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Concerning the Szlenk index

by

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Abstract. We discuss pruning and coloring lemmas on regular families. We discuss several applications of these lemmas to computing the Szlenk index of certain w^* compact subsets of the dual of a separable Banach space. Applications include estimates of the Szlenk index of Minkowski sums, infinite direct sums of separable Banach spaces, constant reduction, and three-space properties.

We also consider using regular families to construct Banach spaces with prescribed Szlenk index. As a consequence, we give a characterization of which countable ordinals occur as the Szlenk index of a Banach space, prove the optimality of a previous universality result, and compute the Szlenk index of the injective tensor product of separable Banach spaces.

1. Introduction. A classical result in Banach space theory is that every separable Banach space embeds isometrically in $\mathcal{C}[0,1]$. One can ask whether other classes of Banach spaces, for example the class of Banach spaces having separable dual, admit a member which contains isomorphic copies of every member of that class. For the case of Banach spaces having separable dual, Szlenk [22] introduced the Szlenk index to prove that there is no Banach space having separable dual which contains isomorphic copies of all Banach spaces having separable dual. Since its inception, the Szlenk index has been the object of significant investigation.

Typically defined in terms of slicings of the unit ball of the dual of a separable Banach space, the Szlenk index of a separable Banach space is equal to the weakly null ℓ_1^+ index of that space in the case that this space does not contain a copy of ℓ_1 [2]. This fact allows for a modification of certain transfinite versions of an argument of James [12] involving equivalence of finite representability and crude finite representability of ℓ_1 in a Banach space. This argument can be used to yield new information about the Szlenk

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index and new methods for estimating it. More generally, regular families play a key role in computing so-called σ indices in separable Banach spaces. Consequently, certain purely combinatorial results concerning colorings of regular families have as easy corollaries strong results about Szlenk index, including that of [2]. Moreover, regular families can be used to construct Banach spaces with prescribed weakly null ℓ_1^+ behavior, which can be used to prove certain existence and non-existence results. For example, we provide a characterization of which countable ordinals occur as the Szlenk index of a Banach space. In [7], it was shown that for each countable ordinal ξ there exists a separable Banach space with Szlenk index $\omega^{\xi+1}$ which contains isomorphic copies of every separable Banach space having Szlenk index not exceeding ω^ξ. Thanks to being able to construct a Banach space with precise control over the weakly null ℓ_1^+ index, we can prove the optimality of that result.

In the first half of the paper, we discuss regular families, colorings and prunings thereof, and applications of these coloring results to computing the Szlenk index of certain subsets of the dual of a separable Banach space. We generalize Alspach, Judd, and Odell's argument that the Szlenk index of a Banach space not containing ℓ_1 is equal to its weakly null ℓ_1^+ index in order to compute the Szlenk index of certain sets $K \subset X^*$, X a separable Banach space. We then deduce as easy applications of this work a number of corollaries, some old and some new. In the second half of the paper, we discuss how to construct Banach spaces with prescribed weakly null ℓ_1^+ structure. As a consequence, we provide a characterization of the countable ordinals which occur as the Szlenk index of a Banach space and use this to prove the optimality of the universality results of [7] and [8]. We also show how one can compute the Szlenk index of a Banach space having separable dual via embeddings into Banach spaces with shrinking basis having subsequential upper block estimates in certain mixed Tsirelson spaces. With this, we prove an optimal result about the Szlenk index of an injective tensor product of two separable Banach spaces.

The paper is arranged as follows. In Section 2, we discuss the necessary definitions concerning Banach spaces and finite-dimensional decompositions. In Section 3, we discuss trees, regular families, and their use in computing ordinal indices. In that section we also give two useful pruning lemmas which will be used throughout. In Section 4, we state and prove the combinatorial lemmas concerning regular families. In Section 5, we define the Szlenk and weakly null ℓ_1^+ indices and provide several examples of applications thereof. In Section 6, we discuss the use of mixed Tsirelson spaces in constructing Banach spaces with prescribed ℓ_1^+ behavior and the special role played by these families.

2. Banach spaces and finite-dimensional decompositions. If X is a Banach space, we say a sequence $E = (E_n)$ of finite-dimensional subspaces of X is a *finite-dimensional decomposition* (FDD) for X provided that for each $x \in X$, there exists a unique sequence (x_n) such that $x_n \in E_n$ for each $n \in \mathbb{N}$ and $x = \sum x_n$. In this case, for each $n \in \mathbb{N}$, the operator $x = \sum x_m \mapsto x_n$ is a bounded linear operator from X to E_n, called the nth *canonical projection*, denoted P_n^E. For a finite set A, we let $P_A = \sum_{n\in A} P_n$. By the principle of uniform boundedness, the *projection constant of E in X*, given by $\sup_{m\le n} \|P^E_{[m,n]}\|$, is finite. We say E is *bimonotone* for X if the projection constant of E in X is 1. It is well-known that if E is an FDD for X, one can equivalently renorm X to make E a bimonotone FDD for X with the new norm. Throughout, we will assume that for each $n \in \mathbb{N}$, $E_n \neq \{0\}$.

We can consider E_n^* as being embedded in X^* via the adjoint $(P_n^E)^*$, although this embedding is not necessarily isometric unless E is bimonotone. We let $E^* = (E_n^*)$, and consider these as subspaces of X^*. The FDD E is said to be *shrinking* for X if E^* is an FDD for X^*. Since E^* will always be an FDD for the closed span $[E_n^*]_{n\in\mathbb{N}}$ with projection constant in this space not exceeding the projection constant of E in X, we see that E is a shrinking FDD for X if and only if $X^* = [E_n^*]_{n\in\mathbb{N}}$.

If E is an FDD for X and if $0 = s_0 < s_1 < \cdots$, and $F_n = [E_k]_{s_{n-1}<k\le s_n}$, then $F = (F_n)$ is called a *blocking* of E. In this case, F is also an FDD for X with projection constant in X not exceeding the projection constant of E in X. If E is shrinking, any blocking of E will be as well.

If $x \in X$, we let $\mathrm{supp}_E(x) = \{n \in \mathbb{N} : P_n^E x \neq 0\}$. We let $\mathrm{ran}_E(x)$ be the smallest interval in $\mathbb{N}$ which contains $\mathrm{supp}_E(x)$. We let $c_{00}(E) = \{x \in X : |\mathrm{supp}_E(x)| < \infty\}$. We say a (finite or infinite) sequence of non-zero vectors (x_n) is a *block sequence with respect to* E provided $\max \mathrm{supp}_E(x_n) < \min \mathrm{supp}_E(x_{n+1})$ for each appropriate n.

We let $\Sigma(E,X)$ denote all finite block sequences with respect to E in B_X. We say $\mathcal{B} \subset \Sigma(E,X)$ is a *hereditary block tree* in X with respect to E if it contains all subsequences of its members. If $\overline{\varepsilon} = (\varepsilon_i) \subset (0,1)$ and if $\mathcal{B}$ is a hereditary block tree, we let

$$\mathcal{B}^{E,X}_{\overline{\varepsilon}} = \{(x_i)_{i=1}^n \in \Sigma(E,X) : n \in \mathbb{N} \cup \{0\},\ \exists (y_i)_{i=1}^n \in \mathcal{B},\ \|x_i - y_i\| < \varepsilon_i\ \forall 1 \le i \le n\}.$$

If (ε_i) is non-increasing, $\mathcal{B}^{E,X}_{\overline{\varepsilon}}$ is also a hereditary block tree in X with respect to E.

Given (finite or infinite) sequences $(e_n), (f_n)$ of the same length in (possibly different) Banach spaces, we say (e_n) *C-dominates* (f_n), or that (f_n)

is C-*dominated by* (e_n), provided that for each $(a_n) \in c_{00}$,

$$\Big\|\sum a_n f_n\Big\| \leq C\Big\|\sum a_n e_n\Big\|.$$

If E is an FDD for a Banach space X and if (e_n) is a normalized 1-unconditional basis for a Banach space U, we say E *satisfies subsequential* C-U *upper block estimates in* X provided that for any normalized block sequence (x_n) with respect to E, if $m_n = \max \operatorname{supp}_E(x_n)$, then (x_n) is C-dominated by (e_{m_n}). This idea has occurred in other works, such as [18], [10], and [7], where m_n was taken to be $\min \operatorname{supp}_E(x_n)$ rather than the maximum. Our definition is chosen for convenience within this work, and it does not affect the main theorems contained herein, or the main theorems contained in the cited works. This is because for each basis (e_n) considered in the main theorems of the cited works, and for each pair of sequences of natural numbers $k_1 < k_2 < \cdots$, $l_1 < l_2 < \cdots$ such that $\max\{k_n, l_n\} < \min\{k_{n+1}, l_{n+1}\}$, (e_{k_n}) and (e_{l_n}) are equivalent, and hence equivalent with uniform constant.

Proposition 2.1. *Let X be a Banach space not containing ℓ_1.*

(i) *Suppose $Y \leq X$ is a closed subspace, $(x_n) \subset B_X$ is weakly null, and $\delta \in (0, 1/2)$ is such that $\|x_n\|_{X/Y} < \delta$ for all $n \in \mathbb{N}$. Then there exists a weakly null sequence $(y_n) \subset B_Y$ and a subsequence (x_{k_n}) of (x_n) such that $\|x_{k_n} - y_n\| < 4\delta$ for each $n \in \mathbb{N}$.*
(ii) *If $Q : X \to Z$ is a quotient map and $(z_n) \subset B_Z$ is weakly null, then for any $\delta > 0$, there exists a weakly null sequence $(x_n) \subset 3B_X$ and a subsequence (z_{k_n}) of (z_n) such that $\|z_{k_n} - Qx_n\| < \delta$ for all $n \in \mathbb{N}$.*

Proof. Several times, we will use Rosenthal's ℓ_1 dichotomy [20], which states that any bounded sequence in a Banach space has either a subsequence equivalent to the canonical ℓ_1 basis or a subsequence which is weakly Cauchy.

(i) For each n, choose some $u_n \in Y$ so that $\|x_n - u_n\| < \delta$. By passing to a subsequence, we can assume that (u_n) is weakly Cauchy. Choose a convex block defined by $v_n = \sum_{i \in I_n} a_i x_i$ so that $\|v_n\| < \delta - \|x_n - u_n\|$ and $\min I_n \to \infty$ as $n \to \infty$. Let $w_n = \sum_{i \in I_n} a_i u_i$. Then $(u_n - w_n)$ is weakly null in Y and

$$\|u_n - w_n\| \leq \|x_n\| + \|x_n - u_n\| + \|v_n\| + \sum_{i \in I_n} a_i \|x_i - u_i\| \leq 1 + 2\delta.$$

Moreover,

$$\begin{aligned}\|x_n - (u_n - w_n)\| &\leq \|x_n - v_n - (u_n - w_n)\| + \|v_n\| \\ &\leq \|x_n - u_n\| + \|v_n\| + \sum_{i \in I_n} a_i \|x_i - u_i\| < 2\delta.\end{aligned}$$

Now if $y_n = u_n - w_n$ for those $n \in \mathbb{N}$ such that $\|u_n - w_n\| \leq 1$ and if y_n is the normalization of $u_n - w_n$ for those $n \in \mathbb{N}$ such that $\|u_n - w_n\| > 1$, then

$$\|x_n - y_n\| \leq \|x_n - (u_n - w_n)\| + \|y_n - (u_n - w_n)\| < 4\delta.$$

Since $(u_n - w_n)$ is weakly null and $y_n = a_n(u_n - w_n)$ for some sequence $(a_n) \subset [0,1]$, (y_n) is also weakly null.

(ii) Choose $\varepsilon > 0$ to be determined. For each $n \in \mathbb{N}$, choose $u_n \in X$ with $\|u_n\| < 1 + \varepsilon$ so that $Qu_n = z_n$. By passing to a subsequence, we can assume (u_n) is weakly Cauchy. Choose a convex block $v_n = \sum_{i \in I_n} a_i z_i$ so that $\|v_n\| < \varepsilon$ and $\min I_n \to \infty$ as $n \to \infty$. Let $w_n = \sum_{i \in I_n} a_i u_i$. Then

$$\|u_n - w_n\| \leq 1 + \varepsilon + \sum_{i \in I_n} a_i(1 + \varepsilon) = 2 + 2\varepsilon < 3$$

for appropriate ε. Moreover, this sequence is weakly null. Last,

$$\|z_n - Q(u_n - w_n)\| = \|Qw_n\| = \|v_n\| < \varepsilon < \delta.$$

Thus taking $\varepsilon < \min\{1/2, \delta\}$ suffices. ∎

3. Trees, derivatives, and indices

3.1. Trees on sets. Throughout, if P, Q are partially ordered sets, we say $f : P \to Q$ is *order preserving* provided that $x, y \in P$ with $x <_P y$ implies $f(x) <_Q f(y)$. We say $f : P \to Q$ is an *embedding* if it is an injection such that for $x, y \in P$, $x <_P y$ if and only if $f(x) <_Q f(y)$.

Given a set S, we let S^ω (resp. $S^{<\omega}$) denote the set of all infinite (resp. finite) sequences in S. We include the sequence of length zero, denoted $\emptyset$, in $S^{<\omega}$. For $s \in S^{<\omega}$, we let $|s|$ denote the length of s. For $s, t \in S^{<\omega}$, we let $s^\frown t$ denote the concatenation of s with t. Given $s = (x_i)_{i=1}^n \in S^{<\omega}$, we let $s|_m = (x_i)_{i=1}^m$ for $0 \leq m \leq n$. We define the partial order $\prec$ on $S^{<\omega}$ by $s \prec s'$ provided $|s| < |s'|$ and $s = s'|_{|s|}$. If $s \prec s'$, we say s is a *predecessor* of s', and s' is a *successor* of s. If $|s'| = |s| + 1$, we say s is the *immediate predecessor* of s', and s' is an *immediate successor* of s. Given a set $U \subset S^{<\omega}$, we let $C(U)$ denote the set of all finite, non-empty chains in $U \setminus \{\emptyset\}$. We define a partial order $<$ on $C(U)$ by $c < c'$ provided $s \prec s'$ for all $s \in c$ and $s' \in c'$.

If $T \subset S^{<\omega}$ is downward closed with respect to the order $\prec$, we call T a *tree*, and we let $\mathrm{MAX}(T)$ denote the maximal elements of T with respect to the order $\prec$. We let $\widehat{T} = T \setminus \{\emptyset\}$. If T contains all subsequences of its members, we say T is *hereditary*. If $T \subset S^{<\omega}$, we let

$$T(s) = \{t \in S^{<\omega} : s^\frown t \in T\},$$

and note that if T is a tree (resp. hereditary tree), then $T(s)$ is a tree (resp. hereditary tree) as well. If T is a tree, we call linearly ordered subsets of T *chains* of T, and chains which are maximal with respect to inclusions will be called *branches* of T. If T is a tree on a vector space, we say T is *convex* provided it contains all convex blockings of its members. We recall that for a sequence $(x_i)_{i=1}^n$ in a vector space, $(y_i)_{i=1}^m$ is a *convex blocking* of $(x_i)_{i=1}^n$ provided there exist $0 = k_0 < \cdots < k_m = n$ and non-negative scalars $(a_i)_{i=1}^n$ such that for each j, $\sum_{i=k_{j-1}+1}^{k_j} a_i = 1$ and $y_j = \sum_{i=k_{j-1}+1}^{k_j} a_i x_i$.

Given a tree T, we let $T' = T \setminus \mathrm{MAX}(T)$, and note that this is a tree as well. We define the countable transfinite derivations as follows. Throughout, ω and ω_1 will denote the first infinite and uncountable ordinals, respectively. We let

$$T^0 = T, \quad T^{\xi+1} = (T^\xi)', \quad \xi < \omega_1,$$

and

$$T^\xi = \bigcap_{\zeta<\xi} T^\zeta, \quad \xi < \omega_1 \text{ a limit ordinal.}$$

Finally, we define the order o of the tree T by

$$o(T) = \min\{\xi < \omega_1 : T^\xi = \emptyset\}$$

provided such a ξ exists, and $o(T) = \omega_1$ otherwise.

3.2. Regular trees on $\mathbb{N}$. Throughout, if M is any infinite subset of $\mathbb{N}$, we let $[M]^\omega$ (resp. $[M]^{<\omega}$) denote the infinite (resp. finite) subsets of M. We identify the subsets of $\mathbb{N}$ in the natural way with strictly increasing sequences in $\mathbb{N}$. We topologize the power set of $\mathbb{N}$ by identifying it with the Cantor set. A set $\mathcal{F} \subset [\mathbb{N}]^{<\omega}$ is called compact if it is compact with respect to this topology. For $E, F \subset \mathbb{N}$, we write $E < F$ to denote $\max E < \min F$. For $n \in \mathbb{N}$ and $E \subset \mathbb{N}$, we write $n \leq E$ to denote $n \leq \min E$. By convention, we let $\emptyset < E < \emptyset$ for any E. Throughout, we will write $E^\frown F$ in place of $E \cup F$ when $E < F$. We write $n^\frown E$ (resp. $E^\frown n$) in place of $(n)^\frown E$ (resp. $E^\frown(n)$).

Given $(k_i)_{i=1}^n, (l_i)_{i=1}^n \in [\mathbb{N}]^{<\omega}$, we say $(l_i)_{i=1}^n$ is a *spread* of $(k_i)_{i=1}^n$ provided $k_i \leq l_i$ for each $1 \leq i \leq n$. We say $\mathcal{F} \subset [\mathbb{N}]^{<\omega}$ is *spreading* provided it contains all spreads of its members. We say $\mathcal{F}$ is *hereditary* if it contains all subsets of its members. With the identification of sets with sequences, we can naturally identify a hereditary family with a (hereditary) tree on $\mathbb{N}$. We call a family $\mathcal{F} \subset [\mathbb{N}]^{<\omega}$ *regular* provided it is compact, spreading, and hereditary.

We say that a sequence $(E_i)_{i=1}^n \subset [\mathbb{N}]^{<\omega}$ is $\mathcal{F}$-*admissible* if it is *successive* (that is, $E_1 < \cdots < E_n$), $n \geq 0$, $E_i \neq \emptyset$, and $(\min E_i)_{i=1}^n \in \mathcal{F}$. Given a regular family $\mathcal{G}$ and a set E, we say the successive sequence $(E_i)_{i=1}^n$ is the *standard decomposition of E with respect to $\mathcal{G}$* provided that $E = \bigcup_{i=1}^n E_i$ and for each $j \leq n$, E_j is the maximal initial segment of $\bigcup_{i=j}^n E_i$ which is a member of $\mathcal{G}$. Note that E admits a standard decomposition with respect to $\mathcal{G}$ if and only if $E = \emptyset$ (in which case $n = 0$) or $E \neq \emptyset$ and $(\min E) \in \mathcal{G}$. In each case the standard decomposition is unique.

If $(m_n) = M \in [\mathbb{N}]^\omega$, the bijection $n \mapsto m_n$ induces a natural bijection between the power sets of $\mathbb{N}$ and M, which we also denote M. That is, $M(E) = (m_n : n \in E)$. For $\mathcal{F} \subset [\mathbb{N}]^{<\omega}$, we let $M(\mathcal{F}) = \{M(E) : E \in \mathcal{F}\}$. If $M \in [\mathbb{N}]^\omega$ and if $\mathcal{F} \subset [\mathbb{N}]^{<\omega}$, we let $M^{-1}(\mathcal{F}) = \{E : M(E) \in \mathcal{F}\}$.

Given regular families $\mathcal{F}, \mathcal{G}$, we define

$$\begin{aligned}
\mathcal{F}^\frown\mathcal{G} &= \{F^\frown G : F \in \mathcal{F},\, G \in \mathcal{G}\},\\
\mathcal{F}[\mathcal{G}] &= \Big\{\bigcup_{i=1}^{n} E_i : n \ge 0,\, E_1 < \cdots < E_n,\, \emptyset \neq E_i \in \mathcal{G},\, (\min E_i)_{i=1}^n \in \mathcal{F}\Big\}\\
&= \Big\{\bigcup_{i=1}^{n} E_i : n \ge 0,\, (E_i)_{i=1}^n \subset \mathcal{G} \text{ is } \mathcal{F}\text{-admissible}\Big\}.
\end{aligned}$$

We observe that a set E is in $\mathcal{F}[\mathcal{G}]$ if and only if E has an $\mathcal{F}$-admissible standard decomposition $(E_i)_{i=1}^n$ with respect to $\mathcal{G}$. For a given $\mathcal{F}$, we let $[\mathcal{F}]^1 = \mathcal{F}$ and $[\mathcal{F}]^{n+1} = \mathcal{F}[[\mathcal{F}]^n]$ for $n \in \mathbb{N}$.

If $(\mathcal{G}_n)$ is a sequence of regular families, we let

$$\mathcal{D}(\mathcal{G}_n) = \{E : \exists n \le E \in \mathcal{G}_n\}.$$

We think of $\mathcal{F}^\frown\mathcal{G}$ as the sum of the trees $\mathcal{F}, \mathcal{G}$, of $\mathcal{F}[\mathcal{G}]$ as the product of $\mathcal{F}, \mathcal{G}$, and of $\mathcal{D}(\mathcal{G}_n)$ as the diagonalization of the families $\mathcal{G}_n$.

For each $1 \le n$, let $\mathcal{A}_n = \{E \in [\mathbb{N}]^{<\omega} : |E| \le n\}$ and $\mathcal{S} = \mathcal{D}(\mathcal{A}_n)$. If $\zeta \le \omega_1$ is a limit ordinal, we say that the family $(\mathcal{G}_\xi)_{0\le\xi<\zeta}$ is *additive* if for each $\xi < \zeta$, $\mathcal{G}_{\xi+1} = \mathcal{A}_1^\frown\mathcal{G}_\xi$, and for each limit ordinal $\xi < \zeta$, there exists $\xi_n \uparrow \xi$ such that $\mathcal{G}_\xi = \mathcal{D}(\mathcal{G}_{\xi_n})$. We say $(\mathcal{G}_\xi)_{0\le\xi<\zeta}$ is *multiplicative* if $\mathcal{G}_{\xi+1} = \mathcal{S}[\mathcal{G}_\xi]$ for each $\xi < \zeta$, $(1) \in \mathrm{MAX}(\mathcal{G}_0)$, and for every limit ordinal $\xi < \zeta$, there exists a sequence $\xi_n \uparrow \xi$ such that $\mathcal{G}_\xi = \mathcal{D}(\mathcal{G}_{\xi_n})$. Observe in this case that $(1) \in \mathrm{MAX}(\mathcal{G}_\xi)$ for every $\xi < \zeta$.

If $\mathcal{F}$ is regular, we observe that $\mathcal{F}'$ is also regular, and $\mathrm{MAX}(\mathcal{F})$ is the set of isolated points in $\mathcal{F}$. Thus $\mathcal{F}'$ is the Cantor–Bendixson derivative of $\mathcal{F}$. In place of the Cantor–Bendixson index, we define the index

$$\iota(\mathcal{F}) = \min\{\xi < \omega_1 : \mathcal{F}^\xi \subset \{\emptyset\}\}.$$

It is easy to see that for $\mathcal{F}$ hereditary, this set of ordinals is non-empty if and only if $\mathcal{F}$ is compact, which is equivalent to $\mathcal{F}$ not containing any infinite chain. Moreover, if $\mathcal{F} \neq \emptyset$, then $\iota(\mathcal{F}) + 1$ coincides with the Cantor–Bendixson derivative of $\mathcal{F}$. The justification for using the index ι in place of the Cantor–Bendixson index is evident in the following proposition.

