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A note on weighted bounds for singular operators with nonsmooth kernels

by

THE ANH BUI (North Ryde), JOSÉ M. CONDE-ALONSO (Madrid), XUAN THINH DUONG (North Ryde) and MAHDI HORMOZI (Gothenburg and Shiraz)

Abstract. Let T be a multilinear integral operator which is bounded on certain products of Lebesgue spaces on $\mathbb{R}^n$. We assume that its associated kernel satisfies some mild regularity condition which is weaker than the usual Hölder continuity of kernels of multilinear Calderón–Zygmund singular integral operators. In this paper, given a suitable multiple weight $\vec{w}$, we obtain a bound for the weighted norm of T in terms of $\vec{w}$. As applications, we obtain new weighted bounds for certain singular integral operators such as linear and multilinear Fourier multipliers and the Riesz transforms associated to Schrödinger operators on $\mathbb{R}^n$.

1. Introduction. In the past decades, weighted inequalities have been a very attractive realm in harmonic analysis. One basic problem concerning them consists in determining conditions for a given operator to be bounded in $L^p(w)$ with an appropriate weight w. A sustained research period started with the famous work of Muckenhoupt [Mu] in the seventies. In that work he characterized the class of weights u, v such that the following weak inequality for the Hardy–Littlewood maximal operator M and for $1 \le p < \infty$ holds:

$$\|M(f)\|_{L^{p,\infty}(u)} \le C\|f\|_{L^p(v)}. \tag{1.1}$$

When $u = v = w$, this condition on the weights is known as the A_p *condition*:

$$[w]_{A_p} := \sup_Q \bigg(\frac{1}{|Q|}\int_Q w(x)\,dx\bigg)\bigg(\frac{1}{|Q|}\int_Q w(x)^{-1/(p-1)}\,dx\bigg)^{p-1} < \infty, \qquad p > 1,$$

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where the supremum is taken over all cubes (or balls) in $\mathbb{R}^n$. For $p > 1$, Muckenhoupt proved that the strong estimate

$$\int_{\mathbb{R}^n} (Mf(x))^p w(x)\,dx \le C \int_{\mathbb{R}^n} |f(x)|^p w(x)\,dx, \qquad f \in L^p(w),$$

holds if and only if w satisfies the A_p condition.

After that, harmonic analysts focused on studying weighted inequalities for many different classical operators such as the Hilbert and Riesz transforms and other singular integral operators, leading to a vast literature. However, the classical results did not reflect the quantitative dependence of the $L^p(w)$ operator norm on the relevant constant involving the weight. The question of the sharp dependence of the norm estimates of a given operator on the A_p constant of the weight was specifically raised by S. Buckley [Bu], who proved the following optimal bound for the Hardy–Littlewood operator:

$$\|M\|_{L^p(w)} \le C_p\,[w]_{A_p}^{1/(p-1)}, \tag{1.2}$$

where C_p is a dimensional constant that also depends on p, but not on w. The estimate in (1.2) is sharp in the sense that the exponent $1/(p-1)$ cannot be replaced by a smaller one.

On the other hand, it turned out that for singular integral operators the question was much more complicated. Linear bounds for the Hilbert and Riesz transforms were addressed by Petermichl [P1, P2]. Since then, the so-called A_2 conjecture attracted much attention. This conjecture states that the sharp dependence of the $L^2(w)$ norm of a Calderón–Zygmund operator on the A_2 constant of the weight w is linear. Finally, in 2012 T. Hytönen [Hyt1] proved the so-called A_2 theorem, which confirmed that conjecture. This, in combination with the extrapolation theorem of [DGPP], gives the sharp dependence of the $L^p(w)$ norm for Calderón–Zygmund operators with $1 < p < \infty$. More precisely, if T is a Calderón–Zygmund operator then

$$\|T\|_{L^p(w)} \le C_{T,n,p}[w]_{A_p}^{\max(1,1/(p-1))}, \qquad 1 < p < \infty,\ w \in A_p. \tag{1.3}$$

