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AN ALGEBRA WHICH IS A MAXIMAL COMMUTATIVE SUBALGEBRA IN VERY FEW ALGEBRAS

BY

WIESŁAW ŻELAZKO (Warszawa)

Abstract. "Very few" in the title means two. We show that a unital real or complex algebra generated by a nilpotent of order two can be a maximal abelian subalgebra only in two algebras. One of them is of dimension three and the other of dimension four.

All algebras considered in this paper are unital and over the field of complex or real numbers. This field will be denoted by $\mathbb{K}$, and the unity by e. A commutative subalgebra $\mathcal{A}$ of such a non-commutative algebra A is called a *maximal abelian subalgebra* (briefly m.a.s.) if there is no strictly larger abelian subalgebra of A. It is clear that every m.a.s. in A must contain its unity. The fact that every commutative Banach algebra A is an m.a.s. in the algebra $L(A)$ of all bounded operators of the Banach space A was already known to Gelfand and used in [2]. Here the embedding of $\mathcal{A}$ in A is given by $x \mapsto T_x$, where the operator T_x is given by the left regular representation: $T_x y = xy$, $x, y \in A$. This fact is also true for any commutative algebra A if we take as $L(A)$ the algebra of all endomorphisms of the linear space A.

This paper was originally intended to deal with Banach algebras, but the proofs work for all algebras.

In this paper we shall consider an algebra $\mathcal{A}$ generated by an element ξ satisfying

$$\xi^2 = 0. \tag{1}$$

Thus

$$\mathcal{A} = \{\alpha e + \beta\xi : \alpha, \beta \in \mathbb{K}\}. \tag{2}$$

By the above mentioned result, it is an m.a.s. in the algebra $\mathbf{M}_2$ of all 2×2

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matrices with scalar entries, where it can be represented as

$$\tilde{\mathcal{A}} = \left\{ \begin{pmatrix} \alpha & \beta \\ 0 & \alpha \end{pmatrix} : \alpha, \beta \in \mathbb{K} \right\}. \tag{3}$$

It is also an m.a.s. in the algebra

$$\tilde{A}_0 = \left\{ \begin{pmatrix} \alpha & \beta \\ 0 & \gamma \end{pmatrix} : \alpha, \beta, \gamma \in \mathbb{K} \right\}. \tag{4}$$

The result below gives the converse. Thus $\mathbf{M}_2$ and $\tilde{A}_0$ are the only (up to isomorphism) algebras in which $\mathcal{A}$ is an m.a.s.

Theorem. *Let $\mathcal{A}$ be the algebra defined by* (1) *and* (2). *Let A be a* (*non-commutative unital*) *algebra in which $\mathcal{A}$ is a maximal abelian subalgebra. Then A is isomorphic either to the algebra $\mathbf{M}_2$ of all 2×2 matrices with scalar entries, or to the algebra $\tilde{A}_0$ given by* (4).

Proof. Suppose that $\mathcal{A}$ is an m.a.s. in a (unital) algebra A. Define

$$A_0 = \{\eta \in A : \xi\eta\xi = 0\}.$$

Clearly A_0 is a linear subspace of A containing $\mathcal{A}$. We shall show that A_0 is a (proper or not) subalgebra of A of dimension three in which $\mathcal{A}$ is an m.a.s. First we show that A_0 is a subalgebra of A. Let $\eta \in A_0$. Since $\xi\eta$ commutes with ξ, it belongs to $\mathcal{A}$, and so $\xi\eta = \alpha e + \beta(\eta)\xi$ for some scalars α and $\beta(\eta)$. Multiplying both sides by ξ on the left we obtain $\alpha = 0$, and so

$$\xi\eta = \beta(\eta)\xi.$$

Thus for any η_1, η_2 in A_0 we have

$$\xi\eta_1\eta_2\xi = \beta(\eta_1)\xi\eta_2\xi = 0,$$

and our claim follows.

We now consider two cases.

Case (i): $A = A_0$. In this case $A_0 \neq \mathcal{A}$, and there is an element η in $A \setminus \mathcal{A}$. Since η does not commute with ξ, at least one of the products $\xi\eta$ and $\eta\xi$ is non-zero. As in the above proof, we show that $\eta\xi = \beta_1\xi$ and $\xi\eta = \beta_2\xi$ with $\beta_1 \neq \beta_2$. If $\beta_1 \neq 0$, we can assume

$$\eta\xi = \xi. \tag{5}$$

If $\beta_1 = 0$, we also have (5) with $e - \eta$ instead of η. Thus we can assume that (5) holds true. We can also assume

