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Enrichments of Boolean algebras by Presburger predicates

by

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Abstract. We give a unified treatment of the model theory of various enrichments of infinite atomic Boolean algebras, with special attention to quantifier eliminations, complete axiomatizations and decidability. Our main enrichment is by a predicate for the ideal of finite sets and predicates for congruence conditions on the cardinalities of finite sets, but we also give new proofs of some classical results. We then classify and compare the expressive power of the enriched theories.

1. Introduction. A classical result of Tarski (cf. [8, Theorem 16, p. 70]) proves quantifier elimination and decidability for the theory of infinite atomic Boolean algebras in a natural enrichment by certain definable predicates (see Theorem 2.1 below for the exact statement). Later, Feferman and Vaught (see [6] and Theorem 2.3) extended this result to the case when one adds a predicate for the ideal of finite sets (see also [5]). In this paper we extend this by adding predicates giving congruence conditions on cardinalities of finite sets. We then compare and classify the expressive power of these enrichments.

Our motivation for defining and studying this enrichment comes from our works on the model theory of the ring of adeles of number fields [4, 3]. The results in this paper together with those in [4, 3] give quantifier elimination and decidability of new enrichments of the ring of adeles of a number field which have more expressive power than the language of rings (see [3]).

2. Axiomatizations, quantifier elimination, and decidability. Let T denote the theory of infinite atomic Boolean algebras, in the Boolean language with signature $\{0, 1, \cap, \cup, \neg\}$. The main models are $Powerset(I)$

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(which denotes the powerset of I), for I an infinite set. These are clearly not the only models, since no countable model is a full powerset algebra. A complete set of axioms for T_0 is given by saying that our models are infinite Boolean algebras such that every nonzero element has an atom below it (see [2]).

We find it slightly more enlightening to work in the equivalent formalism of Boolean rings (see [1]), using the dictionary

$$x.y = x \cap y, \quad x + y = (x \cap \neg y) \cup (\neg x \cap y)$$
$$x \cap y = x.y, \quad x \cup y = x + y + xy.$$

Since all these "definitions" are quantifier-free, we can prove our quantifier-elimination by working in the categories of enriched Boolean rings. An ordering on a Boolean ring is defined by $x \leq y$ if and only if $x.y = x$.

The predicates needed for the quantifier elimination for T, as added by Tarski, are $C_n(x)$, $n \geq 1$, with the interpretation that there are at least n distinct atoms α with $\alpha \leq x$. Tarski [8] proved the following.

THEOREM 2.1. *The theory T of infinite atomic Boolean algebras in the enriched Boolean language with all the C_n ($n \geq 1$) is complete, decidable, has quantifier elimination, and is axiomatized by sentences stating that the models are infinite Boolean algebras and every nonzero element has an atom below it.*

We start by giving a new proof of this theorem. We shall then build on our proof to prove similar results for other enrichments. For this and the other enrichments, we prove quantifier elimination by a variant of the standard back-and-forth criterion, in a form given in Hodges' book [7, Exercise 4, p. 389], especially well suited to our situation. To apply this criterion it is crucial to note that Boolean algebras are locally finite. We shall demonstrate the forth-stages of the argument, since the back-stages are completely analogous.

Now suppose $\mathbb{B}_1$ and $\mathbb{B}_2$ are ω-saturated models of T, $\{\alpha_1, \dots, \alpha_m\}$, $\{\beta_1, \dots, \beta_m\}$ are finite Boolean subrings R_1, R_2 of $\mathbb{B}_1, \mathbb{B}_2$ respectively, and

$$F(\alpha_j) = \beta_j$$

is an isomorphism of Boolean rings, in addition respecting all C_n and $\neg C_n$ (interpreted respectively in $\mathbb{B}_1, \mathbb{B}_2$). Now in fact $m = 2^k$ for some $k \geq 1$. R_1 and R_2 are each atomic, but their atoms need not be atoms of $\mathbb{B}_1, \mathbb{B}_2$. If $k = 1$, then

$$R_1 = \{0, 1\} \subset \mathbb{B}_1, \quad R_2 = \{0, 1\} \subset \mathbb{B}_2.$$

Note that if some α_j is an atom of $\mathbb{B}_1$, then

$$\mathbb{B}_1 \models C_1(\alpha_j) \wedge \neg C_2(\alpha_j), \quad \text{so} \quad \mathbb{B}_2 \models C_1(\beta_j) \wedge \neg C_2(\beta_j),$$

so β_j is an atom of $\mathbb{B}_2$.

We use systematically the following function:

$$\sharp(x) = \begin{cases} n & \text{if } C_n(x) \wedge \neg C_{n+1}(x), \\ \infty & \text{if no such } n \text{ exists.} \end{cases}$$

Note that any map respecting each C_n and $\neg C_n$ preserves $\sharp$.

Now we do the back-and-forth argument. Let α be an element of $\mathbb{B}_1$ not in R_1. We try to extend F to the Boolean ring

$$R_1[\alpha] = \{r_1 + s_1.\alpha : r_1, s_1 \in R_1\}$$

of cardinals between 2^k and 2^{2k}.

Note. In any atomic Boolean algebra, every nonzero element is the supremum of the atoms below it (see [1]).

In particular, $R_1[\alpha]$ has atoms not in R_1. We get to $R_1[\alpha]$ from R_1 by successive adjunctions of the atoms of $R_1[\alpha]$, and so without loss of generality (for the extension problem) we can assume that α is an atom of $R_1[\alpha]$. We assume this henceforward.

Case 1: $k = 1$. Now, α and $1 - \alpha$ are atoms of $R_1[\alpha]$, though not necessarily of $\mathbb{B}_1$. Note that not both of $\sharp(\alpha)$ and $\sharp(1-\alpha)$ can be finite, but that and being nonzero is the only restriction on the pair $(\sharp(\alpha), \sharp(1-\alpha))$.

