ACTA ARITHMETICA
177.2 (2017)

On the numbers of prime factors of square free amicable pairs

by

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0. Introduction. For any positive integer n, we denote by $\sigma(n)$ the sum of all positive divisors of n.

DEFINITION 0.1. A pair (m,n) of integers greater than 1 is said to be *amicable* if m and n are distinct and

$$\sigma(m) = \sigma(n) = m + n, \tag{1}$$

or equivalently

$$\sigma(m) - m = n \quad \text{and} \quad \sigma(n) - n = m. \tag{2}$$

In this paper, we pay a special attention to amicable pairs (m,n) such that m and n are *square free*.

DEFINITION 0.2. Let $s \le t$ be positive integers. We say that a square free amicable pair (m,n) is *of type* $\{s,t\}$ if one of m,n has s prime factors and the other has t prime factors.

EXAMPLE 0.1. In 1986, the list of all 1427 amicable pairs which are smaller than 10^{10} was published by te Riele [5]. Then he and his coauthors [6] published the table of 10445 known amicable pairs which have smaller members between 10^{10} and 10^{52}.

The minimum square free amicable pair is the 18th pair

$$(142310, 168730)$$

in the list, which is of type $\{4,5\}$ with prime factorizations

$$142310 = 2 \times 5 \times 7 \times 19 \times 107, \quad 168730 = 2 \times 5 \times 47 \times 359.$$

REMARK 0.1. Note that the notion of "type $\{s,t\}$" in Definition 0.2 is different from the notion of "regular of type (i,j)" given by te Riele [4]. For

2010 *Mathematics Subject Classification*: 11A99, 11P99.
Key words and phrases: number of prime factors, amicable pairs.
Received 13 October 2015; revised 24 May 2016.
Published online 18 January 2017.

DOI: 10.4064/aa8327-7-2016

example, the pair $(142310, 168730)$ is regular of type $(3, 2)$ with the greatest common divisor $2 \times 5 = 10$.

There are 788 known square free amicable pairs which have smaller members below 10^{52}. The numbers of pairs of any given type are as follows:

Type	Number	Type	Number	Type	Number
$\{4,5\}$	4	$\{5,7\}$	61	$\{6,9\}$	47
		$\{6,6\}$	116	$\{7,8\}$	89
$\{4,6\}$	9	$\{5,8\}$	13	$\{8,8\}$	5
$\{5,5\}$	14	$\{6,7\}$	142		
$\{4,7\}$	3	$\{6,8\}$	32	$\{8,9\}$	119
$\{5,6\}$	107	$\{7,7\}$	27		

Concerning the list above, we can ask the following:

QUESTIONS. (1) Are there infinitely many square free amicable pairs?
(2) What percentage of amicable pairs are square free?
(3) Is there any square free amicable pair of type $\{s, t\}$ with $s + t \leq 8$?
(4) Is there any square free amicable pair of type $\{s, t\}$ with $s + t \geq 18$?
(5) Is there any square free amicable pair of type $\{s, t\}$ with $t - s \geq 4$?
(6) Is there any square free amicable pair of type $\{s, t\}$ with $s \leq 3$?
(7) Is there a formula, in terms of s and t, giving the percentage of square free amicable pairs of type $\{s, t\}$ among all square free amicable pairs?

As a partial answer to Questions (3) and (6) above, we shall prove the following

THEOREM 0.1. *There are no square free amicable pairs of the following types:*

(1) $\{1, t\}$ *with arbitrary* t,
(2) $\{2, t\}$ *with* $2 \leq t \leq 8$,
(3) $\{3, 3\}$.

In particular, for any square free amicable pair (m, n), *the sum of the numbers of prime factors of* m *and* n *is at least* 7.

REMARK 0.2. (1) If m and n are square free positive integers greater than 1, then (m, n) is amicable if and only if (m, n) is *unitary* amicable by [1, Theorem 6], so the assertion obtained by replacing "amicable" in Theorem 0.1 with "unitary amicable" also holds.

(2) Kanold [3] has proved that there are no amicable pairs of the form $(p_1^{\alpha_1} p_2^{\alpha_2}, q_1^{\beta_1} q_2^{\beta_2})$ with any primes p_1, p_2, q_1, q_2 and any positive integers $\alpha_1, \alpha_2, \beta_1, \beta_2$. His result includes Theorem 0.1(2) with $t = 2$.

(3) Concerning Theorem 0.1(2) and Questions (6) and (7) above, the author conjectures that one can prove the non-existence of amicable pairs of type $\{2, t\}$ with any $9 \le t \le 19$ by the same calculations as in Example 2.2 using Lemma 2.2 of Section 2.4.

(4) Concerning Question (3) above, the author hopes that the method of Section 3 will also work for types $\{3, 4\}$, $\{3, 5\}$ and $\{4, 4\}$, although the calculations may be very complicated.