Shortly thereafter, A. K. Lerner [Ler4] gave a much simpler proof of the A_2 theorem proving that every Calderón–Zygmund operator is bounded from above by a supremum of sparse operators. Namely, if $\mathbb{X}$ is a Banach function space, then

$$\|T(f)\|_{\mathbb{X}} \le C \sup_{\mathscr{D},\mathcal{S}} \|\mathcal{A}_{\mathscr{D},\mathcal{S}}(f)\|_{\mathbb{X}}, \tag{1.4}$$

where the supremum is taken over arbitrary dyadic grids $\mathscr{D}$ and sparse families $\mathcal{S} \subset \mathscr{D}$, and

$$\mathcal{A}_{\mathscr{D},\mathcal{S}}(f) = \sum_{Q\in\mathcal{S}} \Big(\fint_Q f\Big)\chi_Q.$$

The interested readers can consult [Hyt2] for a survey on the history of the proof. Lerner's techniques were used in [DLP] to extend (1.4) and the A_2 theorem to multilinear Calderón–Zygmund operators. Later on, Li, Moen and Sun [LMS] proved the corresponding sharp weighted $A_{\vec{P}}$ bounds for multilinear sparse operators (here $\vec{P} = (p_1, \ldots, p_m)$). Namely, if $1 < p_1, \ldots, p_m < \infty$ with $1/p_1 + \cdots + 1/p_m = 1/p$ and $\vec{w} = (w_1, \ldots, w_m) \in A_{\vec{P}}$, then

$$\|\mathcal{A}_{\mathscr{D},\mathcal{S}}(\vec{f})\|_{L^p(\nu_{\vec{w}})} \lesssim [\vec{w}]_{A_{\vec{P}}}^{\max\{1, p_1'/p, \ldots, p_m'/p\}} \prod_{i=1}^{m} \|f_i\|_{L^{p_i}(w_i)} \tag{1.5}$$

for all tuples $\vec{f} = (f_1, \ldots, f_m)$. Here $\mathcal{A}_{\mathscr{D},S}$ denotes the multilinear sparse operator

$$\mathcal{A}_{\mathscr{D},S}(\vec{f})(x) = \sum_{Q} \Big(\prod_{i=1}^{m} (f_i)_Q \Big) \chi_Q(x).$$

From (1.5), we can derive the multilinear $A_{\vec{P}}$ theorem for $1/m < p < \infty$ (see [CR, LMS]). More precisely, if T is a multilinear Calderón–Zygmund operator, $1 < p_1, \ldots, p_m < \infty$, $1/p_1 + \cdots + 1/p_m = 1/p$ and $\vec{w} \in A_{\vec{P}}$, then

$$\|T(\vec{f})\|_{L^p(\nu_{\vec{w}})} \le C_{n,m,\vec{P},T} [\vec{w}]_{A_{\vec{P}}}^{\max\{1, p_1'/p, \ldots, p_m'/p\}} \prod_{i=1}^{m} \|f_i\|_{L^{p_i}(w_i)}. \tag{1.6}$$

For further details on the theory of multilinear Calderón–Zygmund operators, we refer to [G, GT] and the references therein.

In this paper, we study weighted bounds for certain multilinear singular integral operators on products of weighted Lebesgue spaces. It is important to note that the multilinear singular integral operators considered here are beyond the Calderón–Zygmund class of multilinear singular integral operators studied in [GT]. More precisely, we assume that T is a multilinear operator initially defined on the m-fold product of Schwartz spaces and taking values in the space of tempered distributions,

$$T : \mathcal{S}(\mathbb{R}^n) \times \cdots \times \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n).$$

The associated kernel $K(x, y_1, \ldots, y_m)$ is a function defined away from the diagonal $x = y_1 = \cdots = y_m$ in $(\mathbb{R}^n)^{m+1}$, satisfying

$$T(f_1, \ldots, f_m)(x) = \int\limits_{(\mathbb{R}^n)^m} K(x, y_1, \ldots, y_m) f_1(y_1) \ldots f_m(y_m)\, dy_1 \ldots dy_m$$

for all $f_j \in \mathcal{S}(\mathbb{R}^n)$ and all $x \notin \bigcap_{j=1}^{m} \operatorname{supp} f_j$, $j = 1, \ldots, m$.

