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Experimental investigation on the uniqueness of a center of a body

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Abstract. The object of our investigation is a point that gives the maximum value of a potential with a strictly decreasing radially symmetric kernel. It defines a center of a body in $\mathbb{R}^m$. When the kernel is the Riesz kernel or the Poisson kernel, such a center is called an $r^{\alpha-m}$-center or an illuminating center, respectively.

The existence of a center is easily shown but uniqueness does not always hold. The main results in this paper are some new sufficient conditions for the uniqueness.

1. Introduction. Let Ω be a body (the closure of a bounded open set) in $\mathbb{R}^m$. We consider a potential of the form

$$K_\Omega(x) = \int_\Omega k(r)\,dy, \qquad x \in \mathbb{R}^m,\ r = |x-y|. \tag{1.1}$$

If the kernel $k : (0,\infty) \to \mathbb{R}$ is strictly decreasing and satisfies condition (C^0_α) (see Section 2), then K_Ω is continuous on $\mathbb{R}^m$ (Proposition 2.1) and all of its maximum points are in the convex hull of Ω (Proposition 2.2). We call a maximum point of K_Ω a *k-center* of Ω. This is the object of investigation in this paper.

Analytically, the study of k-centers is related to the investigation of the shape of a solution of a partial differential equation. When $k(r)$ is the Gauss kernel $(4\pi t)^{-m/2}\exp(-r^2/(4t))$ with a positive parameter t, we obtain the unique bounded solution of the Cauchy problem for the heat equation with initial datum χ_Ω. A (spatial) maximum point of the solution of the heat equation is called a *hot spot*. The existence, asymptotic behavior, uniqueness and location of a hot spot are well-studied, for example, in [BL, BMS, CK, JS, MS]. When the kernel $k(r)$ is the Poisson kernel $h(r^2+h^2)^{-(m+1)/2}$

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with a positive parameter h, we obtain the Poisson integral for the upper half-space (up to a constant multiple). The Poisson integral is a solution of the Laplace equation for the upper half-space. Maximum points of Poisson integrals were studied in [Sak].

Geometrically, the study of k-centers is related to Moszyńska's radial centers. In [M1], she introduced a *radial center* of a star body A induced by a function ϕ as a maximum point of the function

$$\Phi_A(x) = \int_{S^{m-1}} \phi(\rho_{A-x}(v))d\sigma(v), \qquad x \in \operatorname{Ker} A. \tag{1.2}$$

Here, $\rho_{A-x}(v) = \max\{\lambda \geq 0 \mid \lambda v + x \in A\}$ is the *radial function* of A with respect to x, and $\operatorname{Ker} A = \{p \in A \mid \forall q \in A, \overline{pq} \subset A\}$ is the *kernel* of A. Her motivation for the study of radial centers comes from the optimal position of the origin for the *intersection body* of a star body. Intersection bodies were introduced by Lutwak [L] to solve Busemann and Petty's problem [BP]. We refer to Moszyńska's textbook [M2, pp. 185–201] for historical background in convex geometry. The paper [HMP] is also a good reference for the physical meaning of radial centers.

Using polar coordinates, we rewrite the function $\Phi_A(x)$ as

$$\Phi_A(x) = \int_A \phi'(r)r^{1-m}\,dy + \phi(0)\sigma(S^{m-1}), \qquad x \in \operatorname{Ker} A,\ r = |x-y|. \tag{1.3}$$

Setting $k(r) = \phi'(r)r^{1-m}$, we obtain the potential K_A. Since K_A is defined on $\mathbb{R}^m$ even if A is NOT star-shaped, we see that the notion of k-centers is an extension of radial centers.

When the kernel $k(r)$ is the monomial $r^{\alpha-m}$, k-centers are well-studied. When $\phi(\rho) = \rho^\alpha$ in (1.2), Moszyńska [M1] called a maximum point of Φ_A a *radial center of order* α and showed that if $m \geq 2$ and $0 < \alpha \leq 1$, then every convex body has a unique such center. For $\alpha > 1$, the uniqueness of a radial center of a convex body was studied by Herburt [H1] but the argument included an error. In [H2], Herburt studied the location of a radial center of order 1. She showed that every smooth convex body has every radial center of order 1 in its interior. O'Hara [O1] investigated the potential

$$V_\Omega^{(\alpha)}(x) = \begin{cases} \operatorname{sign}(m-\alpha)\int_\Omega r^{\alpha-m}\,dy & (0 < \alpha \neq m), \\ -\int_\Omega \log r\,dy & (\alpha = m), \end{cases} \qquad r = |x-y|. \tag{1.4}$$

He called it the $r^{\alpha-m}$*-potential* and defined an $r^{\alpha-m}$*-center* of Ω as a maximum point of $V_\Omega^{(\alpha)}$. In other words, he extended the notion of [M1] to a non-star-shaped case. He showed that if $m \geq 2$ and $\alpha \geq m+1$, then each body has a unique $r^{\alpha-m}$-center.

To get the uniqueness of a k-center in [M1, O1], the common idea is to show the strict concavity of the potential K_Ω on the convex hull of Ω (where k-centers lie). But using *Aleksandrov's reflection principle* or the *moving plane method* [GNN, Ser], we can restrict the region containing all k-centers to be smaller than the convex hull of Ω. We call a certain such region the *minimal unfolded region* of Ω, denoted by $\mathrm{Uf}(\Omega)$; it was introduced by O'Hara [O1]. When Ω is a convex body, the minimal unfolded region was independently defined by Brasco, Magnanini and Salani [BMS] as the *heart* of Ω, denoted by $\heartsuit(\Omega)$. Hence, in order to show the uniqueness of a k-center, it is sufficient to show the strict concavity of K_Ω on the minimal unfolded region.

The minimal unfolded region $\mathrm{Uf}(\Omega)$ is obtained by the following procedure: Fix a direction $v \in S^{m-1}$ and a parameter $b \in \mathbb{R}$. Let $\mathrm{Refl}_{v,b}$ denote the reflection of $\mathbb{R}^m$ in the hyperplane $\{z \in \mathbb{R}^m \mid z \cdot v = b\}$. We denote by $\Omega^+_{v,b} = \{z \in \Omega \mid z \cdot v \geq b\}$ the set of all points in Ω whose height in direction v is not smaller than b. We continue to fold the set $\Omega^+_{v,b}$ by the reflection $\mathrm{Refl}_{v,b}$ and to gradually decrease $b \in \mathbb{R}$ until the image protrudes from Ω. Let $l(v)$ be the *maximal folding function* for v, that is,

$$l(v) = \min\{a \in \mathbb{R} \mid \forall b \geq a,\ \mathrm{Refl}_{v,b}(\Omega^+_{v,b}) \subset \Omega\} \tag{1.5}$$

(see Figure 1). Define the *minimal unfolded region* of Ω by

$$\mathrm{Uf}(\Omega) = \bigcap_{v \in S^{m-1}} \{z \in \mathbb{R}^m \mid z \cdot v \leq l(v)\}. \tag{1.6}$$

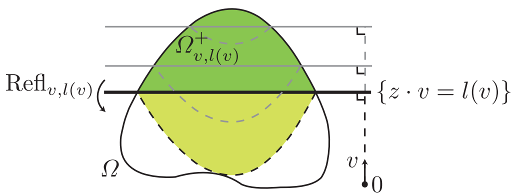

Fig. 1. The recipe for the minimal unfolded region

For example, in $\mathbb{R}^2$, the minimal unfolded region of the union of the two same-sized discs

$$\begin{aligned} &D_1 \cup D_2 \\ &\quad = \{(y_1, y_2) \mid (y_1 + 1)^2 + y_2^2 \leq 1\} \cup \{(y_1, y_2) \mid (y_1 - 1)^2 + y_2^2 \leq 1\} \end{aligned} \tag{1.7}$$

is the line segment $\{(y_1, 0) \mid -1 \leq y_1 \leq 1\}$. Therefore, when we investigate the number of k-centers of $D_1 \cup D_2$, we should consider the graph of the function $K_{D_1 \cup D_2}(\lambda, 0)$ for $-1 \leq \lambda \leq 1$. Then, for a specific kernel, we can draw the graph of $K_{D_1 \cup D_2}(\lambda, 0)$ using Maple. In such a manner, we give some examples of the graphs of $r^{\alpha-m}$-potentials. To be precise, we produce the following examples:

(1) The union (1.7) of the discs has two $r^{-1/2}$-centers (Example 3.1).
(2) The set of $r^{-1/2}$-centers of the annulus $\{(y_1, y_2) \mid 1 \le y_1^2 + y_2^2 \le 4\}$ is a circle (Example 3.2).
(3) The isosceles triangle $\{(y_1, y_2) \mid 0 \le y_1 \le 1,\ |y_2| \le \tan(\pi/10) y_1\}$ has a unique $r^{-1/2}$-center (Example 3.3).
(4) The cone $\{(y_1, y_2, y_3) \mid 0 \le y_1 \le 1,\ y_2^2 + y_3^2 \le \tan^2(\pi/10) y_1^2\}$ has a unique $r^{-1/2}$-center (Example 3.4).
(5) The body of revolution of a parabola $\{(y_1, y_2, y_3) \mid 0 \le y_1 \le 1,\ y_2^2 + y_3^2 \le \tan^2(\pi/10) y_1\}$ has a unique $r^{-1/2}$-center (Example 3.5).

