteq \theta$ , we have  $a \cap d = (a \cap \theta) \cap d$ . Since  $d$  is countably approximated by  $H(\theta)^{M[G \times H][I]}$ ,  $a \cap d = (a \cap \theta) \cap d$  is in  $H(\theta)^{M[G \times H][I]}$ , and hence is in  $M[G \times H][I]$ . Thus  $d$  is countably approximated by  $M[G \times H][I]$ . Since  $(M[G \times H][I], M[J][K])$  has the  $\omega_1$ -approximation property by Lemma 8.6, it follows that  $d \in M[G \times H][I]$ . But  $d$  is a bounded subset of  $\theta$  and  $\theta$  is regular in  $M[G \times H][I]$ , so that  $d \in H(\theta)^{M[G \times H][I]}$ .

We conclude the paper with two questions:

- (1) In Corollary 3.5, we proved that IGMP implies SCH. Does GMP imply SCH?
- (2) At the end of Section 2, we noted that GMP is consistent with  $\mathfrak{p} = \omega_1$ . Does IGMP imply  $\mathfrak{p} > \omega_1$ ?

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