In Section 1, we shall prove Theorem 0.1(1). In Subsections 2.1–2.3, we shall prove Theorem 0.1(2) with $2 \le t \le 6$. Then we shall make an observation on type $\{2, t\}$ with $7 \le t \le 19$ and prove Theorem 0.1(2) with $t = 7, 8$ in Subsection 2.4. Finally, we shall prove Theorem 0.1(3) in Section 3.

Although the proofs for types $\{1, t\}$ and $\{2, 2\}$ in Theorem 0.1 are very easy, to prove the other cases we shall need Hagis' theorem [2, Theorem] which asserts that for any *coprime* amicable pair (m, n), the sum of the numbers of prime factors of m and n is at least 22. Thanks to Hagis' theorem, it suffices to prove that for any square free amicable pair (m, n) of the type in question, m and n must be coprime.

1. The case of $s = 1$. To prove Theorem 0.1(1), we assume that there exists a square free amicable pair (m, n) of type $\{1, t\}$ for some t.

Since m is a prime number, $\sigma(m) = m + 1$. By the amicability condition (2) in Definition 0.1, we see that

$$1 = \sigma(m) - m = n,$$

which is a contradiction, since $n > 1$.

2. The case of $s = 2$. In this section, we shall prove Theorem 0.1(2) with $t = 2, 3, 4, 5, 6$ (Subsections 2.1–2.3). Then we shall make an observation on type $\{2, t\}$ with $7 \le t \le 19$ and prove Theorem 0.1(2) with $t = 7, 8$ in Subsection 2.4.

2.1. Type $\{2, 2\}$. To prove Theorem 0.1(2) with $t = 2$, we assume that there exists a square free amicable pair (m, n) of type $\{2, 2\}$ with prime factorizations

$$m = p_1 p_2, \quad n = q_1 q_2,$$

where $p_1 \ne p_2$ and $q_1 \ne q_2$ are prime numbers.

Since $\sigma(m) = (p_1 + 1)(p_2 + 1)$ and $\sigma(n) = (q_1 + 1)(q_2 + 1)$, by the amicability condition (1) in Definition 0.1, we see that

$$(p_1 + 1)(p_2 + 1) = (q_1 + 1)(q_2 + 1).$$

This implies that if $p_i = q_j$ for some $1 \le i, j \le 2$, then $m = n$, which is a contradiction, since m and n are distinct. Therefore (m, n) is a coprime amicable pair such that the sum of the numbers of prime factors of m and n is 4. This contradicts the following theorem proved by Hagis:

THEOREM 2.1 ([2, Theorem]). *For any coprime amicable pair* (m, n), *the sum of the numbers of prime factors of* m *and* n *is at least* 22.

2.2. Type $\{2, 3\}$**.** To prove Theorem 0.1(2) with $t = 3$, we assume that there exists a square free amicable pair (m, n) of type $\{2, 3\}$ with prime factorizations

$$m = p_1 p_2, \quad n = q_1 q_2 q_3,$$

where $p_1 < p_2$ and $q_1 < q_2 < q_3$ are prime numbers.

Since $\sigma(m) = (p_1 + 1)(p_2 + 1)$ and $\sigma(n) = (q_1 + 1)(q_2 + 1)(q_3 + 1)$, by (2) of Definition 0.1 we have

$$1 + p_1 + p_2 = q_1 q_2 q_3, \tag{3}$$

$$1 + q_1 + q_2 + q_3 + q_1 q_2 + q_1 q_3 + q_2 q_3 = p_1 p_2. \tag{4}$$

Thanks to Theorem 2.1, it suffices to prove that the sets $\{p_1, p_2\}$ and $\{q_1, q_2, q_3\}$ are disjoint.

(i) Firstly, we assume that $p_2 \in \{q_1, q_2, q_3\}$. Then

$$1 + p_1 \equiv 0 \pmod{p_2}$$

by the mod p_2 reduction of (3). Since $p_1 < p_2$ are prime numbers, this implies that

$$p_1 = 2 \quad \text{and} \quad p_2 = 3.$$

Then $m = 6$ is a *perfect* number and n is also equal to 6. This is a contradiction, since m and n are distinct.

(ii) Secondly, we assume that $q_3 \in \{p_1, p_2\}$. Then

$$(1 + q_1)(1 + q_2) \equiv 0 \pmod{q_3}$$

by the mod q_3 reduction of (4). This is a contradiction, since $q_1 < q_2 < q_3$ are prime numbers.