From the third example, we see that in general the $r^{\alpha-m}$-potential is not concave on the convex hull of a body for $1 < \alpha < m+1$. Hence it seems difficult to give a sufficient condition for the uniqueness of an $r^{\alpha-m}$-center for $1 < \alpha < m+1$.

Our main result in this paper is a sufficient condition for the uniqueness of a k-center, covering examples (3) and (5). More precisely, if the kernel k satisfies condition (C_α^1) for some $\alpha > 1$ (see Section 2), and $k'(r)/r$ is increasing, then the body of revolution

$$\Omega = \{(y_1, \bar{y}) \in \mathbb{R} \times \mathbb{R}^{m-1} \mid 0 \le y_1 \le 1,\ |\bar{y}| \le \omega(y_1)\}, \tag{1.8}$$

where $\omega : [0,1] \to [0,\infty)$ is a piecewise C^1 function with ω^{m-1} concave, has a unique k-center. This immediately implies the uniqueness of an $r^{\alpha-m}$-center of (1.8) for $1 < \alpha < m+1$. In the same manner, we also show that a non-obtuse triangle in $\mathbb{R}^2$ has a unique k-center if $k'(r)/r$ is increasing. We remark that these results cannot be obtained by using a power-concavity argument as in [BL].

Throughout this paper, $\operatorname{conv} X$, $\operatorname{diam} X$, $\mathring{X}$ (or X°) and X^c denote the convex hull, diameter, interior and complement of a set X in $\mathbb{R}^m$, respectively. We denote the spherical Lebesgue measure of any N-dimensional space by σ_N.

2. Preliminaries. In this section, we recall some results of [BMS, BM, O1, Sak], necessary for our study.

For $\alpha > 0$ and $k : (0,\infty) \to \mathbb{R}$, consider the following condition:

(C_α^0) k is continuous on $(0,\infty)$, and

$$k(r) = \begin{cases} O(r^{\alpha-m}) & (\alpha < m), \\ O(\log r) & (\alpha = m), \\ O(1) & (\alpha > m), \end{cases} \tag{2.1}$$

as r tends to 0^+.

For $\alpha > 1$ and $k : (0,\infty) \to \mathbb{R}$, we consider

(C^1_α) k is once continuously differentiable on $(0,\infty)$, and

$$k(r) = \begin{cases} O(r^{\alpha-m}) & (\alpha < m), \\ O(\log r) & (\alpha = m), \\ O(1) & (\alpha > m), \end{cases} \tag{2.2}$$
$$k'(r) = \begin{cases} O(r^{\alpha-m-1}) & (\alpha < m+1), \\ O(\log r) & (\alpha = m+1), \\ O(1) & (\alpha > m+1), \end{cases}$$

as r tends to 0^+.

Let Ω be a *body* (the closure of a bounded open set) in $\mathbb{R}^m$, and

$$K_\Omega(x) = \int_\Omega k(r)\,dy, \qquad x \in \mathbb{R}^m,\ r = |x-y|. \tag{2.3}$$

We always assume that the kernel k satisfies (C^0_α) or (C^1_α). We denote a point x in $\mathbb{R}^m$ by $x = (x_1,\ldots,x_m)$ and a point y in Ω by $y = (y_1,\ldots,y_m)$. The letter r is always used for $r = |x-y|$.

2.1. Properties of K_Ω. We recall some properties of the potential K_Ω from [Sak] without proofs.

PROPOSITION 2.1 ([Sak, Propositions 2.3, 2.6, 4.1 and Corollary 4.2]). *Let Ω be a body in $\mathbb{R}^m$.*

(1) *If the kernel k satisfies condition (C^0_α) for some $\alpha > 0$, then the potential K_Ω is continuous on $\mathbb{R}^m$.*
(2) *If k satisfies condition (C^1_α) for some $\alpha > 1$, then K_Ω is of class C^1 on $\mathbb{R}^m$, and*
$$\frac{\partial K_\Omega}{\partial x_j}(x) = \int_\Omega \frac{\partial}{\partial x_j} k(r)\,dy, \qquad x \in \mathbb{R}^m.$$
(3) *If Ω has a piecewise C^1 boundary, and if k satisfies condition (C^1_α) for some $\alpha > 1$, then K_Ω is of class C^2 on $\mathbb{R}^m \setminus \partial\Omega$, and*
$$\frac{\partial K_\Omega}{\partial x_j}(x) = -\int_{\partial\Omega} k(r) e_j \cdot n(y)\,d\sigma(y), \qquad x \in \mathbb{R}^m,$$
$$\frac{\partial^2 K_\Omega}{\partial x_i \partial x_j}(x) = -\int_{\partial\Omega} \frac{\partial}{\partial x_i} k(r) e_j \cdot n(y)\,d\sigma(y), \qquad x \in \mathbb{R}^m \setminus \partial\Omega,$$
where n is the outer unit normal vector field on $\partial\Omega$, and e_j is the jth unit vector of $\mathbb{R}^m$.

Proposition 2.2 ([Sak, Proposition 3.2]). *Let Ω be a body in $\mathbb{R}^m$. Suppose that the kernel k is strictly decreasing and satisfies condition (C^0_α) for some $\alpha > 0$. Then the potential K_Ω has a maximum point, and any maximizer of K_Ω belongs to the convex hull of Ω.*

Definition 2.3 ([Sak, Definition 3.3]). Let Ω be a body in $\mathbb{R}^m$. A point x is called a *k-center* of Ω if it gives the maximum value of K_Ω.

2.2. Properties of minimal unfolded regions. Let $\mathrm{Uf}(\Omega)$ be the minimal unfolded region of a body Ω as in (1.6). We introduce some of its properties from [BM, BMS, O1, Sak] with slight modifications. ([BM] does not require the regularity of k but requires the boundedness of $k(r)$ at $r = 0^+$.) Geometric properties of the minimal unfolded region were also studied in [O2].

Remark 2.4 ([O1, p. 381]). Let Ω be a body in $\mathbb{R}^m$.

(1) The centroid (the center of mass) of Ω is contained in $\mathrm{Uf}(\Omega)$. Hence $\mathrm{Uf}(\Omega)$ is not empty.

(2) $\mathrm{Uf}(\Omega)$ is contained in $\operatorname{conv}\Omega$ but in general not in Ω (see Figure 2).

(3) $\mathrm{Uf}(\Omega)$ is compact and convex.

Example 2.5 ([BM, Lemma 5], [O1, Example 3.4]). (1) The minimal unfolded region of a non-obtuse triangle is given by the polygon formed by the mid-perpendicular of edges and the bisectors of angles (see Figure 3). In particular, it is contained in the triangle formed by joining the middle points of the edges.

(2) The minimal unfolded region of an obtuse triangle is given by the polygon formed by the largest edge, its mid-perpendicular and the bisectors of angles (see Figure 4).

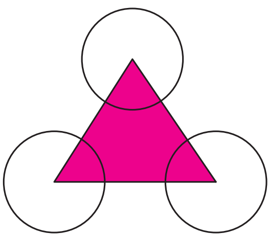

Fig. 2

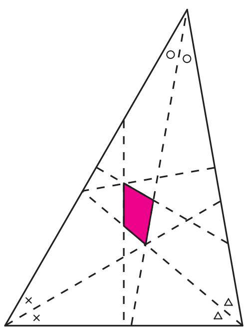

Fig. 3

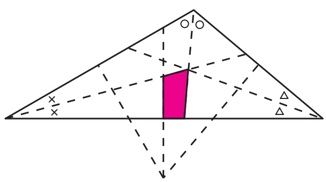

Fig. 4

Proposition 2.6 ([Sak, Proposition 4.9]). *Let Ω be a body in $\mathbb{R}^m$. If k is strictly decreasing, then every k-center of Ω belongs to* $\mathrm{Uf}(\Omega)$.

We give a relation between Ω and $\mathrm{Uf}(\Omega)$. The idea of the proof is due to [BM, Theorem 1]. To be precise, in [BM], Brasco and Magnanini studied the geometry of the minimal unfolded region (heart) of a *convex* body, but their argument works for a *non-convex* body with slight modifications.