Clearly the extension problem is solved once one has a $\beta \in \mathbb{B}_2$ with $\beta \notin \{0, 1\}$ and

$$(\sharp(\beta), \sharp(1-\beta)) = (\sharp(\alpha), \sharp(1-\alpha)).$$

If $\sharp(\alpha)$ is finite, it is trivial to get β with $\sharp(\beta) = \sharp(\alpha)$ (just take β a sum of $\sharp(\alpha)$ atoms), and then $\sharp(1-\beta) = \sharp(1-\alpha)$ automatically).

If $\sharp(1-\alpha)$ is finite, a dual argument works. If

$$\sharp(\alpha) = \sharp(1-\alpha) = \infty,$$

we use ω-saturation of $\mathbb{B}_2$ to get β with

$$\sharp(\beta) = \sharp(1-\beta) = \infty.$$

Case 2: $k > 1$. Now $1 = \gamma_1 + \cdots + \gamma_k$ where the γ_i's are the atoms of R_1. It follows that for some $i_0 \in \{1, \ldots, k\}$ we must have

$$0 < \alpha.\gamma_{i_0} < \gamma_{i_0}.$$

Indeed, if $\alpha.\gamma_i = \gamma_i$ for all the $\alpha.\gamma_i$ which are nonzero, where $i \in \{1, \ldots, k\}$, then

$$\alpha = \alpha.1 = \alpha\Big(\sum_{1\le i\le k} \gamma_i\Big) = \sum_{1\le i\le k} \alpha.\gamma_i = \sum_{1\le i\le k,\, \alpha.\gamma_i \neq 0} \gamma_i \in R_1,$$

a contradiction.

Let $A = \alpha.\gamma_{i_0}$. Note that $A.\gamma_j = 0$ for all $j \neq i_0$ in $\{1, \ldots, k\}$ since $\gamma_j.\gamma_{i_0} = 0$ for all $j \in \{1, \ldots, k\}$. Since α is an atom of $R_1[\alpha]$, we must have

$A = \alpha.\gamma_{i_0} = \alpha$. So we have shown that α lies below a unique atom γ_{i_0} of R_1. In the following we shall write γ for γ_{i_0}.

It follows that the atoms of $R_1[\alpha]$ are (i) the atoms of R_1 distinct from γ, and (ii) α and $\gamma - \alpha$. Hence an arbitrary element of $R_1[\alpha]$ can be represented uniquely in the form

$$\epsilon_1.\alpha + \epsilon_2.(\gamma - \alpha) + \sum_\tau \epsilon_\tau.\tau$$

where the τ-summation is over all atoms of R_1 different from γ, and the ϵ's are each 0 or 1. Note that the three summands are pairwise disjoint.

Now clearly

$$\sharp\Big(\epsilon_1.\alpha + \epsilon_2.(\gamma - \alpha) + \sum_\tau \epsilon_\tau.\tau\Big) = \epsilon_1\sharp(\alpha) + \epsilon_2\sharp(\gamma - \alpha) + \sum_\tau \epsilon_\tau\sharp(\tau).$$

So the extension problem this time is to find β with

$$0 < \beta < F(\gamma), \quad \sharp(\beta) = \sharp(\alpha), \quad \sharp(F(\gamma) - \beta) = \sharp(\gamma - \alpha).$$

Now the key issue is $\sharp(\gamma)$ $(= \sharp(F(\gamma)))$.

Subcase 1: $\sharp(\gamma)$ *finite.* Then $\sharp(\gamma) = \sharp(\alpha) + \sharp(\gamma - \alpha)$, and both $\sharp(\alpha)$ and $\sharp(\gamma - \alpha)$ are greater than 0. To solve the extension problem we simply choose $0 < \beta < F(\gamma)$ with $\sharp(\beta) = \sharp(\alpha)$, and then it is automatic that

$$\sharp(F(\gamma) - \beta) = \sharp(\gamma - \alpha).$$

Subcase 2: $\sharp(\gamma)$ *infinite.* Then (cf. the slightly different Case 1) not both $\sharp(\alpha)$ and $\sharp(\gamma - \alpha)$ can be finite, but there is no other constraint except that each is positive.

The argument goes exactly as in Case 1, with an appeal to ω-saturation when both $\sharp(\alpha)$ and $\sharp(\gamma - \alpha)$ are infinite.

The proof of Theorem 2.1 is complete.

Note 2.2. We build on this proof to get analogous results for several enriched formalisms. The essential point will be that the choice of β will now involve more constraints than above, and all our work will be to show these constraints can be met.

2.1. A predicate for the ideal of finite sets. We enrich the language of T by a unary predicate Fin, and extend the axioms of T by axioms saying that Fin is a proper ideal, and, for each $n < \omega$,

$$(\forall x)(\sharp(x) \leq n \Rightarrow Fin(x)).$$

We call these the *basic ideal axioms.* They can be stated for any ideal J (in our case $J = Fin$).

In addition we add the crucial

Main Axiom. $(\forall x)(\neg\, Fin(x) \Rightarrow (\exists y)(y < x \wedge \neg\, Fin(y) \wedge \neg\, Fin(x - y)))$.

In this way we get a theory T^{fin}. We interpret *Fin* in *Powerset*(I) as the ideal of *Finite* sets. Note that Theorem 3.1 in Section 3 shows that *Fin* is not definable in the language of the theory T. So this is not a definitional expansion of the theory of infinite atomic Boolean algebras.

We will prove a quantifier elimination theorem as for T, using the predicates C_n and *Fin*. This result is due to Feferman and Vaught [6]. We give a new proof based on our proof of Theorem 2.1.

THEOREM 2.3. *The theory* T^{fin} *of infinite atomic Boolean algebras with the set of finite sets distinguished is complete, decidable and has quantifier elimination with respect to all the* C_n $(n \geq 1)$ *and Fin. The axioms required for completeness are the axioms of* T *together with sentences expressing that Fin is a proper ideal, the sentence*

$$(\forall x)(\neg\, Fin(x) \Rightarrow (\exists y)(y < x \wedge \neg\, Fin(y) \wedge \neg\, Fin(x - y))).$$

and, for each $n < \omega$, *the sentence* $(\forall x)(\sharp(x) \leq n \Rightarrow Fin(x))$.