(iii) Finally, we assume that $p_1 \in \{q_1, q_2\}$. We set $r := p_1 = q_i$ with suitable i, and $q := q_j$ with $j = 1, 2,\ j \neq i$. By (3) and (4), we see that

$$1 + r + p_2 = rqq_3, \tag{5}$$

$$(1 + r)(1 + q + q_3) + qq_3 = rp_2. \tag{6}$$

If $q = 2$, then the mod 2 reduction of (5) implies that $r = 2$, since the inequality $p_2 > r$ forces p_2 to be odd. This is a contradiction, since $r \neq q$ by the definition. Therefore $q \neq 2$, i.e., $q \geq 3$.

By eliminating p_2 from (5) and (6) above, we find that

$$(r - 1)qq_3 - q - q_3 = r + 1,$$

or equivalently

$$((r-1)q-1)((r-1)q_3-1) = r^2. \tag{7}$$

Since $q < q_3$ and $r - 1 > 0$, we infer that

$$(r-1)q - 1 < r,$$

or equivalently

$$r < 1 + \frac{2}{q-1}. \tag{8}$$

This leads to a contradiction, since $r \geq 2$ and $q \geq 3$.

2.3. Type $\{2, t\}$ with $t = 4, 5, 6$

(i) To prove Theorem 0.1(2) with $t = 4$, we assume that there exists a square free amicable pair (m, n) of type $\{2, 4\}$ with prime factorizations

$$m = p_1 p_2, \quad n = q_1 q_2 q_3 q_4,$$

where $p_1 < p_2$ and $q_1 < q_2 < q_3 < q_4$ are prime numbers.

Thanks to Theorem 2.1, it suffices to prove that the sets $\{p_1, p_2\}$ and $\{q_1, q_2, q_3, q_4\}$ are disjoint. By the same argument as in Section 2.2(i)&(ii), we see that

$$p_2 \notin \{q_1, q_2, q_3, q_4\} \quad \text{and} \quad q_4 \notin \{p_1, p_2\}.$$

Now we assume that $p_1 \in \{q_1, q_2, q_3\}$. Then we set $r := p_1 = q_i$ with suitable i and denote by $q < q'$ the prime numbers in $\{q_1, q_2, q_3\} \setminus \{r\}$. By the same argument as in Section 2.2(iii), we see that $q \geq 3$.

By the same argument as used in Section 2.2(iii) to obtain (7), we deduce that

$$\begin{aligned} &\big((rq-(q+1))q' - (q+1)\big)\big((rq-(q+1))q_4 - (q+1)\big) \\ &\qquad = rq\big(r + (q-1)(1+1/q)\big). \end{aligned} \tag{9}$$

On the right hand side, we have

$$rq \geq r + (q-1)(1+1/q),$$

since

$$rq - (r + (q-1)(1+1/q)) = (q-1)(r - (1+1/q))$$

and

$$r - (1+1/q) \geq r - 4/3 > 0,$$

because $r \geq 2$, $q \geq 3$.

Since $q' < q_4$ and $rq - (q+1) > 0$ on the left hand side of (9), we see that

$$(rq - (q+1))q' - (q+1) < rq,$$

or equivalently

$$r < \left(1+\frac{1}{q}\right)\left(1+\frac{2}{q'-1}\right). \tag{10}$$

This leads a contradiction, since $r \geq 2$ and $3 \leq q < q'$ are prime numbers.

(ii) To prove Theorem 0.1(2) with $t = 5$, we assume that there exists a square free amicable pair (m, n) of type $\{2, 5\}$ with prime factorizations

$$m = p_1 p_2, \quad n = q_1 q_2 q_3 q_4 q_5,$$

where $p_1 < p_2$ and $q_1 < q_2 < q_3 < q_4 < q_5$ are prime numbers.

Thanks to Theorem 2.1, it suffices to prove that $\{p_1, p_2\}$ and $\{q_1, q_2, q_3, q_4, q_5\}$ are disjoint. By the same argument as in Section 2.2(i)&(ii), we find that

$$p_2 \notin \{q_1, q_2, q_3, q_4, q_5\} \quad \text{and} \quad q_5 \notin \{p_1, p_2\}.$$

Now we assume that $p_1 \in \{q_1, q_2, q_3, q_4\}$. We set $r := p_1 = q_i$ with suitable i and denote by $q < q' < q''$ the prime numbers in $\{q_1, q_2, q_3, q_4\} \setminus \{r\}$. By the same argument as in Section 2.2(iii), we see that $q \geq 3$.