Proposition 3.1. *Let $\mathcal{F}$, $\mathcal{G}$, and $\mathcal{G}_n$ be non-empty regular families.*

(i) *For $0 \le \zeta, \xi < \omega_1$, we have $(\mathcal{F}^\zeta)^\xi = \mathcal{F}^{\zeta+\xi}$.*
(ii) *$\mathcal{F}^\frown\mathcal{G}$ is regular and $\iota(\mathcal{F}^\frown\mathcal{G}) = \iota(\mathcal{G}) + \iota(\mathcal{F})$.*
(iii) *$\mathcal{F}[\mathcal{G}]$ is regular and $\iota(\mathcal{F}[\mathcal{G}]) = \iota(\mathcal{G})\iota(\mathcal{F})$.*
(iv) *For any $M \in [\mathbb{N}]^\omega$, $M^{-1}(\mathcal{F})$ is regular and $\iota(M^{-1}(\mathcal{F})) = \iota(\mathcal{F})$.*

(v) *For any* $M \in [\mathbb{N}]^\omega$, *we have* $M^{-1}(\mathcal{F}[\mathcal{G}]) = M^{-1}(\mathcal{F})[M^{-1}(\mathcal{G})]$.
(vi) $\mathcal{D}(\mathcal{G}_n)$ *is regular and* $\iota(\mathcal{D}(\mathcal{G}_n)) = \sup_n \iota(\mathcal{G}_n)$.
(vii) *If* $M \in [\mathbb{N}]^\omega$ *and* $\iota(\mathcal{F}) \le \iota(\mathcal{G})$, *then there exists* $N \in [M]^\omega$ *such that* $N(\mathcal{F}) \subset \mathcal{G}$.
(viii) *If* $\zeta \le \omega_1$ *is a limit ordinal and* $(\mathcal{G}_\xi)_{0\le\xi<\zeta}$ *is either additive or multiplicative, then for each* $0 \le \xi \le \eta < \zeta$, *there exist* $m, n \in \mathbb{N}$ *such that* $\mathcal{G}_\xi \cap [[m,\infty)]^{<\omega} \subset \mathcal{G}_\eta$ *and* $\mathcal{G}_\xi \subset \mathcal{G}_{\eta+n}$.

Proof. (i) We use induction on ξ for ζ fixed. The $\xi = 0$ and successor cases are trivial. If ξ is a limit ordinal, $\zeta + \xi$ is also a limit, so

$$(\mathcal{F}^\zeta)^\xi = \bigcap_{\eta<\xi} (\mathcal{F}^\zeta)^\eta = \bigcap_{\eta<\xi} \mathcal{F}^{\zeta+\eta} = \bigcap_{\eta<\zeta+\xi} \mathcal{F}^\eta = \mathcal{F}^{\zeta+\xi}.$$

Here we have used the facts that $\eta \mapsto \zeta + \eta$ is continuous and the Cantor–Bendixson derivatives of $\mathcal{F}$ are decreasing.

(ii) It is clear that a subset (resp. spread) of $F^\frown G$, for $F \in \mathcal{F}$ and $G \in \mathcal{G}$, can be written in the form $F_0^\frown G_0$ where F_0 (resp. G_0) is a subset (resp. spread) of F (resp. G). Thus $\mathcal{F}^\frown\mathcal{G}$ is spreading and hereditary. If $N|_n \in \mathcal{F}^\frown\mathcal{G}$ for all $n \in \mathbb{N}$, let $m \in \mathbb{N} \cup \{0\}$ be maximal such that $N|_m \in \mathcal{F}$. Then choose $n \in \mathbb{N} \cup \{0\}$ maximal with $(N \setminus N|_m)|_n \in \mathcal{G}$. It is clear that $N|_k \notin \mathcal{F}^\frown\mathcal{G}$ for any $k > n+m$. This is because if $F^\frown G = N|_k$, then either F is a proper extension of $N|_m$, or G has a subset which is a proper extension of $(N \setminus N|_m)|_n$, either of which contradicts the maximality of either m or n.

Next, we note that $(\mathcal{F}^\frown\mathcal{G})(F) = \mathcal{G} \cap [(\max F, \infty)]^{<\omega}$ for $F \in \mathrm{MAX}(\mathcal{F})$. Since $\iota(\mathcal{G} \cap [(\max F,\infty)]^{<\omega}) = \iota(\mathcal{G})$, we have $(\emptyset) = (\mathcal{F}^\frown\mathcal{G})(F)^{\iota(\mathcal{G})}$, which means $F \in \mathrm{MAX}((\mathcal{F}^\frown\mathcal{G})^{\iota(\mathcal{G})})$. If $E \in (\mathcal{F}^\frown\mathcal{G}) \setminus \mathcal{F}$, write $E = F^\frown G$ where F is the maximal initial segment of E which lies in $\mathcal{F}$, and $\emptyset \ne G \in \mathcal{G}$. Then, $(\mathcal{F}^\frown\mathcal{G})^\zeta(E) = \mathcal{G}^\zeta(G)$ for any ordinal ζ. Since $\iota(\mathcal{G}(G)) < \iota(\mathcal{G})$, we have $(\mathcal{F}^\frown\mathcal{G})^{\iota(\mathcal{G})}(E) = \mathcal{G}^{\iota(\mathcal{G})}(G) = \emptyset$. This means $E \notin (\mathcal{F}^\frown\mathcal{G})^{\iota(\mathcal{G})}$. Therefore $\mathcal{F} = (\mathcal{F}^\frown\mathcal{G})^{\iota(\mathcal{G})}$, and $\iota(\mathcal{F}^\frown\mathcal{G}) = \iota(\mathcal{G}) + \iota(\mathcal{F})$.

(iii) Any spread (resp. subset) of $\bigcup_{i=1}^n E_i$ is an $\mathcal{F}$-admissible union of spreads (resp. subsets) F_i of E_i. If $N|_n \in \mathcal{F}[\mathcal{G}]$ for all $n \in \mathbb{N}$, choose recursively $n_0, n_1, n_2, \ldots$ maximal such that $n_0 = 0$ and $(N \setminus N|_{n_{i-1}})|_{n_i} \in \mathcal{G}$ for all $i \in \mathbb{N}$. Let $m_i = \min(N \setminus N|_{n_{i-1}})$ and choose k so that $(m_i)_{i=1}^k \notin \mathcal{F}$. Then $N|_s \notin \mathcal{F}[\mathcal{G}]$ for any $s > \sum_{i=1}^k n_i$. Indeed, if $N|_s \in \mathcal{F}[\mathcal{G}]$, let $(E_i)_{i=1}^t$ be the standard decomposition of $N|_s$ with respect to $\mathcal{G}$. Then $\mathcal{F} \ni (\min E_i)_{i=1}^t$ is a proper extension of $(m_i)_{i=1}^k$, a contradiction.

We prove by induction that $\mathcal{F}[\mathcal{G}]^{\iota(\mathcal{G})\xi} = \mathcal{F}^\xi[\mathcal{G}]$ for all $\xi \le \iota(\mathcal{F})$. The result is clear if $\mathcal{F} = \{\emptyset\}$ or $\mathcal{G} = \{\emptyset\}$, so assume $\iota(\mathcal{F}), \iota(\mathcal{G}) > 0$. The base case is true by definition. If $(E_i)_{i=1}^n \subset \mathcal{G}$ is $\mathcal{F}$-admissible with $F := (\min E_i)_{i=1}^n \in \mathcal{F}'$, then there exists $m > \max E_n$ such that $F^\frown i \in \mathcal{F}$ for each $i \ge m$. Then $\mathcal{G} \cap (m,\infty)^{<\omega} \subset \mathcal{F}[\mathcal{G}]\big(\bigcup_{i=1}^n E_i\big)$. This means $\bigcup_{i=1}^n E_i \in \mathcal{F}[\mathcal{G}]^{\iota(\mathcal{G})}$, whence

$\mathcal{F}'[\mathcal{G}] \subset \mathcal{F}[\mathcal{G}]^{\iota(\mathcal{G})}$. Next, fix $E \in \mathcal{F}[\mathcal{G}]$ and let $(E_i)_{i=1}^n$ be the standard decomposition of E with respect to $\mathcal{G}$. Suppose that $(\min E_i)_{i=1}^n \in \mathrm{MAX}(\mathcal{F})$. Then $\mathcal{F}[\mathcal{G}]\big(\bigcup_{i=1}^n E_i\big) = \mathcal{G}(E_n)$. But $\iota(\mathcal{G}(E_n)) < \iota(\mathcal{G})$, which means $\bigcup_{i=1}^n E_i \notin \mathcal{F}[\mathcal{G}]^{\iota(\mathcal{G})}$. Therefore $\mathcal{F}[\mathcal{G}]^{\iota(\mathcal{G})} \subset \mathcal{F}'[\mathcal{G}]$, and these sets are equal. Applying this argument again to $\mathcal{F}^\xi$ in place of $\mathcal{F}$ yields the successor case. Last, for a limit ordinal ξ, $\iota(\mathcal{G})\xi$ is also a limit ordinal. Then

$$\mathcal{F}[\mathcal{G}]^{\iota(\mathcal{G})\xi} = \bigcap_{\zeta<\iota(\mathcal{G})\xi} \mathcal{F}[\mathcal{G}]^{\zeta} = \bigcap_{\zeta<\xi} \mathcal{F}[\mathcal{G}]^{\iota(\mathcal{G})\zeta} = \bigcap_{\zeta<\xi} \mathcal{F}^{\zeta}[\mathcal{G}] = \mathcal{F}^{\xi}[\mathcal{G}].$$

The last equality follows from the fact that E will lie in either of the two sets if and only if E has a standard decomposition $(E_i)_{i=1}^n$ with respect to $\mathcal{G}$ and that this sequence is $\mathcal{F}^\xi$-admissible, while this second property is equivalent to being $\mathcal{F}^\eta$-admissible for every $\zeta < \xi$.

(iv) If $E \in M^{-1}(\mathcal{F})$ and F is a subset (resp. spread) of E, then $M(F)$ is a subset (resp. spread) of $M(E)$. Therefore $M(F) \in \mathcal{F}$, whence $F \in M^{-1}(\mathcal{F})$. If $N \in [\mathbb{N}]^\omega$ is such that $N|_n \in M^{-1}(\mathcal{F})$ for all $n \in \mathbb{N}$, then $M(N|_n) \in \mathcal{F}$ for all $n \in \mathbb{N}$, contradicting the compactness of $\mathcal{F}$. Thus $M^{-1}(\mathcal{F})$ is regular. It is easy to see that $M^{-1}(\mathcal{F})^\xi = M^{-1}(\mathcal{F}^\xi)$ for any $0 \le \xi < \omega_1$, so $\iota(M^{-1}(\mathcal{F})) = \iota(\mathcal{F})$.

(v) Let $F \in M^{-1}(\mathcal{F}[\mathcal{G}])$. Then write $M(F) = \bigcup_{i=1}^n E_i$, where $(E_i)_{i=1}^n \subset \mathcal{G}$ is $\mathcal{F}$-admissible. Note that for each $1 \le i \le n$, $E_i = M(F_i)$ for some F_i, which necessarily lies in $M^{-1}(\mathcal{G})$. Moreover, $M((\min F_i)_{i=1}^n) = (\min E_i)_{i=1}^n \in \mathcal{F}$, and $(\min F_i)_{i=1}^n \in M^{-1}(\mathcal{F})$. Note that $F = \bigcup_{i=1}^n F_i \in M^{-1}(\mathcal{F})[M^{-1}(\mathcal{G})]$, so that $M^{-1}(\mathcal{F}[\mathcal{G}]) \subset M^{-1}(\mathcal{F})[M^{-1}(\mathcal{G})]$.

If $E \in M^{-1}(\mathcal{F})[M^{-1}(\mathcal{G})]$, write $E = \bigcup_{i=1}^n E_i$, $(\min E_i)_{i=1}^n \in M^{-1}(\mathcal{F})$, $E_i \in M^{-1}(\mathcal{G})$. Then $(\min M(E_i))_{i=1}^n = M((\min E_i)_{i=1}^n) \in \mathcal{F}$ and $M(E_i) \in \mathcal{G}$. Therefore $M(E) = \bigcup_{i=1}^n M(E_i) \in \mathcal{F}[\mathcal{G}]$, and $E \in M^{-1}(\mathcal{F}[\mathcal{G}])$.

(vi) Suppose $E \in \mathcal{D}(\mathcal{G}_n)$ and fix $m \le E \in \mathcal{G}_m$. If F is a subset (resp. spread) of E, then $m \le F \in \mathcal{G}_m$, so $F \in \mathcal{D}(\mathcal{G}_n)$. If $N|_m \in \mathcal{D}(\mathcal{G}_n)$ for all $m \in \mathbb{N}$, then we can choose for each $m \in \mathbb{N}$ some $k_m \in \mathbb{N}$ so that $k_m \le N$ and $N|_m \in \mathcal{G}_{k_m}$. We can, of course, assume that for some $k \le N$, $k_m = k$ for all m. Then $N|_m \in \mathcal{G}_k$ for all m, a contradiction.

Let $\mathcal{D} = \mathcal{D}(\mathcal{G}_n)$ and $\xi = \sup_n \iota(\mathcal{G}_n)$. It is clear that $\iota(\mathcal{D}) \ge \sup_n \iota(\mathcal{G}_n \cap [[n,\infty)]^{<\omega}) = \xi$. For any $n \in \mathbb{N}$, we have $\mathcal{D}(n) = \bigcup_{i=1}^n \mathcal{G}_i(n)$ and $\mathcal{D}^\xi(n) = \bigcup_{i=1}^n \mathcal{G}^\xi(n) = \emptyset$. From this it follows that $\mathcal{D}^\xi \subset \{\emptyset\}$, and $\iota(\mathcal{D}) = \xi$.

(vii) First, we observe that for any regular $\mathcal{F}$, $(\iota(\mathcal{F}(n)))_{n\in\mathbb{N}}$ is a nondecreasing sequence. This is because $\mathcal{F}(n)$ is homeomorphic to a subset of $\mathcal{F}(m)$ for $n \le m$ via the map $E \mapsto (k+m : k \in E)$. We next observe that if $\iota(\mathcal{F}) = \xi+1$, then $\iota(\mathcal{F}(n)) = \xi$ eventually. First, if $\iota(\mathcal{F}(n)) > \xi$ for some $n \in \mathbb{N}$, then $(n) \in \mathcal{F}^{\xi+1}$, which means $\iota(\mathcal{F}) > \xi+1$. If $\iota(\mathcal{F}(n)) < \xi$ for all $n \in \mathbb{N}$, then $\mathcal{F}^\xi$ contains no singletons, and therefore $\iota(\mathcal{F}) \le \xi$.

Next, if ξ is a limit ordinal and $\iota(\mathcal{F}) = \xi$, then $\iota(\mathcal{F}(n)) \nearrow \xi$. We know $\iota(\mathcal{F}(n)) < \xi$ for all $n \in \mathbb{N}$ by the same argument as in the successor case. We know this sequence is non-decreasing, again by the same reasoning as in the successor case. If $\iota(\mathcal{F}(n)) \leq \zeta < \xi$ for all $n \in \mathbb{N}$, then $\iota(\mathcal{F}) \leq \zeta + 1 < \xi$.

Before completing (vii), we prove the following.

Claim. *Suppose $\mathcal{F}, \mathcal{G}$ are regular families with $\iota(\mathcal{G}) \geq 1$. Suppose also that for any $n \in \mathbb{N}$ and any $M \in [\mathbb{N}]^\omega$, there exist $k_n \in \mathbb{N}$ and $N \in [M]^\omega$ such that $N(\mathcal{F}(n)) \subset \mathcal{G}(k_n)$. Then for any $M \in [\mathbb{N}]^\omega$, there exists $N \in [M]^\omega$ such that $N(\mathcal{F}) \subset \mathcal{G}$.*

Proof of claim. If $\iota(\mathcal{G}) \geq 1$, then $\{(k) : k \geq k_0\} \subset \mathcal{G}$ for some k_0. Let $M_0 = M$ and choose $M_1 \in [M_0]^\omega$, and $k_1 \in \mathbb{N}$ so that $M_1(\mathcal{F}(1)) \subset \mathcal{G}(k_1)$. By replacing M_1 with a subset of M_1, we can assume $k_0, k_1 \leq M_1$. We can do this since if $M' \in [M_1]^\omega$, each member of $(M')(\mathcal{F}(1))$ is a spread of $M_1(\mathcal{F}(1))$, so the desired containment is preserved by passing to M'.

Next, assume that for $1 \leq i < n$, we have chosen $M_i \in [M_0]^\omega$ and $k_i \in \mathbb{N}$ so that $M_i \in [M_{i-1}]^\omega$, $M_i(\mathcal{F}(i)) \subset \mathcal{G}(k_i)$, and $k_i \leq M_i$. Then choose $k_n \in \mathbb{N}$ and $M_n \in [M_{n-1}]^\omega$ so that $M_n(\mathcal{F}(n)) \subset \mathcal{G}(k_n)$, and again assume that $k_n \leq M_n$. This completes the recursive choices of k_n and M_n.

Let $M_n = (m^n_i)_i$ and let $N = (m^n_n)$. Note that $m^1_1 < m^2_2 < \cdots$ and $k_n \leq m^n_n$. We claim that $N(\mathcal{F}) \subset \mathcal{G}$. To see this, fix $E \in \mathcal{F}$. If $|E| = 0$, then $N(E) = \emptyset \in \mathcal{G}$. If $|E| = 1$, then for some $n \in \mathbb{N}$, we have $N(E) = (m^n_n) \in \{(k) : k \geq k_0\} \subset \mathcal{G}$. Last, if $|E| > 1$, we can write $E = n^\frown F$ for some $n \in \mathbb{N}$ and $F \in \mathcal{F}(n)$. Since $n < F$, $N(F)$ is a spread of $M_n(F) \in M_n(\mathcal{F}(n)) \subset \mathcal{G}(k_n)$. Therefore $N(F) \in \mathcal{G}(k_n)$ and $k_n^\frown N(F) \in \mathcal{G}$. But since $k_n \leq m^n_n$ and $N(E) = m^{n\frown}_n N(F)$ is a spread of $k_n^\frown N(F)$, it follows that $N(E) \in \mathcal{G}$. ∎

We return to (vii). If the result were false, we could choose $\zeta < \omega_1$ minimal such that there exists $\eta \leq \zeta$ and regular families $\mathcal{F}, \mathcal{G}$ such that $\iota(\mathcal{F}) = \eta$, $\iota(\mathcal{G}) = \zeta$, and $M \in [\mathbb{N}]^\omega$ such that $N(\mathcal{F}) \not\subset \mathcal{G}$ for each $N \in [M]^\omega$. Next, we could choose $\xi \leq \zeta$ to be a minimal value of η such that the indicated $\mathcal{F}, \mathcal{G}$, and $M \in \mathbb{N}$ exist. We assume we have fixed such $\mathcal{F}, \mathcal{G}, M$. We consider several cases.

First, if $\iota(\mathcal{G}) = 0$, then $\mathcal{G} = \{\emptyset\} = \mathcal{F}$. Clearly this cannot be.

If ζ is a successor, say $\zeta = \beta + 1$, then there exists $n \in \mathbb{N}$ such that $\iota(\mathcal{G}(m)) = \beta$ for each $m \geq n$. If $\xi \leq \beta$, then there exists $N \in [M]^\omega$ such that $N(\mathcal{F}) \subset \mathcal{G}(n) \subset \mathcal{G}$, which also cannot be. Thus if $\zeta = \beta + 1$, it must be true that $\xi = \beta + 1 = \zeta$. Then for each $m \in \mathbb{N}$, $\iota(\mathcal{F}(m)) \leq \beta$, and by the hypothesis for any $M' \in [\mathbb{N}]^\omega$ there exist $N' \in [M']^\omega$ such that $N'(\mathcal{F}(m)) \subset \mathcal{G}(n)$. By the Claim, we deduce that there exists $N \in [M]^\omega$ such that $N(\mathcal{F}) \subset \mathcal{G}$, and this contradiction means that ζ cannot be a successor.

Last, suppose ζ is a limit ordinal. Then $\iota(\mathcal{G}(n)) \nearrow \zeta$. If ξ is a successor, then $\xi < \zeta$ and $\iota(\mathcal{F}(n)) \leq \xi < \zeta$ for each $n \in \mathbb{N}$. If ξ is a limit, then for each $n \in \mathbb{N}$, by our remarks above, $\iota(\mathcal{F}(n)) < \xi \leq \zeta$. Therefore we can choose a sequence $(k_n) \in [\mathbb{N}]^\omega$ such that $\iota(\mathcal{F}(n)) \leq \iota(\mathcal{G}(k_n))$. Then by the inductive hypothesis, for $n \in \mathbb{N}$ and any $M' \in [\mathbb{N}]^\omega$, there exists $N' \in [M']^\omega$ such that $N'(\mathcal{F}(n)) \subset \mathcal{G}(k_n)$. Again, our Claim implies that there exists $N \in [M]^\omega$ such that $N(\mathcal{F}) \subset \mathcal{G}$, and this contradiction exhausts the possibilities of ways that (vii) could fail.

(viii) First assume $(\mathcal{G}_\xi)_{0\leq\xi<\zeta}$ is either additive or multiplicative. We prove the first part by induction on η with ξ held fixed. The $\eta = \xi$ case is clear. Suppose that for a given $\xi \leq \eta < \zeta$, the conclusion holds. Choose $m \in \mathbb{N}$ so that $\mathcal{G}_\xi \cap [[m,\infty)]^{<\omega} \subset \mathcal{G}_\eta$. Since $\mathcal{G}_\eta \subset \mathcal{G}_{\eta+1}$, we have $\mathcal{G}_\xi \cap [[m,\infty)]^{<\omega} \subset \mathcal{G}_{\eta+1}$. Last, suppose $\xi < \eta < \zeta$ is a limit ordinal and the conclusion holds for each $\xi \leq \gamma < \eta$. Fix $\xi < \eta$ and let $\eta_n \uparrow \eta$ be such that $\mathcal{G}_\eta = \mathcal{D}(\mathcal{G}_{\eta_n})$. Choose some $n \in \mathbb{N}$ so that $\xi < \eta_n$ and $k \in \mathbb{N}$ so that $\mathcal{G}_\xi \cap [[k,\infty)]^{<\omega} \subset \mathcal{G}_{\eta_n}$. Let $m = \max\{k, n\}$. Then

$$\mathcal{G}_\xi \cap [[m,\infty)]^{<\omega} \subset \mathcal{G}_{\eta_n} \cap [[n,\infty)]^{<\omega} \subset \mathcal{G}_\eta.$$

This completes the first statement in both the additive case and the multiplicative case.

Next, assume $(\mathcal{G}_\xi)_{0\leq\xi<\zeta}$ is additive. Observe that if $\mathcal{G}_\xi \cap [[m,\infty)]^{<\omega} \subset \mathcal{G}_\eta$, then $\mathcal{G}_\xi \cap [[m-1,\infty)]^{<\omega} \subset \mathcal{A}_1 ^\frown \mathcal{G}_\eta = \mathcal{G}_{\eta+1}$. By induction, $\mathcal{G}_\xi = \mathcal{G}_\xi \cap [[1,\infty)]^{<\omega} \subset \mathcal{G}_{\eta+m-1}$.