LEMMA 2.7. *Let Ω be a body in $\mathbb{R}^m$. The maximal folding function $l : S^{m-1} \to \mathbb{R}$ is lower semicontinuous.*

Proof. We show that the set $\{v \in S^{m-1} \mid l(v) > b\}$ is open in S^{m-1} for any $b \in \mathbb{R}$. Fix $b \in \mathbb{R}$. Let $w \in S^{m-1}$ be a direction with $l(w) > b$.

We first show that the non-empty intersection $\mathrm{Refl}_{w,b}(\Omega^+_{w,b}) \cap \Omega^c$ has an interior point. We take a point x from the intersection. Since Ω^c is open in $\mathbb{R}^m$, there exists an $\varepsilon_1 > 0$ such that the ε_1-neighborhood of x is contained in Ω^c. Since $\mathrm{Refl}_{w,b}(\Omega^+_{w,b})$ is the closure of an open set, x is in its interior or on its boundary. We only consider the latter case. We can choose a point x' from the ε_1-neighborhood of x such that $x' \in (\mathrm{Refl}_{w,b}(\Omega^+_{w,b}))^\circ$. There exists an $\varepsilon_2 > 0$ such that the ε_2-neighborhood of x' is contained in the interior of $\mathrm{Refl}_{w,b}(\Omega^+_{w,b}) \cap B_{\varepsilon_1}(x)$. Hence the ε_2-neighborhood of x' is contained in $\mathrm{Refl}_{w,b}(\Omega^+_{w,b}) \cap \Omega^c$, that is, x' is an interior point of the intersection.

Next, we complete the proof. Let x be an interior point of $\mathrm{Refl}_{w,b}(\Omega^+_{w,b}) \cap \Omega^c$, and $\varepsilon > 0$ be such that $B_\varepsilon(x) \subset \mathrm{Refl}_{w,b}(\Omega^+_{w,b}) \cap \Omega^c$. Let $\xi = \mathrm{Refl}^{-1}_{w,b}(x)$. Then the ε-neighborhood of ξ is contained in $\Omega^+_{w,b}$. The continuity of the map

$$S^{m-1} \ni u \mapsto \mathrm{Refl}_{u,b}(\xi) = \xi + 2(b - \xi \cdot u)u \in \mathbb{R}^m$$

implies the existence of a positive constant δ such that, for any $u \in B_\delta(w) \cap S^{m-1}$, the ball $B_{\varepsilon/2}(\xi)$ is contained in $\Omega^+_{u,b}$, and we have

$$\mathrm{Refl}_{u,b}(B_{\varepsilon/2}(\xi)) \subset B_\varepsilon(x) \subset \Omega^c,$$

which completes the proof. ∎

For $v \in S^{m-1}$, we denote its orthogonal complement vector space by $v^\perp$, that is,

$$v^\perp = \{z \in \mathbb{R}^m \mid z \cdot v = 0\}. \tag{2.4}$$

We say that Ω is *convex in direction* v if $\Omega \cap (\mathrm{Span}\langle v\rangle + z)$ is connected for any $z \in v^\perp$.

PROPOSITION 2.8. *Let Ω be a body in $\mathbb{R}^m$.*

(1) *If there exist p $(1 \le p \le m)$ independent directions $v_1, \dots, v_p \in S^{m-1}$ such that Ω is symmetric with respect to the hyperplanes $v_1^\perp, \dots, v_p^\perp$ and convex in directions $v_1, \dots, v_p$, then*

$$\mathrm{Uf}(\Omega) \subset \bigcap_{j=1}^{p} v_j^\perp.$$

(2) *If the dimension of the minimal unfolded region of Ω is p $(0 \leq p \leq m-1)$, then there exists a direction $w \in S^{m-1}$ orthogonal to* $\mathrm{Uf}(\Omega)$ *such that Ω is symmetric with respect to a hyperplane parallel to $w^\perp$, and convex in direction w.*

Proof. (1) We remark that $l(v_j) = l(-v_j)$ for any $1 \leq j \leq p$. Let us show $l(v_j) = 0$ for any $1 \leq j \leq p$, which implies

$$\begin{aligned}\mathrm{Uf}(\Omega) &\subset \bigcap_{j=1}^{p} \big(\{z \in \mathbb{R}^m \mid z \cdot v_j \leq l(v_j)\} \cap \{z \in \mathbb{R}^m \mid z \cdot (-v_j) \leq l(-v_j)\}\big) \\ &= \bigcap_{j=1}^{p} v_j^\perp .\end{aligned}$$

Suppose that $l(v_j) > 0$ for some j. There exists a height b $(0 < b < l(v_j))$ such that $\mathrm{Refl}_{v_j,b}(\Omega^+_{v_j,b}) \cap \Omega^c \neq \emptyset$. Choose $x \in \Omega^+_{v_j,b}$ such that $x' = \mathrm{Refl}_{v_j,b}(x) \in \Omega^c$.

From the symmetry of Ω with respect to the hyperplane $v_j^\perp$, we have $x'' = \mathrm{Refl}_{v_j,0}(x) \in \Omega$. By the convexity of Ω for v_j,

$$\emptyset \neq \overline{xx'} \cap \Omega^c \subset \overline{xx''} \cap \Omega^c \subset \Omega \cap \Omega^c = \emptyset,$$

which is a contradiction.

(2) Since $\mathrm{Uf}(\Omega)$ is compact and convex, we may assume that it is contained in the p-dimensional vector space $\mathbb{R}^p \times \{0\}^{m-p} \subset \mathbb{R}^p \times \mathbb{R}^{m-p} = \mathbb{R}^m$. By a translation, we may also assume that the centroid of $\mathrm{Uf}(\Omega)$, denoted by G_Ω, coincides with the origin.

We first show that the minimum value of l is zero. Suppose that $l(v)$ is positive for any $v \in S^{m-1}$. By the lower semicontinuity of l, we have

$$\rho = \inf_{v \in S^{m-1}} l(v) = \min_{v \in S^{m-1}} l(v) > 0.$$

Then the m-dimensional ball $B_\rho(0)$ is contained in $\mathrm{Uf}(\Omega)$, which is a contradiction. Hence there exists a direction $w \in S^{m-1}$ such that $l(w) = 0$.

In order to show the symmetry of Ω with respect to the hyperplane orthogonal to w, we show that $\Omega = \Omega^+_{w,0} \cup \mathrm{Refl}_{w,0}(\Omega^+_{w,0})$. Suppose that the set $\Omega \setminus (\Omega^+_{w,0} \cup \mathrm{Refl}_{w,0}(\Omega^+_{w,0}))$ is not empty. Since Ω is a body, this set has an interior point. By a reflection argument, we have

$$0 > \int_{\Omega \setminus (\Omega^+_{w,0} \cup \mathrm{Refl}_{w,0}(\Omega^+_{w,0}))} y \cdot w \, dy = \int_{\Omega} y \cdot w \, dy = \mathrm{Vol}(\Omega) G_\Omega \cdot w = 0,$$

which is a contradiction. Hence Ω is symmetric with respect to the hyperplane $w^\perp$.

Finally, we show the convexity of Ω in direction w. Assume there exist $x, x' \in \Omega$ such that $\overline{xx'}$ is parallel to $\mathrm{Span}\langle w\rangle$ and contains a point ξ in Ω^c. We may assume $(x+\xi)\cdot w > 0$. Let $b = ((x+\xi)\cdot w)/2$. Then $\xi = \mathrm{Refl}_{w,b}(x)$, which contradicts $l(w) = 0 < b$.

Furthermore, (1) implies that $w \perp \mathrm{Uf}(\Omega)$. ■

3. Examples of graphs. Let Ω be a body in $\mathbb{R}^m$ $(m \geq 2)$ with a piecewise C^1 boundary. In this section, in order to investigate the number of k-centers of Ω, using Maple, we produce some examples of the graphs of the $r^{\alpha-m}$-potentials

$$V_\Omega^{(\alpha)}(x) = \begin{cases} \mathrm{sign}(m-\alpha)\int_\Omega r^{\alpha-m}\,dy & (0<\alpha\neq m),\\ -\int_\Omega \log r\,dy & (\alpha = m),\end{cases} \tag{3.1}$$

and their second derivatives. When we use Maple to draw the graph of the $r^{\alpha-m}$-potential, it is useful to use the boundary integral expression

$$\begin{aligned} V_\Omega^{(\alpha)}(x) &\\ = &\begin{cases} -\dfrac{\mathrm{sign}(m-\alpha)}{\alpha}\displaystyle\int_{\partial\Omega} r^{\alpha-m}(x-y)\cdot n(y)\,d\sigma(y) & (0<\alpha\neq m),\\ \dfrac{1}{m}\displaystyle\int_{\partial\Omega}\left(\log r - \frac{1}{m}\right)(x-y)\cdot n(y)\,d\sigma(y) & (\alpha = m),\end{cases}\end{aligned} \tag{3.2}$$

for $x \in \mathbb{R}^m \setminus \partial\Omega$ [O1, Theorem 2.8].