By the same argument as used in (i) to obtain (9), we get

$$\begin{aligned}\big((rqq' - (q+1)(q'+1))q'' - (q+1)(q'+1)\big) \\ \times \big((rqq' - (q+1)(q'+1))q_5 - (q+1)(q'+1)\big) \\ = rqq'\big(r + (qq'-1)(1+1/q)(1+1/q')\big).\end{aligned} \tag{11}$$

On the right hand side, we have

$$rqq' \geq r + (qq'-1)(1+1/q)(1+1/q'),$$

since

$$\begin{aligned} rqq' - (r + (qq'-1)(1+1/q)(1+1/q')) \\ = (qq'-1)\big(r - (1+1/q)(1+1/q')\big)\end{aligned}$$

and

$$r - (1+1/q)(1+1/q') \geq r - 8/5 > 0,$$

because $r \geq 2,\ 3 \leq q < q'$ are prime numbers.

Since $q'' < q_5$ and $rqq' - (q+1)(q'+1) > 0$ on the left hand side of (11), we see that

$$\big(rqq' - (q+1)(q'+1)\big)q'' - (q+1)(q'+1) < rqq',$$

or equivalently

$$r < \left(1+\frac{1}{q}\right)\left(1+\frac{1}{q'}\right)\left(1+\frac{2}{q''-1}\right). \tag{12}$$

Since $r \geq 2$ and $3 \leq q < q' < q''$ are prime numbers, (12) implies that $(r, q, q', q'') = (2, 3, 5, 7)$. Then (11) yields $q_5 = 97/9$, a contradiction.

(iii) To prove Theorem 0.1(2) with $t = 6$, we assume that there exists a square free amicable pair (m, n) of type $\{2, 6\}$ with prime factorizations

$$m = p_1 p_2, \quad n = q_1 q_2 q_3 q_4 q_5 q_6,$$

where $p_1 < p_2$ and $q_1 < q_2 < q_3 < q_4 < q_5 < q_6$ are prime numbers.

Thanks to Theorem 2.1, it suffices to prove that $\{p_1, p_2\}$ and $\{q_1, q_2, q_3, q_4, q_5, q_6\}$ are disjoint. By the same argument as in Section 2.2(i)&(ii), we see that

$$p_2 \notin \{q_1, q_2, q_3, q_4, q_5, q_6\} \quad \text{and} \quad q_6 \notin \{p_1, p_2\}.$$

Now we assume that $p_1 \in \{q_1, q_2, q_3, q_4, q_5\}$. Then we set $r := p_1 = q_i$ with suitable i and denote by $q < q' < q'' < q'''$ the prime numbers in $\{q_1, q_2, q_3, q_4, q_5\} \setminus \{r\}$. By the same argument as in Section 2.2(iii), we see that $q \geq 3$.

By the same argument as used in (ii) to obtain (11), we get

$$\begin{aligned}(13)\quad &\big((rqq'q'' - (q+1)(q'+1)(q''+1))q''' - (q+1)(q'+1)(q''+1)\big)\\ &\times \big((rqq'q'' - (q+1)(q'+1)(q''+1))q_6 - (q+1)(q'+1)(q''+1)\big)\\ &= rqq'q''\big(r + (qq'q'' - 1)(1 + 1/q)(1 + 1/q')(1 + 1/q'')\big).\end{aligned}$$

On the right hand side, we have

$$rqq'q'' \geq r + (qq'q'' - 1)(1 + 1/q)(1 + 1/q')(1 + 1/q''),$$

since

$$\begin{aligned}rqq'q'' - \big(r + (qq'q'' - 1)(1 + 1/q)(1 + 1/q')(1 + 1/q'')\big)\\ = (qq'q'' - 1)\big(r - (1 + 1/q)(1 + 1/q')(1 + 1/q'')\big)\end{aligned}$$

and

$$r - (1 + 1/q)(1 + 1/q')(1 + 1/q'') > 0,$$

because $r \geq 2$ and $3 \leq q < q' < q''$ are prime numbers.

Since $q''' < q_6$ and $rqq'q'' - (q+1)(q'+1)(q''+1) > 0$ on the left hand side of (13), we see that

$$\big(rqq'q'' - (q+1)(q'+1)(q''+1)\big)q''' - (q+1)(q'+1)(q''+1) < rqq'q'',$$

or equivalently

$$(14)\qquad r < \left(1 + \frac{1}{q}\right)\left(1 + \frac{1}{q'}\right)\left(1 + \frac{1}{q''}\right)\left(1 + \frac{2}{q''' - 1}\right).$$

Since $r \geq 2$ and $3 \leq q < q' < q'' < q'''$ are prime numbers, (14) implies that (r, q, q', q'', q''') is one of the following:

$$(2, 3, 5, 7, q''') \text{ with } q''' = 11, 13, 17, 19, \quad \text{or} \quad (2, 3, 5, 11, 13).$$

Then (13) implies that

$$q_6 = 363,\ \frac{1345}{21},\ \frac{91}{3},\ \frac{1921}{75},\ \frac{2017}{129}$$

in the respective cases, which is a contradiction.