We use the same notation and formalism, except that in addition the map F now respects *Fin* and $\neg Fin$.

CASE 1: $k = 1$. We add α and $1 - \alpha$ and want to extend F to $R_1[\alpha]$. We already know how to handle the various possibilities for $\sharp(\alpha)$ and $\sharp(1 - \alpha)$.

Note that one cannot have both $Fin(\alpha)$ and $Fin(1 - \alpha)$.

In the case $\sharp(\alpha) < \infty$, we have $Fin(\alpha)$ and $\neg\, Fin(1 - \alpha)$. If we choose $\beta \notin \{0, 1\}$ with $\sharp(\alpha) = \sharp(\beta)$ (as we can do by the proof of Theorem 2.1), it is automatic that $Fin(\beta)$ and $\neg\, Fin(1 - \beta)$.

Similarly if $\sharp(1 - \alpha) < \infty$, we have $Fin(1 - \alpha)$ and $\neg\, Fin(\alpha)$, and again choosing β as in the proof of Theorem 2.1 gives $\sharp(1-\beta) = \sharp(1-\alpha)$, $Fin(1-\beta)$, and $\neg\, Fin(\beta)$.

So the remaining case is

$$\sharp(\alpha) = \sharp(1 - \alpha) = \infty.$$

In this case more care is required as it leaves open the possibility that $Fin(\alpha) \wedge \neg\, Fin(1-\alpha)$, or $\neg\, Fin(\alpha) \wedge Fin(1-\alpha)$ (can happen by compactness in a nonstandard model).

SUBCASE 1: $Fin(\alpha)$. We simply have to choose β so that

$$Fin(\beta), \quad \sharp(\beta) = \infty, \quad \sharp(1 - \beta) = \infty, \quad \neg\, Fin(1 - \beta).$$

This is trivial by ω-saturation.

SUBCASE 2: $\neg\, Fin(\alpha)$. There are two subcases:

SUBSUBCASE 2.1: $Fin(1-\alpha)$. We have to use ω-saturation *and* the Main Axiom. First use the Main Axiom to find some δ in $\mathbb{B}_2$ with

$$\neg\, Fin(\delta), \quad \neg\, Fin(1 - \delta).$$

Now use ω-saturation to get μ with

$$\mu \leq 1-\delta, \qquad Fin(\mu), \qquad \sharp(\mu)=\infty.$$

Now take β as $1-\mu$. Clearly $\neg\, Fin(\beta)$ but $Fin(1-\beta)$, and $\sharp(1-\beta)=\infty$.

Subsubcase 2.2: $\neg\, Fin(1-\alpha)$. Just use the Main Axiom to get β with $\neg\, Fin(\beta)$ and $\neg\, Fin(1-\beta)$.

Now we get to

Case 2: $k>1$. As before we need only make minor changes to the procedure in the proof of Theorem 2.1. We preserve the notation (especially for the atom γ), and try to extend F to preserve Fin (and $\neg Fin$) as well. As in the proof of Theorem 2.1, we can assume that α is an atom of $R_1[\alpha]$.

So we have

$$Fin\Big(\epsilon_1.\alpha+\epsilon_2.(\gamma-\alpha)+\sum_\tau \epsilon_\tau.\tau\Big) \Leftrightarrow Fin(\epsilon_1.\alpha)\wedge Fin(\epsilon_2.(\gamma-\alpha)\wedge Fin\Big(\sum_\tau \epsilon_\tau.\tau\Big),$$

where the ϵ's are either 0 or 1. Note that the summands are disjoint.

Thus it is clear that Fin and $\neg\, Fin$ are preserved by the choice of β if and only if

$$Fin(\alpha)\Leftrightarrow Fin(\beta) \quad \text{and} \quad Fin(\gamma-\alpha)\Leftrightarrow Fin(F(\gamma)-\beta)$$

(provided $0<\beta<F(\gamma)$).

If $Fin(\gamma)$ then clearly $Fin(\alpha)$ and $Fin(\gamma-\alpha)$, with the same for β and $F(\gamma)-\beta$ if chosen as in the proof of Theorem 2.1.

If $\neg\, Fin(\gamma)$ then at least one of α and $\gamma-\alpha$ satisfies $\neg\, Fin$, with no other constraint except that $\sharp(\alpha)$ and $\sharp(\gamma-\alpha)$ are each nonzero.

Subcase 1: *$Fin(\alpha)$ and $\sharp(\alpha)<\infty$.* This is handled just as in the proof of Theorem 2.1.

Subcase 2: *$Fin(\alpha)$ and $\sharp(\alpha)=\infty$.* Then automatically $\neg\, Fin(\gamma-\alpha)$. So we need

$$\beta\leq F(\gamma), \qquad Fin(\beta), \qquad \sharp(\beta)=\infty.$$

This is easily done by ω-saturation.

Subcase 3: *$Fin(\gamma-\alpha)$ and $\sharp(\gamma-\alpha)<\infty$.* Exactly like Subcase 1.

Subcase 4: *$Fin(\gamma-\alpha)$ and $\sharp(\gamma-\alpha)=\infty$.* Exactly like Subcase 2.

Subcase 5: *$\neg\, Fin(\alpha)$ and $\neg\, Fin(\gamma-\alpha)$.* By the Main Axiom applied below $F(\gamma)$, there exists $\beta\in\mathbb{B}_2$ such that $\neg\, Fin(\beta)$ and $\neg\, Fin(F(\gamma)-\beta)$.

This concludes the proof of quantifier elimination.

Remark 2.4. Note that T^{fin} is not complete if we remove the Main Axiom since in that case the finite-cofinite algebra on an index set I (defined as the set of finite and cofinite subsets of I, and denoted $\mathbb{B}_{\mathrm{fin/cofin}}(I)$)

and the powerset $Powerset(I)$ are both models which are not elementarily equivalent.

Note that in the Boolean language with signature $\{0, 1, \cap, \cup, \neg\}$, $\mathbb{B}_{\text{fin/cofin}}(I)$ is an elementary substructure of $Powerset(I)$. This follows from Theorem 2.1 since $\mathbb{B}_{\text{fin/cofin}}(I)$ and $Powerset(I)$ have the same atoms.