Example 3.1. Let $m = 2$ and

$$\Omega = \{(y_1,y_2) \mid (y_1+1)^2 + y_2^2 \leq 1\} \cup \{(y_1,y_2) \mid (y_1-1)^2 + y_2^2 \leq 1\}.$$

Then

$$\begin{aligned} \partial\Omega &= \{(\cos\theta - 1, \sin\theta) \mid 0 \leq \theta \leq 2\pi\} \cup \{(\cos\theta+1,\sin\theta) \mid 0 \leq \theta \leq 2\pi\},\\ \mathrm{Uf}(\Omega) &= \{(y_1,0) \mid -1 \leq y_1 \leq 1\},\\ V_\Omega^{(\alpha)}(\lambda,0) &= -\frac{1}{\alpha}\int_0^{2\pi}((\lambda-\cos\theta+1)^2+\sin^2\theta)^{(\alpha-2)/2}((\lambda+1)\cos\theta - 1)\,d\theta\\ &\quad -\frac{1}{\alpha}\int_0^{2\pi}((\lambda-\cos\theta-1)^2+\sin^2\theta)^{(\alpha-2)/2}((\lambda-1)\cos\theta - 1)\,d\theta, \end{aligned}$$

and the graph of $V_\Omega^{(3/2)}(\lambda,0)$ for $-1 \leq \lambda \leq 1$ is in Figure 5. Hence, in this case, Ω has two $r^{-1/2}$-centers.

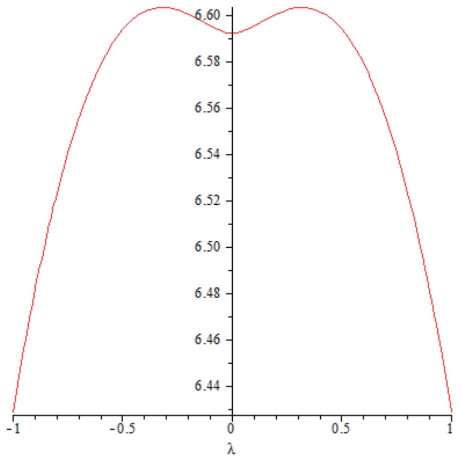

Fig. 5. The graph of $V_{\Omega}^{(3/2)}(\lambda, 0)$ when Ω is the union of two discs

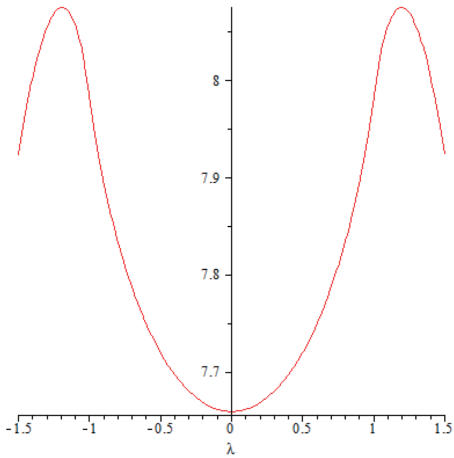

Fig. 6. The graph of $V_{\Omega}^{(3/2)}(\lambda, 0)$ when Ω is an annulus

Example 3.2. Let $m = 2$ and

$$\Omega = \{(y_1, y_2) \mid 1 \leq y_1^2 + y_2^2 \leq 4\}.$$

Then

$$\partial\Omega = \{(2\cos\theta, 2\sin\theta) \mid 0 \leq \theta \leq 2\pi\} \cup (-\{(\cos\theta, \sin\theta) \mid 0 \leq \theta \leq 2\pi\}),$$
$$\mathrm{Uf}(\Omega) = \{(y_1, y_2) \mid y_1^2 + y_2^2 \leq 9/4\},$$
$$V_{\Omega}^{(\alpha)}(\lambda, 0) = \frac{1}{\alpha}\int_0^{2\pi} (\lambda^2 - 2\lambda\cos\theta + 1)^{(\alpha-2)/2}(\lambda\cos\theta - 1)\, d\theta$$
$$- \frac{2}{\alpha}\int_0^{2\pi} (\lambda^2 - 4\lambda\cos\theta + 4)^{(\alpha-2)/2}(\lambda\cos\theta - 2)\, d\theta,$$

and the graph of $V_{\Omega}^{(3/2)}(\lambda, 0)$ for $-3/2 \leq \lambda \leq 3/2$ is in Figure 6. Hence, in this case, the set of $r^{-1/2}$-centers of Ω is a circle.

Example 3.3 ([O1, Remark 3.13]). Let $m = 2$ and

$$\Omega = \{(y_1, y_2) \mid 0 \leq y_1 \leq 1,\ 0 \leq |y_2| \leq \tan(\pi/10)y_1\}.$$

Then

$$\partial\Omega = \{(y_1, y_2) \mid 0 \leq y_1 \leq 1,\ y_2 = -\tan(\pi/10)y_1\}$$
$$\cup \{(1, y_2) \mid -\tan(\pi/10) \leq y_2 \leq \tan(\pi/10)\},$$
$$\cup (-\{(y_1, y_2) \mid 0 \leq y_1 \leq 1,\ y_2 = \tan(\pi/10)y_1\}),$$
$$(1/2, 0) \in \mathrm{Uf}(\Omega) \subset \{(y_1, 0) \mid 1/2 \leq y_1 \leq 1\},$$

$$\frac{\partial^2 V_\Omega^{(\alpha)}}{\partial x_1^2}(\lambda,0) = -2(2-\alpha)\tan(\pi/10)\int_0^1 \bigl((\lambda-t)^2+(\tan(\pi/10)t)^2\bigr)^{(\alpha-4)/2}(\lambda-t)\,dt$$
$$+\,2(2-\alpha)(\lambda-1)\int_0^{\tan(\pi/10)}((\lambda-1)^2+t^2)^{(\alpha-4)/2}\,dt,$$

and the graph of the second derivative of $V_\Omega^{(3/2)}(\lambda,0)$ for $0\le\lambda\le1$ is in Figure 7. Moreover, the contribution of the slopes to the boundary integral (the first integral) is shown in Figure 8. Hence, in this case, Ω has a unique $r^{-1/2}$-center.

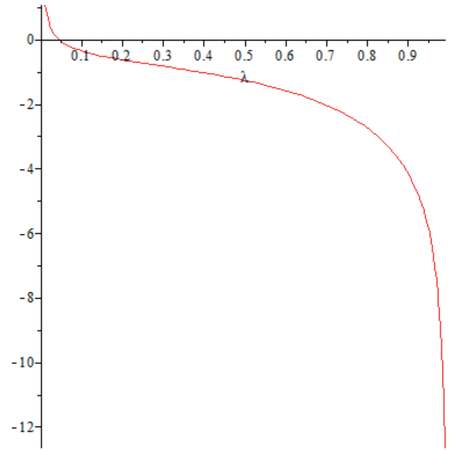

Fig. 7. The graph of $(\partial^2 V_\Omega^{(3/2)}/\partial x_1^2)(\lambda,0)$ when Ω is an isosceles triangle

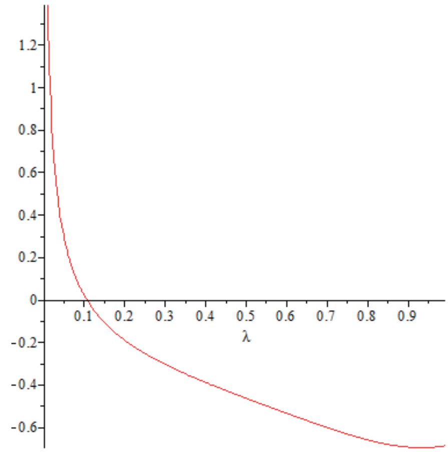

Fig. 8. The contribution of the slopes to the boundary integral

Example 3.4. Let $m=3$ and

$$\Omega=\{(y_1,y_2,y_3)\mid 0\le y_1\le1,\ y_2^2+y_3^2\le\tan^2(\pi/10)y_1^2\}.$$