In what follows, we denote $dy_1 \ldots dy_m$ by $d\vec{y}$. For the rest of this paper, we assume that there exist $p_0 \ge 1$ and a constant $C > 0$ such that:

(H1) T maps $L^{p_0} \times \cdots \times L^{p_0}$ into $L^{p_0/m,\infty}$.

(H2) There exists $\delta > n/p_0$ such that for the conjugate exponent p_0' of p_0, one has

$$\Big(\int_{S_{j_m}(Q)} \cdots \int_{S_{j_1}(Q)} |K(x,y_1,\dots,y_m) - K(\overline{x},y_1,\dots,y_m)|^{p_0'}\,d\vec{y}\Big)^{1/p_0'} \le C\frac{|x-\overline{x}|^{m(\delta-n/p_0)}}{|Q|^{m\delta/n}}2^{-m\delta j_0} \tag{1.7}$$

for all balls Q, all $x,\overline{x}\in\frac{1}{2}Q$ and $(j_1,\dots,y_m)\neq(0,\dots,0)$, where $j_0=\max\{j_k : k=1,\dots,m\}$ and $S_j(Q)=2^jQ\setminus 2^{j-1}Q$ if $j\ge 1$, otherwise $S_j(Q)=Q$.

Note that we do not require any size condition on the kernel of T. Considering the class of operators satisfying (H1) and (H2) is motivated by the recent works [KW, BD, GT, LO$^+$, LRT, LMRT, LMPR]. In the linear case of $m=1$, the class of such operators is contained implicitly in [KW]. Condition (H2) is similar to the L^r-Hörnander conditions considered in [LRT, LMRT, LMPR]. In the multilinear case, this kind of operators were considered by the first and third authors [BD] in studying weighted norm inequalities for multilinear Fourier multiplier operators with symbols of limited smoothness. More importantly, the class of operators satisfying (H1) and (H2) includes the class of multilinear Calderón–Zygmund singular integral operators (see [GT, LO$^+$] for the precise definition). More precisely, if T is a multilinear Calderón–Zygmund singular integral operator then it is easy to see that T satisfies (H1) and (H2) with $p_0=1$.

The main goal of this paper is to obtain weighted bounds for multilinear singular integrals which satisfy (H1) and (H2). According to a standard approach, it is natural to consider the following multi-sublinear operators. Fix $p_0\in[1,\infty)$ and a dyadic grid $\mathscr{D}\subset\mathbb{R}^n$. Define, for any cube Q,

$$\langle f\rangle_{Q,p_0} := \Big(\frac{1}{|Q|}\int_Q |f(x)|^{p_0}\,dx\Big)^{1/p_0}.$$

For $k\ge 0$, denote by $\mathcal{A}^{k,p_0}_{\mathscr{D},\mathcal{S}}$ the m-sublinear sparse operator

$$\mathcal{A}^{k,p_0}_{\mathscr{D},\mathcal{S}}(\vec{f})(x) = \sum_{Q\in\mathcal{S}}\Big[\prod_{i=1}^m \langle f_i\rangle_{Q^{(k)},p_0}\Big]\chi_Q(x),$$

which acts on measurable m-tuples $\vec{f}=(f_1,\dots,f_m)$. Here $Q^{(k)}$ denotes the kth dyadic ancestor of Q in $\mathscr{D}$. Also, we define an operator $\mathcal{T}^{k,p_0}_{\mathcal{S}}$ by

$$\mathcal{T}^{k,p_0}_{\mathcal{S}}(\vec{f})(x) = \sum_{Q\in\mathcal{S}}\Big[\prod_{i=1}^m \langle f_i\rangle_{2^kQ,p_0}\Big]\chi_Q(x).$$

We shall also work with the localized versions of the operators above, in which the sum in the definition ranges over cubes Q contained in some fixed cube P. We will denote them respectively by $\mathcal{A}^{k,p_0}_{\mathcal{S},P}$ and $\mathcal{T}^{k,p_0}_{\mathcal{S},P}$.