NOTE 2.5. There are many complete extensions of the basic ideal axioms (for an ideal J). The Main Axiom gives a unique one, as does the axiom $\mathbb{B}/J \cong \{0, 1\}$ (true in the finite-cofinite algebra). There are also examples where $\mathbb{B}/J \cong \mathbb{B}_k$, where $\mathbb{B}_k$ is any fixed finite k-element Boolean algebra.

A construction of such a Boolean algebra can be given as follows. Let $\mathbb{B} = \mathbb{B}_k^\omega$, where $\mathbb{B}_k$ is a k-element Boolean algebra, i.e. the functions $f : \omega \to \mathbb{B}_k$. Note that $\mathbb{B}$ is atomic with atoms the functions which are 0 except at one $n \in \omega$, where the value is an atom. Let

$$J = \{f \in \mathbb{B} : f(0) = 0\}.$$

Then $\mathbb{B}/J \cong \mathbb{B}_k$.

2.2. An enrichment by predicates for the ideal of finite sets and for congruence conditions on cardinalities of finite sets. We now add to the language of the theory T^{fin} unary predicates $Res(n, r)(x)$ for $n, r \in \mathbb{Z}$, $n > 0$, with the intended interpretation, in $Powerset(I)$, that $Fin(x)$ and the cardinal of x is congruent to r modulo n. There are various "arithmetic" axioms aside from

$$(\forall x)(Res(n, r)(x) \Rightarrow Fin(x))$$

for all n, r. For example, one clearly wants an axiom scheme stating that if $Fin(x)$ holds and $\sharp(x) = m$ where m is congruent to r modulo n, then $Res(n, r)(x)$ holds. Note that this implies

$$Res(n, 0)(0).$$

Also, we need

$$(\forall x)(Res(n, r)(x) \wedge r \equiv s \pmod{n} \Rightarrow Res(n, s)(x))$$

and

$$(\forall x)(Res(n, r)(x) \wedge r \not\equiv s \pmod{n} \Rightarrow \neg\, Res(n, s)(x)$$

for all n, r, s. One also needs

$$(\forall x)(Res(m, r)(x) \Rightarrow Res(n, r)(x)),$$

if $n \mid m$, and

$$(\forall x)\Big(Fin(x) \Rightarrow \bigvee_{0 \leq r < n} Res(n, r)(x)\Big)$$

for all m, n.

Finally, we need "finite additivity" axioms, namely

$$(\forall x)(\forall y)(x \cap y = 0 \wedge \mathit{Res}(n,r)(x) \wedge \mathit{Res}(n,s)(y) \Rightarrow \mathit{Res}(n,r+s)(x \cup y))$$

for all n, r, s, and

$$(\forall x)(\forall y)\Big(x \cap y = 0 \wedge \mathit{Res}(n,r)(x \cup y) \Rightarrow \bigvee_{\substack{0 \le s < n\\ 0 \le t < n\\ s+t \equiv r \pmod n}} \mathit{Res}(n,s)(x) \wedge \mathit{Res}(n,t)(y)\Big)$$

for all n, r.

[It is easy to deduce from this the extension to the case of more than two variables, in inclusion/exclusion style.]

We call these the *Boolean–Presburger axioms*. Adding them to the axioms of T^{fin} we get a theory that we denote by $T^{\mathrm{fin,res}}$. Our main result is the following.

Theorem 2.6. *The theory $T^{\mathrm{fin,res}}$ of infinite atomic Boolean algebras in the enriched language with all the C_n $(n \ge 1)$ Fin, and all $\mathit{Res}(r,n)$ $(n, r \in \mathbb{Z},\ n > 0)$, is complete, decidable, and has quantifier elimination. The axioms needed to get the elimination are the axioms of T^{fin} together with the Boolean–Presburger axioms as follows:*

$$(\forall x)(\mathit{Res}(n,r)(x) \Rightarrow \mathit{Fin}(x)),$$
$$(\forall x)(\mathit{Fin}(x) \wedge \sharp(x) = m \wedge m \equiv r \pmod n \Rightarrow \mathit{Res}(n,r)(x)),$$
$$(\forall x)(\mathit{Res}(n,r)(x) \wedge r \equiv s \pmod n \Rightarrow \mathit{Res}(n,s)(x)),$$
$$(\forall x)(\mathit{Res}(n,r)(x) \wedge r \not\equiv s \pmod n \Rightarrow \neg\, \mathit{Res}(n,s)(x),$$
$$(\forall x)(\mathit{Res}(m,r)(x) \wedge n \,|\, m \Rightarrow \mathit{Res}(n,r)(x)),$$
$$(\forall x)\Big(\mathit{Fin}(x) \Rightarrow \bigvee_{0 \le r < n} \mathit{Res}(n,r)(x)\Big)$$

for all n, r, s, m,

$$(\forall x)(\forall y)(x \cap y = 0 \wedge \mathit{Res}(n,r)(x) \wedge \mathit{Res}(n,s)(y) \Rightarrow \mathit{Res}(n,r+s)(x \cup y)),$$

for all n, r, s, and

$$(\forall x)(\forall y)\Big(x \cap y = 0 \wedge \mathit{Res}(n,r)(x \cup y) \Rightarrow \bigvee_{\substack{0 \le s < n\\ 0 \le t < n\\ s+t \equiv r \pmod n}} \mathit{Res}(n,s)(x) \wedge \mathit{Res}(n,t)(y)\Big)$$

for all n, r.

Now we try to elaborate the back-and-forth of the proof of Theorem 2.3, with the initial assumption that F on R_1 respects all the C_n, Fin, and all $\mathit{Res}(n,r)$.

Case 1: $k = 1$. If neither $\mathit{Fin}(\alpha)$ nor $\mathit{Fin}(1-\alpha)$ there is nothing to prove, as all $\mathit{Res}(n,r)(\alpha)$ and $\mathit{Res}(n,r)(1-\alpha)$ are false, and the same will be true for the matching β used in the proof of Theorem 2.3.