Last, assume $(\mathcal{G}_\xi)_{0\leq\xi<\zeta}$ is multiplicative. Observe that $\mathcal{G}_0 \subset \mathcal{G}_\xi$ and $(1) \in \mathrm{MAX}(\mathcal{G}_\xi)$ for each $0 \leq \xi < \zeta$. We claim that if $\mathcal{G}_\xi \cap [[m,\infty)]^{<\omega} \subset \mathcal{G}_\eta$ for $m > 2$, then $\mathcal{G}_\xi \cap [[m-1,\infty)]^{<\omega} \subset \mathcal{G}_{\eta+1}$. This is because if $E = (m-1)^\frown F \in \mathcal{G}_\xi \cap [[m-1,\infty)]^{<\omega}$, then $F \in \mathcal{G}_\xi \cap [[m,\infty)]^{<\omega} \subset \mathcal{G}_\eta$. Hence $(m-1, \min F) \in \mathcal{S}$ and $E = (m-1)^\frown F \in \mathcal{S}[\mathcal{G}_\eta] = \mathcal{G}_{\eta+1}$. Therefore, since $\mathcal{G}_\xi \cap [[m,\infty)]^{<\omega} \subset \mathcal{G}_\eta$, we obtain $\mathcal{G}_\xi \cap [[2,\infty)]^{<\omega} \subset \mathcal{G}_{\eta+m-2}$. But since $(1) \in \mathrm{MAX}(\mathcal{G}_\xi) \cap \mathcal{G}_{\eta+m-2}$, we conclude that $\mathcal{G}_\xi = \{(1)\} \cup (\mathcal{G}_\xi \cap [[2,\infty)]^{<\omega}) \subset \mathcal{G}_{\eta+m-2}$. ■

We are now ready to define the *fine Schreier families* $(\mathcal{F}_\xi)_{0\leq\xi<\omega_1}$. These families were defined in [18], and are a finer version of the more familiar Schreier families defined in [1]. We let $\mathcal{F}_0 = \{\emptyset\}$. Next, if $\mathcal{F}_\xi$ has been defined, we let $\mathcal{F}_{\xi+1} = \mathcal{A}_1 ^\frown \mathcal{F}_\xi$. If $\xi < \omega_1$ is a limit ordinal and $\mathcal{F}_\zeta$ has been defined for each $\zeta < \xi$ so that $(\mathcal{F}_\zeta)_{0\leq\zeta<\xi}$ is additive, fix $\eta_n \uparrow \xi$. By Proposition 3.1(viii), we can choose recursively some natural numbers m_n so that $\mathcal{F}_{\eta_n+m_n} \subset \mathcal{F}_{\eta_{n+1}+m_{n+1}}$ for each $n \in \mathbb{N}$. We let $\xi_n = \eta_n + m_n$ and $\mathcal{F}_\xi = \mathcal{D}(\mathcal{F}_{\xi_n})$.

We next define the *Schreier families* $(\mathcal{S}_\xi)_{0\leq\xi<\omega_1}$. We let $\mathcal{S}_0 = \mathcal{F}_1$, $\mathcal{S}_{\xi+1} = \mathcal{S}[\mathcal{S}_\xi]$, and if $\mathcal{S}_\zeta$ has been defined for each ζ less than the countable limit ordinal ξ, we fix $\xi_n \uparrow \xi$ and define $\mathcal{S}_\xi = \mathcal{D}(\mathcal{S}_{\xi_n})$. Proposition 3.1 and our construction yield the following.

PROPOSITION 3.2. *For each $0 \leq \xi < \omega_1$, $\mathcal{F}_\xi$ is regular with $\iota(\mathcal{F}_\xi) = \xi$. Moreover, for each limit $\xi < \omega_1$, there exists $\xi_n \uparrow \xi$ such that $\mathcal{F}_\xi = \mathcal{D}(\mathcal{F}_{\xi_n})$ and $\mathcal{F}_{\xi_n} \subset \mathcal{F}_{\xi_{n+1}}$ for each $n \in \mathbb{N}$. For each $0 \leq \xi < \omega_1$, $\mathcal{S}_\xi$ is regular with $\iota(\mathcal{S}_\xi) = \omega^\xi$.*

A straightforward induction proof shows that if $0 \leq \xi < \omega_1$ and $E \in \mathcal{F}'_\xi$, then $E^\frown(1+\max E) \in \mathcal{F}_\xi$. We will implicitly use this fact in our proofs, but it is inessential.

We recall the following dichotomies for subsets of $[\mathbb{N}]^{<\omega}$.

THEOREM 3.3 ([11]). *For $\mathcal{F}, \mathcal{G} \subset [\mathbb{N}]^{<\omega}$ hereditary, for any $N \in [\mathbb{N}]^\omega$ there exists $M \in [N]^\omega$ such that either*

$$\mathcal{F} \cap [M]^{<\omega} \subset \mathcal{G} \quad \text{or} \quad \mathcal{G} \cap [M]^{<\omega} \subset \mathcal{F}.$$

THEOREM 3.4 ([19]). *For a regular family $\mathcal{F}$, if $\mathcal{A}, \mathcal{B} \subset \mathrm{MAX}(\mathcal{F})$ are such that $\mathcal{A} \cup \mathcal{B} = \mathrm{MAX}(\mathcal{F})$, then there exists $M \in [\mathbb{N}]^\omega$ such that either*

$$\mathrm{MAX}(\mathcal{F}) \cap [M]^{<\omega} \subset \mathcal{A} \quad \text{or} \quad \mathrm{MAX}(\mathcal{F}) \cap [M]^{<\omega} \subset \mathcal{B}.$$

3.3. The pruning lemmas and applications. In this section, we discuss two useful lemmas involving prunings. The notion of a pruning is the regular family analogue of passing to a subsequence of a sequence. The statement and proof of the pruning lemma require notations which belie the simplicity of the underlying idea, so we say a word about the content before stating it. Let $\mathcal{F} \subset [\mathbb{N}]^{<\omega}$ be a regular family. For each $E \in \mathcal{F}'$, suppose that the sequence of immediate successors of E in $\mathcal{F}$ has a subsequence with some desired property P_E which is allowed to depend on E. Then beginning at the root $\emptyset$ of $\mathcal{F}$, we can pass to a subsequence of the immediate successors of $\emptyset$ (while "pruning" the rest from the tree) so that the remaining sequence has the desired property $P_\emptyset$. For each immediate successor E of $\emptyset$ which survives the pruning, we pass to a subsequence of the immediate successors of E in $\mathcal{F}$ which have the desired property P_E, and so on. Thus, beginning with the root of the tree, we recursively prune the levels of the tree so that in the pruned tree $\mathcal{G}$, for each $E \in \mathcal{G}'$, the sequence of immediate successors of E in $\mathcal{G}$ has the desired property. All this is done so that, although we have passed to subsequences, $\mathcal{F}$ and $\mathcal{G}$ have the same "size."

We will say that a function $\phi : \mathcal{F} \to \mathcal{F}$ is a *pruning* provided $\phi(\emptyset) = \emptyset$ and for each $E \in \mathcal{F}'$, if $s(E) = \min\{n \in \mathbb{N} : E^\frown n \in \mathcal{F}\}$, then there exists a strictly increasing function $\psi_E : [s(E), \infty) \to [s(\phi(E)), \infty)$ such that $\phi(E^\frown n) = \phi(E)^\frown\psi_E(n)$ for each $n \geq s(E)$. The first lemma is essentially contained in [2, Lemma 2.8], so we omit the proof.

LEMMA 3.5 ([2]). *Let $\mathcal{F}$ be a regular family. For each $E \in \mathcal{F}'$, suppose $P_E \subset ([\mathbb{N}]^{<\omega})^\omega$ is such that some subsequence $(E^\frown m)_{m\in M}$ of $(E^\frown m)_{m\geq s(E)}$*

lies in P_E. *Then there exists a pruning* $\phi : \mathcal{F} \to \mathcal{F}$ *such that for each* $E \in \mathcal{F}'$, $(\phi(E^\frown n))_{n \geq s(E)} \in P_{\phi(E)}$.

For convenience, in the examples below we freely relabel and denote a pruned tree the same way as the original tree. In these examples, we will say $(x_E)_{E\in\widehat{\mathcal{F}}}$ is a *weakly null tree* (resp. w^* *null tree, block tree*) if for each $E \in \mathcal{F}'$, the sequence (x_{E_n}) is weakly null (resp. w^* null, a block sequence), where (E_n) is the sequence of immediate successors of E in $\mathcal{F}$ with the natural enumeration. Recall that $\widehat{\mathcal{F}} = \mathcal{F} \setminus \{\emptyset\}$.

EXAMPLE 3.6. If X is a Banach space with FDD F and $(x_E)_{E\in\widehat{\mathcal{F}}} \subset X$ is a weakly null tree such that $\inf_{E\in\widehat{\mathcal{F}}} \|x_E\| = c > 0$, then for fixed $\varepsilon > 0$, for each $E \in \widehat{\mathcal{F}}$ we can find $z_E \in c_{00}(F)$ such that $\|z_E\| = \|x_E\|$, $\|x_E - z_E\| < \varepsilon_{|E|}$, and so that $\operatorname{supp}_F(E^\frown n) \to \emptyset$ for each $E \in \mathcal{F}'$. Here $(\varepsilon_n) \subset (0,1)$ is decreasing to zero at a rate which depends on c, ε, and the projection constant of F in X. If P_E consists of sequences (E_n) of immediate successors of E in $\mathcal{F}$ such that (z_{E_n}) is a seminormalized sequence of successively supported vectors with $\max\operatorname{supp}_F(z_E) < \min\operatorname{supp}_F(z_{E_1})$, we can prune to obtain a pruned tree $(y_E)_{E\in\widehat{\mathcal{F}}}$ of $(x_E)_{E\in\widehat{\mathcal{F}}}$ and $(u_E)_{E\in\widehat{\mathcal{F}}}$ of $(z_E)_{E\in\widehat{\mathcal{F}}}$ such that $\|y_E - u_E\| < \varepsilon_{|E|}$ for each $E \in \widehat{\mathcal{F}}$ and moreover for each $E \in \mathcal{F}'$, $(u_{E^\frown n})$ is a block sequence with respect to F, and for each $E \in \widehat{\mathcal{F}}$, $(u_{E|_i})_{i=1}^{|E|}$ is a block sequence with respect to F. With an auspicious choice of (ε_n), for each $E \in \widehat{\mathcal{F}}$, $(y_{E|_i})_{i=1}^{|E|}$ and $(u_{E|_i})_{i=1}^{|E|}$ will be $(1+\varepsilon)$-equivalent.

EXAMPLE 3.7. Fix a function $f : [\mathbb{N}]^{<\omega} \to (0,1)$ with $\sum_{E\in[\mathbb{N}]^{<\omega}} f(E) < \infty$. Suppose $g : \mathcal{F} \to \mathbb{R}$ is any function such that $g(E^\frown n) \to 0$ for each $E \in \mathcal{F}'$. Then we can find a pruning $\phi : \mathcal{F} \to \mathcal{F}$ such that $g(\phi(E)) < f(E)$ for each $E \in \widehat{\mathcal{F}}$. We will use this in two cases.

Suppose $\emptyset \neq K \subset X^*$. If $(x_E)_{E\in\widehat{\mathcal{F}}} \subset B_X$ is such that $x^*(x_{E^\frown n}) \to 0$ for all $E \in \mathcal{F}'$ and $x^* \in K$, we say $(x_E)_{E\in\widehat{\mathcal{F}}}$ is a K *null tree*. Note that if $(c_k) \subset C(\mathcal{F})$ is a sequence of pairwise disjoint chains and (x_k) is a sequence such that x_k is a convex combination of $(x_E)_{E\in c_k}$, then (x_k) need not be pointwise null on K. We wish to overcome this, which we can easily do under the assumption that K is norm separable. Let (x_n^*) be a dense sequence in K and let $d(x) = \sum d_n|x_n^*(x)|$, where (d_n) is any sequence of positive numbers such that $\sum d_n\|x_n^*\| < \infty$. Note that $(x_n) \subset B_X$ is pointwise null on K if and only if $d(x_n) \to 0$. Suppose that $(x_E)_{E\in\widehat{\mathcal{F}}} \subset B_X$ is a K null tree, $g(E) = d(x_E)$ for $E \in \widehat{\mathcal{F}}$, and let $g(\emptyset) = 0$. After pruning, we may assume $d(x_E) < f(E)$ for each $E \in \widehat{\mathcal{F}}$. Now suppose $(c_k)_k$ are pairwise disjoint members of $C(\mathcal{F})$ and $y_k \in \operatorname{co}\{x_E : E \in c_k\} \subset B_X$.

Then

$$\sum_k d(y_k) \le \sum_k \sum_{E\in c_k} d(x_E) \le \sum_k \sum_{E\in c_k} f(E) \le \sum_{E\in[\mathbb{N}]^{<\omega}} f(E) < \infty.$$

Thus $d(y_k) \to 0$, which means (y_k) is pointwise null on K. In what follows, any K null tree $(x_E)_{E\in\widehat{\mathcal{F}}}$ in a Banach space X such that any sequence (x_k) with $x_k \in \mathrm{co}\{x_E : E \in c_k\}$, $(c_k) \subset C(\mathcal{F})$ pairwise disjoint, is pointwise null on K will be called a *strongly K null tree*. In the case $K = B_{X^*}$, we call a K null tree a *weakly null tree* and a strongly K null tree a *strongly weakly null tree*.

Example 3.8. (B, d) is a metric space and $(b_E)_{E\in\mathcal{F}} \subset B$ is a tree such that $b_{E^\frown n} \to b_E$ for each $E \in \mathcal{F}'$. We call such a tree a *convergent tree*. For $E \in \widehat{F}$, let $g(E) = d(b_E, b_{E|_{|E|-1}})$. Then by passing to a pruning and relabeling, we can assume $d(b_E, b_{E|_{|E|-1}}) < f(E)$. We claim that the resulting tree, which we also denote by $(b_E)_{E\in\mathcal{F}}$, is such that $E \mapsto b_E$ is continuous. To see this, it is sufficient to show that if $E < E_k$, $k \in \mathbb{N}$, are such that $\min E_k$ strictly increases and $F_k := E^\frown E_k \in \mathcal{F}$ for each $k \in \mathbb{N}$, then $b_{F_k} \to b_E$. Let $c_k = \{F : E \prec F \preceq F_k\}$, so (c_k) are pairwise disjoint chains. Therefore

$$\sum_k d(b_{F_k}, b_E) \le \sum_k \sum_{i=|E|+1}^{|F_k|} d(b_{F_k|_i}, b_{F_k|_{i-1}}) < \sum_k \sum_{F\in c_k} f(F) < \infty.$$

In what follows any tree $(b_E)_{E\in\mathcal{F}} \subset B$ such that $E \mapsto b_E$ is continuous will be called a *continuous tree*. In the case where $B = B_{X^*}$ for some separable Banach space X and d is a metric compatible with the w^* topology on B_{X^*}, we refer to these trees as *w^* convergent* and *w^* continuous*, respectively.

Example 3.9. Suppose that X is a Banach space and $S, K \subset B_{X^*}$ are norm separable, non-empty sets. Suppose that $(x_n^*) \subset K - K$ is a w^* null sequence so that $\|x_n^*\| > \varepsilon$ for all $n \in \mathbb{N}$. First we can choose for each $n \in \mathbb{N}$ some $x_n \in B_X$ so that $x_n^*(x_n) > \varepsilon$. By passing to subsequences, we can assume the sequence (x_n) is pointwise convergent on $S \cup K$. For $\delta > 0$, we can pass to further subsequences and assume that $|x_n^*(x_m)| < \delta$ for any $m < n$. We let $y_n = (x_{2n} - x_{2n-1})/2$ and $y_n^* = x_{2n}^*$. Then $y_n^*(y_n) \ge \varepsilon/2 - \delta/2$ and (y_n) is pointwise null on $S \cup K$.

Next, suppose $(x_E^*)_{E\in\widehat{\mathcal{F}}} \subset K - K$ is a w^* null tree such that $\|x_E^*\| > \varepsilon$ for all $E \in \widehat{\mathcal{F}}$. We can choose for each $E \in \widehat{\mathcal{F}}$ some $x_E \in B_X$ so that $x_E^*(x_E) > \varepsilon$. By using the previous paragraph and pruning, we can assume that for some $\varepsilon' \in (0, \varepsilon/2)$, $(x_E)_{E\in\widehat{\mathcal{F}}}$ is an $S \cup K$ null tree and $x_E^*(x_E) > \varepsilon'$ for each $E \in \widehat{\mathcal{F}}$. Next, we fix decreasing $(\varepsilon_n) \subset (0, 1)$ and prune $(x_E)_{E\in\widehat{\mathcal{F}}}$ using the rule that a sequence (u_n) in X has property P_E provided $|x_F^*(u_n)| < \varepsilon_{|E|+1}$ for all $\emptyset \preceq F \preceq E$ and $n \in \mathbb{N}$. Of course, we pass to the corresponding pruning of $(x^*_E)_{E\in\widehat{\mathcal{F}}}$. The result is a pair of trees $(x_E)_{E\in\widehat{\mathcal{F}}}$ and $(x^*_E)_{E\in\widehat{\mathcal{F}}}$ such that $(x_E)_{E\in\widehat{\mathcal{F}}}$ is $S\cup K$ null, $(x^*_E)_{E\in\widehat{\mathcal{F}}}$ is w^* null, and if $\emptyset\preceq E\prec F\in\mathcal{F}$ then $|x^*_E(x_F)|<\varepsilon_{|F|}$. We last pass to a pruning of $(x^*_E)_{E\in\widehat{\mathcal{F}}}$ using the rule that a sequence (u^*_n) has property P_E provided $|u^*_n(x_F)|<\varepsilon_{|E|+1}$ for $\emptyset\prec F\preceq E$. After passing to the corresponding pruning of (x_E), we have obtained $S\cup K$ null and w^* null trees $(x_E)_{E\in\widehat{\mathcal{F}}}\subset B_X$ and $(x^*_E)_{E\in\widehat{\mathcal{F}}}\subset K-K$ such that $x^*_E(x_E)\ge\varepsilon'$ for each $E\in\widehat{\mathcal{F}}$, and $|x^*_E(x_F)|<\min\{\varepsilon_{|E|},\varepsilon_{|F|}\}$ for any comparable, distinct E,F.

Note that this example is also true without the assumption that S and K are norm separable as long as X does not contain a copy of ℓ_1. This is because norm separability was used here to deduce that if $(x_n)\subset B_X$, we can pass to a sequence which is pointwise convergent on $S\cup K$. If ℓ_1 does not embed into X, we can use Rosenthal's ℓ_1 theorem to pass to a weakly Cauchy subsequence of (x_n), and the rest of the argument goes through unchanged.

The pruning method defined above is a "bottom up" pruning, since it begins at the root of the tree. We will also want to use a "top down" pruning which begins with the leaves of the tree.

LEMMA 3.10. *Let K,L be compact metric spaces, $\mathcal{F}$ a regular family, and $k_0:\mathrm{MAX}(\mathcal{F})\to K$ and $l_0:\mathrm{MAX}(\mathcal{F})\to L$ be any functions. Then there exist functions $k:\mathcal{F}\to K$ and $l:\mathcal{F}\to L$ extending k_0 and l_0, respectively, and a pruning $\phi:\mathcal{F}\to\mathcal{F}$ such that $k\circ\phi$ and $l\circ\phi$ are continuous.*

Proof. Recall that for each $E\in\mathcal{F}'$, we let $s(E)=\min\{n\in\mathbb{N}:E^\frown n\in\mathcal{F}\}$. We will define $k(E),l(E)$ for $E\in\mathrm{MAX}(\mathcal{F}^\xi)$ by induction on ξ for $0\le\xi\le\iota(\mathcal{F})$, and $\psi_E:[s(E),\infty)\to[s(E),\infty)$ for $E\in\mathrm{MAX}(\mathcal{F}^\xi)$ by induction on ξ for $0<\xi\le\iota(\mathcal{F})$. Then for $E=(k_1,\dots,k_n)$, we let

$$\phi(E)=\phi(E|_{n-1})^\frown\psi_{E|_{n-1}}\big(k_n-s(E|_{n-1})+s(\phi(E|_{n-1}))\big)$$

so that the resulting tree is convergent. A second pruning as in the example above will yield a continuous tree.

For $\xi=0$, we set $k(E)=k_0(E)$ and $l(E)=l_0(E)$.

Next, suppose that for some ξ with $\xi+1\le\iota(\mathcal{F})$, $k(E)$ and $l(E)$ have been defined for each $E\in\bigcup_{0\le\zeta\le\xi}\mathrm{MAX}(\mathcal{F}^\zeta)$ and ψ_E has been defined for each $E\in\bigcup_{1\le\zeta\le\xi}\mathrm{MAX}(\mathcal{F}^\zeta)$. Choose $E\in\mathrm{MAX}(\mathcal{F}^{\xi+1})$. By compactness, we can choose a set $(m^E_n)\in[[s(E),\infty)]^\omega$ so that $(k(E^\frown m^E_n))$ and $(l(E^\frown m^E_n))$ converge to some $k(E)\in K$ and $l(E)\in L$, respectively. Let $\psi_E(s(E)+n)=m^E_{n+1}$ for $n=0,1,\dots$.

Last, suppose that for some limit ordinal $\xi\le\iota(\mathcal{F})$, $k(E)$ and $l(E)$ have been defined for each $E\in\bigcup_{0\le\zeta<\xi}\mathrm{MAX}(\mathcal{F}^\zeta)$ and ψ_E has been defined for each $E\in\bigcup_{1\le\zeta<\xi}\mathrm{MAX}(\mathcal{F}^\zeta)$. The further steps in this case are the same as in the successor case. ∎

4. Coloring theorems for regular trees. If $\xi < \omega_1$ is an ordinal, there exist $k \in \mathbb{N}$, non-negative integers $n_1, \ldots, n_k$, and $\omega_1 > \alpha_1 > \cdots > \alpha_k$ such that

$$\xi = \omega^{\alpha_1} n_1 + \cdots + \omega^{\alpha_k} n_k.$$

If $\xi > 0$, there is a unique representation of this form such that each n_i is non-zero. This is called the *Cantor normal form* of ξ. Let ξ and ζ be countable ordinals and $\alpha_1 > \cdots > \alpha_k$, n_i, m_i non-negative integers with

$$\xi = \omega^{\alpha_1} m_1 + \cdots + \omega^{\alpha_k} m_k \quad \text{and} \quad \zeta = \omega^{\alpha_1} n_1 + \cdots + \omega^{\alpha_k} n_k.$$

By allowing m_i or n_i to be zero, we can assume that the same ordinals α_i are used in the representations of both. Then we define the *Hessenberg* (or *natural*) *sum* of ξ and ζ by

$$\xi \oplus \zeta = \omega^{\alpha_1}(m_1 + n_1) + \cdots + \omega^{\alpha_k}(m_k + n_k).$$

Note that including extra zero terms does not change the value of this sum.

We also note that for each $\xi < \omega_1$, $\{(\alpha, \beta) : \alpha \oplus \beta = \xi\}$ is finite. This sum is not continuous, since $n \oplus n = 2n \to \omega$, while $\omega \oplus \omega = \omega 2$. But for each $\eta < \omega_1$ and each pair $(\xi_n), (\zeta_n)$ of sequences,

$$\sup_n \xi_n \oplus \zeta_n = \omega^\eta \;\Rightarrow\; \Big(\sup_n \xi_n\Big) \vee \Big(\sup_n \zeta_n\Big) = \omega^\eta.$$

This is because for natural numbers $n_1, \ldots, n_k$,

$$\omega^\eta > \omega^{\alpha_1} n_1 + \cdots + \omega^{\alpha_k} n_k$$

if and only if $\eta > \alpha_1$. Therefore if $\xi = \sup_n \xi_n$ and $\zeta = \sup_n \zeta_n < \omega^\eta$, then $\sup_n \xi_n \oplus \zeta_n \le \xi \oplus \zeta < \omega^\eta$.

Moreover, suppose that $\zeta_m \oplus \eta_m \nearrow \xi$ for a limit ordinal ξ. We can write

$$\xi = \omega^{\alpha_1} n_1 + \cdots + \omega^{\alpha_k}(n_k + 1)$$

for $n_i \ge 0$, where $\alpha_k > 0$. Let $\alpha = \omega^{\alpha_1} n_1 + \cdots + \omega^{\alpha_k} n_k$ and $\beta = \omega^{\alpha_k}$. By passing to a subsequence, assume that $\zeta_m \oplus \eta_m = \alpha + \beta_m > \alpha$ for each $m \in \mathbb{N}$ and note that $\beta_m \nearrow \beta$. Then for each $m \in \mathbb{N}$, there exist $s_{1,m}, \ldots, s_{k,m}, t_{1,m}, \ldots, t_{k,m} \ge 0$ with $s_{i,m} + t_{i,m} = n_i$ for each $1 \le i \le k$ and ζ'_m, η'_m such that $\zeta'_m \oplus \eta'_m = \beta_m$, $\zeta_m = \omega^{\alpha_1} s_{1,m} + \cdots + \omega^{\alpha_k} s_{k,m} + \zeta'_m$ and $\eta_m = \omega^{\alpha_1} t_{1,m} + \cdots + \omega^{\alpha_k} t_{1,k} + \eta'_m$.