Our first main result reads as follows:

THEOREM 1.1. *Assume that the tuple $\vec{f}$ is compactly supported. Then for each sparse family $\mathcal{S}$ there exist sparse families $\mathcal{S}_j$, $1 \le j \le c_n$, such that*

$$\mathcal{T}^{k,p_0}_{\mathscr{D},\mathcal{S}}(\vec{f})(x) \le C(k+1)\sum_{j=1}^{c_n} \mathcal{A}^{0,p_0}_{\mathscr{D}^j,\mathcal{S}_j}(\vec{f})(x) \qquad a.e.,$$

for some constant C that may depend on m, but not on k or $\vec{f}$.

The proof of Theorem 1.1 follows the scheme of [CR], where the case $p_0 = 1$ is considered. The main new difficulty is that the operator $\mathcal{A}^{m,p_0}_{\mathcal{S}}$ is not linear for $p_0 \neq 1$. Of course, $\mathcal{A}^{m,p}_{\mathcal{S}}\vec{f} \ge \mathcal{A}^{m,q}_{\mathcal{S}}\vec{f}$ for positive tuples $\vec{f}$ and $p \ge q$. Therefore, bounding the operators $\mathcal{A}^{m,p_0}_{\mathcal{S}}$ for $p_0 > 1$ leaves some space for estimates involving Calderón–Zygmund operators with rough kernels. On the other hand, the operators $\mathcal{A}^{p_0}_{\mathcal{S}} := \mathcal{A}^{0,p_0}_{\mathcal{S}}$ have nice quantitative properties:

THEOREM 1.2. *Suppose that $p_0 < p_1, \dots, p_m < \infty$ with $1/p_1 + \dots + 1/p_m = 1/p$ and $\vec{w} \in A_{\vec{P}/p_0}$. Then*

$$\|\mathcal{A}^{p_0}_{\mathcal{S}}(\vec{f})\|_{L^p(\nu_{\vec{w}})} \lesssim [\vec{w}]_{A_{\vec{P}/p_0}}^{\max\{1,(p_1/p_0)'/p,\dots,(p_m/p_0)'/p\}} \prod_{i=1}^m \|f_i\|_{L^{p_i}(w_i)}.$$

This is all we need to get weighted bounds of our operators with non-smooth kernels.

THEOREM 1.3. *Let T satisfy* (H1) *and* (H2). *If f has compact support inside a cube Q_0, then we have the pointwise bound*

$$|T(\vec{f})(x)| \le C\sum_{j=1}^{c_n} \mathcal{A}^{0,p_0}_{\mathscr{D}^j,\mathcal{S}_j}(\vec{f})(x) \qquad \text{for a.e. } x \in Q_0.$$

Moreover, let $\mathbb{X}$ be a quasi Banach function space (*in the sense of* [CR]). *Then*

$$\|T(\vec{f})\|_{\mathbb{X}} \le C \sup_{\mathscr{D},\mathcal{S}} \|\mathcal{A}^{0,p_0}_{\mathscr{D},\mathcal{S}}(\vec{f})\|_{\mathbb{X}}$$

(*the supremum runs over dyadic systems $\mathscr{D}$ and sparse families $\mathcal{S}$*). *In particular, suppose that $p_0 < p_1, \dots, p_m < \infty$ with $1/p_1 + \dots + 1/p_m = 1/p$ and $\vec{w} \in A_{\vec{P}/p_0}$. Then*

$$\|T(\vec{f})\|_{L^p(\nu_{\vec{w}})} \lesssim [\vec{w}]_{A_{\vec{P}/p_0}}^{\max\{1,(p_1/p_0)'/p,\dots,(p_m/p_0)'/p\}} \prod_{i=1}^m \|f_i\|_{L^{p_i}(w_i)}.$$

We point out that our results are new even for the linear case. Although our conjecture is that these bounds are sharp, we have not been able to prove this and leave it as an open problem.