Then

$$\partial\Omega=\bigl\{(t,\tan(\pi/10)t\cos\theta,\tan(\pi/10)t\sin\theta)\bigm| 0\le t\le1,\ 0\le\theta\le2\pi\bigr\}$$
$$\cup\,\{(1,r\cos\theta,r\sin\theta)\mid 0\le r\le\tan(\pi/10),\ 0\le\theta\le2\pi\},$$

$$(1/2,0,0)\in\mathrm{Uf}(\Omega)\subset\{(y_1,0,0)\mid 1/2\le y_1\le1\},$$

$$\frac{\partial^2 V_\Omega^{(\alpha)}}{\partial x_1^2}(\lambda,0,0)$$
$$=-2\pi(3-\alpha)\tan^2(\pi/10)\int_0^1\bigl((\lambda-t)^2+(\tan(\pi/10)t)^2\bigr)^{(\alpha-5)/2}(\lambda-t)t\,dt$$
$$+\,2\pi(3-\alpha)(\lambda-1)\int_0^{\tan(\pi/10)}((\lambda-1)^2+r^2)^{(\alpha-5)/2}r\,dr,$$

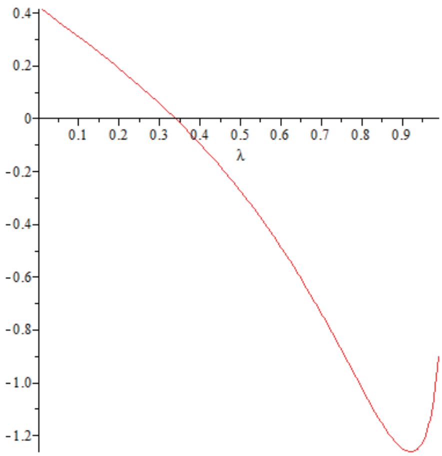

Fig. 9. The graph of $(\partial^2 V_\Omega^{(5/2)}/\partial x_1^2)(\lambda,0)$ when Ω is a cone

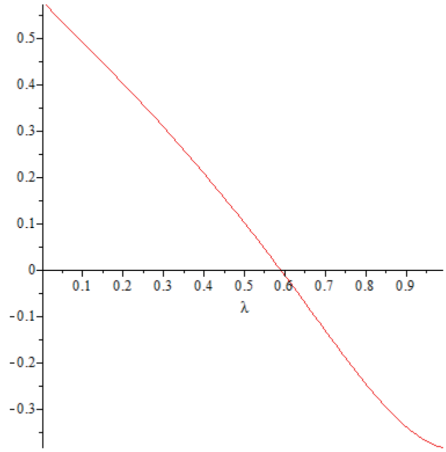

Fig. 10. The contribution of the side to the boundary integral

and the graph of the second derivative of $V_\Omega^{(5/2)}(\lambda,0,0)$ for $0\le\lambda\le 1$ is in Figure 9. Moreover, the contribution of the side to the boundary integral (the first integral) is shown in Figure 10. Hence, in this case, Ω has a unique $r^{-1/2}$-center.

Example 3.5. Let $m=3$ and

$$\Omega=\{(y_1,y_2,y_3)\mid 0\le y_1\le 1,\ y_2^2+y_3^2\le \tan^2(\pi/10)y_1\}.$$

Then

$$\begin{aligned}
\partial\Omega&=\{\big(t,\tan(\pi/10)\sqrt{t}\cos\theta,\ \tan(\pi/10)\sqrt{t}\sin\theta\big)\mid 0\le t\le 1,\ 0\le\theta\le 2\pi\}\\
&\qquad\cup\{(1,r\cos\theta,r\sin\theta)\mid 0\le r\le\tan(\pi/10),\ 0\le\theta\le 2\pi\},\\
&(1/2,0,0)\in\mathrm{Uf}(\Omega)\subset\{(y_1,0,0)\mid 1/2\le y_1\le 1\},\\
&\frac{\partial^2 V_\Omega^{(\alpha)}}{\partial x_1^2}(\lambda,0,0)\\
&=-\pi(3-\alpha)\tan^2(\pi/10)\int_0^1\big((\lambda-t)^2+\tan^2(\pi/10)t\big)^{(\alpha-5)/2}(\lambda-t)\,dt\\
&\qquad+2\pi(3-\alpha)(\lambda-1)\int_0^{\tan(\pi/10)}((\lambda-1)^2+r^2)^{(\alpha-5)/2}r\,dr,
\end{aligned}$$

and the graph of the second derivative of $V_\Omega^{(5/2)}(\lambda,0,0)$ for $0\le\lambda\le 1$ is in Figure 11. Moreover, the contribution of the side to the boundary integral (the first integral) is shown in Figure 12. Hence, in this case, Ω has a unique $r^{-1/2}$-center.

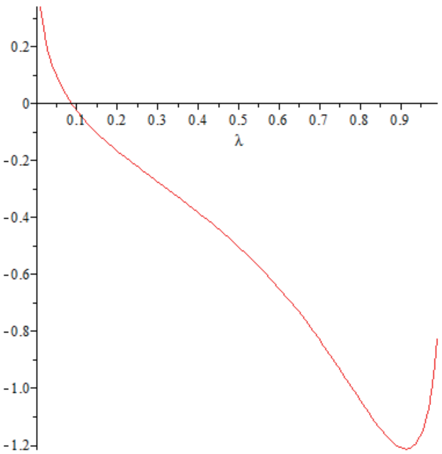

Fig. 11. The graph of $(\partial^2 V_\Omega^{(5/2)}/\partial x_1^2)(\lambda,0)$ when Ω is the body of revolution of a parabola

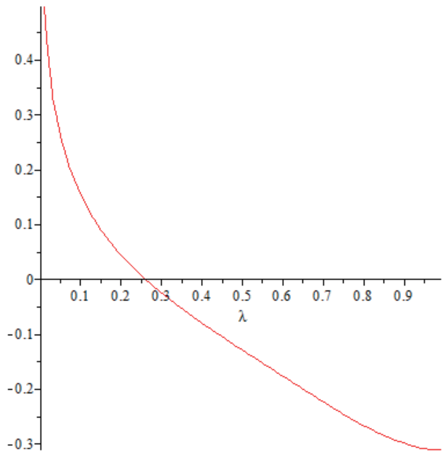

Fig. 12. The contribution of the side to the boundary integral

4. Uniqueness of a k-center. Let Ω be a body in $\mathbb{R}^m$. In this section, we investigate the uniqueness of a k-center of Ω. Set

$$\begin{aligned} d(\Omega) &= \min\{|z-w| \mid z\in \mathrm{Uf}(\Omega),\, w\in\partial\Omega\},\\ D(\Omega) &= \max\{|z-w| \mid z\in \mathrm{Uf}(\Omega),\, w\in\partial\Omega\}.\end{aligned} \tag{4.1}$$

4.1. Uniqueness of a center of an axially symmetric convex body

THEOREM 4.1. *Let $\omega:[0,1]\to[0,\infty)$ be a piecewise C^1 function such that the function $\omega^{m-1}: t\mapsto \omega(t)^{m-1}$ is concave. Let*

$$\Omega = \{y=(y_1,\bar{y})\in\mathbb{R}\times\mathbb{R}^{m-1} \mid 0\le y_1\le 1,\ |\bar{y}|\le\omega(y_1)\}.$$

Suppose that the kernel k is strictly decreasing and satisfies condition (C_α^1) for some $\alpha>1$. If $k'(r)/r$ is increasing on the interval $(d(\Omega),D(\Omega))$, then the potential K_Ω is strictly concave on the minimal unfolded region.

Proof. Set

$$a = \min\Big\{t\in[0,1]\ \Big|\ \omega(t) = \max_{0\le\tau\le 1}\omega(\tau)\Big\},$$
$$b = \max\Big\{t\in[0,1]\ \Big|\ \omega(t) = \max_{0\le\tau\le 1}\omega(\tau)\Big\}.$$

Proposition 2.8 and the concavity of ω imply that $\mathrm{Uf}(\Omega)$ is contained in the line segment

$$\{(y_1,0)\in\mathbb{R}\times\mathbb{R}^{m-1} \mid a/2\le y_1\le (1+b)/2\}.$$

Therefore, we will show the negativity of $(\partial^2 K_\Omega/\partial x_1^2)(\lambda,0)$ for any $a/2\le \lambda\le(1+b)/2$.