If (exactly) one satisfies *Fin*, say α, we consider two subcases.

Subcase 1: $\sharp(\alpha) < \infty$. In this subcase, the truth of $Res(n,r)(\alpha)$ is determined by whether

$$\sharp(\alpha) \equiv r \pmod{n}.$$

This transfers automatically to the matching β of Theorem 2.3.

Subcase 2: $\sharp(\alpha) = \infty$. Note that any condition $\neg\, Res(n,r)(\alpha)$ is equivalent to a finite disjunction of various $Res(n,s)(\alpha)$, and so by saturation we need only get, for any $m \geq 1$, a matching $\beta_{\Sigma,m}$ satisfying

$$\sharp(\beta_{\Sigma,m}) \geq m \quad \text{and} \quad Res(n,r)(\beta_{\Sigma,m}),\ (n,r) \in \Sigma,$$

for any finitely many conditions $Res(n,r)(x)$, where $(n,r) \in \Sigma$ (where Σ is a finite set), satisfied by α.

The argument needed is a slight variant of a corresponding argument used in the proof of Theorem 2.3 (which depends on a similar argument in the proof of Theorem 2.1). All we need is $\sharp(\beta_{\Sigma,m}) \geq m$ and $\sharp(\beta_{\Sigma,m})$ in the nonempty set (of nonnegative integers)

$$\{l : l \equiv r \pmod{n},\ (n,r) \in \Sigma\}.$$

Here all we need is that any Presburger definable nonempty set of the form

$$\{l : l \equiv r \pmod{n},\ (n,r) \in \Sigma\}$$

has arbitrarily large members. This is obvious.

This, with saturation, gives the required β.

Case 2: $k > 1$. Again we preserve the notation from the proof of Theorem 2.3 (so γ is an atom of R_1 and $0 < \alpha < \gamma$).

We do the usual argument representing an arbitrary element of $R_1[\alpha]$ as a (disjoint) sum

$$\epsilon_1.\alpha + \epsilon_2.(\gamma - \alpha) + \sum_{\delta} \epsilon_\delta.\delta$$

(see proofs of Theorems 2.1 and 2.3).

By the disjointness, we see that just as *Fin* (and $\neg$ *Fin*) for such an element is determined by $Fin(\alpha)$ and $Fin(\gamma - \alpha)$, it is clear that then $Res(n,r)$ is determined by the $Res(n,r)(\alpha)$ and $Res(n,r)(\gamma - \alpha)$.

So a choice of β will preserve the basic relations and functions if and only if

$$Fin(\alpha) \Leftrightarrow Fin(\beta),$$
$$Fin(\gamma - \alpha) \Leftrightarrow Fin(F(\gamma) - \beta) \quad (\text{provided } 0 < \beta < F(\gamma)),$$

and

$$Res(n,r)(\alpha) \Leftrightarrow Res(n,r)(\beta),$$
$$Res(n,r)(\gamma - \alpha) \Leftrightarrow Res(n,r)(F(\gamma) - \beta).$$

We first consider the case when $\mathit{Fin}(\gamma)$. Then clearly $\mathit{Fin}(\alpha)$ and $\mathit{Fin}(\gamma-\alpha)$, with the same for β and $F(\gamma)-\beta$ if chosen as in the proof of Theorem 2.1 (where there are subcases). But what about $\mathit{Res}(n,r)(\beta)$, which must match $\mathit{Res}(n,r)(\alpha)$? We have to go back and look at the subcases:

Subcase 1: *$\sharp(\gamma)$ is finite.* As in the proof of Theorem 2.1, it is necessary to choose $0<\beta<F(\gamma)$ with $\sharp(\beta)=\sharp(\alpha)$. It is then automatic that $\mathit{Res}(n,r)(\beta)$ matches $\mathit{Res}(n,r)(\alpha)$.

Subcase 2: *$\sharp(\gamma)$ is infinite (and $\mathit{Fin}(\gamma)$).* Then (cf. Case 2 in the proof of Theorem 2.1) not both $\sharp(\alpha)$ and $\sharp(\gamma-\alpha)$ can be finite, but there is no other constraint except that each is positive.

If $\sharp(\alpha)$ is finite, then $\mathit{Res}(n,r)(\alpha)$ is determined by $\sharp(\alpha)$, and $\mathit{Res}(n,r)(\gamma-\alpha)$ is determined by disjointness. So in this case we need only match

$$\sharp(\beta)=\sharp(\alpha),$$

as in the proof of Theorem 2.1.

The case that $\sharp(\gamma-\alpha)$ is finite is dual.

The crucial case is when $\sharp(\alpha)$ and $\sharp(\gamma-\alpha)$ are both infinite. The matching problem is to get $\beta<F(\gamma)$ satisfying

$$\begin{aligned} \sharp(\beta) &\geq m_1 && \text{for all } m_1\in\mathbb{N},\\ \sharp(F(\gamma)-\beta) &\geq m_2 && \text{for all } m_2\in\mathbb{N},\\ \mathit{Res}(n,r)(\alpha) &\Rightarrow \mathit{Res}(n,r)(\beta) && \text{for all } n,r\in\mathbb{N}. \end{aligned}$$

This is like Case 1, Subcase 2. The saturation argument follows as before by the argument about Presburger definable sets.

Next we have to consider the situation when $\neg\,\mathit{Fin}(\gamma)$ holds. Then at least one of α and $\gamma-\alpha$ satisfies $\neg \mathit{Fin}$, with no other constraint except that $\sharp(\alpha)$ and $\sharp(\gamma-\alpha)$ are each nonzero.

Note that if $\neg\,\mathit{Fin}(\alpha)$ and $\neg\,\mathit{Fin}(\gamma-\alpha)$ both hold then the only way to have

$$\mathit{Fin}\Big(\epsilon_1.\alpha+\epsilon_2.(\gamma-\alpha)+\sum_{\delta}\epsilon_\delta.\delta\Big)$$

is that $\epsilon_1=\epsilon_2=0$, thus in the case $\neg\,\mathit{Fin}(\alpha)$ and $\neg\,\mathit{Fin}(\gamma-\alpha)$ both hold, the only elements of $R_1[\alpha]$ satisfying Fin are in R_1, and so the $\mathit{Res}(n,r)$ are determined. So one just has to get β with $\neg\,\mathit{Fin}(\beta)$ and $\neg\,\mathit{Fin}(F(\gamma)-\beta)$ as in Subcase 5 in the proof of Theorem 2.3.