2.4. Type $\{2, t\}$ with $7 \le t \le 19$. Summing up the arguments in Subsection 2.3, we obtain the following explicit criterion:

LEMMA 2.2. *Let $4 \le t \le 19$ be an integer. There is no square free amicable pair of type $\{2, t\}$ if the following argument* (A) *is valid:*

(A) *Let r and $q_1 < \cdots < q_{t-1}$ be t distinct prime numbers with $q_1 \ge 3$. Assume that*

$$\begin{aligned}(15)\qquad &(rq_1 \cdots q_{t-3}q_{t-2} - (q_1+1)\cdots(q_{t-3}+1)(q_{t-2}+1))\\ &\quad\times(rq_1\cdots q_{t-3}q_{t-1} - (q_1+1)\cdots(q_{t-3}+1)(q_{t-1}+1))\\ &\qquad = r(rq_1\cdots q_{t-3} + (q_1\cdots q_{t-3}-1)(q_1+1)\cdots(q_{t-3}+1)).\end{aligned}$$

Then

$$r \ge \left(1+\frac{1}{q_1}\right)\cdots\left(1+\frac{1}{q_{t-3}}\right) \tag{16}$$

holds, and the inequality

$$r < \left(1+\frac{1}{q_1}\right)\cdots\left(1+\frac{1}{q_{t-3}}\right)\left(1+\frac{2}{q_{t-2}-1}\right) \tag{17}$$

induced by (15) *and* (16) *leads to a contradiction.*

Proof. For $t = 4, 5, 6$, this is already proved in Subsection 2.3. The proof for $7 \le t \le 19$ is by the same arguments. Theorem 2.1 forces t to be at most 19.

Note that the notation of prime numbers q_i in Lemma 2.2 is different from that used in Subsection 2.3. Namely, prime numbers $q_1 < \cdots < q_{t-2}$ and q_{t-1} in the lemma play the same respective roles as $q < q' < \cdots$ and q_t in the previous subsections. ∎

EXAMPLE 2.1. As an example of the use of Lemma 2.2, we shall prove Theorem 0.1(2) with $t = 7$. For the first four odd prime numbers $3, 5, 7, 11$, we see that

$$\left(1+\frac{1}{3}\right)\left(1+\frac{1}{5}\right)\left(1+\frac{1}{7}\right)\left(1+\frac{1}{11}\right) = \frac{768}{385} < 2.$$

Therefore (16) with $t = 7$ holds.

Next inequality (17) with $t = 7$ implies that $(r, q_1, q_2, q_3, q_4, q_5)$ is one of the following 188 tuples, divided into two types:

$$\begin{array}{lll} \text{Type I}: & (2,3,5,7,11,q_5) \ \text{ with } \ 13 \le q_5 \le 761, \\ & (2,3,5,7,17,q_5) \ \text{ with } \ 19 \le q_5 \le 61, \\ & (2,3,5,7,23,q_5) \ \text{ with } \ 29 \le q_5 \le 41, \\ & (2,3,5,7,29,31), \ (2,3,5,11,17,19), \ (2,3,5,11,17,23), \\ & (2,3,5,11,19,23), \ (2,3,5,13,17,19); \\ \text{Type II}: & (2,3,5,7,13,q_5) \ \text{ with } \ 17 \le q_5 \le 127, \\ & (2,3,5,7,19,q_5) \ \text{ with } \ 23 \le q_5 \le 47, \\ & (2,3,5,11,13,q_5) \ \text{ with } \ 17 \le q_5 \le 31, \\ & (2,3,7,11,13,17). \end{array}$$

For any $(r, q_1, q_2, q_3, q_4, q_5)$ of type I, we have $r = 2$, $q_1 = 3$ and two primes among (q_2, q_3, q_4) are congruent to 2 modulo 3. This implies that the left hand side of (15) is divisible by 3^2, although the right hand side is not, a contradiction.

On the other hand, for any $(r, q_1, q_2, q_3, q_4, q_5)$ of type II, by direct calculations we can see that no prime number q_6 satisfies (15).

In this way, Theorem 0.1(2) with $t = 7$ is proved by applying Lemma 2.2.

Example 2.2. As another example, we shall prove Theorem 0.1(2) with $t = 8$. In contrast to Example 2.1, we see that

$$2 \le \left(1 + \frac{1}{q_1}\right) \cdots \left(1 + \frac{1}{q_5}\right) < 3$$

for any quintuplet $(q_1, q_2, q_3, q_4, q_5)$ of odd prime numbers which is equal to one of the following 90 tuples, divided into two types:

$$\begin{array}{lll} \text{Type I}: & (3,5,7,11,q_5) \ \text{ with } \ 13 \le q_5 \le 383, \\ & (3,5,7,13,q_5) \ \text{ with } \ q_5 = 17,23,29,41,47,53,59, \\ & (3,5,7,17,q_5) \ \text{ with } \ 19 \le q_5 \le 29, \\ & (3,5,7,19,23); \\ \text{Type II}: & (3,5,7,13,q_5) \ \text{ with } \ q_5 = 19,31,37,43,61. \end{array}$$

For any $(r, q_1, q_2, q_3, q_4, q_5)$ of type I, we have $q_1 = 3$ and two primes among (q_2, q_3, q_4, q_5) are congruent to 2 modulo 3. This implies that if $r = 2$, then the left hand side of (15) is divisible by 3^2, although the right hand side is not, a contradiction. Therefore in all cases of type I, (16) holds with $r \ge 3$.