As applications of Theorem 1.3, we prove the following new results (see Section 5):

(a) Weighted bounds for the linear Fourier multipliers T_m with a symbol m of limited smoothness as in [KW]—see Theorem 5.3. Note that in this situation, T_m is not a Calderón–Zygmund operator.
(b) Weighted bounds for the Riesz transforms $\nabla L^{-1/2}$, where $L = -\Delta + V$ is a Schrödinger operator with potential $V \in RH_q$, $n/2 \leq q < n$—see Theorem 5.6. It is worth noticing that if $V \geq q \geq n$ then $\nabla L^{-1/2}$ is a Calderón–Zygmund operator. However, this is not so far $V \in RH_q, n/2 \leq q < n$ (see for example [S]);
(c) Weighted bounds for the multilinear Fourier multiplier T_m as in [BD]—see Theorem 5.10. Note that in this situation, the multilinear Fourier multiplier T_m cannot be a multilinear Calderón–Zygmund singular integral due to the limited smoothness imposed on m.

The outline of the rest of the paper is the following: In the next section we recall the definition of multiple weights and Lerner's local oscillation formula. Section 3 is devoted to proving Theorem 1.1. The proofs of Theorems 1.2 and 1.3 are given in Section 4. Finally, in Section 5, we apply Theorem 1.3 to obtain weighted bounds for certain singular integral operators such as linear and multilinear Fourier multipliers and Riesz transforms associated to Schrödinger operators.

Throughout, $A \lesssim B$ will denote $A \leq CB$, where C is a positive constant independent of the weight which may change from line to line. Moreover, $A \lesssim_{a,b} B$ will denote $A \leq CB$, where C is a positive constant depending on a and b.

2. Preliminaries

2.1. Multiple weight theory. For a general account on multiple weights and related results we refer the interested reader to [LO$^+$]. In this section we briefly introduce some definitions and results that we will need. Consider m weights $w_1, \dots, w_m$ and denote $\vec{w} = (w_1, \dots, w_m)$. Also let $1 < p_1, \dots, p_m < \infty$ and $1/m < p < \infty$ be numbers such that $1/p = 1/p_1 + \dots + 1/p_m$, and denote $\vec{P} = (p_1, \dots, p_m)$. Set

$$\nu_{\vec{w}} := \prod_{i=1}^{m} w_i^{p/p_i}.$$

We say that $\vec{w}$ satisfies the $A_{\vec{P}}$ *condition* if

$$(2.1)\qquad [\vec{w}]_{A_{\vec{P}}} := \sup_Q \left(\frac{1}{|Q|}\int_Q \nu_{\vec{w}}\right)\prod_{i=1}^m \left(\frac{1}{|Q|}\int_Q w_i^{1-p_i'}\right)^{p/p_i'} < \infty.$$

When $p_i = 1$ for some i, $(|Q|^{-1}\int_Q w_i^{1-p_i'})^{p/p_i'}$ is understood as $(\inf_Q w_i)^{-p}$. This condition, introduced in [LO$^+$], was shown to characterize the classes of weights for which the multilinear maximal function $\mathcal{M}$ is bounded from $L^{p_1}(w_1)\times\cdots\times L^{p_m}(w_m)$ into $L^p(\nu_{\vec{w}})$ (see [LO$^+$, Thm. 3.7]). We also denote by $A_p, 1 \le p < \infty$, and $RH_q, 1 < q \le \infty$, the classes of Muckenhoupt weights and of reverse Hölder weights on $\mathbb{R}^n$, respectively. For $w \in A_p, 1 \le p < \infty$, the quantity $[w]_{A_p}$ is defined by