By our above remarks, either $\zeta'_m \nearrow \beta$ or $\eta_m \nearrow \beta$. Assume that $\zeta'_m \nearrow \beta$. By passing to a further subsequence, we can assume that there exist $s_1, \ldots, s_k$, $t_1, \ldots, t_k$ such that for each $m \in \mathbb{N}$ and $1 \le i \le k$, $s_{i,m} = s_i$ and $t_{i,m} = t_i$. In this case, with $\zeta'' = \omega^{\alpha_1} s_1 + \cdots + \omega^{\alpha_k} s_k$ and $\eta'' = \omega^{\alpha_1} t_1 + \cdots + \omega^{\alpha_k} t_k$, we have $\zeta_m = \zeta'' + \zeta'_m \nearrow \zeta'' + \beta$, $\eta_m \ge \eta''$, and $(\zeta'' + \beta) \oplus \eta'' = \xi$. We will use this observation in the limit ordinal case of the proof of our next lemma.

If we give each member of a set S of cardinality n at least one of the two colors 0 and 1, of course we can find numbers i, j such that $i + j = n$ and subsets A, B of S with cardinality i, j, respectively, such that each member of A gets color 0, and each member of B gets color 1. We wish to generalize this to colorings of regular families, in which case the analogous result, where addition is the Hessenberg sum, is true for colorings of regular families. Here, we consider the case in which each member of $\mathrm{MAX}(\mathcal{F})$ colors each of its non-empty predecessors with at least one, but possibly both, of the colors $0, 1$. If $\mathcal{F}$ is a regular family, we say a collection $(\mathcal{A}_E^0, \mathcal{A}_E^1)_{E\in\widehat{\mathcal{F}}}$ of subsets of $\mathrm{MAX}(\mathcal{F})$ is a *coloring* of $\mathcal{F}$ if $\mathcal{A}_E^0 \cup \mathcal{A}_E^1 = \{F \in \mathrm{MAX}(\mathcal{F}) : E \preceq F\}$ for each $E \in \widehat{\mathcal{F}}$.

For the sake of simplifying the following proof, we introduce some more terminology. Given regular families $\mathcal{F}, \mathcal{G}$, we say the pair (i, e) is an *extended embedding* of $\mathcal{F}$ into $\mathcal{G}$ if $i : \widehat{\mathcal{F}} \to \widehat{\mathcal{G}}$ is an embedding and $e : \mathrm{MAX}(\mathcal{F}) \to \mathrm{MAX}(\mathcal{G})$ is a function such that $i(E) \preceq e(E)$ for each $E \in \mathrm{MAX}(\mathcal{F})$. If $(\mathcal{A}_E^0, \mathcal{A}_E^1)_{E\in\widehat{\mathcal{G}}}$ is a coloring of $\mathcal{G}$ and (i, e) is an extended embedding of $\mathcal{F}$ into $\mathcal{G}$, we define for $j = 0, 1$ and $E \in \widehat{\mathcal{F}}$ the set

$$\mathcal{B}_E^j = \{F \in \mathrm{MAX}(\mathcal{F}) : e(F) \in \mathcal{A}_{i(E)}^j\}.$$

We refer to $(\mathcal{B}_E^0, \mathcal{B}_E^1)$ as the *induced coloring of $\mathcal{F}$ by (i, e) and $(\mathcal{A}_E^0, \mathcal{A}_E^1)$*, or, if no confusion can arise, simply the *induced coloring*. It is easy to see that this is indeed a coloring of $\mathcal{F}$. We say that the induced coloring $(\mathcal{B}_E^0, \mathcal{B}_E^1)$ is *monochromatically* j provided that for each $E \in \mathrm{MAX}(\mathcal{F})$,

$$e(E) \in \bigcap_{k=1}^{|E|} \mathcal{A}_{i(E|_k)}^j.$$

We observe that if $\mathcal{E}, \mathcal{F}, \mathcal{G}$ are regular families, $(\mathcal{A}_E^0, \mathcal{A}_E^1)_{E\in\widehat{\mathcal{G}}}$ is a coloring of $\mathcal{G}$, (i, e) is any extended embedding of $\mathcal{E}$ into $\mathcal{F}$, and (i', e') is an extended embedding of $\mathcal{F}$ into $\mathcal{G}$ such that the induced coloring of $\mathcal{F}$ by (i', e') and $(\mathcal{A}_E^0, \mathcal{A}_E^1)$ is monochromatically j, then $(i' \circ i, e' \circ e)$ is an extended embedding of $\mathcal{E}$ into $\mathcal{G}$ such that the induced coloring of $\mathcal{E}$ by $(i' \circ i, e' \circ e)$ and $(\mathcal{A}_E^0, \mathcal{A}_E^1)$ is monochromatically j.

Lemma 4.1 (Coloring lemma for sums). *Suppose $\mathcal{F}$ is a regular family with $\iota(\mathcal{F}) > 0$. If $(\mathcal{A}_E^0, \mathcal{A}_E^1)$ is a coloring of $\mathcal{F}$, then for $j = 0, 1$, there exist an ordinal ξ_j and an extended embedding (i_j, e_j) of $\mathcal{F}_{\xi_j}$ into $\mathcal{F}$ such that the induced coloring of $\mathcal{F}_{\xi_j}$ is monochromatically j and $\xi_0 \oplus \xi_1 = \iota(\mathcal{F})$.*

Here, it should be understood that if either $\xi_j = 0$ for $j = 0$ or 1, we take i_j and e_j to be the empty maps to satisfy the conclusion for that j.

Proof of Lemma 4.1. We prove the result by induction on $\iota(\mathcal{F})$. Fix $0 \le \xi < \omega_1$, and in the case $\xi > 0$ assume the result holds for all families $\mathcal{F}$ with $\iota(\mathcal{F}) \le \xi$ and all colorings $(\mathcal{A}_E^0, \mathcal{A}_E^1)$ of $\mathcal{F}$. Fix a regular family $\mathcal{F}$ with $\iota(\mathcal{F}) = \xi + 1$ and a coloring $(\mathcal{A}_E^0, \mathcal{A}_E^1)$ of $\mathcal{F}$. There exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, $\iota(\mathcal{F}(n)) = \xi$. For each $n \geq n_0$, $E \in \mathcal{F}(n)$, and $j \in \{0,1\}$, let

$$\mathcal{A}^j_E(n) = \{F \in \mathrm{MAX}(\mathcal{F}(n)) : n^\frown F \in \mathcal{A}^j_{n^\frown E}\}.$$

This defines a coloring of $\mathcal{F}(n)$, and in fact is the induced coloring of $\mathcal{F}(n)$ corresponding to the extended embedding $E \mapsto n^\frown E$. Note that for each $n \geq n_0$, we have $\mathcal{A}^0_\emptyset(n) \cup \mathcal{A}^1_\emptyset(n) = \mathrm{MAX}(\mathcal{F}(n))$. By Theorem 3.4, there exists $M_n \in [\mathbb{N}]^\omega$ such that either

$$\mathrm{MAX}(\mathcal{F}(n)) \cap [M_n]^{<\omega} \subset \mathcal{A}^0_\emptyset(n) \quad \text{or} \quad \mathrm{MAX}(\mathcal{F}(n)) \cap [M_n]^{<\omega} \subset \mathcal{A}^1_\emptyset(n).$$

Without loss of generality, we can find $n_0 \leq N \in [\mathbb{N}]^\omega$ such that for each $n \in N$, we have $\mathrm{MAX}(\mathcal{F}(n)) \cap [M_n]^{<\omega} \subset \mathcal{A}^0_\emptyset(n)$. Next, for each $n \in N$, choose a function

$$f_n : \mathrm{MAX}(\mathcal{F}(n)) \to \mathrm{MAX}(\mathcal{F}(n)) \cap [M_n]^{<\omega}$$

so that $M_n(F) \preceq f_n(F)$ for each $F \in \mathrm{MAX}(\mathcal{F}(n))$. We can do this because $\mathcal{F}(n)$ is regular, which means any member of $\mathcal{F}(n) \cap [M_n]^{<\omega}$ has an extension in $\mathrm{MAX}(\mathcal{F}(n)) \cap [M_n]^{<\omega}$. Let $(\mathcal{B}^0_E(n), \mathcal{B}^1_E(n))$ be the coloring of $\mathcal{F}(n)$ given by

$$\mathcal{B}^j_E(n) = \{F \in \mathrm{MAX}(\mathcal{F}(n)) : f_n(F) \in \mathcal{A}^j_{M_n(E)}(n)\}.$$

It is easy to check that this is indeed a coloring. In fact, this is the induced coloring corresponding to the extended embedding of $\mathcal{F}(n)$ into itself given by $E \mapsto M_n(E)$, and for $E \in \mathrm{MAX}(\mathcal{F}(n))$, $E \mapsto f_n(E)$. Now apply the inductive hypothesis to find some $\xi_{0,n}, \xi_{1,n}$ with $\xi_{0,n} \oplus \xi_{1,n} = \iota(\mathcal{F}(n)) = \xi$ and an extended embedding $(i_{j,n}, e_{j,n})$ of $\mathcal{F}_{\xi_{j,n}}$ into $\mathcal{F}(n)$ which is monochromatically j with respect to the coloring $(\mathcal{B}^0_E(n), \mathcal{B}^1_E(n))$. By passing to an infinite subset of N, we can assume that we have some $n_0 \leq N \in [\mathbb{N}]^\omega$ and ξ_0, ξ_1 such that for each $n \in N$, $\xi_{0,n} = \xi_0$ and $\xi_{1,n} = \xi_1$. It is clear that for $n \in N$ and $j = 0$ or 1,

$$i'_{j,n}(E) = M_n(i_{j,n}(E)), \qquad e'_{j,n}(E) = f_n(e_{j,n}(E))$$

defines an extended embedding of $\mathcal{F}_{\xi_j}$ into $\mathcal{F}(n)$ such that the induced coloring of $\mathcal{F}_{\xi_j}$ by $(\mathcal{A}^0_E(n), \mathcal{A}^1_E(n))$ is monochromatically j.

For convenience, set $i'_{0,n}(\emptyset) = \emptyset$ and let $e'_{0,n}(\emptyset) = \emptyset$ if $\emptyset \in \mathrm{MAX}(\mathcal{F}(n))$. Define $i_0 : \widehat{\mathcal{F}}_{\xi_0+1} \to \widehat{\mathcal{F}}$, $e_0 : \mathrm{MAX}(\mathcal{F}_{\xi_0+1}) \to \mathrm{MAX}(\mathcal{F})$, $i_1 : \widehat{\mathcal{F}}_{\xi_1} \to \widehat{\mathcal{F}}$ and $e_1 : \mathrm{MAX}(\mathcal{F}_{\xi_1}) \to \mathrm{MAX}(\mathcal{F})$ by

$$i_0(k^\frown E) = n_k^\frown i'_{0,n_k}(E), \qquad e_0(k^\frown E) = n_k^\frown e'_{0,n_k}(E),$$
$$i_1(E) = n_1^\frown i'_{1,n_1}(E), \qquad e_1(E) = n_1^\frown e'_{1,n_1}(E),$$

where $N = (n_k)$. The coloring induced by (i_1, e_1) is monochromatically 1 with respect to $(\mathcal{A}^0_E, \mathcal{A}^1_E)$. To see that the coloring induced by (i_0, e_0) is monochromatically 0, fix $F \in \mathrm{MAX}(\mathcal{F}_{\xi_0+1})$. Write $F = k^\frown E$. By our choices and the definition of $(\mathcal{A}^0_G(n_k))_G$,

$$e_0(F) = n_k^\frown e'_{0,n_k}(E) \in \bigcap_{i=1}^{|E|} \mathcal{A}^0_{n_k^\frown i'_{0,n_k}(E|_i)} = \bigcap_{i=2}^{|F|} \mathcal{A}^0_{i_0(F|_i)}.$$

But by our choices, $e_0(F) \in \mathrm{MAX}(\mathcal{F}_{n_k}) \cap [M_{n_k}]^{<\omega} \subset \mathcal{A}^0_{(n_k)}$, so

$$e_0(F) \in \bigcap_{i=1}^{|F|} \mathcal{A}^0_{i_0(F|_i)}.$$

Thus the coloring on $\mathcal{F}_{\xi_0+1}$ induced by (i_0, e_0) and $(\mathcal{A}^0_E, \mathcal{A}^1_E)$ is monochromatically 0. Since $(\xi_0 + 1) \oplus \xi_1 = \xi_0 \oplus \xi_1 + 1 = \xi + 1$, this finishes the $\xi + 1$ case.

Suppose ξ is a limit ordinal and the result holds for every coloring of every regular family with ι index less than ξ. Fix $\mathcal{F}$ with $\iota(\mathcal{F}) = \xi$ and a coloring $(\mathcal{A}^0_E, \mathcal{A}^1_E)$ of $\mathcal{F}$. Fix $n_0 \in \mathbb{N}$ so that $(n_0) \in \mathcal{F}$. For $n \geq n_0$, define the coloring $(\mathcal{A}^0_E(n), \mathcal{A}^1_E(n))$ as in the successor case. Recall that $\iota(\mathcal{F}(n)) \nearrow \xi$. For each $n \geq n_0$, choose $\xi_{j,n}$ so that $\xi_{0,n} \oplus \xi_{1,n} = \iota(\mathcal{F}(n))$, and extended embeddings $(i_{j,n}, e_{j,n})$ of $\mathcal{F}_{\xi_{j,n}}$ into $\mathcal{F}(n)$ so that the induced coloring is monochromatically j. Recall by our separation technique that we can pass to a subsequence $N = (n_k) \in [\mathbb{N}]^\omega$, find ordinals $\alpha, \beta, \beta_k, \gamma$, and find $j \in \{0, 1\}$ (which we assume without loss of generality is equal to 0) such that

(i) $\xi_{0,n_k} = \alpha + \beta_k$,
(ii) $\beta_k \nearrow \beta$,
(iii) β is a limit ordinal,
(iv) $(\alpha + \beta) \oplus \gamma = \xi$,
(v) $\gamma \leq \xi_{1,n_k}$ for all $k \in \mathbb{N}$.

Fix $\zeta_k \uparrow \alpha + \beta$ so that $\mathcal{F}_{\alpha+\beta} = \mathcal{D}(\mathcal{F}_{\zeta_k})$ and $\mathcal{F}_{\zeta_k} \subset \mathcal{F}_{\zeta_{k+1}}$ for all $k \in \mathbb{N}$. By passing to a further subsequence of N, we can assume without loss of generality that $\zeta_k \leq \alpha + \beta_k$ for all $k \in \mathbb{N}$. Choose an extended embedding (i', e') of $\mathcal{F}_\gamma$ into $\mathcal{F}_{\xi_{1,n_1}}$, and for each $k \in \mathbb{N}$, an extended embedding (i'_k, e'_k) of $\mathcal{F}_{\zeta_k}$ into $\mathcal{F}_{\alpha+\beta_k} = \mathcal{F}_{\xi_{0,n_k}}$. We define extended embeddings (i_0, e_0) and (i_1, e_1) of $\mathcal{F}_{\alpha+\beta}$ and $\mathcal{F}_\gamma$, respectively, into $\mathcal{F}$ such that the coloring induced by (i_j, e_j) is monochromatically j by

$$i_1(E) = n_1^\frown (i_{1,n_1} \circ i')(E), \quad e_1(E) = n_1^\frown (e_{1,n_1} \circ e')(E),$$

and if $E \in \widehat{\mathcal{F}}_{\alpha+\beta}$ with $k = \min E$,

$$i_0(E) = n_k^\frown (i_{0,n_k} \circ i'_k)(E), \quad e_0(E) = n_k^\frown (e_{0,n_k} \circ e'_k)(E). \ \blacksquare$$

LEMMA 4.2 (Coloring lemma for products). *Let $\mathcal{F}, \mathcal{G}$ be non-empty regular families. Suppose $f : C(\mathcal{F}[\mathcal{G}]) \to \{0, 1\}$ is a function such that for any embedding $j : \mathcal{G} \to \mathcal{F}[\mathcal{G}]$, there exists $c \in C(j(\mathcal{G}))$ with $f(c) = 0$. Then there* *exists an order preserving* $j : \widehat{\mathcal{F}} \to C(\mathcal{F}[\mathcal{G}])$ *such that* $f \circ j \equiv 0$ *and the sets* $\{j(E) : E \in \widehat{\mathcal{F}}\}$ *are pairwise disjoint.*

Proof. We first recursively define $r : \widehat{\mathcal{F}} \to C(\mathcal{G})$ so that for $E \in \widehat{\mathcal{F}}$, if we let $F_i = \max r(E|_i) \in \mathcal{G}$ for $1 \leq i \leq |E|$, we have:

(i) $(\min F_i)_{i=1}^{|E|}$ is a spread of E, hence is a member of $\mathcal{F}$,
(ii) $(F_i)_{i=1}^{|E|}$ is successive,
(iii) $f(\{(\bigcup_{i=1}^{|E|-1} F_i)^\frown F : F \in r(E)\}) = 0$.

Then $j(E) = \{(\bigcup_{i=1}^{|E|-1} F_i)^\frown F : F \in r(E)\}$ gives the desired function.

To perform the base step and inductive step simultaneously, we only need to demonstrate how to perform the construction on the sequence of immediate successors of any $E \in \mathcal{F}'$. Suppose that $E \in \mathcal{F}'$ is such that $r(E|_i)$ has been defined for each $1 \leq i \leq |E|$. Let F_i be as above. Let $m_0 > E$ be minimal such that $E^\frown m_0 \in \mathcal{F}$. Choose $m_0 \leq m_1 \in \mathbb{N}$ so that $F_{|E|} < m_1$ and $((\min F_i)_{i=1}^{|E|})^\frown m_1 \in \mathcal{F}$. Since $(\min F_i)_{i=1}^{|E|}$ is a spread of E, which is non-maximal in $\mathcal{F}$, such an m_1 exists. If there exists $n \geq m_1$ with

$$f\Big(\Big\{\Big(\bigcup_{i=1}^{|E|} F_i\Big)^\frown F : F \in c\Big\}\Big) = 1$$

for all $c \in C(\mathcal{G} \cap [(n, \infty)]^{<\omega})$, we obtain a contradiction. This is because in this case the embedding $j(G) = \big(\bigcup_{i=1}^{|E|} F_i\big)^\frown(k + n : k \in G)$ is such that $f|_{j(\mathcal{G})} \equiv 1$. This is indeed an embedding by our choice of m_1 and the fact that $F_i \in \mathcal{G}$ for each $1 \leq i \leq |E|$. We can choose chains $c_{m_0}, c_{m_0+1}, \dots$ so that for each $m \geq m_0$, $c_m \in C(\mathcal{G} \cap [(m_1, \infty)]^{<\omega})$, $\min\min c_m$ is strictly increasing with m, and

$$f\Big(\Big\{\Big(\bigcup_{i=1}^{|E|} F_i\Big)^\frown F : F \in c_m\Big\}\Big) = 0.$$

Setting $r(E^\frown m) = c_m$ for each $m \geq m_0$ we easily see that (i)–(iii) are satisfied. ∎

5. The Szlenk and weakly null ℓ_1^+ indices

5.1. Definition and remarks. Let X be a Banach space and let $L \subset X^*$ be a bounded set. For $\varepsilon > 0$, we let

$$s_\varepsilon(L) = \{x^* : \forall\, w^* \text{ neighborhood } V \text{ of } x^*,\ \mathrm{diam}_{\|\cdot\|}(V \cap L) > \varepsilon\}.$$

As usual, we define the transfinite derivatives

$$s_\varepsilon^0(L) = L, \quad s_\varepsilon^{\xi+1}(L) = s_\varepsilon(s_\varepsilon^\xi(L)),$$

and if ξ is a limit ordinal,

$$s_\varepsilon^\xi(L) = \bigcap_{\zeta<\xi} s_\varepsilon^\zeta(L).$$

It is easy to see that if L is w^* compact, then so is $s_\varepsilon^\xi(L)$ for each ξ.

We define the *Szlenk index* $\mathrm{Sz}_\varepsilon(L) = \min\{\xi < \omega_1 : s_\varepsilon^\xi(L) = \emptyset\}$ provided this set is non-empty, and $\mathrm{Sz}_\varepsilon(L) = \omega_1$ otherwise. Last, we define $\mathrm{Sz}(L) = \sup_{\varepsilon>0} \mathrm{Sz}_\varepsilon(L)$. We set $\mathrm{Sz}(X) = \mathrm{Sz}(B_{X^*})$.

PROPOSITION 5.1 ([22, 14]). *Let X, Y be separable Banach spaces, and let $\emptyset \neq K \subset X^*$ be w^* compact.*

(i) *If X is isomorphic to a subspace of Y, then* $\mathrm{Sz}(X) \leq \mathrm{Sz}(Y)$.
(ii) $\mathrm{Sz}(K) < \omega_1$ *if and only if K is norm separable.*
(iii) *If K is convex, then either* $\mathrm{Sz}(K) = \omega_1$ *or there exists $\xi < \omega_1$ such that* $\mathrm{Sz}(K) = \omega^\xi$.
(iv) *If K is convex and not norm compact, then the supremum* $\sup_\varepsilon \mathrm{Sz}_\varepsilon(K)$ *is not attained.*
(v) $\mathrm{Sz}(K) = 1$ *if and only if K is compact.*

5.2. Weakly null and general σ indices. For a given set S and a given $\sigma \subset S^\omega$, we can define the σ derivatives and σ indices for general hereditary trees on S. Given a tree $\mathcal{H}$ on S, we let

$$(\mathcal{H})'_\sigma = \{t \in \mathcal{H} : \exists (s_i) \in \sigma,\ t^\frown s_i \in \mathcal{H}\ \forall i \in \mathbb{N}\}.$$

If $\mathcal{H}$ is a hereditary tree on S, then $(\mathcal{H})'_\sigma$ is also a hereditary tree on S. It is not hard to see that if $\mathcal{H}$ is not hereditary, $(\mathcal{H})'_\sigma$ need not be a tree. As usual, we define the *transfinite σ derivatives* and *σ index* by

$$(\mathcal{H})_\sigma^0 = \mathcal{H}, \qquad (\mathcal{H})_\sigma^{\xi+1} = ((\mathcal{H})_\sigma^\xi)'_\sigma$$

and

$$(\mathcal{H})_\sigma^\xi = \bigcap_{\zeta<\xi} (\mathcal{H})_\sigma^\zeta, \qquad \xi < \omega_1 \text{ is a limit ordinal.}$$

We define $I_\sigma(\mathcal{H}) = \min\{\xi < \omega_1 : (\mathcal{H})_\sigma^\xi = \emptyset\}$ provided this set is non-empty, and $I_\sigma(\mathcal{H}) = \omega_1$ otherwise. We say σ *contains diagonals* if any subsequence of a member of σ is also a member of σ, and if $(s_{i,j})_i \in \sigma$ for each $j \in \mathbb{N}$ implies there exists a sequence (i_j), $i_1 < i_2 < \cdots$, such that $(s_{i_j,j})_j \in \sigma$. A standard induction proof gives the following.

PROPOSITION 5.2 ([18]). *Let $\mathcal{H}$ be a non-empty hereditary tree on S, and suppose $\sigma \subset S^\omega$ contains diagonals. Then for $0 \leq \xi < \omega_1$, $I_\sigma(\mathcal{H}) > \xi$ if and only if there exists $(t_E)_{E\in\widehat{\mathcal{F}_\xi}} \subset S$ such that*

(i) $(t_{E|_i})_{i=1}^{|E|} \in \mathcal{H}$ *for each* $E \in \widehat{\mathcal{F}_\xi}$,
(ii) $(t_{E^\frown n})_{E<n} \in \sigma$ *for each* $E \in \mathcal{F}'_\xi$.

Observe that in place of $\mathcal{F}_\xi$, we can use any regular family $\mathcal{F}$ with $\iota(\mathcal{F}) = \xi$, since there exists $M \in [\mathbb{N}]^\omega$ such that $M(\mathcal{F}) \subset \mathcal{F}_\xi$ and $M(\mathcal{F}_\xi) \subset \mathcal{F}$.