By Proposition 2.1, we have

$$
\begin{aligned}
\frac{\partial^2 K_\Omega}{\partial x_1^2}(\lambda,0) &= -\int\limits_{\partial\Omega} \frac{k'\big(\sqrt{(\lambda-y_1)^2+|\bar{y}|^2}\big)}{\sqrt{(\lambda-y_1)^2+|\bar{y}|^2}}(\lambda-y_1)e_1\cdot n(y)\,d\sigma(y) \\
&= \lambda\sigma_{m-2}(S^{m-2})\int\limits_0^{\omega(0)} \frac{k'(\sqrt{\lambda^2+r^2})}{\sqrt{\lambda^2+r^2}} r^{m-2}\,dr \\
&\quad + \frac{\sigma_{m-2}(S^{m-2})}{m-1}\int\limits_0^1 \frac{k'\big(\sqrt{(\lambda-t)^2+\omega(t)^2}\big)}{\sqrt{(\lambda-t)^2+\omega(t)^2}}(\lambda-t)\,d\omega(t)^{m-1} \\
&\quad - (\lambda-1)\sigma_{m-2}(S^{m-2})\int\limits_0^{\omega(1)} \frac{k'\big(\sqrt{(\lambda-1)^2+r^2}\big)}{\sqrt{(\lambda-1)^2+r^2}} r^{m-2}\,dr.
\end{aligned}
$$

For any $a/2 \le \lambda \le (1+b)/2$, the first and third terms are obviously negative. Therefore, it is sufficient to show the negativity of the second integral.

We first consider the case of $a/2 \le \lambda \le a$. We decompose the second integral into

$$
\Big(\int\limits_0^{2\lambda-a} + \int\limits_{2\lambda-a}^{\lambda} + \int\limits_{\lambda}^{a}\Big) \frac{k'\big(\sqrt{(\lambda-t)^2+\omega(t)^2}\big)}{\sqrt{(\lambda-t)^2+\omega(t)^2}}(\lambda-t)\,d\omega(t)^{m-1}.
$$

For any $0 \le \delta \le a-\lambda$, the concavity of ω^{m-1} implies $0 \le (\omega^{m-1})'(\lambda+\delta) \le (\omega^{m-1})'(\lambda-\delta)$, and the monotonicity of ω implies $0 \le \omega(\lambda-\delta) \le \omega(\lambda+\delta)$. Hence we obtain

$$
\begin{aligned}
&\Big(\int\limits_{2\lambda-a}^{\lambda} + \int\limits_{\lambda}^{a}\Big) \frac{k'\big(\sqrt{(\lambda-t)^2+\omega(t)^2}\big)}{\sqrt{(\lambda-t)^2+\omega(t)^2}}(\lambda-t)\,d\omega(t)^{m-1} \\
&\qquad = \int\limits_0^{a-\lambda} \frac{k'\big(\sqrt{(\delta^2+\omega(\lambda-\delta)^2}\big)}{\sqrt{\delta^2+\omega(\lambda-\delta)^2}}(\omega^{m-1})'(\lambda-\delta)\delta\,d\delta \\
&\qquad\quad - \int\limits_0^{a-\lambda} \frac{k'\big(\sqrt{\delta^2+\omega(\lambda+\delta)^2}\big)}{\sqrt{\delta^2+\omega(\lambda+\delta)^2}}(\omega^{m-1})'(\lambda+\delta)\delta\,d\delta \\
&\qquad \le 0.
\end{aligned}
$$

Furthermore, we can easily get

$$
\int\limits_0^{2\lambda-a} \frac{k'\big(\sqrt{(\lambda-t)^2+\omega(t)^2}\big)}{\sqrt{(\lambda-t)^2+\omega(t)^2}}(\lambda-t)\,d\omega(t)^{m-1} < 0,
$$

which completes the proof in the case of $a/2 \le \lambda \le a$.

The same argument works for $b \le \lambda \le (1+b)/2$. Furthermore, the negativity of $(\partial^2 K_\Omega/\partial x_1^2)(\lambda,0)$ for $a \le \lambda \le b$ is obvious. ∎

Corollary 4.2. *Let Ω and k be as in Theorem* 4.1. *Then Ω has a unique k-center.*

Remark 4.3. When $\omega(t) = t^p$, the assumption "ω^{m-1} is concave" corresponds to $0 \le p \le 1/(m-1)$.

Remark 4.4. In the proof of Theorem 4.1, in order to show the negativity of $(\partial^2 K_\Omega/\partial x_1^2)(\lambda, 0)$, we decomposed the boundary integral expression of $(\partial^2 K_\Omega/\partial x_1^2)(\lambda, 0)$ into three integrals, over the left base, the side and the right base. The integrals over the bases were obviously negative, and we showed the negativity of the integral over the side.

Unfortunately, this argument does not work for any axially symmetric convex body Ω. When we apply this argument to the cone of Example 3.4, the boundary integral over the side is not negative on the minimal unfolded region. In other words, in order to show the negativity of $(\partial^2 K_\Omega/\partial x_1^2)(\lambda, 0)$ for any axially symmetric convex body Ω, we have to estimate the boundary integrals over the bases in more detail. We have not been able to do it and leave the following problem as a conjecture: *Does an axially symmetric convex body Ω have a unique k-center? More generally, does a convex body Ω have a unique k-center? Assume some conditions on the kernel k if necessary.*

4.2. Uniqueness of a center of a non-obtuse triangle

Theorem 4.5. *Let Ω be a non-obtuse triangle in $\mathbb{R}^2$. Suppose that the kernel k is strictly decreasing and satisfies condition (C^1_α) for some $\alpha > 1$. If $k'(r)/r$ is increasing on the interval $(d(\Omega), D(\Omega))$, then the potential K_Ω is strictly concave on the minimal unfolded region of Ω.*

Proof. For $-\pi/2 \le \theta \le \pi/2$, let

$$R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

We will show that the second derivative $\partial^2 K_{R_\theta \Omega}/\partial x_1^2$ is negative on the minimal unfolded region of $R_\theta\Omega$ for any $-\pi/2 \le \theta \le \pi/2$.

Let O be the origin, P the point $(1, 0)$, and Q a point (a, b) with

$$1/2 \le a \le 1, \qquad b > 0, \qquad (a - 1/2)^2 + b^2 \ge 1/4.$$

By an orthogonal transformation of $\mathbb{R}^2$, we may assume that $\Omega = \triangle OPQ$. Let A, B and C be the middle points of the line segments OP, PQ and QO, respectively. We remark that the minimal unfolded region of Ω is contained in $\triangle ABC$ (see Example 2.5).

We identify the notation z_j for the jth coordinate with the function $z_j : \mathbb{R}^2 \ni (z_1, z_2) \mapsto z_j \in \mathbb{R}$. We denote the point $R_\theta P$ by P_θ for short, and so on.