$$[w]_{A_p} := \sup_Q \left(\frac{1}{|Q|}\int_Q w(x)\,dx\right)\left(\frac{1}{|Q|}\int_Q w(x)^{-1/(p-1)}\,dx\right)^{p-1},$$

with the usual modification when $p = 1$. The supremum above is taken over all cubes (or balls) in $\mathbb{R}^n$. For $w \in RH_q$, $1 < q \le \infty$, we define

$$[w]_{RH_q} := \sup_Q \left(\frac{1}{|Q|}\int_Q w(x)^q\,dx\right)^{1/q}\left(\frac{1}{|Q|}\int_Q w(x)\,dx\right)^{-1},$$

with the usual modification when $q = \infty$. Again, the supremum is taken over all cubes (or balls) in $\mathbb{R}^n$.

Let $\sigma \in A_\infty = \bigcup_{p\ge1} A_p$. The *dyadic maximal function* with respect to σ is defined as

$$(2.2)\qquad M_\sigma^{\mathscr{D}}(f)(x) = \sup_{\substack{x\in Q\\ Q\in\mathscr{D}}} \frac{1}{\sigma(Q)}\int_Q |f|\sigma.$$

It is well-known (see e.g. [Mo]) that

$$(2.3)\qquad \|M_\sigma^{\mathscr{D}} f\|_{L^p(\sigma)} \le p'\|f\|_{L^p(\sigma)}, \qquad 1 < p < \infty.$$

2.2. A local mean oscillation formula. For the notion of a *general dyadic grid* $\mathscr{D}$ we refer to previous works (e.g. [Ler2] and [Hyt2]). A collection $\mathcal{S} = \{Q\} \subset \mathscr{D}$ is called a *sparse family* of cubes if there exist pairwise disjoint subsets $E_Q \subset Q$ with $|Q| \le 2|E_Q|$ for each $Q \in \mathcal{S}$.

The major tool to prove our main results is Lerner's local oscillation formula from [Ler2]. To formulate it we need to introduce several notions. By a *median value* of a measurable function f on a set Q we mean a possibly nonunique, real number $m_f(Q)$ such that

$$\max\big\{|\{x\in Q : f(x) > m_f(Q)\}|, |\{x \in Q : f(x) < m_f(Q)\}|\big\} \le |Q|/2.$$

The *decreasing rearrangement* of a measurable function f on $\mathbb{R}^n$ is defined by

$$f^*(t) = \inf\{\alpha > 0 : |\{x \in \mathbb{R}^n : |f(x)| > \alpha\}| < t\}, \qquad 0 < t < \infty.$$

The *local mean oscillation* of f is

$$\omega_\lambda(f; Q) = \inf_{c \in \mathbb{R}} ((f - c)\chi_Q)^*(\lambda|Q|), \qquad 0 < \lambda < 1.$$

Then it follows from the definitions that

$$|m_f(Q)| \le (f\chi_Q)^*(|Q|/2). \tag{2.4}$$

The following theorem was proved by Hytönen [Hyt2, Theorem 2.3] in order to improve Lerner's original formula given in [Ler1, Ler2].

THEOREM 2.1. *Let f be a measurable function on $\mathbb{R}^n$ and let Q_0 be a fixed cube. Then there exists a (possibly empty) sparse family $\mathcal{S}$ of cubes $Q \in \mathscr{D}(Q_0)$ such that for a.e. $x \in Q_0$,*

$$|f(x) - m_f(Q_0)| \le 2 \sum_{Q \in \mathcal{S}} \omega_{1/2^{n+2}}(f; Q)\chi_Q(x). \tag{2.5}$$