We go quickly through the other cases.

Subcase 1: *$\mathit{Fin}(\alpha)$ and $\sharp(\alpha)<\infty$.* Then $\sharp(\alpha)$ determines all $\mathit{Res}(n,r)(\alpha)$, and the choice of β as in the proof of Theorem 2.3 gives the required correspondence.

Subcase 2: *Fin*(α) *and* $\sharp(\alpha) = \infty$. This is easily done by saturation, in the style of Subcase 2 of Case 1.

Subcase 3: *Fin*$(\gamma - \alpha)$ *and* $\sharp(\gamma - \alpha) < \infty$. Dual to Subcase 1.

Subcase 4: *Fin*$(\gamma - \alpha)$ *and* $\sharp(\gamma - \alpha) = \infty$. Dual to Subcase 2.

This concludes the proof.

We remark that it may be possible to find a strengthening of the language of $T^{\mathrm{fin,res}}$ using well-behaved strengthenings of Presburger arithmetic (see [9]).

3. Relative strength of the theories T, T^{fin}, and $T^{\mathrm{fin,res}}$.

Note that for each enrichment we have given a complete set of axioms in the appropriate formalism. Now we show that each enrichment is more expressive than its predecessor.

Theorem 3.1. *In no model of the theory of infinite atomic Boolean algebras can we define, in the formalism of the theory* T, *a predicate Fin satisfying the axioms of the theory* T^{fin}.

Proof. It suffices to show this for *Powerset*(ω). Suppose $\Phi(x)$ defines the intended interpretation of *Fin* in *Powerset*(ω). $\Phi(x)$ can be taken as a Boolean combination of conditions

$$p(x) = 0, \qquad C_k(q(v)) \quad \text{for } k \le N \in \mathbb{N},$$

where $p(x)$ and $q(x)$ are Boolean ring polynomials.

Going to disjunctive normal form we see that $\Phi(x)$ can be taken as a finite disjunction of conditions

$$p_1(x) = 0 \wedge \cdots \wedge p_k(x) = 0 \wedge p_{k+1}(x) \ne 0 \wedge \cdots \wedge p_{k+l}(x) \ne 0 \wedge \\ C_{s_1}(t_1(x)) \wedge \cdots \wedge C_{s_d}(t_d(x)) \wedge \neg C_{k_1}(r_1(x)) \wedge \cdots \wedge \neg C_{k_m}(r_m(x)).$$

Note that $k_i, s_j \ge 1$ and all the polynomials occurring are of one of the forms

$$0,\ 1,\ 1 + x,\ x.$$

Note that $C_k(0)$ is false and $C_k(1)$ is true. Also

$$1 + x = 0 \Leftrightarrow x = 1.$$

So our conjunction can be taken as Boolean combination of

$$x = 1, \quad x = 0, \qquad C_l(x),\ C_m(1 + x).$$

Only finitely many l, m occur in the disjunctive normal form.

We need only consider conjunctions of the form

$$C_l(x) \wedge C_m(1 + x) \wedge \neg C_r(x) \wedge \neg C_s(1 + x),$$

where not all of l, m, r, s need occur. Note only one of r, s can occur.

Consider each conjunction separately. Those which contain some $\neg C_r(x)$ can define only the set of elements a with $\sharp(a) < r$. So we need consider only conjunctions which contain no $\neg C_r(x)$.

If in such a conjunction some $\neg C_s(1+x)$ occurs, the conjunction can define only sets a with $\sharp(1+a)<s$, in particular only elements with $\sharp(a)=\infty$.

So we need only consider conjunctions

$$C_l(x) \wedge C_m(1+x),$$

where one of C_l, C_m may be missing. Any $a \in \mathit{Powerset}(\omega)$ with $\neg\, \mathit{Fin}(a) \wedge \neg\, \mathit{Fin}(1+a)$ will not satisfy this, contradiction. ■

The case of the theory $T^{\text{fin,res}}$ is harder. We will show the following.

THEOREM 3.2. *If p is a prime, the predicates $\mathit{Res}(p,r)(x)$ are not definable in $\mathit{Powerset}(\omega)$ from Fin, C_k and any $\mathit{Res}(q^m,r)$ for primes $q \neq p$.*

NOTE. $\mathit{Res}(p,r)$ is definable from the $\mathit{Res}(p^k,s)$ where $s \equiv r \pmod p$, for any $k > 1$.

Proof of Theorem 3.2. We give the proof for $p = 2$ and no other primes, and explain the general method at the end. We work in a nonstandard model $\mathbb{B}$ of $\mathit{Th}(\mathit{Powerset}(\omega))$ in the formalism of Theorem 2.6. What we need in this model is a $b \in \mathit{Fin}$ such that $C_k(b)$ for all $k \in \mathbb{N}$, and $\mathit{Res}(2,0)(b)$. For example, use compactness, or an ultrapower of $\mathit{Powerset}(\omega)$.

As usual we replace $\mathbb{B}$ by the corresponding Boolean ring R. Think of b as a nonstandard finite even element. In R we have the ideal *Fin*, and the filter *Cofin* (i.e. the c such that $\mathit{Fin}(1+c)$). However, we need to consider also

$$\mathit{Cofin}^{\mathrm{ST}} = \{c : \neg C_k(1+c) \text{ for some } k\},$$

i.e. the "standard" cofinite sets.

CLAIM 1. *$\{b\} \cup \mathit{Cofin}^{\mathrm{ST}}$ has the finite intersection property.*

Proof. Clearly $\mathit{Cofin}^{\mathrm{ST}}$ has the finite intersection property, and if $b.\tau = 0$ for some $\tau \in \mathit{Cofin}^{\mathrm{ST}}$, we have $b \leq 1 + \tau$, so b is standard finite. ■

So we get a nonprincipal ultrafilter D containing b. We now extend R by an element γ, with the conditions

$$\gamma.\alpha = \gamma \quad \text{if } \alpha \in D, \qquad \gamma.\alpha = 0 \quad \text{if } \alpha \notin D.$$

We use compactness to show that the conditions on γ are finitely satisfiable in R.