Note also that only one direction of Proposition 5.2 requires that S contains diagonals. Indeed, if $(t_E)_{E\in\widehat{\mathcal{F}}_\xi} \subset S$ is as in the statement, then $I_S(\mathcal{H}) > \xi$.

Example 5.3. If X is a Banach space and $\emptyset \neq K \subset B_{X^*}$ is norm separable, and if σ denotes all sequences in B_X which are pointwise null on K, then σ contains diagonals. This is because $(x_n) \subset B_X$ is pointwise null on K if and only if $d(x_n) \to 0$, where $d(x) = \sum c_n|x^*(x_n)|$, (x^*_n) is dense in K, and $c_n > 0$ is chosen so that $\sum c_n\|x^*_n\| < \infty$. In this, for any hereditary tree $\mathcal{H}$ on B_X, we denote the *pointwise null on K derivative* by $(\mathcal{H})'_K$ and the *pointwise null on K index* by $I_K(\mathcal{H})$. In the case $K = B_{X^*}$, we refer to this derivative as the *weakly null derivative*, denoted by $(\mathcal{H})'_w$, and the *weakly null index*, denoted by $I_w(\mathcal{H})$.

Example 5.4. Let X be a Banach space and $\emptyset \neq K \subset X^*$. For $r > 0$, we say $(x_n) \subset B_X$ has *K radius r* if for any $x^* \in K$, $\limsup |x^*(x_n)| \leq r$. If K is norm separable and if σ is the collection of sequences $(x_n) \subset B_X$ having K radius r, then σ contains diagonals. Clearly any subsequence of a member of σ is a member of σ. If (x^*_n) is a dense sequence in K, and for each $i \in \mathbb{N}$, $(x^i_n)_n \in \sigma$, we can choose $i_1, i_2, \dots$ so that $|x^*_k(x^n_{i_n})| < r + 1/n$ for all $n \in \mathbb{N}$ and $1 \leq k \leq n$. Then $(x^n_{i_n}) \in \sigma$. In this case, for any hereditary tree $\mathcal{H}$ on B_X, we let $(\mathcal{H})'_{K,r}$ denote the derivative when σ consists of all sequences in B_X with K radius r, and $I_{K,r}(\mathcal{H})$ denotes the σ index in this case.

Example 5.5. If X is a Banach space with FDD E, and if σ denotes all infinite block sequences in B_X with respect to E, then σ contains diagonals. In this case, for any hereditary block tree $\mathcal{H}$ on B_X, we denote the block derivative by $(\mathcal{H})'_{\mathrm{bl}}$ and the block index by $I_{\mathrm{bl}}(\mathcal{H})$.

Example 5.6. If σ consists of all sequences $(B_n) \subset [\mathbb{N}]^{<\omega}$ such that $B_n \to_n \emptyset$, then σ contains diagonals. In this case, for any hereditary tree $\mathcal{H}$ on $[\mathbb{N}]^{<\omega}$ consisting of successive sets, we also denote the derivative by $(\mathcal{H})'_{\mathrm{bl}}$ and the index by $I_{\mathrm{bl}}(\mathcal{H})$. We think of this as a discretized version of the block index for FDDs.

Proposition 5.7 ([18]). *Suppose X is a Banach space with FDD E. Let $\mathcal{B}$ be a hereditary block tree on B_X with respect to E. Let the* compression $\tilde{\mathcal{B}}$ *of $\mathcal{B}$ be defined by*

$$\tilde{\mathcal{B}} = \{(\max \mathrm{supp}_E(x_i))_{i=1}^k : (x_i)_{i=1}^k \in \mathcal{B}\}.$$

Then for any non-increasing $\overline{\varepsilon} = (\varepsilon_n) \subset (0,1)$,

$$\iota(\tilde{\mathcal{B}}) \leq 2I_{\mathrm{bl}}(\mathcal{B}^{E,X}_{\overline{\varepsilon}}).$$

In particular, if λ is a limit ordinal and $I_{\mathrm{bl}}(\mathcal{B}^{E,X}_{\overline{\varepsilon}}) < \lambda$, then $\iota(\tilde{\mathcal{B}}) < \lambda$.

REMARK 5.8. The compression was defined in [18] using minima of supports rather than maxima of supports. We include a sketch of proof to outline how to obtain the version of the statement made here.

We note that any ordinal ξ can be written as $\xi = \lambda + n$ where λ is either zero or a limit ordinal and n is either zero or a natural number. We recall that in this case, $2\xi = \lambda + 2n$.

Sketch of proof of Proposition 5.7. First, one defines for any $\mathcal{B} \subset \Sigma(E, X)$ the *support tree*

$$\mathrm{supp}(\mathcal{B}) = \{(\mathrm{supp}_E(z_i))_{i=1}^n : (z_i)_{i=1}^n \in \mathcal{B}\}$$

and proves by induction on ξ that for any non-increasing $\overline{\varepsilon} \subset (0,1)$,

$$(\mathrm{supp}(\mathcal{B}))_{\mathrm{bl}}^{\xi} \subset \mathrm{supp}((\mathcal{B}_{\overline{\varepsilon}}^{E,X})_{\mathrm{bl}}^{\xi}).$$

This part of the proof is unchanged.

Next, one proves a discretized version of the statement. For each collection $\mathcal{B}$ of finite, non-empty, successive sequences of finite subsets of $\mathbb{N}$, one defines

$$\max(\mathcal{B}) = \{(\max A_i)_{i=1}^n : (A_i)_{i=1}^n \in \mathcal{B}\}.$$

Then one shows by induction that if $\mathcal{B} \subset [\mathbb{N}]^{<\omega}$ is a hereditary collection of finite, non-empty, successive sequences of finite subsets of $\mathbb{N}$, then for any ordinal $\xi = \lambda + n$, where λ is either zero or a limit and n is either 0 or a natural number,

$$(\max \mathcal{B})^{\xi} \subset \max((\mathcal{B})_{\mathrm{bl}}^{\lambda+2n}).$$

Since $\tilde{\mathcal{B}} = \max(\mathrm{supp}(\mathcal{B}))$ for any $\mathcal{B} \subset \Sigma(E, X)$, one applies these two facts to $\tilde{\mathcal{B}}$ to obtain

$$\iota(\tilde{\mathcal{B}}) = \iota(\max(\mathrm{supp}(\mathcal{B}))) \le 2I_{\mathrm{bl}}(\mathrm{supp}(\mathcal{B})) \le 2I_{\mathrm{bl}}(\mathcal{B}_{\overline{\varepsilon}}^{E,X}).$$

The difference lies in the discretized version. The key part of the proof lies in the successor case. If one supposes that $(n_1, \dots, n_r) \in \max(\mathcal{B})''$ and $p_j, q_{jk} \to \infty$ satisfy $p_j < q_{jk}$ and $(n_1, \dots, n_r, p_j, q_{jk}) \in \max(\mathcal{B})$ for all $j, k \in \mathbb{N}$, we can choose for any $j, k \in \mathbb{N}$ some successive $A_1^j, \dots, A_r^j, C_j, D_{jk} \in [\mathbb{N}]^{<\omega}$ so that $\max A_i^j = n_i$, $\max C_j = p_j$, $\max D_{jk} = q_{jk}$ and $(A_1^j, \dots, A_r^j, C_j, D_{jk}) \in \mathcal{B}$. Since $A_i^j \subset \{1, \dots, n_r\}$ for all $j \in \mathbb{N}$ and $1 \le i \le n$, we can pass to some subsequence and assume we have successive $A_1, \dots, A_r$ such that $A_i^j = A_i$ for all $j \in \mathbb{N}$ and $1 \le i \le r$. Since $\max C_j \underset{j}{\to} \infty$, we may fix a sequence k_j such that $\min D_{jk_j}$ tends to infinity with j, and by heredity, $(A_1, \dots, A_r, D_{jk_j}) \in \mathcal{B}$ for all j. Since $\min D_{jk_j} \underset{j}{\to} \infty$, we deduce $(A_1, \dots, A_r) \in (\mathcal{B})_{\mathrm{bl}}'$. ■

In what follows, for a Banach space and $\emptyset \neq K \subset B_{X^*}$, we let

$$\mathcal{H}_{\varepsilon}^{K} = \{(x_i)_{i=1}^n \in B_X^{<\omega} : \exists x^* \in K,\ x^*(x_i) \ge \varepsilon\ \forall 1 \le i \le n\}.$$

We let $\mathcal{H}_{\varepsilon}^{X} = \mathcal{H}_{\varepsilon}^{B_{X^*}}$.

5.3. Dualization for separable spaces. In [2], it was shown that the weakly null ℓ_1^+ index is equal to the Szlenk index of any separable Banach space not containing ℓ_1. Here we discuss how to modify this result to compute the Szlenk index of certain subsets of the dual of a separable Banach space.

LEMMA 5.9. *If X is a separable Banach space and if $\emptyset \neq K \subset X^*$ is w^* compact and norm separable, then the following are equivalent for any $0 < \xi < \omega_1$:*

(i) *There exists $\varepsilon > 0$ such that $\mathrm{Sz}_\varepsilon(K) > \xi$.*
(ii) *There exists $\varepsilon > 0$ such that for every norm separable $\emptyset \neq S \subset X^*$, there exists an S null $(x_E)_{E\in\widehat{\mathcal{F}}_\xi} \subset B_X$ and a w^* continuous $(x^*_E)_{E\in\mathcal{F}_\xi} \subset K$ such that $x^*_E(x_F) \geq \varepsilon$ for all $E \in \mathcal{F}_\xi$ and $\emptyset \prec F \preceq E$.*
(iii) *There exists $\varepsilon > 0$ such that for every norm separable $\emptyset \neq S \subset X^*$, there exists an S null $(x_E)_{E\in\widehat{\mathcal{F}}_\xi} \subset B_X$ and $(x^*_E)_{E\in\mathrm{MAX}(\mathcal{F}_\xi)} \subset K$ such that $x^*_E(x_F) \geq \varepsilon$ for all $E \in \mathrm{MAX}(\mathcal{F}_\xi)$ and $\emptyset \prec F \preceq E$.*
(iv) *There exists $\varepsilon > 0$ such that $I_S(\mathcal{H}^K_\varepsilon) > \xi$ for every norm separable $\emptyset \neq S \subset X^*$.*
(v) *There exist $0 < r < \varepsilon$ such that $I_{S,r}(\mathcal{H}^K_\varepsilon) > \xi$ for every norm separable $\emptyset \neq S \subset X^*$.*

Moreover, if ℓ_1 does not embed into X, the result is true without the assumption that K or S is norm separable.

Proof. (i)⇒(ii). An easy proof by induction shows that if $x^* \in s^\xi_\varepsilon(K)$, there must exist some tree $(x^*_E)_{E\in\mathcal{F}_\xi} \subset K$ such that, setting $x^*_\emptyset = x^*$, for each $E \in \mathcal{F}'_\xi$,

(a) $x^*_{E^\frown n} \underset{w^*}{\to} x^*_E$,
(b) $\|x^*_E - x^*_{E^\frown n}\| > \varepsilon/2$ for all $n > E$.

Suppose $\mathrm{Sz}_\varepsilon(K) > \xi$ and fix $x^* \in s^\xi_\epsilon(K)$. Fix $\emptyset \neq S \subset B_{X^*}$ norm separable. Let $(x^*_E)_{E\in\mathcal{F}} \subset K$ be as above with $x^*_\emptyset = x^*$. For each $E \in \widehat{\mathcal{F}}_\xi$, let $y^*_E = x^*_E - x^*_{E|_{|E|-1}}$. Let $y^*_\emptyset = x^*_\emptyset$. Then $(y^*_E)_{E\in\widehat{\mathcal{F}}_\xi} \subset K-K$ is a w^* null tree in X^* so that $\|y^*_E\| > \varepsilon/2$ for all $\emptyset \prec E \in \mathcal{F}_\xi$. Fix $0 < \delta < \varepsilon' < \varepsilon/4$ and $(\varepsilon_n) \subset (0,1)$ so that $\delta > n\varepsilon_n + \sum_{i>n}\varepsilon_i$ for each $n \in \mathbb{N}$. By Lemma 3.5 and Example 3.9, we can pass to a pruning and assume $(y^*_E)_{E\in\widehat{\mathcal{F}}_\xi}, (x_E)_{E\in\widehat{\mathcal{F}}_\xi} \subset B_X$ are such that $(x_E)_{E\in\widehat{\mathcal{F}}_\xi}$ is S null (actually $S \cup K$ null), $y^*_E(x_E) > \varepsilon'$ for all $E \in \widehat{\mathcal{F}}_\xi$, and for any $E \in \widehat{\mathcal{F}}_\xi$, $F \in \mathcal{F}_\xi$ comparable and distinct,

$$|y^*_F(x_E)| < \min\{\varepsilon_{|E|}, \varepsilon_{|F|}\}.$$

Then for all $E \in \widehat{\mathcal{F}}$ and $\emptyset \prec F \preceq E$,

$$\Big(y_\emptyset^* + \sum_{i=1}^{|E|} y_{E|_i}^*\Big)(x_F) \geq y_F^*(x_F) - \sum_{i=0}^{|F|-1} |y_{F|_i}^*(x_F)| - \sum_{i=|F|+1}^{|E|} |y_{E|_i}^*(x_F)|$$
$$> \varepsilon' - |F|\varepsilon_{|F|} - \sum_{i>|F|} \varepsilon_i > \varepsilon' - \delta.$$

But

$$y_\emptyset^* + \sum_{i=1}^{|E|} y_{E|_i}^* = x_\emptyset^* + \sum_{i=1}^{|E|}(x_{E|_i}^* - x_{E|_{i-1}}^*) = x_E^* \in K.$$

Note that $(x_E^*)_{E\in\mathcal{F}_\xi}$ is w^* convergent, so by pruning once more (as in Example 3.8) and passing to the appropriate pruning of $(x_E)_{E\in\widehat{\mathcal{F}}_\xi}$ (which is still $S\cup K$ null), we can assume $(x_E^*)_{E\in\mathcal{F}_\xi}\subset K$ is w^* continuous.

(ii)⇒(iii). This is trivial.

(iii)⇒(iv). Suppose $\varepsilon>0$ is such that for each norm separable $\emptyset\neq S\subset X^*$, there exists an S null tree $(x_E)_{E\in\widehat{\mathcal{F}}_\xi}\subset B_X$ with branches lying in $\mathcal{H}_\varepsilon^K$. By Proposition 5.2, this tree witnesses the fact that $I_S(\mathcal{H}_\varepsilon^K)>\xi$.

(iv)⇒(v). This is trivial, since if σ denotes all sequences in B_X pointwise null on S and $\sigma(r)$ denotes all sequences in B_X with S radius r, then $\sigma\subset\sigma(r)$. Thus for any $r>0$, we have $I_S(\mathcal{H})\leq I_{S,r}(\mathcal{H})$ for any $\mathcal{H}$.

(v)⇒(i). We apply (v) with $S=K$. We claim that for any $\zeta\leq\xi$, any $0<\delta<\varepsilon-r$, and any sequence $t\in(\mathcal{H}_\varepsilon^K)_{K,r}^\zeta$, there exists $x^*\in s_\delta^\zeta(K)$ such that $x^*(x)\geq\varepsilon$ for each $x\in t$. Applying this with $\zeta=\xi$ yields the non-emptiness of $s_\delta^\xi(K)$. The $\zeta=0$ case follows by the definition of $\mathcal{H}_\varepsilon^K$. Assume the result holds for some $\zeta<\xi$ and $t\in(\mathcal{H}_\varepsilon^K)_{K,r}^{\zeta+1}$. Then there exists a sequence $(x_n)\subset B_X$ having K radius r such that $t^\frown x_n\in(\mathcal{H}_\varepsilon^K)_{K,r}^\zeta$ for every $n\in\mathbb{N}$. For each n, fix $x_n^*\in s_\delta^\zeta(K)$ such that $x^*(x)\geq\varepsilon$ for every $x\in t^\frown x_n$. By passing to a subsequence, we may assume $x_n^*\underset{w^*}{\to}x^*\in s_\delta^\zeta(K)$. Note that

$$\liminf_n\|x_n^*-x^*\|\geq n\liminf_n(x_n^*-x^*)(x_n)\geq\varepsilon-r>\delta.$$

Hence $x^*\in s_\delta^{\zeta+1}(K)$. Of course, $x^*(x)=\lim_n x_n^*(x)\geq\varepsilon$ for any $x\in t$. Last, suppose that $\zeta\leq\xi$ is a limit ordinal and the result holds for every $\gamma<\zeta$. Suppose that $t\in(\mathcal{H}_\varepsilon^K)_{K,r}^\zeta$. For every $\gamma<\zeta$, there exists $x_\gamma^*\in s_\delta^\gamma(K)$ such that $x_\gamma^*(x)\geq\varepsilon$ for every $x\in t$. Any w^* limit x^* of a subnet of $(x_\gamma^*)_{\gamma<\zeta}$ lies in $s_\delta^\zeta(K)$ and satisfies $x^*(x)\geq\varepsilon$ for every $x\in t$.

To see the last statement, we leave it to the reader to check that the only cited results here which depend upon separability of either K or S are Example 3.9 and Proposition 5.2, guaranteeing that σ contains diagonals. However, the last paragraph of Example 3.9 indicates that the example holds without the assumption of separability whenever X does not contain ℓ_1, and that we have only used the direction of Proposition 5.2 which does not require that σ contains diagonals. ■

Corollary 5.10. *Let X be a separable Banach space. If $\emptyset \neq K \subset X^*$ is w^* compact and separable, then for any separable $S \supset K$, $\mathrm{Sz}(K) = \sup_{\varepsilon>0} I_S(\mathcal{H}_\varepsilon^K)$. If ℓ_1 does not embed into X, then $\mathrm{Sz}(K) = \sup_{\varepsilon>0} I_w(\mathcal{H}_\varepsilon^K)$.*

Proof. If $\xi < I_S(\mathcal{H}_\varepsilon^K)$ for $S \supset K$ norm separable, or if $I_w(\mathcal{H}_\varepsilon^K) > \xi$, then the proof of (v)⇒(i) above shows that $\mathrm{Sz}(K) > \xi$. Therefore $\mathrm{Sz}(K) \geq \sup_{\varepsilon>0} I_S(\mathcal{H}_\varepsilon^K)$ and $\mathrm{Sz}(K) \geq \sup_{\varepsilon>0} I_w(\mathcal{H}_\varepsilon^K)$. But (i)⇒(iv) above implies that if $\xi < \mathrm{Sz}_\delta(K)$, then $\xi < \sup_{\varepsilon>0} I_S(\mathcal{H}_\varepsilon^K)$, and $\mathrm{Sz}(K) \leq \sup_{\varepsilon>0} I_S(\mathcal{H}_\varepsilon^K)$. The implication (i)⇒(iv) in the "moreover" case of Lemma 5.9 yields $\mathrm{Sz}(K) \leq \sup_{\varepsilon>0} I_w(\mathcal{H}_\varepsilon^K)$ whenever ℓ_1 does not embed into X. ■

5.4. First application: Minkowski sums

Theorem 5.11. *For any separable Banach space X, any $\varepsilon > 0$, and $\emptyset \neq K, L, S \subset X^*$ norm separable such that K, L are w^* compact,*

$$I_S(\mathcal{H}_\varepsilon^{K+L}) \leq I_S(\mathcal{H}_{\varepsilon/2}^K) \oplus I_S(\mathcal{H}_{\varepsilon/2}^L).$$

If K, L are also assumed to be convex, then $\mathrm{Sz}(K+L) = \max\{\mathrm{Sz}(K), \mathrm{Sz}(L)\}$. In particular, for any separable Banach spaces Y, Z,

$$\mathrm{Sz}(Y \oplus Z) = \max\{\mathrm{Sz}(Y), \mathrm{Sz}(Z)\}.$$

Remark 5.12. The third part of the statement was shown in [18], using slicings of the dual ball.

Proof of Theorem 5.11. Let $K_0 = K$ and $K_1 = L$. Suppose $\xi < I_S(\mathcal{H}_\varepsilon^{K_0+K_1})$. Fix a strongly S null tree $(x_E)_{E\in\widehat{\mathcal{F}}_\xi} \subset B_X$ with branches lying in $\mathcal{H}_\varepsilon^{K_0+K_1}$. For each $F \in \mathrm{MAX}(\mathcal{F}_\xi)$, choose $x_F^*(0) \in K_0$ and $x_F^*(1) \in K_1$ so that $(x_F^*(0) + x_F^*(1))(x_E) \geq \varepsilon$ for all $\emptyset \prec E \preceq F$. For $E \in \widehat{\mathcal{F}}_\xi$ and $j = 0$ or 1, let

$$\mathcal{A}_E^j = \{F \in \mathrm{MAX}(\mathcal{F}_\xi) : E \preceq F,\, x_F^*(j)(x_E) \geq \varepsilon/2\}.$$

By Lemma 4.1, we can find ξ_0, ξ_1 with $\xi_0 \oplus \xi_1 = \xi$ and for $j = 0$ or 1 an extended embedding (i_j, e_j) of $\mathcal{F}_{\xi_j}$ into $\mathcal{F}_\xi$ such that the induced coloring is monochromatically j. But this means that for each $E \in \mathrm{MAX}(\mathcal{F}_{\xi_j})$,

$$e(E) \in \bigcap_{k=1}^{|E|} \mathcal{A}_{i_j(E|_k)}^j,$$

so $x_{e(E)}^*(j)(x_{i_j(E|_k)}) \geq \varepsilon/2$ for $0 < k \leq |E|$. Thus the S null tree $(x_{i_j(E)})_{E\in\widehat{\mathcal{F}_{\xi_j}}}$ witnesses the fact that $\xi_j < I_S(\mathcal{H}_{\varepsilon/2}^{K_j})$. Then

$$\xi = \xi_0 \oplus \xi_1 < I_S(\mathcal{H}^K_{\varepsilon/2}) \oplus I_S(\mathcal{H}^L_{\varepsilon/2}).$$

Since $\xi < I_S(\mathcal{H}^{K+L}_\varepsilon)$ was arbitrary, $I_S(\mathcal{H}^{K+L}_\varepsilon) \le I_S(\mathcal{H}^K_{\varepsilon/2}) \oplus I_S(\mathcal{H}^L_{\varepsilon/2})$.

For the second statement, $\max\{\mathrm{Sz}(K), \mathrm{Sz}(L)\} = \omega^\xi$ for some $0 \le \xi < \omega_1$. If $\xi = 0$, both K and L, and therefore $K + L$, must be norm compact. This gives the result for $\xi = 0$. Suppose $\xi > 0$. Then

$$I_{K\cup L}(\mathcal{H}^{K+L}_\varepsilon) \le I_{K\cup L}(\mathcal{H}^K_{\varepsilon/2}) \oplus I_{K\cup L}(\mathcal{H}^L_{\varepsilon/2}) \le I_K(\mathcal{H}^K_{\varepsilon/2}) \oplus I_L(\mathcal{H}^L_{\varepsilon/2}) < \omega^\xi.$$

Here we have used the fact that $I_K(\mathcal{H}^K_{\varepsilon/2}), I_L(\mathcal{H}^L_{\varepsilon/2})$ are successors, and therefore strictly less than $\mathrm{Sz}(K), \mathrm{Sz}(L)$, respectively. Since any sequence pointwise null on $K \cup L$ is pointwise null on $K + L$, we can take the supremum over ε and deduce $\mathrm{Sz}(K + L) \le \max\{\mathrm{Sz}(K), Sz(L)\}$. Since $K + L$ contains translates of K and L, and since the Szlenk index is translation invariant, we deduce that $\mathrm{Sz}(K + L) \ge \max\{\mathrm{Sz}(K), \mathrm{Sz}(L)\}$.