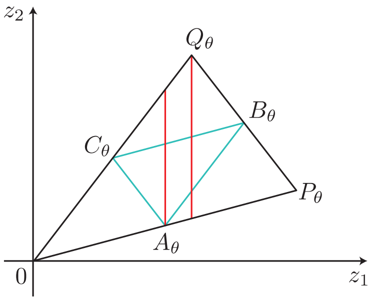

Fig. 13. Case I.1

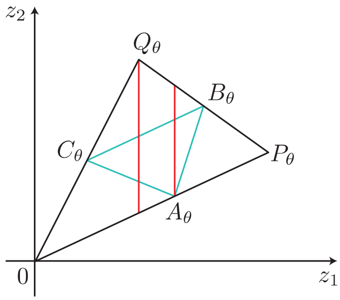

Fig. 14. Case I.2

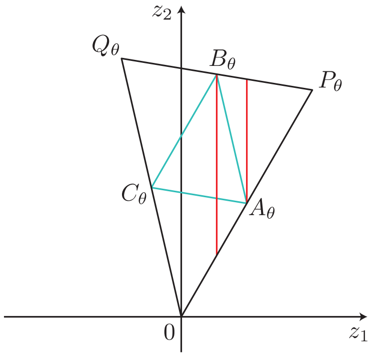

Fig. 15. Case I.3.1

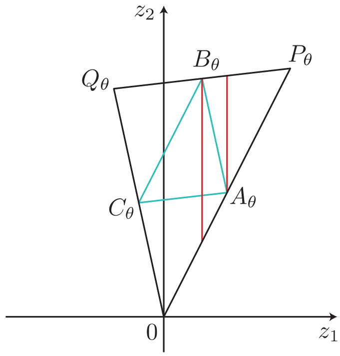

Fig. 16. Case I.3.2

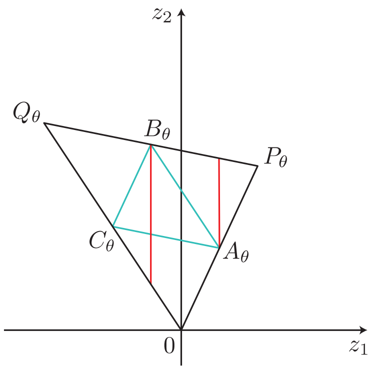

Fig. 17. Case I.4.1

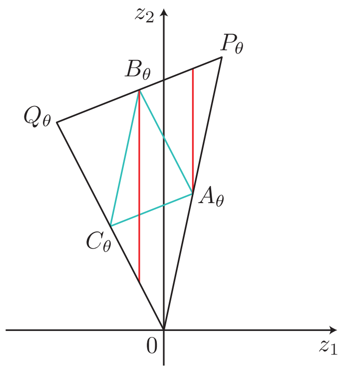

Fig. 18. Case I.4.2

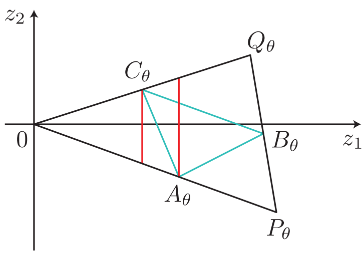

Fig. 19. Case II.1

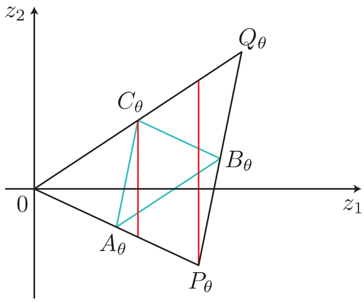

Fig. 20. Case II.2.1

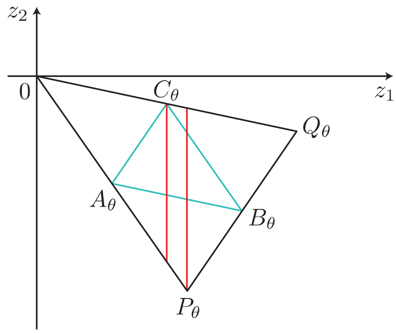

Fig. 21. Case II.2.2

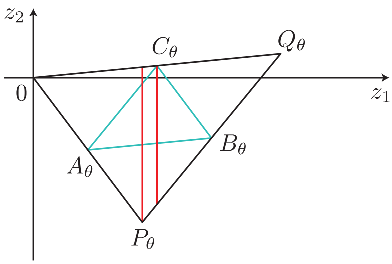

Fig. 22. Case II.3.1

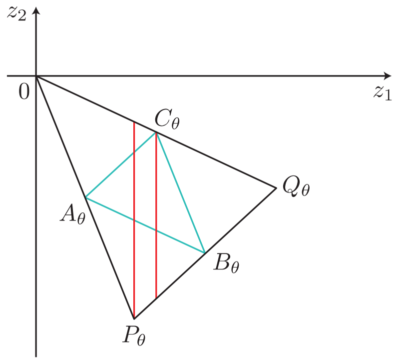

Fig. 23. Case II.3.2

We have to consider the following eleven cases of the position of $R_\theta \Omega$ (see Figures 13 to 23):

CASE I: The rotation angle θ is non-negative.

I.1: $z_1(A_\theta) \le z_1(Q_\theta) \le z_1(B_\theta)$.
I.2: $z_1(Q_\theta) \le z_1(A_\theta) \le z_1(B_\theta)$.
I.3.1: $0 \le z_1(B_\theta) \le z_1(A_\theta)$ and $\mathrm{slope}(P_\theta Q_\theta) \le 0$.
I.3.2: $0 \le z_1(B_\theta) \le z_1(A_\theta)$ and $\mathrm{slope}(P_\theta Q_\theta) \ge 0$.
I.4.1: $z_1(B_\theta) \le 0 \le z_1(A_\theta)$ and $\mathrm{slope}(P_\theta Q_\theta) \le 0$.
I.4.2: $z_1(B_\theta) \le 0 \le z_1(A_\theta)$ and $\mathrm{slope}(P_\theta Q_\theta) \ge 0$.

CASE II: The rotation angle θ is non-positive.

II.1: $z_1(C_\theta) \le z_1(A_\theta)$.
II.2.1: $z_1(A_\theta) \le z_1(C_\theta) \le z_1(P_\theta)$ and $\mathrm{slope}(OQ_\theta) \ge 0$.
II.2.2: $z_1(A_\theta) \le z_1(C_\theta) \le z_1(P_\theta)$ and $\mathrm{slope}(OQ_\theta) \le 0$.
II.3.1: $z_1(P_\theta) \le z_1(C_\theta)$ and $\mathrm{slope}(OQ_\theta) \ge 0$.
II.3.2: $z_1(P_\theta) \le z_1(C_\theta)$ and $\mathrm{slope}(OQ_\theta) \le 0$.

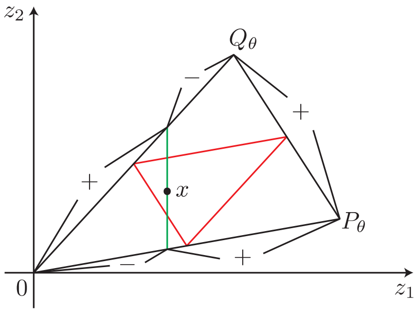

Fig. 24

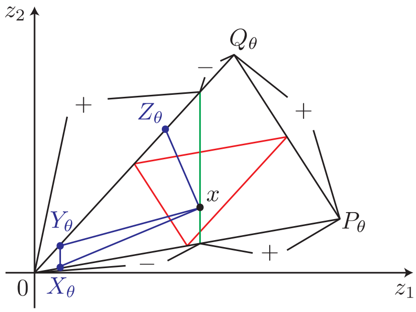

Fig. 25

We show the negativity of

$$\frac{\partial^2 K_{R_\theta \Omega}}{\partial x_1^2}(x) = -\int\limits_{\partial R_\theta \Omega} \frac{k'(r)}{r}(x_1 - y_1) e_1 \cdot n(y)\, d\sigma(y)$$
$$= -\int\limits_{\partial R_\theta \Omega} \frac{k'(r)}{r}(x_1 - y_1)\, dy_2$$

for any $x \in R_\theta(\triangle ABC)$ only in Case I.1. The other cases are analogous. Fix x in $R_\theta(\triangle ABC)$.

Suppose $z_1(C_\theta) \le x_1 \le z_1(A_\theta)$. Then we obtain the following inequalities in the same manner as in Theorem 4.2 (see Figure 24):

$$\int\limits_{OP_\theta} \frac{k'(r)}{r}(x_1 - y_1)\, dy_2 > 0, \qquad \int\limits_{Q_\theta O} \frac{k'(r)}{r}(x_1 - y_1)\, dy_2 > 0.$$

Thus the second derivative of $K_{R_\theta \Omega}$ is negative at x.

Suppose $z_1(A_\theta) \le x_1 \le z_1(Q_\theta)$. Set

$$\begin{aligned} X_\theta &= \big(2x_1 - z_1(P_\theta), \mathrm{slope}(OP_\theta)(2x_1 - z_1(P_\theta))\big), \\ Y_\theta &= \big(2x_1 - z_1(P_\theta), \mathrm{slope}(OQ_\theta)(2x_1 - z_1(P_\theta))\big), \\ Z_\theta &= \big(2x_1 - z_1(Q_\theta), \mathrm{slope}(OQ_\theta)(2x_1 - z_1(Q_\theta))\big). \end{aligned}$$

We obtain the following inequalities in the same manner as in Theorem 4.1 (see Figure 25):

$$\int\limits_{X_\theta P_\theta} \frac{k'(r)}{r}(x_1 - y_1)\, dy_2 > 0, \qquad \int\limits_{Q_\theta Z_\theta} \frac{k'(r)}{r}(x_1 - y_1)\, dy_2 > 0.$$

Let us show the positivity of the contour integral along the line segments $Y_\theta O$ and OX_θ. We remark that, for any $0 \le \delta \le z_1(X_\theta)$, we have

$$\begin{aligned} &(\mathrm{slope}(OP_\theta) + \mathrm{slope}(OQ_\theta))(z_1(X_\theta) - \delta) - 2x_2 \\ &\qquad \le (\mathrm{slope}(OP_\theta) + \mathrm{slope}(OQ_\theta))(z_1(X_\theta) - \delta) \\ &\qquad\quad - 2\big(\mathrm{slope}(OQ_\theta)(x_1 - z_1(A_\theta)) + z_2(A_\theta)\big) \end{aligned}$$

$$
\begin{aligned}
&= 2x_1 \operatorname{slope}(OP_\theta) - 2z_2(P_\theta) - \delta(\operatorname{slope}(OP_\theta) + \operatorname{slope}(OQ_\theta)) \\
&\le -\delta(\operatorname{slope}(OP_\theta) + \operatorname{slope}(OQ_\theta)) \le 0,
\end{aligned}
$$

where the first and the second inequalities follow from the fact that x lies above the line $A_\theta B_\theta$ and from $x_1 \le z_1(P_\theta)$, respectively. This implies

$$
\begin{aligned}
&\left| \begin{pmatrix} z_1(X_\theta) - \delta \\ \operatorname{slope}(z_1(X_\theta) - \delta) \end{pmatrix} - \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \right|^2 - \left| \begin{pmatrix} z_1(Y_\theta) - \delta \\ \operatorname{slope}(OQ_\theta)(z_1(Y_\theta) - \delta) \end{pmatrix} - \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \right|^2 \\
&\qquad = ((\operatorname{slope}(OP_\theta) + \operatorname{slope}(OQ_\theta))(z_1(OP_\theta) - \delta) - 2x_2) \\
&\qquad\quad \times (\operatorname{slope}(OP_\theta) - \operatorname{slope}(OQ_\theta))(z_1(X_\theta) - \delta) \ge 0
\end{aligned}
$$

for any $0 \le \delta \le z_1(X_\theta)$. Hence

$$
\Big(\int_{Y_\theta O} + \int_{OX_\theta} \Big) \frac{k'(r)}{r}(x_1 - y_1)\, dy_2 > 0 \tag{4.2}
$$

as in Theorem 4.1 (see also Figure 25). Therefore the second derivative of $K_{R_\theta \Omega}$ is negative at x.