If we have finitely many conditions

$$\gamma.\alpha_1 = \gamma, \ldots, \gamma.\alpha_r = \gamma, \qquad \gamma.\beta_1 = 0, \ldots, \gamma.\beta_s = 0$$

with $\alpha_1, \ldots, \alpha_r \in D,\ \beta_1, \ldots, \beta_s \notin D$, then

$$\alpha_1 \cap \cdots \cap \alpha_r \in D,$$

and is infinite, and

$$\beta_1 \cup \cdots \cup \beta_r \notin D.$$

Therefore there exists

$$\delta \in (\alpha_1 \cap \cdots \cap \alpha_r) \setminus (\beta_1 \cup \cdots \cup \beta_s),$$

since otherwise

$$\alpha_1 \cap \cdots \cap \alpha_r \subset \beta_1 \cup \cdots \cup \beta_s,$$

which contradicts D being an ultrafilter. This proves finite satisfiability. Note that the argument shows there are in fact infinitely many such δ. Thus we get the extension $R[\gamma]$.

Note that $\gamma \notin R$: if $\gamma \in R$, then there is an atom s of R below γ. Since s is a standard finite set,

$$1 + s \in \mathit{Cofin}^{\mathrm{ST}},$$

so $1 + s \in D$, hence $s \notin D$, thus $\gamma.s = 0$.

Now $R[\gamma] = \{r + s.\gamma : r, s \in R\}$.

Claim 2. *γ is an atom in $R[\gamma]$.*

Proof. Assume that $(r + s.\gamma).\gamma = r + s.\gamma$. Then $r.\gamma + s.\gamma = r + s.\gamma$, so $r.\gamma = r$.

If $r \in D$, then $r.\gamma = \gamma$, hence $\gamma = r \in R$, a contradiction. Hence $r \notin D$, so $0 = r.\gamma = r$, so $r = 0$. Now $s.\gamma = \gamma$ or $s.\gamma = 0$. So γ is an atom of $R[\gamma]$. ∎

Claim 3. *All the atoms of R are also atoms of $R[\gamma]$.*

Proof. Let t be an atom of R. Assume that $(r + s.\gamma).t = r + s.\gamma$. Then since γ is an atom (which is by assumption different from t), we have

$$r.t = r + s.\gamma.$$

We have two cases. The first case is when $r \in D$, hence $r.t = 0$. In this case $r + s.\gamma = 0$. The second case is when $r \notin D$, hence $r.t = t$. In this case we have $t = r + s.\gamma$. ∎

Claim 4. *Any atom of $R[\gamma]$ is either an atom of R or γ.*

Proof. Suppose $r + s.\gamma$ is an atom of $R[\gamma]$. Since γ is an atom, $s.\gamma = 0$ or $s.\gamma = \gamma$. So $r + s.\gamma$ is either r or $r + \gamma$.

In the former case, it is an old atom. In the latter, $r + \gamma$ must be an atom. But this is a contradiction since $\gamma.(r + \gamma) = \gamma.r + \gamma = \gamma$. Hence $r + \gamma = \gamma$ since $\gamma \neq 0$ and $r + \gamma$ is an atom. ∎

Claim 5. *$R[\gamma]$ is atomic.*

Proof. Consider $r + \gamma \in R[\gamma]$. Let s be an atom of R below r. Then

$$1 + s \in \mathit{Cofin}^{\mathrm{ST}} \subset D,$$

so $s.\gamma = 0$. Hence $s.(r + \gamma) = s$, thus $s \leq r + \gamma$. ∎

Now we can finish the proof of the theorem. $R[\gamma]$ is infinite atomic and a model of the axioms of the theories T^{fin} and $T^{\mathrm{fin,res}}$. The predicates C_k thus have an interpretation in $R[\gamma]$.

Note that for every element a of R satisfying Fin, we have $a \notin D$, hence $\gamma.a = 0$, so

$$\gamma = \gamma.1 = \gamma(a + (1 - a)) = \gamma.(1 - a),$$

so γ lies below every cofinite element of R.

We give an interpretation Fin^* of the predicate Fin in $R[\gamma]$ as

$$\begin{aligned} &Fin^*(r + \gamma) \Leftrightarrow Fin(r), \\ &Fin^*(r) \Leftrightarrow Fin(r) \quad \text{for } r \in R. \end{aligned}$$

Note that Fin^* is closed under addition since Fin is so. Moreover,

$$\begin{aligned} &r.(s + \gamma) = r.s + \gamma, \\ &(r + \gamma)(s + \gamma) = r.s + (r + s).\gamma + \gamma = r.s + \gamma, \end{aligned}$$

hence Fin^* is closed under multiplication and is an ideal in $R[\gamma]$ since Fin is so.

As for the predicates C_k we define

$$\begin{aligned} &C_k(r + \gamma) \Leftrightarrow C_k(r) && \text{if } r \notin D, \\ &C_k(r + \gamma) \Leftrightarrow C_{k-1}(r) && \text{if } r \in D. \end{aligned}$$

We need to show that the predicates Fin, C_k and $\neg C_k$ are preserved in passing from R to $R[\gamma]$. This is clear for Fin. If $r \in R$ satisfies C_k in R, then it clearly satisfies C_k in $R[\gamma]$. Suppose $\neg C_l(r)$ holds in R. Then r is a standard finite element. So

$$1 + r \in Cofin^{\mathrm{ST}} \subset D,$$

hence $\gamma(1 + r) = \gamma$, so $\gamma.r = 0$, therefore γ does not lie below r, thus

$$R[\gamma] \models \neg C_l(r).$$

Since γ is a new atom below any element $r \in D$ and $b \in D$, b is not an even element in $R[\gamma]$, and

$$R[\gamma] \not\models Res(2, 0)(b).$$

This proves that $Res(2, 0)$ is not definable from C_k and Fin.