On the other hand, for any $(r, q_1, q_2, q_3, q_4, q_5)$ of type II, we can see that (15) leads a contradiction. Indeed, if $q_5 = 19$, the left hand side of (15) is divisible by 5^2, while the right hand side is not.

If $q_5 = 31, 37, 61$, the left hand side of (15) is divisible by 7^2, while the right hand side is not.

If $q_5 = 43$, the left hand side of (15) is not divisible by 3^3, while the right hand side is divisible.

In this way, we see that no case of type II can occur.

Moreover, for any quintuplet $(q_1, q_2, q_3, q_4, q_5)$ of odd prime numbers other than those of types I and II, we have

$$\left(1+\frac{1}{q_1}\right)\cdots\left(1+\frac{1}{q_5}\right) < 2,$$

so (16) with $t = 8$ holds with any prime number r.

Next (17) implies that $r = 2$ and $q_1 = 3$, since

$$2<\left(1+\frac{1}{3}\right)\left(1+\frac{1}{5}\right)\left(1+\frac{1}{7}\right)\left(1+\frac{1}{11}\right)\left(1+\frac{1}{13}\right)\left(1+\frac{2}{17-1}\right) = 2.416\ldots < 3,$$

$$\left(1+\frac{1}{5}\right)\left(1+\frac{1}{7}\right)\left(1+\frac{1}{11}\right)\left(1+\frac{1}{13}\right)\left(1+\frac{1}{17}\right)\left(1+\frac{2}{19-1}\right) = 1.895\ldots < 2.$$

In particular, no case of type I above with $r \geq 3$ satisfies (17).

Thus $q_2 = 5$ or 7, and if $q_2 = 7$, then $q_3 = 11$, since

$$\left(1+\frac{1}{3}\right)\left(1+\frac{1}{7}\right)\left(1+\frac{1}{11}\right)\left(1+\frac{1}{13}\right)\left(1+\frac{1}{17}\right)\left(1+\frac{2}{19-1}\right) = 2.106\ldots > 2,$$

$$\left(1+\frac{1}{3}\right)\left(1+\frac{1}{11}\right)\left(1+\frac{1}{13}\right)\left(1+\frac{1}{17}\right)\left(1+\frac{1}{19}\right)\left(1+\frac{2}{23-1}\right) = 1.904\ldots < 2,$$

$$\left(1+\frac{1}{3}\right)\left(1+\frac{1}{7}\right)\left(1+\frac{1}{13}\right)\left(1+\frac{1}{17}\right)\left(1+\frac{1}{19}\right)\left(1+\frac{2}{23-1}\right) = 1.995\ldots < 2.$$

By similar arguments, any quintuplet $(q_1, q_2, q_3, q_4, q_5)$ satisfying (17) is one of the following with suitable prime numbers q_4, q_5:

Type III: $(3, 5, 11, q_4, q_5)$, $(3, 5, 13, 17, q_5)$,
$(3, 5, 13, 19, 23)$, $(3, 5, 13, 19, 29)$,
$(3, 7, 11, 17, q_5)$, $(3, 7, 11, 13, 17)$, $(3, 7, 11, 13, 23)$;

Type IV: $(3, 7, 11, 13, 19)$.

For any $(q_1, q_2, q_3, q_4, q_5)$ of type III, we have $r = 2$, $q_1 = 3$ and two primes among (q_2, q_3, q_4, q_5) are congruent to 2 modulo 3. On the other hand, for $(q_1, q_2, q_3, q_4, q_5) = (3, 7, 11, 13, 19)$ of type IV, the right hand side of (15) is $12269550000 \equiv 3 \pmod 9$.

This implies that in each type, the left hand side of (15) is divisible by 3^2, while the right hand side is not, a contradiction.

Therefore Theorem 0.1(2) with $t = 8$ is proved in view of Lemma 2.2.

3. Type $\{3,3\}$. To prove Theorem 0.1(3), we assume that there exists a square free amicable pair (m,n) of type $\{3,3\}$ with prime factorizations

$$m = p_1 p_2 p_3, \quad n = q_1 q_2 q_3,$$

where $p_1 < p_2 < p_3$ and $q_1 < q_2 < q_3$ are prime numbers.