3. Proof of Theorem 1.1. This section is entirely devoted to the proof of Theorem 1.1. To that end, we first make some reductions. First, since the operator $\mathcal{T}_{\mathcal{S}}^{k,p_0}$ is (multi-)sublinear, we may assume that $f_i \ge 0$ for $1 \le i \le m$. Second, by a well known variation of the one-third trick (see, for example, [HLP]), we may replace centered dilations by dyadic ancestors. More precisely, we may write

$$\mathcal{T}_{\mathcal{S}}^{k,p_0} \vec{f}(x) \lesssim_{p_0,n} \sum_{j=1}^{c_n} \mathcal{A}_{\mathscr{D}^j,\mathcal{S}_j}^{k,p_0} \vec{f}(x)$$

for certain dyadic systems $\mathscr{D}^1, \ldots, \mathscr{D}^{c_n}$, sparse families $\mathcal{S}_j \subset \mathscr{D}^j$ and some dimensional constant c_n. Therefore, we may just concentrate on one such operator $\mathcal{A}_{\mathscr{D},\mathcal{S}}^{k,p_0}$. However, we will consider a slightly more general operator. Namely, given a dyadic system $\mathscr{D}$ (that we remove from the notation from now on), we will study operators of the form

$$\mathcal{A}_\alpha^{k,p_0} \vec{f}(x) = \sum_{Q \in \mathscr{D}} \alpha_Q \Big[\prod_{i=1}^m \langle f_i \rangle_{Q^{(k)},p_0}\Big] \chi_Q(x),$$

where the sequence $\alpha = (\alpha_Q)_Q$ is *Carleson* and normalized:

$$\sup_{Q \in \mathscr{D}} \sum_{T \in \mathscr{D}(Q)} \alpha_T \frac{|T|}{|Q|} = 1.$$

Finally, by the usual density arguments, we may assume that the sequence α is finite, which in particular implies that there exists some cube $P_0 \in \mathscr{D}$

such that $\mathcal{A}_{\alpha}^{k,p_0} = \mathcal{A}_{\alpha,P_0}^{k,p_0}$, that is,

$$\mathcal{A}_{\alpha}^{k,p_0}\vec{f}(x) = \sum_{Q\in\mathscr{D},\, Q^{(k)}\subset P_0} \alpha_Q \Big[\prod_{i=1}^{m}\langle f_i\rangle_{Q^{(k)},p_0}\Big]\chi_Q(x).$$

Also, recall that we assume that each function f_i is supported in the cube P_0. The rest of the proof consists in obtaining a pointwise estimate of $\mathcal{A}_{\alpha,P_0}^{k,p_0}$ and follows the lines of [CR]. Since it is a bit lengthy, we divide it into several steps. We will skip some details at the points where our argument does not differ substantially from that of [CR].

Step 1. *Slicing: reduction to separated scales.* We start the proof by separating the scales of $\mathcal{A}_{\alpha,P_0}^{k,p_0}$ as follows:

$$\begin{aligned}\mathcal{A}_{\alpha,P_0}^{k,p_0}\vec{f}(x) &= \sum_{\ell=0}^{k-1}\sum_{j=1}^{\infty}\sum_{Q\in\mathscr{D}_{jk+\ell}(P_0)} \alpha_Q\Big(\prod_{i=1}^{m}\langle f_i\rangle_{Q^{(k)},p_0}\Big)\chi_Q(x)\\ &=: \sum_{\ell=0}^{k-1}\mathcal{A}_{\alpha,P_0}^{k,p_0;\ell}\vec{f}(x).\end{aligned}$$

Now, as in [CR], we rewrite $\mathcal{A}_{\alpha,P_0}^{k,p_0;\ell}$ as a sum of disjointly supported operators of the form $\mathcal{A}_{\alpha,P}^{k,p_0;0}$. Indeed, we have the expression

$$\mathcal{A}_{\alpha,P_0}^{k,p_0}\vec{f}(x) = \sum_{\ell=0}^{k-1}\sum_{P\in\mathscr{D}_\ell(P_0)}\mathcal{A}_{\alpha,P}^{k,p_0;0}\vec{f}(x).$$