In general, we give the predicates $Res(n, j)$ interpretations in Fin^* as follows:

$$\begin{aligned} &Res^*(n, j)(r) = Res(n, j + 1)(r) && \text{if } r.\gamma = \gamma, \\ &Res^*(n, j)(r) = Res(n, j)(r) && \text{if } r.\gamma = 0, \\ &Res^*(n, j)(r + \gamma) = Res(n, j + 1)(r) && \text{if } r.\gamma = \gamma, \\ &Res^*(n, j)(r + \gamma) = Res(n, j)(r) && \text{if } r.\gamma = 0. \end{aligned}$$

We now prove that it is not possible to define $\mathit{Res}(p,s)$ from the predicates Fin, C_k, $k \geq 1$ and $\mathit{Res}(q^l, r)$ for primes $q \neq p$. If there was such a definition, the defining formula would involve only finitely many predicates

$$\mathit{Res}(q_1^{l_1}, r_1), \ldots, \mathit{Res}(q_m^{l_m}, r_m),$$

with each $q_j \neq p$. Let $\pi = \prod_{1 \leq j \leq m} q_j^{l_j}$.

By the Chinese remainder theorem there is $N \in \mathbb{Z}$ satisfying the congruences

$$N \equiv 0 \pmod{\pi}, \qquad N \equiv 1 \pmod{p}.$$

Applying the above method we get an element $b \in D$ such that

$$R \models \mathit{Res}(p,s)(b),$$

and new atoms $\gamma_1, \ldots, \gamma_N$ such that

$$\gamma_i.r = \gamma_i \quad \text{if } r \in D, \qquad \gamma_i.r = 0 \quad \text{if } r \notin D.$$

The first congruence implies

$$R[\gamma_1, \ldots, \gamma_N] \models \mathit{Res}(q_j^{l_j}, r_j)(b),$$

and the second congruence shows that $\mathit{Res}(p,s)(b)$ does not hold in $R[\gamma_1, \ldots, \gamma_N]$. The proof is complete. ■

4. Some applications to the model theory of restricted products. In this section, we briefly state some applications of Theorem 2.6 to the model theory of direct and restricted products, extending results of Feferman–Vaught [6]. We shall also state applications to the model theory of the ring of adeles of a number field following [4, 3].

We first recall the notion of a restricted product of a family of structures with respect to a formula (see [4, 6]). Let $\mathcal{L}$ be a language and $\mathcal{M}_i$ (for $i \in I$) a family of $\mathcal{L}$-structures. Let $\psi(x)$ denote an $\mathcal{L}$-formula in the single variable x. Let $\prod_{i\in I} \mathcal{M}_i$ denote the Cartesian product of the $\mathcal{M}_i$. The restricted product of $\mathcal{M}_i$ with respect to $\psi(x)$, denoted $\prod_{i\in I}^{(\psi)} \mathcal{M}_i$, is defined to be the set of all $(a_i)_i \in \prod_{i\in I} \mathcal{M}_i$ such that

$$\mathcal{M}_i \models \psi(a_i) \quad \text{for almost all } i \in I.$$

Examples are direct sums of abelian groups (with respect to the formula $x = 0$), and the ring of adeles $\mathbb{A}_K$ of a number field K defined as the restricted product with respect to the formula $v(x) \geq 0$ of the completions K_v of K, where v runs over all the valuations of K (it is a result that such a formula exists [4, 3]).

Given an $\mathcal{L}$-formula $\Phi(x_1, \ldots, x_n)$, and $f_1, \ldots, f_n \in \prod_{i\in I} \mathcal{M}_i$, the Boolean value is defined by

$$[[\Phi(f_1, \ldots, f_n)]] = \{i \in I : \mathcal{M}_i \models \Phi(f_1(i), \ldots, f_n(i))\}.$$

Now let $\mathfrak{L}^+$ be an enrichment of the language $\mathfrak{L}$ of Boolean algebras. Given an $\mathfrak{L}^+$-formula $\Psi(z_1,\ldots,z_m)$, and $\mathcal{L}$-formulas

$$\Phi_1(x_1,\ldots,x_n),\ldots,\Phi_m(x_1,\ldots,x_n),$$

enrich the restricted direct product $\prod_{i\in I}^{(\psi)}\mathcal{M}_i$ by n-place relations defined by

$$\begin{aligned}&\Theta_{(\Psi,\Phi_1,\ldots,\Phi_m)}(x_1,\ldots,x_n)\\&\qquad\Leftrightarrow \mathcal{P}(I)^+\models\Psi([[\Phi_1(x_1,\ldots,x_n)]],\ldots,[[\Phi_m(x_1,\ldots,x_n)]]),\end{aligned}$$

where $\mathcal{P}(I)^+$ is the enrichment of the Boolean algebra structure on $\mathcal{P}(I)$ to an $\mathfrak{L}^+$-structure. These n-place relations, for all n, yield a language for the restricted product extending the language $\mathcal{L}$ (the predicates of $\mathcal{L}$ are expressible in this language). See [6, 4] for details.

By results from [6, 4], restricted products have quantifier elimination in the language with all the predicates $\Theta_{(\Psi,\Phi_1,\ldots,\Phi_m)}$. This reduces the study of the elementary theory of the restricted direct product $\prod_{i\in I}^{(\psi)}\mathcal{M}_i$ and its definable subsets to that of the enriched Boolean algebra $\mathcal{P}(I)^+$ and the factors $\mathcal{M}_i$. The choice of enrichment of the Boolean algebra $\mathcal{P}(I)$ thus has applications to the elementary theory and study of definable subsets of restricted products.

The enrichments of $\mathcal{P}(I)$ by the predicates in the language of the theory $T^{\mathrm{fin,res}}$ and Theorem 2.6 on quantifier elimination and decidability, imply decidability and quantifier elimination for the ring of adeles $\mathbb{A}_K$ in a language that is stronger than the language of rings, relevant to such matters as the product formula for Hilbert symbol and other reciprocity laws in number theory. See [3] for details.

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