For the last part, it is sufficient to assume Y^*, Z^* are separable, since otherwise both sides of the equation are ω_1. It is clear that $\mathrm{Sz}(Y \oplus Z) \ge \max\{\mathrm{Sz}(Y), \mathrm{Sz}(Z)\}$ and $\mathrm{Sz}(Y \oplus Z) = \mathrm{Sz}(Y \oplus_1 Z)$, so we assume $Y \oplus Z = Y \oplus_1 Z$. We identify Y, Z in the natural way with subspaces of $Y \oplus_1 Z$ and note that with this identification, $B_{(Y\oplus_1 Z)^*} = B_{Y^*} + B_{Z^*}$. The previous paragraph now gives the conclusion. ■

5.5. Second application: Szlenk index of an operator. Given an operator $T : X \to Y$ with X separable, the *Szlenk index* $\mathrm{Sz}(T)$ of T is defined to be $\mathrm{Sz}(T^*B_{Y^*})$. The next theorem was shown in [5] for the usual definition of the Szlenk index, while what we show uses our dualization of the Szlenk index. What we have already done easily yields the following:

THEOREM 5.13. *For $\xi < \omega_1$, and separable Banach spaces X, Y, let*

$$\mathcal{SZ}_\xi(X, Y) = \{T \in \mathcal{L}(X, Y) : \mathrm{Sz}(T) \le \omega^\xi\}.$$

Then for any separable Banach spaces W, X, Y, Z, any $\xi < \omega_1$, and any $S \in \mathcal{SZ}_\xi(X, Y)$, $T \in \mathcal{L}(W, X)$, and $R \in \mathcal{L}(Y, Z)$, we have $RST \in \mathcal{SZ}_\xi(W, Z)$. Moreover, $\mathcal{SZ}_\xi(X, Y)$ is a closed subspace of $\mathcal{L}(X, Y)$.

Proof. Note that $\mathcal{SZ}_0(X, Y)$ is simply the compact operators, so the result is well-known. Assume $\xi > 0$. We first note that in this case, $S^*B_{Y^*}$ is norm separable and w^* compact for every $S \in \mathcal{SZ}_\xi(X, Y)$.

Note that $\mathrm{Sz}(0) = 1$, so $0 \in \mathcal{SZ}_\xi$ for any $0 \le \xi < \omega_1$. If $S \in \mathcal{SZ}_\xi$, then for any $\varepsilon > 0$ and non-zero scalar c,

$$I_{(cS^*)B_{Y^*}}(\mathcal{H}^{(cS^*)B_{Y^*}}_\varepsilon) = I_{S^*B_{Y^*}}(\mathcal{H}^{(cS^*)B_{Y^*}}_\varepsilon) = I_{S^*B_{Y^*}}(\mathcal{H}^{S^*B_{Y^*}}_{|c^{-1}|\varepsilon}) \le \mathrm{Sz}(S).$$

Therefore $\mathrm{Sz}(cS) \le \mathrm{Sz}(S)$. Since $c \ne 0$ was arbitrary, $\mathrm{Sz}(S) = \mathrm{Sz}(cS)$.

If $\mathrm{Sz}(S) > \omega^\xi$, there exists $\varepsilon > 0$ such that $I_{S^*B_{Y^*}}(\mathcal{H}^{S^*B_{Y^*}}_\varepsilon) > \omega^\xi$. If $S^*B_{Y^*}$ is non-separable, obviously S is not the norm limit of any sequence

$T_n : X \to Y$ such that $T_n^* Y^*$ is separable. From this it follows that S is not the norm limit of a sequence in $\mathcal{SZ}_\xi(X,Y)$. If $S^*B_{Y^*}$ is separable and $\mathrm{Sz}(S) > \omega^\xi$, then there exists $(x_E)_{E\in\widehat{\mathcal{F}}_\xi} \subset B_X$ which is $S^*B_{Y^*}$ null and has branches lying in $\mathcal{H}_\varepsilon^{S^*B_{Y^*}}$. If $\|S-U\| < \varepsilon/3$, then any member of $\mathcal{H}_\varepsilon^{S^*B_{Y^*}}$ is a member of $\mathcal{H}_{2\varepsilon/3}^{U^*B_{Y^*}}$. Moreover, any $S^*B_{Y^*}$ null sequence $(x_n) \subset B_X$ is a $U^*B_{Y^*}$ radius $\varepsilon/3$ sequence. Therefore $(x_E)_{E\in\widehat{\mathcal{F}}_\xi} \subset B_X$ witnesses the fact that $I_{U^*B_{Y^*},\varepsilon/3}(\mathcal{H}_{2\varepsilon/3}^{U^*B_{Y^*}}) > \xi$. Hence $\mathrm{Sz}(U) > \omega^\xi$. Thus S cannot be the norm limit of a sequence lying in $\mathcal{SZ}_\xi$, and $\mathcal{SZ}_\xi$ is a norm closed subset of $\mathcal{L}(X,Y)$.

Using the fact that for $S,U \in \mathcal{SZ}_\xi(X,Y)$ we have $(S^*+U^*)B_{Y^*} \subset S^*B_{Y^*} + U^*B_{Y^*}$ and Theorem 5.11, we get

$$\begin{aligned}\mathrm{Sz}(S+U) &= \mathrm{Sz}((S^*+U^*)B_{Y^*})\\ &\le \mathrm{Sz}(S^*B_{Y^*}+U^*B_{Y^*}) = \max\{S^*B_{Y^*}, U^*B_{Y^*}\},\end{aligned}$$

whence $\mathcal{SZ}_\xi(X,Y)$ is closed under finite sums.

Suppose $S \in \mathcal{SZ}_\xi(X,Y)$ and $R \in \mathcal{L}(Y,Z)$ has norm not exceeding 1. Then

$$\mathcal{H}_\varepsilon^{S^*R^*B_{Z^*}} \subset \mathcal{H}_\varepsilon^{S^*B_{Y^*}},$$

since $S^*R^*B_{Z^*} \subset S^*B_{Y^*}$. Thus

$$I_{S^*B_{Y^*}}(\mathcal{H}_\varepsilon^{S^*R^*B_{Z^*}}) \le I_{S^*B_{Y^*}}(\mathcal{H}_\varepsilon^{S^*B_{Y^*}}) \le \mathrm{Sz}(S).$$

Since $S^*R^*B_{Z^*} \subset S^*B_{Z^*}$, Lemma 5.9 gives

$$\mathrm{Sz}(SR) \le \sup_{\varepsilon>0} I_{S^*B_{Y^*}}(\mathcal{H}_\varepsilon^{S^*R^*B_{Z^*}}) \le \sup_{\varepsilon>0} I_{S^*B_{Y^*}}(\mathcal{H}_\varepsilon^{S^*B_{Y^*}}) = \mathrm{Sz}(S) \le \omega^\xi.$$

Suppose $S \in \mathcal{SZ}_\xi(X,Y)$ and $T \in \mathcal{L}(W,X)$ has norm not exceeding 1. Note that

$$T(\mathcal{H}_\varepsilon^{T^*S^*B_{Y^*}}) \subset \mathcal{H}_\varepsilon^{S^*B_{Y^*}}.$$

More generally, an easy proof by induction shows that for any ξ,

$$T((\mathcal{H}_\varepsilon^{T^*S^*B_{Y^*}})^\xi_{T^*S^*B_{Y^*}}) \subset (\mathcal{H}_\varepsilon^{S^*B_{Y^*}})^\xi_{S^*B_{Y^*}}.$$

The only non-trivial step is the successor step, for which we note that any sequence $(u_j) \subset B_W$ which is pointwise null on $T^*S^*B_{Y^*}$ is such that $(Tu_j) \subset B_X$ is pointwise null on $S^*B_{Y^*}$. This proves $\mathrm{Sz}(TS) \le \mathrm{Sz}(S)$. ∎

In the next section, we will see a new application of pointwise null indices to computing the Szlenk index of an operator.

5.6. Third application: Direct sums. Suppose (X_n) is a sequence of Banach spaces and U is a Banach space with normalized, 1-unconditional basis (e_n). We denote by $\big(\bigoplus_n X_n\big)_U$ the space all sequences (x_n) such that $x_n \in X_n$ and $\sum \|x_n\| e_n \in U$, and let X denote this space with norm $\|(x_n)\| = \big\|\sum \|x_n\| e_n\big\|$. We also let $P_n : X \to X_n$ denote the operator which takes (x_m)

to x_n. More generally, for each $E \subset \mathbb{N}$, we let $P_E = \sum_{n\in E} P_n$. We have the following:

(i) X is a Banach space with this norm.
(ii) X is separable if and only if X_n is separable for each $n \in \mathbb{N}$.
(iii) If (e_n) is a shrinking basis for U, then $X^* = (\bigoplus_n X_n^*)_{U^*}$ isometrically.
(iv) If (e_n) is shrinking, then a sequence $(s_n) \subset X$ is weakly null if and only if it is bounded and $(P_m s_n)_n$ is weakly null in X_m for each $m \in \mathbb{N}$.

THEOREM 5.14. *If U is a Banach space with normalized 1-unconditional basis (e_n) and if X_n is a sequence of separable spaces, then*

$$\mathrm{Sz}(X) \le \Big(\sup_n \mathrm{Sz}(X_n)\Big)\mathrm{Sz}(U).$$

Proof. If U^* is non-separable or X_n^* is non-separable for some $n \in \mathbb{N}$, the result is clear. Thus it is sufficient to assume that (e_n) is shrinking, which means X^* is separable, and it is sufficient to estimate the weakly null ℓ_1^+ index. Let $\xi = \sup_n \mathrm{Sz}(X_n)$ and $\zeta = I_w(\mathcal{H}^U_{\varepsilon/3})$. Seeking a contradiction, suppose $I_w(\mathcal{H}^X_\varepsilon) > \xi\zeta$. Let $(x_E)_{E\in\widehat{\mathcal{F}_\zeta[\mathcal{F}_\xi]}} \subset B_X$ be a weakly null tree with branches in $\mathcal{H}^X_\varepsilon$. Mimicking the proof of Lemma 4.2, we will recursively construct $r : \widehat{\mathcal{F}}_\zeta \to C(\mathcal{F}_\xi)$, $I : \widehat{\mathcal{F}}_\zeta \to [\mathbb{N}]^{<\omega}$, and $u : \widehat{\mathcal{F}}_\zeta \to B_X$ so that for all $E \in \widehat{\mathcal{F}}_\zeta$, letting $F_i = \max r(E|_i)$ for each $1 \le i \le |E|$, and $F = \bigcup_{i=1}^{|E|-1} F_i$,

- $u(E) \in \mathrm{co}(x_{F^\frown G} : G \in r(E))$,
- $\|u(E) - P_{I(E)}(u_E)\| < 2\varepsilon/3$,
- $(\min F_i)_{i=1}^{|E|}$ is a spread of E,
- if $E \prec H \in \mathcal{F}_\zeta$, then $I(E) < I(H)$,
- if $E^\frown k, E^\frown l \in \mathcal{F}_\zeta$ with $k < l$, then $I(E^\frown k) < I(E^\frown l)$,
- $(F_i)_{i=1}^{|E|}$ is successive.

For a given $E \in \mathcal{F}_\zeta$, we must define $r(E)$, $I(E)$, $u(E)$ assuming that $r(H)$, $I(H)$, $u(H)$ have been defined for each $\emptyset \prec H \prec E$. Let $m_0 \in \mathbb{N}$ be minimal such that $E^\frown m_0 \in \mathcal{F}_\zeta$. We will recursively define $r(E^\frown m), I(E^\frown m)$, $u(E^\frown m)$ for each $m \ge m_0$. Assume that for some $k \ge m_0$, these have been defined for each $m_0 \le m < k$. Let $F_i = \max r(E|_i)$ and $F = \bigcup_{i=1}^{|E|} F_i$. Fix n so that $F < n$, $I(E) < n$, $k < n$, and $((\min F_i)_{i=1}^{|E|})^\frown n \in \mathcal{F}_\zeta$. This can be done since $(\min F_i)_{i=1}^{|E|}$ is a spread of E, which is non-maximal in $\mathcal{F}_\zeta$. If $k > m_0$, assume also that $n > I(E^\frown(k-1))$. Define $j : \mathcal{F}_\xi \to \mathcal{F}_\zeta[\mathcal{F}_\xi]$ by

$$j(G) = F^\frown(n + i : i \in G).$$

If for each $c\in C(\mathcal{F}_\xi\cap(n,\infty)^{<\omega})$,

$$\inf\{\|P_{[1,n)}x\| : x\in \mathrm{co}(x_{F^\frown G} : G\in c)\}\geq \varepsilon/3,$$

then $(P_{[1,n)}x_{j(G)})_{G\in\widehat{\mathcal{F}}_\xi}\subset B_{\bigoplus_{i=1}^n X_i}$ is a weakly null tree with branches in $\mathcal{H}^{X'}_{\varepsilon/3}$, where $X'=\bigoplus_{i=1}^n X_i$. But this would mean that

$$\max_{1\leq i\leq n}\mathrm{Sz}(X_i)=\mathrm{Sz}\Big(\bigoplus_{i=1}^n X_i\Big)>\xi,$$

a contradiction. Thus we can find some $c\in C(\mathcal{F}_\xi\cap(n,\infty))$ such that

$$\inf\{\|P_{[1,n)}x\| : x\in \mathrm{co}(x_{F^\frown G}) : G\in c\}<\varepsilon/3.$$

Let $r(E^\frown k)=c$. Let $u(E^\frown k)\in \mathrm{co}\{x_{F^\frown G} : G\in c\}$ be a vector such that $\|P_{[1,n)}u(E^\frown k)\|<\varepsilon/3$. Choose $l\in\mathbb{N}$ so that $\|P_{(l,\infty)}u(E^\frown k)\|<\varepsilon/3$ and let $I(E^\frown k)=[n,l]$. This completes the recursive construction.

We now let $q(E)=\{F^\frown G : G\in r(E)\}$ to obtain an order preserving function. Note that since q is order preserving, $(u(E|_i))_{i=1}^{|E|}$ is a convex block of a member of $\mathcal{H}^X_\varepsilon$, and thus is a member of $\mathcal{H}^X_\varepsilon$. Let $y_E=\sum_{j\in I(E)}\|P_j(u(E))\|e_j$, so that

$$\|y_E\|=\Big\|\sum_{j\in I(E)}\|P_j(u(E))\|e_j\Big\|=\|P_{I(E)}(u(E))\|\leq\|u(E)\|\leq 1,$$

and, for any $(a_i)_{i=1}^{|E|}\subset[0,\infty)$,

$$\begin{aligned}\Big\|\sum_{i=1}^{|E|}a_iy_{E|_i}\Big\|&=\Big\|\sum_{i=1}^{|E|}\sum_{j\in I(E|_i)}a_i\|P_j(u(E|_i))\|e_j\Big\|\\&=\Big\|\sum_j\Big\|P_j\Big(\sum_{i=1}^{|E|}a_iP_{I(E|_i)}(u(E|_i))\Big)\Big\|e_j\Big\|=\Big\|\sum_{i=1}^{|E|}a_iP_{I(E|_i)}(u(E|_i))\Big\|\\&\geq\Big\|\sum_{i=1}^{|E|}a_iu(E|_i)\Big\|-\sum_{i=1}^{|E|}a_i\|u(E|_i)-P_{I(E|_i)}u(E|_i)\|\\&\geq\Big(\varepsilon-2\frac{\varepsilon}{3}\Big)\sum_{i=1}^{|E|}a_i=\frac{\varepsilon}{3}\sum_{i=1}^{|E|}a_i.\end{aligned}$$

But $(y_E)_{E\in\widehat{\mathcal{F}}_\zeta}\subset B_U$ is a block tree, and therefore a weakly null tree. We deduce $I_w(\mathcal{H}^U_{\varepsilon/3})>\zeta$, a contradiction. ∎

Remark 5.15. The result above is optimal in certain cases. Recall that for $\xi<\omega_1$, the *Schreier space of order* ξ, denoted X_ξ, is the completion of

c_{00} under the norm
$$\|x\|_{X_\xi} = \sup_{E\in\mathcal{S}_\xi} \|P_E x\|_{\ell_1}.$$
It is known that $\mathrm{Sz}(X_\xi) = \omega^{\xi+1}$ [7]. Fix $\zeta, \xi < \omega_1$ and let $X = (\bigoplus X_\zeta)_{X_\xi}$. That is, each member of the sequence of spaces is equal to X_ζ. Let $(e_i^n)_i$ denote the basis of the space X_ζ which sits in the nth position in the direct sum. For $E \in \widehat{\mathcal{S}_\xi[\mathcal{S}_\zeta]}$, let $(E_i)_{i=1}^k$ be the standard decomposition of E with respect to S_ζ. Next let $x_E = e_{\max E_k}^{\min E_k}$. Then $(x_E)_{E\in\widehat{\mathcal{S}_\xi[\mathcal{S}_\zeta]}} \subset B_X$ is weakly null. Moreover, if $\emptyset \prec E \in \mathcal{S}_\xi[\mathcal{S}_\zeta]$ and $(a_i)_{i\in E}$ are any scalars, then letting $(E_i)_{i=1}^k$ denote the standard decomposition of E with respect to $\mathcal{S}_\zeta$ and letting $F_i = \bigcup_{j=1}^i E_j$, we get
$$\begin{aligned}\Big\|\sum_{F\preceq E} a_F x_F\Big\|_X &\geq \sum_{i=1}^k \Big\|P_{\min E_i}\sum_{F\preceq E} a_F x_F\Big\|_{X_\zeta} = \sum_{i=1}^k \Big\|\sum_{F_{i-1}\prec F\preceq F_i} a_F e_{\max F}^{\min E_i}\Big\|_{X_\zeta}\\ &= \sum_{i=1}^k \sum_{F_{i-1}\prec F\preceq F_i} |a_F| = \sum_{F\preceq E} |a_F|.\end{aligned}$$
Thus $\mathrm{Sz}(X) > \iota(\mathcal{S}_\xi[\mathcal{S}_\zeta]) = \omega^{\zeta+\xi}$. If ξ is infinite, then $\zeta+1+\xi+1 = \zeta+\xi+1$, so the estimate of $\omega^{\zeta+\xi+1}$ given by Theorem 5.14 is optimal in this case.

Remark 5.16. Suppose U, V are Banach spaces with normalized, shrinking, 1-unconditional bases $(u_n), (v_n)$, respectively, so that the operator $I_{U,V} : U \to V$ defined by $I_{U,V} u_n = v_n$ is bounded. Suppose that we have two sequences X_n, Y_n of separable Banach spaces and a uniformly bounded sequence of operators $T_n : X_n \to Y_n$. Then we can define an operator $T : (\bigoplus X_n)_U \to (\bigoplus Y_n)_V$ by $T(x_n) = (T_n y_n)$. An inessential modification of the preceding proof yields $\mathrm{Sz}(T) \leq (\sup_n \mathrm{Sz}(T_n))\mathrm{Sz}(I_{U,V})$.

5.7. Fourth application: Constant reduction. The following argument is a modification of a well-known argument due to James [12]. Essentially, it is implicitly contained in [2]. However, we need a more precise quantification than the one given there, so we provide a proof. Suppose $(x_i)_{i=1}^{n^2} \subset B_X$ and $\delta, \varepsilon > 0$ are such that each convex combination of these points has norm at least $\delta\varepsilon$. We partition $\{1, \dots, n^2\}$ into successive intervals $I_1 < \dots < I_n$, each having cardinality n, and consider two cases. Either for some $1 \leq i \leq n$, all convex combinations of $(x_j)_{j\in I_i}$ have norm at least ε, or for each $1 \leq i \leq n$, we can find a convex combination $y_i = \sum_{j\in I_i} a_j x_j$ of $(x_j)_{j\in I_i}$ such that $\|y_i\| < \varepsilon$. Then $(\varepsilon^{-1} y_i)_{i=1}^n \subset B_X$, by homogeneity, has the property that each convex combination of this sequence has norm at least δ.

Below, we view a tree of order ξ^2 as being composed of a tree of order ξ, with each vertex being a tree of order ξ. We will again consider two cases:

one of these "interior" trees will already have the lower ε estimate on all of its branches, or we can replace each of these trees with a "bad" convex combination so that, after being multiplied by ε^{-1}, these "bad" combinations will form a tree of size ξ having the appropriate δ lower estimates on all convex combinations of all branches.

THEOREM 5.17. *For $\delta, \varepsilon \in (0,1)$ and a Banach space X having separable dual,*

$$I_w(\mathcal{H}^X_{\varepsilon\delta}) \le I_w(\mathcal{H}^X_\varepsilon) I_w(\mathcal{H}^X_\delta).$$

If $I_w(\mathcal{H}^X_\varepsilon) > \omega^{\omega^\xi}$ for some ξ, then $I_w(\mathcal{H}^X_\delta) > \omega^{\omega^\xi}$. In particular, if $\eta < \omega_1$ is a limit ordinal, then $\mathrm{Sz}(X) \ne \omega^{\omega^\eta}$. Moreover, if $\eta < \omega_1$ is any limit ordinal, and if Y is any Banach space, then $\mathrm{Sz}(Y) \ne \omega^{\omega^\eta}$.

Proof. Let $\xi = I_w(\mathcal{H}^X_\varepsilon)$. Fix $0 < \zeta < \omega_1$. Assume that $I_w(\mathcal{H}^X_{\varepsilon\delta}) > \xi\zeta$. Then we can find a strongly weakly null tree

$$(x_E)_{E \in \widehat{\mathcal{F}_\zeta[\mathcal{F}_\xi]}} \subset B_X$$

whose branches lie in $\mathcal{H}^X_{\varepsilon\delta}$. We define a coloring on $C(\widehat{\mathcal{F}_\zeta[\mathcal{F}_\xi]})$ by letting c have color 0 provided there exists a convex combination of $(x_E)_{E\in c}$ which has norm less than ε, and color 1 otherwise. If there exists an embedding $i : \widehat{\mathcal{F}}_\xi \to \widehat{\mathcal{F}_\zeta[\mathcal{F}_\xi]}$ such that each $c \in C(i(\widehat{\mathcal{F}}_\xi))$ receives color 1, then $(x_{i(E)})_{E\in\widehat{\mathcal{F}}_\xi}$ witnesses the fact that $I_w(\mathcal{H}^X_\varepsilon) > \xi$, a contradiction. Therefore for each embedded tree $i(\widehat{\mathcal{F}}_\xi)$, some of its branches receives color 0. Applying Lemma 4.2, we obtain an order preserving $j : \widehat{\mathcal{F}}_\zeta \to C(\mathcal{F}_\zeta[\mathcal{F}_\xi])$ such that for each $E \in \widehat{\mathcal{F}}_\zeta$, $j(E)$ receives color 0. Letting y_E be a convex combination of $(x_F)_{F\in j(E)}$ with norm less than ε, we obtain a weakly null tree $(y_E)_{E\in\widehat{\mathcal{F}}_\zeta}$. This tree is weakly null because the original tree was strongly weakly null. Since j is order preserving, $(y_{E|_i})_{i=1}^{|E|}$ is a convex block of a member of $\mathcal{H}^X_{\varepsilon\delta}$, and therefore lies in $\mathcal{H}^X_{\varepsilon\delta}$. Then by homogeneity, $(\varepsilon^{-1}y_E)_{E\in\widehat{\mathcal{F}}_\zeta} \subset B_X$ is a weakly null tree with branches in $\mathcal{H}^X_\delta$. This means $I_w(\mathcal{H}^X_\delta) > \zeta$, which proves the first inequality.

Suppose $I_w(\mathcal{H}^X_\varepsilon) > \omega^{\omega^\xi}$ for some ξ. Fix $\zeta < \omega^{\omega^\xi}$. Choose $n \in \mathbb{N}$ so that $\varepsilon^{1/n} > \delta$. Note that $\zeta^n < \omega^{\omega^\xi}$, so $I_w(\mathcal{H}^X_\varepsilon) > \zeta^n$. By applying the first inequality, we deduce $I_w(\mathcal{H}^X_\delta) > \zeta$. Since $\zeta < \omega^{\omega^\xi}$ was arbitrary, $I_w(\mathcal{H}^X_\delta) \ge \omega^{\omega^\xi}$. But since $I_w(\mathcal{H}^X_\delta)$ is always a successor, $I_w(\mathcal{H}^X_\delta) > \omega^{\omega^\xi}$.