$$\xi\eta = 0. \tag{6}$$

Indeed, we have $\beta_2 \neq 1$ (otherwise $\xi\eta$ and $\eta\xi$ commute) and we can again replace η by $(\eta - \beta_2 e)(1 - \beta_2)^{-1}$. Formulas (5) and (6), as well as the formula

$$\eta^2 = \eta, \tag{7}$$

will be needed further. To obtain (7) observe that (5) implies $\eta^2\xi = \eta\xi = \xi$, and so $(\eta^2 - \eta)\xi = 0$. Since, by (6), we have $\xi(\eta^2 - \eta) = 0$ we see that $\eta^2 - \eta = \alpha e + \beta\xi$ for some scalars α and β. Multiplying the last equality by ξ, say on the left, we obtain $\alpha = 0$. Thus

$$\eta^2 - \eta = \beta\xi; \tag{8}$$

multiplying this by η first on the right and then on the left we obtain, by (5) and (6),

$$\eta^3 - \eta^2 = \beta\xi\eta = 0 \quad \text{and} \quad \eta^3 - \eta^2 = \beta\eta\xi = \beta\xi.$$

Thus $\beta = 0$, and so (8) implies (7).

We shall show now that the codimension of $\mathcal{A}$ in A_0 is one, so that the dimension of $A = A_0$ is three. Assume towards a contradiction that there are elements η_1 and η_2 which are linearly independent modulo $\mathcal{A}$, i.e. $\alpha_1\eta_1 + \alpha_2\eta_2 \in \mathcal{A}$ implies $\alpha_1 = \alpha_2 = 0$, $\alpha_i \in \mathbb{K}$. As before (see (5) and (6)), we can assume $\eta_i\xi = \xi$ and $\xi\eta_i = 0$, $i = 1, 2$. Note that the operations leading to these formulas preserve linear independence modulo $\mathcal{A}$. We have now $(\eta_1 - \eta_2)\xi = 0 = \xi(\eta_1 - \eta_2)$. This implies $\eta_1 - \eta_2 \in \mathcal{A}$, which contradicts the linear independence of these element modulo $\mathcal{A}$. Our claim follows.

Case (ii): $A \neq A_0$. Thus there is an element μ in A with $\xi\mu\xi \neq 0$. But then $\xi\mu\xi$ is in $\mathcal{A}$, and so $\xi\mu\xi = \alpha e + \beta\xi$. Multiplying both sides by ξ we can assume as before that

$$\xi\mu\xi = \xi. \tag{9}$$

This implies that both $\xi\mu$ and $\mu\xi$ are non-zero idempotents. They are not equal and different from the unity e. Clearly they are in A_0. Since $\mathcal{A}$ has no idempotents, both belong to $A_0 \setminus \mathcal{A}$. Since the dimension of A_0 is three, these idempotents are not linearly independent modulo $\mathcal{A}$. Consequently, there are scalars α, β, γ such that

$$\xi\mu + \gamma\mu\xi = \alpha e + \beta\xi. \tag{10}$$

Multiplying by ξ, first on the right and then on the left, we obtain, by (9),

$$\xi\mu\xi = \alpha\xi \quad \text{and} \quad \gamma\xi = \alpha\xi,$$

which implies $\alpha = \gamma = 1$. Thus (10) can be rewritten as

$$\xi\mu + \mu\xi = e + \beta\xi,$$

which implies

$$\xi\left(\mu - \tfrac{1}{2}\beta e\right) + \left(\mu - \tfrac{1}{2}\beta e\right)\xi = e.$$

Replacing μ by $\mu - \frac{1}{2}\beta e$ we obtain an element in A satisfying (9) as well as

$$\xi\mu + \mu\xi = e. \tag{11}$$

We now set

$$\eta = \xi\mu, \tag{12}$$

so that

$$\mu\xi = e - \eta. \tag{13}$$

Note that the element η given by (12) satisfies (5), (6), and (7), the first of these in view of (9).

We shall show now that the codimension of A_0 in A is 1. Assume towards a contradiction that there are $\mu_1, \mu_2 \in A \setminus A_0$ which are linearly independent modulo A_0. We have

$$\xi\mu_1\xi = \alpha\xi \neq 0 \neq \beta\xi = \xi\mu_2\xi.$$

This implies

$$\xi(\alpha^{-1}\mu_1 - \beta^{-1}\mu_2)\xi = 0,$$

and so $\alpha^{-1}\mu_1 - \beta^{-1}\mu_2$ is in A_0, which is the desired contradiction. Thus the dimension of A is 4.

We shall show now that the algebra A is isomorphic to $\mathbf{M}_2$, while A_0 is isomorphic to the algebra $\tilde{A}_0$ given by (4).