Thanks to Theorem 2.1, it suffices to prove that $\{p_1, p_2, p_3\}$ and $\{q_1, q_2, q_3\}$ are disjoint. By the same argument as in Section 2.2(i)&(ii), we see that

$$p_3 \notin \{q_1, q_2, q_3\} \quad \text{and} \quad q_3 \notin \{p_1, p_2, p_3\}.$$

Now we assume that $\{p_1, p_2\} \cap \{q_1, q_2\}$ is not empty. We take an element $r := p_i = q_j$ in the intersection with suitable i and j, and denote by p and q the prime numbers in $\{p_1, p_2\} \setminus \{r\}$ and $\{q_1, q_2\} \setminus \{r\}$, respectively.

By (2) of Definition 0.1,

$$\begin{aligned} (r+1)(p_3+p+1) + pp_3 &= rqq_3, \\ (r+1)(q_3+q+1) + qq_3 &= rpp_3. \end{aligned} \tag{18}$$

By eliminating pp_3 or qq_3 from (18), we see that

$$\begin{aligned} ((r-1)q-1)((r-1)q_3-1) &= r\big((r-1)(p+p_3)+r\big), \\ ((r-1)p-1)((r-1)p_3-1) &= r\big((r-1)(q+q_3)+r\big). \end{aligned} \tag{19}$$

Then by eliminating p_3 or q_3 from (19), we obtain the key equalities

$$\big((rp-(p+1))q-(r+p+1)\big)\big((rp-(p+1))q_3-(r+p+1)\big) = 2r(p+1)\cdot\frac{(r-1)p^2+(r+1)}{2}, \tag{20}$$

$$\big((rq-(q+1))p-(r+q+1)\big)\big((rq-(q+1))p_3-(r+q+1)\big) = 2r(q+1)\cdot\frac{(r-1)q^2+(r+1)}{2}. \tag{21}$$

3.1. An observation on (20). Concerning the right hand side of (20), we have

Lemma 3.1. *If $p \geq 11$, then*

$$2r(p+1) \leq \frac{(r-1)p^2+(r+1)}{2}.$$

Proof. The quadratic equation

$$\frac{(r-1)X^2+(r+1)}{2} - 2r(X+1) = 0,$$

or equivalently

$$(r-1)X^2 - 4rX - 3r + 1 = 0,$$

has the solutions

$$X = \frac{2r - \sqrt{7r^2 - 4r + 1}}{r - 1}, \frac{2r + \sqrt{7r^2 - 4r + 1}}{r - 1}.$$

Since $r \geq 2$, we see that

$$\frac{2r - \sqrt{7r^2 - 4r + 1}}{r - 1} = 2 + \frac{2}{r - 1} - \sqrt{7 + \frac{10r - 6}{(r - 1)^2}} < 0$$

and

$$\frac{2r + \sqrt{7r^2 - 4r + 1}}{r - 1} = 2 + \frac{2}{r - 1} + \sqrt{7 + \frac{10r - 6}{(r - 1)^2}} \leq 3 + \sqrt{21} < 11.$$

Therefore if $p \geq 11$, then

$$2r(p + 1) \leq \frac{(r - 1)p^2 + (r + 1)}{2}. \ \blacksquare$$

In the following, we consider five cases:

(i) $p \geq 11$: Since $q < q_3$ and $rp - (p + 1) > 0$, we see that

$$(rp - (p + 1))q - (r + p + 1) < \frac{(r - 1)p^2 + (r + 1)}{2},$$

or equivalently

$$q < \frac{p}{2} + \frac{3(p + r + 1)}{2(rp - (p + 1))} \tag{22}$$

by Lemma 3.1. Since $p \geq 11$, we obtain

$$\begin{aligned} \frac{p}{2} - \frac{3(p + r + 1)}{2(rp - (p + 1))} &= \frac{1}{2}((r - 1)p^2 - 4p - 3r - 3) \\ &= \frac{1}{2}(((r - 1)p^2 - 4rp - 3r + 1) + 4(rp - (p + 1))) > 0 \end{aligned}$$

by Lemma 3.1. Then by (22),

$$q < p. \tag{23}$$

(ii) $p = 2$: We see that $p = 2 < r < p_3$, in particular, $r \geq 3$. If $q = 2$, then $(m, n) = (2rp_3, 2rq_3)$ is amicable and

$$3(r + 1)(p_3 + 1) = 3(r + 1)(q_3 + 1)$$

by (1) of Definition 0.1. This implies that $p_3 = q_3$, a contradiction, since $m \neq n$ by Definition 0.1. Therefore $q \geq 3$. Then $(m, n) = (2rp_3, rqq_3)$ is amicable and

$$(r + 1)(q + 1)(q_3 + 1) = 2rp_3 + rqq_3$$

by (1) of Definition 0.1. This is a contradiction, since the left hand side is even, while the right hand side is odd.