Suppose that $\mathrm{Sz}(X) \ge \omega^{\omega^\eta}$. This means that for $\zeta < \eta$, $I_w(\mathcal{H}^X_\varepsilon) > \omega^{\omega^\zeta}$ for some $\varepsilon \in (0,1)$, and by the preceding part, $I_w(\mathcal{H}^X_{1/2}) > \omega^{\omega^\zeta}$. But since this holds for any $\zeta < \eta$, we have $I_w(\mathcal{H}^X_{1/2}) \ge \sup_{\zeta<\eta} \omega^{\omega^\zeta} = \omega^{\omega^\eta}$. Again, since $I_w(\mathcal{H}^X_{1/2})$ is a successor, this must be a strict inequality, which means $\mathrm{Sz}(X) > \omega^{\omega^\eta}$.

For the last statement, we cite a result of Lancien [14] which states that if the Szlenk index of a Banach space is countable, it is separably determined. Therefore if there existed a Banach space Y with $\mathrm{Sz}(Y) = \omega^{\omega^\eta}$ with η countable, then Y would have a separable subspace X with $\mathrm{Sz}(X) = \omega^{\omega^\eta}$. But this means X^* is separable, hence $\mathrm{Sz}(X) = \omega^{\omega^\eta}$ is impossible. ∎

5.8. Fifth application: Three-space properties. Given our dualization lemma, the following theorem can be shown to be equivalent to [6, Proposition 2.1] in the case of a Banach space having separable dual, up to the value of certain constants. There, however, the result was shown using the usual definition of Szlenk index involving slicing the dual ball, whereas we only use the weakly null ℓ_1^+ index.

THEOREM 5.18. *For any $\varepsilon \in (0, 1/3)$, any Banach space X having separable dual, and any closed subspace $Y \le X$,*

$$I_w(\mathcal{H}_\varepsilon^X) \le I_w(\mathcal{H}_{\varepsilon/5}^{X/Y}) I_w(\mathcal{H}_{\varepsilon/5}^Y).$$

In particular, for any ordinal $\xi < \omega_1$, $\mathrm{Sz}(\cdot) \le \omega^{\omega^\xi}$ and $\mathrm{Sz}(\cdot) < \omega^{\omega^\xi}$ are three-space properties on the class of separable Banach spaces.

Proof. Fix a Banach space X having separable dual, $\varepsilon \in (0, 1/3)$, and $Y \le X$. Let $Q : X \to X/Y$ denote the quotient map. Let $\xi = I_w(\mathcal{H}_{\varepsilon/5}^{X/Y})$ and $\zeta = I_w(\mathcal{H}_{\varepsilon/5}^Y)$. If $I_w(\mathcal{H}_\varepsilon^X) > \xi\zeta$, we can find a strongly weakly null tree $(x_E)_{E \in \widehat{\mathcal{F}_\zeta[\mathcal{F}_\xi]}} \subset S_X$ with branches in $\mathcal{H}_\varepsilon^X$. Define the coloring f on $C(\mathcal{F}_\zeta[\mathcal{F}_\xi])$ by letting $f(c) = 1$ provided that $\|Qx\|_{X/Y} \ge \varepsilon/5$ for each convex combination x of $(x_E)_{E \in c}$, and $f(c) = 0$ otherwise. If there exists an embedding $i : \widehat{\mathcal{F}}_\xi \to \widehat{\mathcal{F}_\zeta[\mathcal{F}_\xi]}$ such that $f(c) = 1$ for all $c \in C(i(\widehat{\mathcal{F}}_\xi))$, then $(Qx_{i(E)})_{E \in \widehat{\mathcal{F}}_\xi} \subset B_{X/Y}$ is a weakly null tree witnessing the fact that $I_w(\mathcal{H}_{\varepsilon/5}^{X/Y}) > \xi$, a contradiction. Therefore we apply Lemma 4.2 to obtain an order preserving $j : \widehat{\mathcal{F}}_\zeta \to C(\widehat{\mathcal{F}_\zeta[\mathcal{F}_\xi]})$ such that $f \circ j \equiv 0$. For each $E \in \widehat{\mathcal{F}}_\zeta$, we let z_E be a convex combination of $(x_F)_{F \in j(E)}$ such that $\|Qz_E\|_{X/Y} < \varepsilon/5$. For each $E \in \widehat{\mathcal{F}}_\zeta$, $(z_{E|_i})_{i=1}^{|E|}$ is a convex block of a member of $\mathcal{H}_\varepsilon^X$, and is therefore also a member of $\mathcal{H}_\varepsilon^X$.

For $E \in \mathcal{F}_\zeta'$, Proposition 2.1 shows that there exists a subsequence $(z_{E^\frown k_n})$ of $(z_{E^\frown n})_{E<n}$ and a weakly null sequence $(y_n)_{E<n} \subset B_Y$ with $\|z_{E^\frown n} - y_n\| < 4\varepsilon/5$. By Lemma 3.5, we can find a pruned subtree $(u_E)_{E \in \widehat{\mathcal{F}}_\zeta}$ of $(z_E)_{E \in \widehat{\mathcal{F}}_\zeta}$ and a weakly null tree $(y_E)_{E \in \widehat{\mathcal{F}}_\zeta} \subset B_Y$ such that $\|u_E - y_E\| < 4\varepsilon/5$ for each $E \in \widehat{\mathcal{F}}_\zeta$. For each $E \in \widehat{\mathcal{F}}_\zeta$, since $(u_{E|_i})_{i=1}^{|E|} \in \mathcal{H}_\varepsilon^X$, there exists $f \in B_{X^*}$ such that $f(u_{E|_i}) \ge \varepsilon$ for each $1 \le i \le |E|$. Then for such i, $f(y_{E|_i}) \ge f(u_{E|_i}) - \|u_E - y_E\| > \varepsilon - 4\varepsilon/5 = \varepsilon/5$. Thus $(y_E)_{E \in \widehat{\mathcal{F}}_\zeta} \subset B_Y$

witnesses the fact that $I_w(\mathcal{H}^Y_{\varepsilon/5}) > \zeta$, a contradiction. This proves the first statement.

For the second and third parts, assume $\mathrm{Sz}(Y), \mathrm{Sz}(X/Y) \leq \omega^{\omega^\xi}$. This means $Y^*, (X/Y)^*$, and therefore X^*, are separable. Moreover, for each $\varepsilon \in (0, 1/3)$,

$$I_w(\mathcal{H}^X_\varepsilon) \leq I_w(\mathcal{H}^{X/Y}_{\varepsilon/5}) I_w(\mathcal{H}^Y_{\varepsilon/5}) < \omega^{\omega^\xi},$$

since $I_w(\mathcal{H}^{X/Y}_{\varepsilon/5}), I_w(\mathcal{H}^Y_{\varepsilon/5}) < \omega^{\omega^\xi}$. Since this holds for all ε, it follows that $\mathrm{Sz}(X) \leq \omega^{\omega^\xi}$. Moreover, if $\mathrm{Sz}(Y), \mathrm{Sz}(X/Y) < \omega^{\omega^\xi}$, then $\mathrm{Sz}(X/Y)\mathrm{Sz}(Y) < \omega^{\omega^\xi}$, and

$$\sup_{\varepsilon \in (0,1/3)} I_w(\mathcal{H}^X_\varepsilon) \leq \mathrm{Sz}(X/Y)\mathrm{Sz}(Y) < \omega^{\omega^\xi}. \ \blacksquare$$

6. Classes of Banach spaces with bounded Szlenk index

6.1. Mixed Tsirelson spaces. For our purposes, mixed Tsirelson spaces are a remarkably useful class of spaces for providing examples with prescribed ℓ_1 behavior. For example, given a sequence of countable ordinals $\xi_n \nearrow \omega^\xi$ and constants $1 \geq \theta_n \searrow 0$, does there exist a Banach space X such that $\omega^\xi > I_w(\mathcal{H}^X_{\theta_n}) \geq \xi_n$ for each $n \in \mathbb{N}$? Theorem 5.17 says this is not possible for arbitrary sequences, since $I_w(\mathcal{H}^X_{\theta^n}) \leq I_w(\mathcal{H}^X_\theta)^n$ for any $\theta \in (0, 1)$. When this estimate is essentially optimal, i.e. we have roughly geometric growth, we encounter this restriction. It is the only restriction, however, as the mixed Tsirelson spaces show.

Let (e_n) denote the canonical c_{00} basis and let P_n, P_E denote the associated canonical coordinate and partial sum projections. Suppose that $1 > \theta_n \searrow 0$ and $(\mathcal{G}_n)_{n\geq 0}$ are regular families such that $\mathcal{G}_0$ contains all singletons. Define the norm $\|\cdot\|_{\mathcal{G}_0}$ on c_{00} by

$$\|x\|_{\mathcal{G}_0} = \max_{E \in \mathcal{G}_0} \|P_E x\|_{\ell_1}.$$

We inductively define norms $|\cdot|_n$, $n = 0, 1, 2, \ldots$, on c_{00} by $|x|_0 = \|x\|_{\mathcal{G}_0}$ and

$$|x|_{n+1} = |x|_n \vee \sup_{m \in \mathbb{N}} \sup\Big\{\theta_m \sum_{i=1}^k |P_{E_i} x|_n : (E_i)_{i=1}^k \text{ is } \mathcal{G}_m\text{-admissible}\Big\}.$$

One can easily prove by induction that $|x|_n \leq \|x\|_{\ell_1}$, so that $\|x\| = \sup_n |x|_n$ is a well-defined norm on c_{00} making the canonical c_{00} basis normalized and 1-unconditional satisfying the implicit equation

$$\|x\| = \|x\|_{\mathcal{G}_0} \vee \sup_{m \in \mathbb{N}} \sup\Big\{\theta_m \sum_{i=1}^k \|P_{E_i} x\| : (E_i)_{i=1}^k \text{ is } \mathcal{G}_m\text{-admissible}\Big\}.$$

We let $T(\mathcal{G}_0, (\theta_n, \mathcal{G}_n))$ denote the completion of c_{00} with respect to this norm.

In the special case where this space is built from a single family $\mathcal{G}$ and a single constant $\theta \in (0,1)$, we denote the resulting space by $T(\theta,\mathcal{G})$. This occurs when $\mathcal{G}_0 = \mathcal{S}_0$ and for each $n \in \mathbb{N}$, $\mathcal{G}_n = \mathcal{G}$ and $\theta_n = \theta$. This space coincides with the usual Tsirelson space $T_{\xi,\theta}$ when $\mathcal{G} = \mathcal{S}_\xi$, and is isomorphic to either c_0 or ℓ_p for some $p > 1$ if $\mathcal{G} = \mathcal{F}_n$ for some $n \in \mathbb{N}$ [3]. We will use the following results. Item (ii) comes from [15], and (iii) comes from [13].

Proposition 6.1. *Fix regular families $(\mathcal{G}_n)_{n\geq 0}$ such that $\mathcal{G}_0$ contains all singletons and constants $1 > \theta_n \searrow 0$. Let $T = (\mathcal{G}_0, (\theta_n, \mathcal{G}_n))$.*

(i) *For any $0 \leq k$ and $m \in \mathbb{N}$, we have $I_w(\mathcal{H}^T_{\theta_m^k}) \geq \iota(\mathcal{G}_0)\iota(\mathcal{G}_m)^k$.*
(ii) *If $\iota(\mathcal{G}_0) \geq \sup_n \iota(\mathcal{G}_n)^\omega$, then $\mathrm{Sz}(T) = \iota(\mathcal{G}_0)\sup_n \iota(\mathcal{G}_n)^\omega$.*
(iii) *For any $\theta \in (0,1)$, any $\xi < \omega_1$, and any $M \in [\mathbb{N}]^\omega$,*
$$\mathrm{Sz}(T(\theta, M^{-1}(\mathcal{S}_\xi))) = \omega^{\xi\omega}.$$
(iv) *For any $\theta \in (0,1)$ and any $n \in \mathbb{N}$, we have $\mathrm{Sz}(T(\theta,\mathcal{F}_n)) = \omega$.*

Proof. (i) One can easily show by induction on k that if $E \in [\mathcal{G}_n]^k[\mathcal{G}_0]$, then for any scalars $(a_i)_{i\in E}$,
$$\Big\|\sum_{i\in E} a_i e_i\Big\| \geq \theta_n^k \sum_{i\in E}|a_i|.$$
Once we establish that the basis of T is shrinking, which we will do below, this will imply that $(e_{\max E})_{E\in\widehat{[\mathcal{G}_n]^k[\mathcal{G}_0]}}$ is a normalized weakly null tree with branches in $\mathcal{H}^T_{\theta_n^k}$. This guarantees that $I_w(\mathcal{H}^T_{\theta_n^k}) > \iota([\mathcal{G}_n]^k[\mathcal{G}_0]) = \iota(\mathcal{G}_0)\iota(\mathcal{G}_n)^k$.

For (ii)–(iv), we must first define the Bourgain ℓ_1 block index of a basis, introduced in [4]. Given a Banach space X with basis (e_i), for $K \geq 1$ we let
$$\begin{aligned}\mathcal{T}(X,(e_i),K) = \Big\{(x_i)_{i=1}^n \in \Sigma((e_i),X) : n \in \mathbb{N},\\ \forall (a_i)_{i=1}^n \subset \mathbb{R},\ K\Big\|\sum_{i=1}^n a_i x_i\Big\| \geq \sum_{i=1}^n |a_i|,\\ \forall 1 \leq i \leq n, \|x_i\| \leq 1\Big\}.\end{aligned}$$
With the order o as defined in Section 3.1, we define
$$B(X,(e_i),K) = o(\mathcal{T}(X,(e_i),K)), \qquad B(X,(e_i)) = \sup_{K\geq 1} B(X,(e_i),K).$$
We recall that ℓ_1 embeds into X if and only if $B(X,(e_i)) = \omega_1$. Moreover, if (e_i) is 1-unconditional, and $I_w(\mathcal{H}^X_\varepsilon) > \xi$, as discussed in Example 3.6, we can replace ε with any strictly smaller number δ and use a standard perturbation argument to find a block tree $(x_E)_{E\in\widehat{\mathcal{F}}_\xi} \subset B_X$ with branches

in $\mathcal{H}_\delta^X$. By 1-unconditionality, for all $E \in \widehat{\mathcal{F}}_\xi$ and scalars $(a_i)_{i=1}^{|E|}$,

$$\delta^{-1}\Big\|\sum_{i=1}^{|E|} a_i x_{E|_i}\Big\| \geq \sum_{i=1}^{|E|} |a_i|.$$

One then shows by induction that for each $0 \leq \zeta \leq \xi$,

$$(x_E)_{E\in\widehat{\mathcal{F}_\xi^\zeta}} \subset d^\zeta(\mathcal{T}(X,(e_i),\delta^{-1})),$$

whence

$$I_w(\mathcal{H}_\varepsilon^X) \leq B(X,(e_i),\delta^{-1}) \leq B(X,(e_i)).$$

By [15], $B(T,(e_i)) < \omega_1$, so that ℓ_1 does not embed into T for any choice $(\mathcal{G}_n)_{n\geq 0}$, $1 > \theta_n \to 0$. By [13], $B(T(\theta,\mathcal{S}_\xi)) = \omega^{\xi\omega}$. Since $T(\theta,\mathcal{F}_n)$ is isomorphic to either c_0 or ℓ_p for some $p > 1$, we deduce that none of these spaces contains ℓ_1, and the basis of each is shrinking. For (ii) and (iv), it remains to note that $B(T,(e_i)) = \iota(\mathcal{G}_0)\sup_{k,n}\iota(\mathcal{G}_n)^k$ [15], and $B(\ell_p,(e_i)) = B(c_0,(e_i)) = \omega$ for $p > 1$. For (iii), we note that $\mathrm{Sz}(T(\theta, M^{-1}(\mathcal{S}_\xi))) \geq \omega^{\xi\omega}$ by (i). It is easy to see that the sequence (e_n) in $T(\theta, M^{-1}(\mathcal{S}_\xi))$ is isometrically equivalent to (e_{m_n}) in $T(\theta,\mathcal{S}_\xi)$ by proving by induction that they are isometrically equivalent with respect to each norm $|\cdot|_n$ in the definitions of these spaces. Therefore

$$\mathrm{Sz}(T(\theta, M^{-1}(\mathcal{S}_\xi))) \leq B(T(\theta, M^{-1}(\mathcal{S}_\xi)),(e_i)) \leq B(T(\theta,S_\xi),(e_i)) = \omega^{\xi\omega}. \ \blacksquare$$

With this, we arrive at a characterization of the countable ordinals which occur as the Szlenk index of a Banach space. We note that in [15], the corresponding result for the Bourgain ℓ_1 index was established, and the result below only requires a minor modification of their result combined with Lancien's result in [14] that the Szlenk index, when countable, is separably determined.

THEOREM 6.2. *Let $1 \leq \xi < \omega_1$ be an ordinal. The following are equivalent:*

(i) *There exists a Banach space X with* $\mathrm{Sz}(X) = \omega^\xi$.
(ii) *There exists a mixed Tsirelson space T with* $\mathrm{Sz}(T) = \omega^\xi$.
(iii) *There does not exist a limit ordinal ζ such that $\xi = \omega^\zeta$.*

Proof. We consider several cases.

CASE 1: $\xi = 0$. Then $\mathrm{Sz}(X) = 1 = \omega^0$ for any finite-dimensional X.

CASE 2: $\xi = 1$. Then $\mathrm{Sz}(T(1/2,\mathcal{F}_1)) = \omega$.

CASE 3: $\xi = \omega^{\zeta+1}$. Then $\mathrm{Sz}(T(1/2,\mathcal{S}_{\omega^\zeta})) = \omega^{\omega^\zeta\omega} = \omega^{\omega^{\zeta+1}}$.

CASE 4: $\xi = \omega^\zeta$, ζ a limit ordinal. There is no Banach space with this Szlenk index by Theorem 5.17.

CASE 5: ξ is not of any of the forms mentioned above. Then write the Cantor normal form of ξ as $\xi = \omega^{\alpha_1} n_1 + \cdots + \omega^{\alpha_k} n_k$. Let $\zeta = \omega^{\alpha_1} n_1 + \cdots + \omega^{\alpha_k}(n_k - 1)$ and let $\eta = \omega^{\omega^{\alpha_k}}$. Then $\omega^\zeta \eta = \omega^\xi$. Moreover, $\beta^\omega \leq \eta$ for any $\beta < \eta$. Take $\beta_n \uparrow \eta$ and note $\zeta \geq \sup_n \beta_n^\omega$, so

$$\mathrm{Sz}(T(\mathcal{S}_\zeta, (2^{-n}, \mathcal{F}_{\beta_n}))) = \iota(\mathcal{S}_\zeta) \sup_n \iota(\mathcal{F}_{\beta_n})^\omega = \omega^\zeta \eta = \omega^\xi. \blacksquare$$

6.2. Mixed Tsirelson spaces as upper envelopes

THEOREM 6.3. *If X is an infinite-dimensional Banach space with shrinking FDD E, then there exists a mixed Tsirelson space T such that* $\mathrm{Sz}(X) = \mathrm{Sz}(T)$ *and a blocking F of E which satisfies subsequential T upper block estimates in X.*

Proof. This is a modification of the proof of [7, Theorem 5.5]. Let $\mathrm{Sz}(X) = \omega^\xi$.

STEP 1: We claim that for any $\rho \in (0,1)$, we can find some $0 = m_0 < m_1 < \cdots$ and regular families $(\mathcal{K}_n)_{n \geq 0}$ such that if $M = (m_n)_{n\geq 1}$ and if $F_n = [E_k]_{m_{n-1} < k \leq m_n}$, then for any $n \in \mathbb{N}$ and any $(x_i)_{i=1}^k \in \Sigma(F, X) \cap \mathcal{H}^X_{\rho^{n-1}}$, we have $[m_n, \infty) \cap (m_{\max \mathrm{supp}_F(x_i)})_{i=1}^k \in \mathcal{K}_n[\mathcal{K}_0]$. Note that if $\mathcal{G}_n = M^{-1}(\mathcal{K}_n)$, this condition implies that if $(x_i)_{i=1}^k \in \Sigma(F, X) \cap \mathcal{H}^X_{\rho^{n-1}}$, then $[n, \infty) \cap (\max \mathrm{supp}_F(x_i))_{i=1}^k \in \mathcal{G}_n[\mathcal{G}_0]$.

STEP 2: We prove that with these choices, if $\theta \in (\rho, 1)$, then F satisfies subsequential T upper block estimates in X.

We first complete Step 2 and then return to Step 1. Let (x_i) be a normalized block sequence with respect to F. Let $l_i = \max \mathrm{supp}_F(x_i)$. Choose $a = (a_i) \in c_{00}$ and let $x = \sum a_i x_i$. Choose $x^* \in S_{X^*}$ so that $x^*(x) = \|x\|$. For each $n \geq 1$, let

$$\begin{aligned} A_n &= \{i \in \mathrm{supp}(a) : i < n,\ \rho^{n-1} \leq |x^*(x_i)| < \rho^{n-2}\}, \\ B_n^+ &= \{i \in \mathrm{supp}(a) : i \geq n,\ \rho^{n-1} \leq x^*(x_i) < \rho^{n-2}\}, \\ B_n^- &= \{i \in \mathrm{supp}(a) : i \geq n,\ \rho^{n-1} \leq -x^*(x_i) < \rho^{n-2}\}. \end{aligned}$$

Since $\rho^{n-1} \leq x^*(x_i)$ for each $i \in B_n^+$, we have $(x_i)_{i\in B_n^+} \in \Sigma(F, X) \cap \mathcal{H}^X_{\rho^{n-1}}$. Since $n \leq B_n^+$ and $l_i \geq i$,

$$(l_i)_{i\in B_n^+} = [n, \infty) \cap (l_i)_{i\in B_n^+} \in \mathcal{G}_n[\mathcal{G}_0].$$

This means

$$\theta^n \sum_{i\in B_n^+} |a_i| \leq \Big\| \sum_{i\in B_n^+} a_i e_{l_i} \Big\|_T \leq \Big\| \sum a_i e_{l_i} \Big\|_T.$$

Similarly,

$$\theta^n \sum_{i\in B_n^-} |a_i| \leq \Big\| \sum a_i e_{l_i} \Big\|_T.$$

Last, since (e_i) is normalized and 1-unconditional, and since $|A_n| < n$,

$$\sum_{i\in A_n} |a_i| \le (n-1)\|a\|_\infty \le (n-1)\Big\|\sum a_i e_{l_i}\Big\|_T.$$

Since $\{A_n, B_n^+, B_n^- : n \in \mathbb{N}\}$ partitions $\{i \in \mathrm{supp}(a) : x^*(x_i) \neq 0\}$, we get

$$\begin{aligned}\|x\| &= \sum_{n=1}^\infty \Big[\sum_{i\in A_n} a_i x^*(x_i) + \sum_{i\in B_n^\pm} a_i x^*(x_i)\Big]\\ &\le \sum_{n=1}^\infty \Big(\sum_{i\in A_n} |a_i|\rho^{n-2} + \sum_{i\in B_n^\pm} |a_i|\rho^{n-2}\Big)\\ &\le \Big\|\sum a_i e_{l_i}\Big\|_T \sum_{n=1}^\infty ((n-1)\rho^{n-2} + 2\rho^{n-2}\theta^{-n})\\ &= \Big(\frac{1}{(1-\rho)^2} + \frac{2\rho^{-1}}{\theta-\rho}\Big)\Big\|\sum a_i e_{l_i}\Big\|_T.\end{aligned}$$

We now complete Step 1. We will choose $(\mathcal{K}_n)_{n\ge 0}$ according to the following cases:

Case 1: $\xi = 1$. Choose $\mathcal{K}_0 = \mathcal{S}_0$ and for $n > 0$, $\mathcal{K}_n = [\mathcal{F}_s]^n$ for some $s \in \mathbb{N}$. In this case, for any $\theta \in (0,1)$, $T = T(\mathcal{G}_0, (\theta^n, \mathcal{G}_n)) = T(\theta, \mathcal{F}_s)$ has $\mathrm{Sz}(T) = \omega$.