Suppose $z_1(Q_\theta) \le x_1 \le z_1(B_\theta)$. As in Theorem 4.1, we obtain (see Figure 26)

$$
\int_{X_\theta P_\theta} \frac{k'(r)}{r}(x_1 - y_1)\, dy_2 > 0, \qquad \int_{P_\theta Q_\theta} \frac{k'(r)}{r}(x_1 - y_1)\, dy_2 > 0.
$$

Since (4.2) also holds in this case, the second derivative of $K_{R_\theta \Omega}$ is negative at x (see also Figure 26). ■

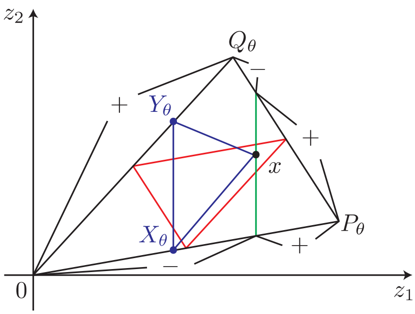

Fig. 26

Corollary 4.6. *Let Ω and k be as in Theorem 4.5. Then Ω has a unique k-center.*

Remark 4.7. In the proof of Theorem 4.5, we showed the concavity of the potential K_Ω on $\triangle ABC$. Since the minimal unfolded region is contained in the triangle, we obtained the conclusion.

Unfortunately, this argument does not work for any obtuse triangle (except isosceles triangles). This is because the minimal unfolded region of an obtuse triangle is not contained in the triangle whose vertices are the middle points of the edges (see Example 2.5).

5. Applications to specific centers. Let Ω be a body in $\mathbb{R}^m$. We consider some applications of the results in the previous section.

Recall that

$$V_\Omega^{(\alpha)}(x) = \begin{cases} \operatorname{sign}(m-\alpha)\int_\Omega r^{\alpha-m}\,dy & (0<\alpha\neq m),\\ -\int_\Omega \log r\,dy & (\alpha=m),\end{cases} \qquad x\in\mathbb{R}^m, \tag{5.1}$$

is called the $r^{\alpha-m}$-potential.

Definition 5.1 ([O1, Definition 3.1]). A point x is called an $r^{\alpha-m}$-*center* of Ω if it gives the maximum value of $V_\Omega^{(\alpha)}$.

Theorem 5.2 ([M1, Theorem 3.1]). *Let $m\ge 2$ and $0<\alpha\le 1$. If Ω is convex, then Ω has a unique $r^{\alpha-m}$-center.*

Theorem 5.3 ([O1, Theorem 3.15]). *If $m\ge 2$ and $\alpha\ge m+1$, then Ω has a unique $r^{\alpha-m}$-center.*

Theorem 5.4 ([O3, Theorem 3.8]). *Let $\tilde\Omega$ be a compact convex set in $\mathbb{R}^m$, $1<\alpha<m+1$, and*

$$\begin{aligned} f(\alpha) = -1 + \frac{\sqrt{m+1-\alpha}}{2}\left(4\sqrt{\frac{m+2-\alpha}{m+1-\alpha}} + \frac{1}{2}\sqrt{\frac{m+1-\alpha}{m+2-\alpha}}\right) \\ \times\left(2 + \frac{3}{\left(1+\left(4\left(4\sqrt{\frac{m+2-\alpha}{m+1-\alpha}}+\frac12\sqrt{\frac{m+1-\alpha}{m+2-\alpha}}\right)^2+1\right)^{-(m+2-\alpha)/2}\right)^{1/(m-2)}-1}\right). \end{aligned}$$

If $\delta\ge f(\alpha)\operatorname{diam}\tilde\Omega$, then the parallel body $\tilde\Omega+\delta B^m = \{\tilde y+\delta w \mid \tilde y\in\tilde\Omega,\ w\in B^m\}$ has a unique $r^{\alpha-m}$-center.

Proposition 5.5. *Let Ω be as in Theorem* 4.1 *or* 4.5. *For any $1<\alpha<m+2$, Ω has a unique $r^{\alpha-m}$-center.*

Proof. If $1<\alpha<m+2$, direct computation shows that the kernel of $V_\Omega^{(\alpha)}$ satisfies the assumptions of Theorem 4.1 or 4.5. ∎

Remark 5.6. The new result in this paper (Proposition 5.5) is the uniqueness of an $r^{\alpha-m}$-center for $1<\alpha<m+1$ when Ω is not a parallel body as in Example 3.3 or 3.5.

Let

$$(5.2) \qquad A_\Omega(x,h) = \int_\Omega \frac{h}{(r^2+h^2)^{(m+1)/2}}\,dy, \qquad x \in \mathbb{R}^m,\ h>0.$$

It is well-known that A_Ω satisfies the Laplace equation for the upper half-space,

$$(5.3) \qquad \Delta A_\Omega(x,h) = \left(\sum_{j=1}^m \frac{\partial^2}{\partial x_j^2} + \frac{\partial^2}{\partial h^2}\right) A_\Omega(x,h) = 0, \qquad x \in \mathbb{R}^m,\ h>0,$$

and the boundary condition

$$(5.4) \qquad \lim_{h\to 0^+} A_\Omega(x,h) = \frac{\sigma_m(S^m)}{2}\chi_\Omega(x), \qquad x \in \mathbb{R}^m \setminus \partial\Omega.$$

The function $A_\Omega(x,h)$ has a geometric meaning: Let $x \in \mathbb{R}^m$ and $h>0$. Define $p_{(x,h)} : \Omega \to S^m$ by

$$(5.5) \qquad p_{(x,h)}(y) = \frac{(y,0)-(x,h)}{|(y,0)-(x,h)|} = \frac{(y,0)-(x,h)}{\sqrt{r^2+h^2}}.$$

The *solid angle* of Ω at (x,h) is defined as the spherical Lebesgue measure of the image $p_{(x,h)}(\Omega)$. Direct calculation shows that $A_\Omega(x,h)$ coincides with the solid angle of Ω at (x,h). In other words, $A_\Omega(x,h)$ gives the "visibility" of Ω at (x,h).

On the other hand, the function $A_\Omega(x,h)$ was introduced by Katsuyuki Shibata [Sh] to give an answer to PISA's problem "Where should a street-light be placed in a triangle-shaped park?". Shibata called a maximizer of $A_\Omega(\cdot,h)$ an *illuminating center* of Ω of height h.

THEOREM 5.7 ([Sak, Theorem 5.32, Proposition 5.33, Theorem 5.36]).

(1) *If $h \geq \sqrt{m+2}\,\tilde{D}(\Omega)$, where $\tilde{D}(\Omega)$ is a slight improvement of $D(\Omega)$, then Ω has a unique illuminating center.*
(2) *If $h \leq \sqrt{2/(m-1)}\,d(\Omega)$, if Ω is convex, and if* $\mathrm{Uf}(\Omega)$ *is contained in the interior of Ω, then Ω has a unique illuminating center.*
(3) *Let $\tilde{\Omega}$ be a compact convex set in $\mathbb{R}^m$. If $\delta \geq \sqrt{(m+2)(m-1)/2} \times \operatorname{diam}\tilde{\Omega}$, then for any h, the parallel body $\tilde{\Omega}+\delta B^m$ has a unique illuminating center.*

PROPOSITION 5.8. *Let Ω be as in Theorem* 4.1 *or* 4.5. *For any $h>0$, Ω has a unique illuminating center.*

Proof. Direct computation shows that the kernel of A_Ω satisfies the assumptions of Theorem 4.1 or 4.5 for any h. ■

REMARK 5.9. The new result in this paper (Proposition 5.8) is the uniqueness of an illuminating center without the assumption on h when Ω is not a parallel body as in Example 3.3 or 3.5.

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