(iii) $p = 3$: In this case, $r = 2$ or $r \geq 5$. By the same argument as in (ii), we see that $q \neq 3$. If $q = 2$, then r is also equal to 2 by the same parity argument as in (ii), since $r < p_3$. This contradicts $q \neq r$. Therefore $q \neq 2$, i.e., $q \geq 5$.

By (20), we see that

$$((3r-4)q-(r+4))((3r-4)q_3-(r+4)) = 4r(10r-8). \tag{24}$$

Since $q < q_3$, and since $3r - 4 > 0$ on the right hand side, and $4r < 10r - 8$ on the left hand side of (24), we deduce that

$$(3r-4)q-(r+4) < 10r-8,$$

or equivalently

$$q < 3 + \frac{2}{3} + \frac{32}{3(3r-2)}.$$

Since $r = 2$ or $r \geq 5$, and $q \geq 5$, this inequality yields $(r, q) = (2, 5)$ or $(2, 7)$. These imply that q_3 is not a prime number by (24) as in Subsection 2.3(iii), which is a contradiction.

(iv) $p = 5$: In this case, $r = 2$, 3 or $r \geq 7$. By the same argument as in (iii), we see that $q \neq 2, 5$. If $q = 3$, then $(m, n) = (5rp_3, 3rq_3)$ is amicable and

$$6(r+1)(p_3+1) = 5rp_3 + 3rq_3$$

by (1) of Definition 0.1. This implies that $5rp_3 \equiv 0 \pmod 3$, a contradiction, since $r \neq q$ and $q_3 > q$ by the definition. Therefore $q \neq 3$, i.e., $q \geq 7$.

By (20), we see that

$$((5r-6)q-(r+6))((5r-6)q_3-(r+6)) = 6r(26r-24). \tag{25}$$

Since $q < q_3$, and since $5r - 6 > 0$ on the left hand side, and $6r < 26r - 24$ on the right hand side of (25), we infer that

$$(5r-6)q-(r+6) < 26r-24,$$

or equivalently

$$q < 5 + \frac{2}{5} + \frac{72}{5(5r-6)}.$$

Since $q \geq 7$ and $r = 2, 3$ or $r \geq 7$, this inequality implies that $(r, q) = (2, 7)$, which yields $q_3 = 31/5$ by (25), a contradiction.

(v) $p = 7$: By the same argument as in (iii), we see that $q \neq 2, 7$. If $q = 3$, then (1) of Definition 0.1 implies that

$$8(r+1)(p_3+1) = 4(r+1)(q_3+1),$$

or equivalently

$$q_3 = 2p_3 + 1.$$

By (2) of Definition 0.1, we then see that

$$8(r+1)(p_3+1) - 7rp_3 = 3rq_3 = 3r(2p_3+1),$$

or equivalently

$$r = \frac{8(p_3+1)}{5(p_3-1)}.$$

Since $p = 7 < p_3$, i.e., $p_3 \geq 11$, we conclude that

$$r < \frac{8}{5}\left(1 + \frac{2}{p_3-1}\right) < 2,$$

which is a contradiction. Therefore $q \neq 3$, i.e., $q = 5$ or $q \geq 11$.

By (20), we see that

$$((7r-8)q - (r+8))((7r-8)q_3 - (r+8)) = 16r(25r-24). \tag{26}$$

If $r = 2$, this implies that

$$(3q-5)(3q_3-5) = 208 = 2^4 \times 13.$$

Since $q < q_3$ and $3q - 5 \equiv 3q_3 - 5 \equiv 1 \pmod 3$, we find that

$$(q, q_3) = (6,7), (3,19) \text{ or } (2,71),$$

each of which leads to a contradiction, since $q \neq 2, 6$ and $2 \times 3 \times 19 = 114$ is not an amicable number. Therefore $r \neq 2$.

Since $q < q_3$, and since $7r-8 > 0$ on the left hand side, and $16r < 25r-24$ on the right hand side of (26) because $r \geq 3$, we see that

$$(7r-8)q - (r+8) < 25r - 24,$$

or equivalently

$$q < 3 + \frac{5}{7} + \frac{96}{7(7r-8)} < 5,$$

which contradicts the fact that $q = 5$ or $q \geq 11$.

Summing up the arguments in (i)–(v), we obtain the inequality $q < p$ from (20).

3.2. The proof of Theorem 0.1(3). By the same argument as in the previous subsection, we obtain the inequality $p < q$ from (21). This contradicts (23), and Theorem 0.1(3) is proved.

Acknowledgements. The author is grateful to the referees for an important comment on Kanold's result.

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