Case 2: $\xi = \omega^{\zeta+1} = \omega^\zeta \omega$. Choose $\mathcal{K}_0 = \mathcal{S}_0$ and for $n > 0$, $\mathcal{K}_n = [\mathcal{S}_{\omega^\zeta s}]^n$ for some $s \in \mathbb{N}$. Then, for any $\theta \in (0,1)$, $T = T(\mathcal{G}_0, (\theta^n, \mathcal{G}_n)) = T(\theta, \mathcal{G}_1) = T(\theta, M^{-1}(\mathcal{S}_{\omega^\zeta s}))$ has $\mathrm{Sz}(T) = \omega^{\omega^\zeta \omega} = \omega^\xi$.

Case 3: ξ is of neither of the forms mentioned above. We write the Cantor normal form of ξ as $\xi = \omega^{\alpha_1} n_1 + \cdots + \omega^{\alpha_k} n_k$. Then we let $\zeta = \omega^{\alpha_1} n_1 + \cdots + \omega^{\alpha_k}(n_k - 1)$ and $\beta = \omega^{\omega^{\alpha_k}}$, so $\omega^\zeta \beta = \omega^\xi$. We choose $\beta_n \uparrow \beta$ and have $\mathcal{K}_0 = \mathcal{S}_\zeta$ and $\mathcal{K}_n = \mathcal{F}_{\beta_n}$ for $n > 0$. Then for any $\theta \in (0,1)$, $T = T(\mathcal{G}_0, (\theta^n, \mathcal{G}_n))$ is such that $\mathrm{Sz}(T) = \omega^\xi$, since $\iota(\mathcal{G}_0) = \omega^\zeta \ge \beta = \sup_{n\in\mathbb{N}} \iota(\mathcal{G}_n)^\omega = \sup_{n\in\mathbb{N}} \beta_n^\omega$. This is because in this case, since $\xi = \omega^\zeta$ with ζ a limit ordinal is impossible, we have $\zeta \ge \beta_0^\omega$ for any $\beta_0 < \beta$.

In each case, $\mathrm{Sz}(T) = \mathrm{Sz}(X)$. Let $2\delta_n = \rho^{n-1} + \rho^n$ and $2\mu_n = \rho^{n-1} - \rho^n$. For each $n \in \mathbb{N}$, let

$$\mathcal{B}_n = \Sigma(E, X) \cap \mathcal{H}^X_{\delta_n}$$

and choose $\overline{\varepsilon}_n = (\varepsilon_{i,n})_i$ non-increasing so that $\sum_i \varepsilon_{i,n} < \mu_n$. Observe that $(\mathcal{B}_n)^{E,X}_{\overline{\varepsilon}_n} \subset \mathcal{H}^X_{\rho^n}$. By Proposition 5.7, this implies

$$\iota(\tilde{\mathcal{B}_n}) \le 2I_{\mathrm{bl}}((\mathcal{B}_n)^{E,X}_{\overline{\varepsilon}_n}) \le 2I_w(\mathcal{H}^X_{\rho^n}) < \mathrm{Sz}(X).$$

Here we have used the fact that E is shrinking, so bounded block sequences in E are weakly null. If $\mathrm{Sz}(X) = \omega$, then choose $s \in \mathbb{N}$ so that $I_w(\mathcal{H}^X_\rho) < s/2$

and note that $I_w(\mathcal{H}^X_{\rho^n}) < (s/2)^n$ for all $n \in \mathbb{N}$ by Theorem 5.17. This means that $2I_w(\mathcal{H}^X_{\rho^n}) < s^n$ for all $n \in \mathbb{N}$. If $\mathrm{Sz}(X) = \omega^{\omega^{\zeta+1}} = \omega^{\omega^\zeta \omega}$, choose $s \in \mathbb{N}$ so that $I_w(\mathcal{H}^X_\rho) < \omega^{\omega^\zeta s}$ and note that $I_w(\mathcal{H}^X_{\rho^n}) < \omega^{\omega^\zeta sn}$ for each $n \in \mathbb{N}$. This means that $2I_w(\mathcal{H}^X_{\rho^n}) < \omega^{\omega^\zeta sn}$ for all $n \in \mathbb{N}$. In the third case, with ζ, β as in Case 3 above, choose any $\beta_n < \beta$ such that $I_w(\mathcal{H}^X_{\rho^n}) < \omega^\zeta \beta_n$. Let $\mathcal{K}_n$ be as given in the cases above. We no longer need to distinguish between the three cases.

Let $M_0 = \mathbb{N}$ and, using Theorem 3.3, recursively choose $M_n \in [\mathbb{N}]^\omega$ so that for each $n \in \mathbb{N}$, $M_n \in [M_{n-1}]^\omega$ and either

$$\tilde{\mathcal{B}_n} \cap [M_n]^{<\omega} \subset \mathcal{K}_n[\mathcal{K}_0] \quad \text{or} \quad \mathcal{K}_n[\mathcal{K}_0] \cap [M_n]^{<\omega} \subset \tilde{\mathcal{B}_n}.$$

But in each case,

$$\iota(\mathcal{K}_n[\mathcal{K}_0] \cap [M_n]^{<\omega}) = \iota(\mathcal{K}_0)\iota(\mathcal{K}_n) > \iota(\tilde{\mathcal{B}_n}),$$

so the first containment always holds. Choose $m_n \in M_n$ with $m_1 < m_2 < \cdots$, set $M = (m_n)_{n\ge1}$, and let $m_0 = 0$. With $F_n = [E_k]_{m_{n-1}<k\le m_n}$, to finish, we only need to show that for $n \in \mathbb{N}$ and $(x_i)_{i=1}^k \in \Sigma(F, X) \cap \mathcal{H}^X_{\rho^{n-1}}$, we have $[m_n, \infty) \cap (m_{\max \mathrm{supp}_F(x_i)})_{i=1}^k \in \mathcal{K}_n[\mathcal{K}_0]$. Again, let $l_i = \max \mathrm{supp}_F(x_i)$. We can find a small perturbation $(y_i)_{i=1}^k \subset B_X$ of $(x_i)_{i=1}^k$ such that

(i) $\|y_i - x_i\| < \mu_n$,
(ii) $\mathrm{ran}_F(y_i) = \mathrm{ran}_F(x_i)$,
(iii) $m_{l_i} = \max \mathrm{supp}_E(y_i)$.

The first two items guarantee that $(y_i)_{i=1}^k \in \Sigma(E, X) \cap \mathcal{H}^X_{\rho^{n-1}-\mu_n} = \Sigma(E, X) \cap \mathcal{H}^X_{\delta_n}$. The last two items guarantee that

$$(m_{l_i})_{i=1}^k = (m_{\max \mathrm{supp}_F(y_i)})_{i=1}^k = (\max \mathrm{supp}_E(y_i))_{i=1}^k \in \tilde{\mathcal{B}_n}.$$

Combining these gives $(m_{l_i})_{i=1}^k \in \tilde{\mathcal{B}_n}$. But $[m_n, \infty) \cap (m_{\max \mathrm{supp}_F(y_i)})_{i=1}^k \in [M_n]^{<\omega}$, so that

$$[m_n, \infty) \cap (m_{\max \mathrm{supp}_F(y_i)})_{i=1}^k \in \tilde{\mathcal{B}_n} \cap [M_n]^{<\omega} \subset \mathcal{K}_n[\mathcal{K}_0]. \ \blacksquare$$

6.3. Universal spaces. If $\mathcal{C}$ is a class of Banach spaces, we say the Banach space U is *universal* for $\mathcal{C}$ if every member of $\mathcal{C}$ is isomorphic to a subspace of U. We say U is *surjectively universal* for $\mathcal{C}$ if every member of $\mathcal{C}$ is isomorphic to a quotient of U.

For each $0 \le \xi < \omega_1$, let $\mathcal{C}_\xi$ denote the class of separable Banach spaces with Szlenk index not exceeding ω^ξ. In [9], it was shown that for each ξ, there exists a Banach space Y_ξ having separable dual which is universal for $\mathcal{C}_\xi$. The results there were obtained using descriptive set theory, without an estimate on $\mathrm{Sz}(Y_\xi)$. In [10], it was shown that for each $\xi < \omega_1$, Y_ξ can be taken to be in $\mathcal{C}_{\zeta+1}$, where $\zeta = \min\{\eta\omega : \eta\omega \ge \zeta\}$. In [7], the following estimate was obtained.

THEOREM 6.4. *For every $1 \leq \xi < \omega_1$, there exists a Banach space $Z_\xi \in \mathcal{C}_{\xi+1}$ which is both universal and surjectively universal for $\mathcal{C}_\xi$.*

It was not stated in [7] that this space is surjectively universal for the class $\mathcal{C}_\xi$, but it is contained within the proof. It was shown there that if X is a separable Banach space with $\mathrm{Sz}(X) \leq \omega^\xi$, then X is isomorphic to a quotient of a Banach space Y which embeds complementably in Z_ξ, so X is isomorphic to a quotient of Z_ξ. Our goal here is to prove that this result is optimal.

THEOREM 6.5. *For any $\xi < \omega_1$, there does not exist a member of $\mathcal{C}_\xi$ which is universal or surjectively universal for $\mathcal{C}_\xi$.*

In [8], it was shown that if $\xi < \omega_1$ and if $\mathcal{CR}_\xi$ denotes the class of separable, reflexive Banach spaces X with $\mathrm{Sz}(X), \mathrm{Sz}(X^*) \leq \omega^\xi$, then there exists a Banach space $Z \in \mathcal{CR}_{\xi+1}$ which is universal and surjectively universal for $\mathcal{CR}_\xi$. In the proof of 6.5, we will show that if $Z \in \mathcal{C}_\xi$, then there exists $X \in \mathcal{C}_\xi$ which is not isomorphic to any quotient of any subspace of Z. If $\xi > 0$, this space will be a mixed Tsirelson space. In the proof, we will have the freedom to choose the families used in the construction of X so that X is reflexive and $\mathrm{Sz}(X^*) = \omega$, so that actually $X \in \mathcal{CR}_\xi$. Therefore we will in fact prove that if Z is either universal or surjectively universal for $\mathcal{CR}_\xi$, then $Z \notin \mathcal{C}_\xi$, and the result in [8] concerning the existence of a member of $\mathcal{CR}_{\xi+1}$ universal for $\mathcal{CR}_\xi$ is also optimal.

Of course, the $\xi = 0$ cases of Theorems 6.4 and 6.5 are trivial, since $\mathrm{Sz}(X) = 1 = \omega^0$ if and only if X is finite-dimensional. We outline the idea for each of the other cases. We note that for each $p > 1$, $\mathrm{Sz}(\ell_p) = \omega$. Moreover, a separable Banach space X has $\mathrm{Sz}(X) = \omega$ if and only if for some $p > 1$, every normalized weakly null tree on X indexed by $\widehat{[\mathbb{N}]^{<\omega}}$ has a branch which is dominated by the ℓ_p basis. This means the ℓ_p spaces, $p > 1$, form a sort of upper envelope for $\mathcal{C}_1$. But among these spaces, no one sits atop all the others. To see how this can be generalized to $\mathcal{C}_\xi$, we use the following

PROPOSITION 6.6. *Let Z be a Banach space having separable dual.*

(i) *If X is isomorphic to a subspace of Z, then there exists $K > 0$ such that $I_w(\mathcal{H}^X_\varepsilon) \leq I_w(\mathcal{H}^Z_{\varepsilon/K})$ for any $\varepsilon \in (0,1)$.*
(ii) *If X is isomorphic to a quotient of Z, then there exists $K > 0$ such that $I_w(\mathcal{H}^X_\varepsilon) \leq I_w(\mathcal{H}^Z_{\varepsilon/K})$ for any $\varepsilon \in (0,1)$.*
(iii) *If X is a Banach space having separable dual and such that $I_w(\mathcal{H}^X_{2^{-n}}) > I_w(\mathcal{H}^Z_{3^{-n}})$ for all $n \in \mathbb{N}$, then X is not isomorphic to any subspace of any quotient of Z.*

Proof. (i) Let $T : X \to Z$ be an isomorphic embedding and fix $a, b > 0$ so that

$$a^{-1}\|x\| \le \|Tx\| \le b\|x\|$$

for all $x \in X$. Let $K = ab$. If $\xi < I_w(\mathcal{H}_\varepsilon^X)$ and if $(x_E)_{E\in\widehat{\mathcal{F}}_\xi} \subset B_X$ is a weakly null tree with branches in $\mathcal{H}_\varepsilon^X$, then one easily checks that $(b^{-1}Tx_E)_{E\in\widehat{\mathcal{F}}_\xi} \subset B_Z$ is a weakly null tree with branches in $\mathcal{H}_{\varepsilon/K}^Z$.

(ii) First we assume X is isometrically a quotient of Z, and then apply part (i). Let $T : Z \to X$ be a quotient map. Then if $\xi < I_w(\mathcal{H}_\varepsilon^X)$, fix $(x_E)_{E\in\widehat{\mathcal{F}}_\xi} \subset B_X$ weakly null with branches in $\mathcal{H}_\varepsilon^X$. By applying Proposition 2.1 and Lemma 3.5, for any $0 < \delta < \varepsilon$, we can find a pruned subtree $(y_E)_{E\in\widehat{\mathcal{F}}_\xi}$ of $(x_E)_{E\in\widehat{\mathcal{F}}_\xi}$ and a weakly null tree $(z_E)_{E\in\widehat{\mathcal{F}}_\xi} \subset 3B_Z$ such that $\|Tz_E - y_E\| < \varepsilon/2$. This implies that $(3^{-1}Tz_E)_{E\in\widehat{\mathcal{F}}_\xi} \subset B_X$ has branches lying in $\mathcal{H}_{\varepsilon/6}^X$. Since T is norm 1, the weakly null tree $(3^{-1}z_E)_{E\in\widehat{\mathcal{F}}_\xi} \subset B_Z$ has branches lying in $\mathcal{H}_{\varepsilon/6}^Z$, and $I_w(\mathcal{H}_{\varepsilon/6}^Z) > \xi$.

(iii) If X is isomorphic to a subspace of a quotient of Z, then there exists K such that $I_w(\mathcal{H}_\varepsilon^X) \le I_w(\mathcal{H}_{\varepsilon/K}^Z)$ for each $\varepsilon \in (0,1)$. If we choose n so large that $2^{-n}K > 3^{-n}$, then

$$I_w(\mathcal{H}_{3^{-n}}^Z) < I_w(\mathcal{H}_{2^{-n}}^X) \le I_w(\mathcal{H}_{2^{-n}/K}^Z) \le I_w(\mathcal{H}_{3^{-n}}^Z),$$

a contradiction. ∎

Proof of Theorem 6.5. Case 1: $\xi = 1$. Suppose $Z \in \mathcal{C}_1$. Then $I_w(\mathcal{H}_{3^{-1}}^Z) < k$ for some $k < \omega$. By Theorem 5.17, for each $n \in \mathbb{N}$, we have $I_w(\mathcal{H}_{3^{-n}}^Z) < k^n$. But the Tsirelson space $T = T(2^{-1}, \mathcal{F}_k)$ has $k^n < I_w(\mathcal{H}_{2^{-n}}^T) < \omega$ for each $n \in \mathbb{N}$. This means $T \in \mathcal{C}_1$ cannot be isomorphic to a subspace of a quotient of Z.

Case 2: $\xi = \omega^{\gamma+1}$. Suppose $Z \in \mathcal{C}_{\omega^{\gamma+1}}$. Then $I_w(\mathcal{H}_{3^{-1}}^Z) < \omega^{\omega^{\gamma+1}} = \omega^{\omega^\gamma \omega}$. This means there exists $k \in \mathbb{N}$ such that $I_w(\mathcal{H}_{3^{-1}}^Z) < \omega^{\omega^\gamma k}$. By Theorem 5.17, $I_w(\mathcal{H}_{3^{-n}}^Z) < \omega^{\omega^\gamma kn}$ for each $n \in \mathbb{N}$. But for any $n \in \mathbb{N}$, the Tsirelson space $T = T(2^{-1}, \mathcal{S}_{\omega^\gamma k})$ has $\omega^{\omega^\gamma kn} < I_w(\mathcal{H}_{2^{-n}}^T)$. This means $T \in \mathcal{C}_{\omega^{\gamma+1}}$ cannot be isomorphic to a subspace of a quotient of Z.

Case 3: $\xi = \omega^\gamma$, γ a limit ordinal. By Theorem 6.2, $\sup_{X\in\mathcal{C}_\xi} \mathrm{Sz}(X) = \omega^\xi$. Therefore if Z is either universal or surjectively universal for $\mathcal{C}_\xi$, then $\mathrm{Sz}(Z) \ge \omega^\xi$. But again by Theorem 6.2, there is no Banach space with this Szlenk index, so $\mathrm{Sz}(Z) > \omega^\xi$.

Case 4: ξ is not of any of the forms above. Then write the Cantor normal form of ξ as $\xi = \omega^{\alpha_1} n_1 + \cdots + \omega^{\alpha_k} n_k$. In this case, let $\zeta = \omega^{\alpha_1} n_1 + \cdots + \omega^{\alpha_k}(n_k - 1)$ and $\eta = \omega^{\alpha_k}$. Fix $Z \in \mathcal{C}_\xi$. Then there exists a sequence $(\beta_n) \subset [0, \omega^\eta)$ such that $I_w(\mathcal{H}_{3^{-n}}^Z) < \omega^\zeta \beta_n$. But for each $n \in \mathbb{N}$, the mixed Tsirelson space $T = T(\mathcal{S}_\zeta, (2^{-n}, \mathcal{F}_{\beta_n}))$ satisfies $\omega^\zeta \beta_n < I_w(\mathcal{H}^T_{2^{-n}})$. Then $T \in \mathcal{C}_\xi$ cannot be isomorphic to a subspace of a quotient of Z. ∎

6.4. Injective tensor products. If X, Y are Banach spaces, we can consider the tensor product $X \otimes Y$ as a collection of finite rank operators mapping Y^* into X. That is, given $u = \sum_{i=1}^n x_i \otimes y_i$, $u(y^*) = \sum_{i=1}^n y^*(y_i)x_i$. Of course, the adjoint u^* of u maps X^* into the image of Y in Y^{**} under the canonical embedding, so we can equally well consider u as a map from X^* into Y. We can endow $X \otimes Y$ with the operator norm and let $X \hat{\otimes}_\varepsilon Y$ denote the completion of $X \otimes Y$ with respect to the operator norm. Of course, $X \hat{\otimes}_\varepsilon Y$ is contained within the space of compact operators from Y^* to X.

If X has an FDD E, then for any compact $u : Y^* \to X$, $P^E_{[1,n]}u \to u$ with respect to the operator norm. This implies that if Y also has an FDD F, $P^E_{[1,n]}u(P^F_{[1,n]})^* \to u$ with respect to the norm topology. If E, F are shrinking FDDs for X, Y, respectively, then

$$H_n = \operatorname{span}\{E_k \otimes F_j : \max\{k, j\} = n\}$$

defines a shrinking FDD for the injective tensor product $X \hat{\otimes}_\varepsilon Y$ [7]. Showing that this forms an FDD is straightforward, while showing that this FDD is shrinking involves a characterization of weak nullity in injective tensor products given in [16]. For $u \in X \hat{\otimes}_\varepsilon Y$, the projection $P^H_{[1,n]}u$ is given by $P^E_{[1,n]}uP^{F^*}_{[1,n]}$. We think about such u as an infinite matrix whose j, k entry is a member of $E_j \otimes F_k$. In this case, the projections $P^H_{[1,n]}$ are the $n \times n$ leading principal minors of this infinite matrix. Then a block sequence (u_n) with respect to H can be considered as a sequence of square matrices such that there exist $0 = k_0 < k_1 < \cdots$ such that u_n is equal to its $k_n \times k_n$ leading principal minor, while its $k_{n-1} \times k_{n-1}$ leading principal minor is zero. In this case, we can write $u_n = r_n + c_n$ so that (r_n) is a sequence of successive rows and (c_n) is a sequence of successive columns. This simple decomposition leads to the following

Proposition 6.7 ([7]). *Suppose T is a Banach space with normalized 1-unconditional basis (e_n). Let X, Y be Banach spaces with shrinking bimonotone FDDs E, F such that E (resp. F) satisfies subsequential C-T upper block estimates in X (resp. Y). Then the FDD H for $X \hat{\otimes}_\varepsilon Y$ satisfies subrequential $2C$-T upper block estimates in $X \hat{\otimes}_\varepsilon Y$.*

Theorem 6.8. *For any separable non-zero Banach spaces X, Y,*

$$\operatorname{Sz}(X \hat{\otimes}_\varepsilon Y) = \max\{\operatorname{Sz}(X), \operatorname{Sz}(Y)\}.$$

For this, we will need the following simple fact.

Proposition 6.9. *If (e_n) is a shrinking, normalized, 1-unconditional basis for the space T, and if F is a shrinking FDD for the Banach space*

Z such that F satisfies subsequential C-T upper block estimates in Z, then $\mathrm{Sz}(Z) \le \mathrm{Sz}(T)$.

If $\xi < \mathrm{Sz}(Z)$, then we can find for some $\varepsilon > 0$ a block tree $(z_E)_{E \in \widehat{\mathcal{F}}_\xi}$ with branches in $\mathcal{H}^Z_\varepsilon$. If $t_E = e_{\max \mathrm{supp}_F(z_E)}$, then $(t_E)_{E \in \widehat{\mathcal{F}_\xi}} \subset B_T$ is a weakly null tree in T with branches lying in $\mathcal{H}^T_{\varepsilon/C}$. This means $\xi < \mathrm{Sz}(T)$.

Proof of Theorem 6.8. If either X^* or Y^* is non-separable, the result is clear. If either space is finite-dimensional, the result is immediate from Theorem 5.11, since in this case the tensor product is isomorphic to a finite direct sum where each summand is either X or Y. Assume $\mathrm{Sz}(X), \mathrm{Sz}(Y) < \omega_1$.

If X is a closed subspace of X_0, and Y is a closed subspace of Y_0, then $X \hat{\otimes}_\varepsilon Y$ is isomorphic to a subspace of $X_0 \hat{\otimes}_\varepsilon Y_0$. By a result of Schlumprecht [21], we can embed X, Y into Banach spaces X_0, Y_0 with shrinking bimonotone bases such that $\mathrm{Sz}(X) = \mathrm{Sz}(X_0)$ and $\mathrm{Sz}(Y) = \mathrm{Sz}(Y_0)$. Thus it suffices to assume that X, Y themselves have shrinking bimonotone bases. Then by Theorem 6.3, we can find Banach spaces T_X, T_Y with normalized 1-unconditional bases $(e^X_n), (e^Y_n)$ such that $\mathrm{Sz}(T_X) = \mathrm{Sz}(X)$ and $\mathrm{Sz}(T_Y) = \mathrm{Sz}(Y)$, and shrinking bimonotone FDDs E, F for X, Y, respectively, such that E satisfies subsequential T_X upper block estimates in X and F satisfies subsequential T_Y upper block estimates in Y. Then if $e_n = e^X_n + e^Y_n \in T_X \oplus_\infty T_Y$, E, F satisfy subsequential $[e_n]$ upper block estimates in X, Y, respectively. Therefore the FDD H is a shrinking FDD for $X \hat{\otimes}_\varepsilon Y$ satisfying subsequential $[e_n]$ upper block estimates in $X \hat{\otimes}_\varepsilon Y$. We deduce that

$$\begin{aligned}\mathrm{Sz}(X \hat{\otimes}_\varepsilon Y) &\le \mathrm{Sz}([e_n]) \le \mathrm{Sz}(T_X \oplus_\infty T_Y) \\ &= \max\{\mathrm{Sz}(T_X), \mathrm{Sz}(T_Y)\} \le \max\{\mathrm{Sz}(X), \mathrm{Sz}(Y)\}.\end{aligned}$$

Since X, Y both embed into $X \hat{\otimes}_\varepsilon Y$, the reverse inequality is clear. ∎

REMARK 6.10. It is unnecessary to take the direct sum $T_X \oplus_\infty T_Y$ in the previous proof. It is easy to see how to modify the proof of Theorem 6.3 to find one mixed Tsirelson space which can play the roles of both T_X and T_Y simultaneously.

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