The linear space A is spanned by the elements ξ, e, η and μ. The element η is defined by (12) and satisfies (5), (6), (7) and (13). We shall need further formulas involving these elements. We have

$$\mu = \mu\eta + \eta\mu. \tag{14}$$

Indeed, by (12) and (13) we have

$$\mu\eta = \mu\xi\mu = (e - \eta)\mu = \mu - \eta\mu.$$

We shall calculate μ^2. By (12)–(14) we have

$$\xi\mu^2 = \eta\mu = \mu - \mu\eta = \mu(e - \eta) = \mu^2\xi.$$

Thus μ^2 commutes with ξ, and so

$$\mu^2 = \alpha e + \beta\xi, \qquad \alpha, \beta \in \mathbb{K}. \tag{15}$$

Multiplying (15) by μ on the left, we obtain

$$\mu^3 = \alpha\mu + \beta\mu\xi,$$

and multiplication by μ on the right gives

$$\mu^3 = \alpha\mu + \beta\xi\mu.$$

Since $\mu\xi \neq \xi\mu$, we have $\beta = 0$, and so (15) implies

$$\mu^2 = \alpha e. \tag{16}$$

If $\alpha = 0$, we have $\mu^2 = 0$. If $\alpha \neq 0$, we replace μ by $\mu - \alpha\xi$. In both cases the "new" μ also generates the linear space A together with ξ, e, and η. Moreover the previous formulas (9), (12) and (13), involving μ, remain unchanged. By (1), (11) and (16) we obtain (for a while we write μ_1 for the "new" μ and μ for the "old" one)

$$\mu_1^2 = (\mu - \alpha\xi)^2 = \mu^2 - \alpha(\xi\mu + \mu\xi) = \mu^2 - \alpha e = 0.$$

Thus we have

$$\mu^2 = 0 \tag{17}$$

with the "new" μ. By (12) and (17) we obtain

$$\eta\mu = \xi\mu^2 = 0, \tag{18}$$

and by (14) and (18) we get

$$\mu\eta = \mu - \eta\mu = \mu. \tag{19}$$

The products of the linear generators ξ, η, μ of the algebra A are given by (1), (5), (6), (7), (12), (13), (17), (18) and (19). We put them in the table below (we omit the unities). Here "l.f" denotes the left factor and "r.f" the right one.

l.f. \ r.f.	ξ	η	μ
ξ	0	0	η
η	ξ	η	0
μ	$e - \eta$	μ	0

This table is the same as for products of the corresponding linear generators of $\mathbf{M}_2$:

$$\tilde{\xi} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \tilde{\eta} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \tilde{\mu} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Thus this correspondence gives an isomorphism between the algebras A and $\mathbf{M}_2$. The restrictions of this isomorphism to A_0 and to $\mathcal{A}$ give isomorphisms between these algebras and the algebras $\tilde{A}_0$, given by (4), and $\tilde{\mathcal{A}}$, given by (3). The conclusion follows.

Remarks. (1) The situation is completely different if we consider another m.a.s. of $\mathbf{M}_2$, namely the algebra $\mathcal{A}_1 = \mathbb{K}^2$ with coordinatewise multiplication. In [4] we constructed an infinite-dimensional Banach algebra which has only one infinite-dimensional m.a.s. (it must be so by the theorem of Laffey [3] stating that any infinite-dimensional algebra must have an infinite-dimensional m.a.s.), while all remaining m.a.s. (uncountably many of them) are isomorphic to $\mathcal{A}_1$. The algebra in question was a subalgebra of $L(H)$ for a separable infinite-dimensional Hilbert space H. Since the construction works also for finite-dimensional spaces, there are infinitely many algebras with $\mathcal{A}_1$ as an m.a.s. In a conversation with the authors of [1] the present author asked whether it is possible to have an infinite-dimensional Banach space X such that $L(X)$ has a finite-dimensional m.a.s. They solved this problem in the negative in [1].

(2) In [5] the author asked when for given two commutative (Banach) algebras there is a (Banach) algebra in which both are m.a.s., and remarked that not every pair has this property. In view of the result of this paper, the algebra $\mathcal{A}$ given by (2) and the commutative algebra $\mathcal{A}'$ are both m.a.s. in some A if and only if $\mathcal{A}'$ is an m.a.s. in $\mathbf{M}_2$.

Also the algebras $\mathcal{A}$ and $\mathcal{A}_1$ have the property that both are m.a.s. in $\tilde{A}_0$ (see (4)) and $\mathbf{M}_2$. This implies that whenever $\mathcal{A}$ is an m.a.s. in some A, then so is $\mathcal{A}_1$. Are there other pairs of algebras with this property?

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Wiesław Żelazko
Institute of Mathematics
Polish Academy of Sciences
Śniadeckich 8
00-656 Warszawa, Poland
E-mail: zelazko@impan.pl