Therefore, it is enough to prove the following claim: Let $k\ge 1$ and α be a normalized Carleson sequence. For nonnegative integrable functions $f_1,\dots,f_m$ on P_0, there exists a sparse family $\mathcal{S}$ of cubes in $\mathscr{D}(P_0)$ such that

$$\mathcal{A}_{\alpha,P_0}^{k,p_0;0}\vec{f}(x) \le C\sum_{Q\in\mathcal{S}}\Big(\prod_{i=1}^{m}\langle f_i\rangle_{Q,p_0}\Big)\chi_Q(x),$$

for some constant C independent of k and the cube P_0.

Step 2. *Construction of the collection $\mathcal{S}$ for the sliced operator.* We now build the family $\mathcal{S}$. The construction is similar to that in [CR, p. 6]. We start by defining

$$C^* := 2^{2(m+1)}\mathrm{W}(p_0,k),$$

where

$$\mathrm{W}(p_0,k) = \sup_{\substack{P\in\mathscr{D},\,\alpha\text{ Carleson}\\ \alpha_Q\neq 0\Rightarrow Q\in\mathscr{D}(P)}}\|\mathcal{A}_{\alpha,P}^{k,p_0;0}\|_{L^{p_0}(P)\times\cdots\times L^{p_0}(P)\to L^{p_0/m,\infty}(P)}.$$

Also, if $Q \in \mathscr{D}_{kn}(P_0)$ for some $n \geq 0$, define

$$\gamma_Q = \max_{R \in \mathscr{D}_k(Q)} \alpha_R.$$

Set also $\varDelta_{P_0} = 0$. Then we inductively implement the following selection procedure, starting with the cube $P = P_0$:

(1) If $\varDelta_P - (\prod_{i=1}^m \langle f_i \rangle_{P,p_0})\gamma_P < 0$, then we choose $P \in \mathcal{S}$ and we set

$$\varDelta_Q = \varDelta_P + (C^* - \alpha_Q) \prod_{i=1}^m \langle f_i \rangle_{P,p_0}$$

for all $Q \in \mathscr{D}_k(P)$.

(2) If $\varDelta_P - (\prod_{i=1}^m \langle f_i \rangle_{P,p_0})\gamma_P \geq 0$, then we choose $P \notin \mathcal{S}$ and we set

$$\varDelta_Q = \varDelta_P - \alpha_Q \prod_{i=1}^m \langle f_i \rangle_{P,p_0}.$$

(3) Go back to (1) for the cubes $Q \in \mathscr{D}_k(P)$.

Since the sequence α is finite, the procedure terminates and yields the family $\mathcal{S}$ that we will use.

Step 3. *The family $\mathcal{S}$ is sparse.* To prove sparsity, we will show the following (stronger) claim: fix $P \in \mathcal{S}$, and denote

$$F(P) := \bigcup_{Q \subsetneq P,\, Q \in \mathcal{S}} Q.$$

Then $|F(P)| \leq \frac{1}{2}|P|$. The claim and its proof are entirely similar to [CR, pp. 7–8]. Let $\mathcal{R}$ be the collection of maximal subcubes of P which belong to $\mathcal{S}$. By maximality, for each $x \in R \in \mathcal{R}$ we have

$$\Big(\prod_{i=1}^m \langle f_i \rangle_{R,p_0}\Big)\gamma_R + \mathcal{A}_{\alpha,P}^{k,p_0;0} \vec{f}(x) > C^* \prod_{i=1}^m \langle f_i \rangle_{P,p_0}.$$

Now, denote

$$\mathcal{G}_{P,p_0} \vec{f} = \sum_{R \in \mathcal{R}} \gamma_R \Big(\prod_{i=1}^m \langle f_i \rangle_{R,p_0}\Big) \chi_R.$$

Then for all $x \in P$,

$$\mathcal{G}_{P,p_0} \vec{f}(x) + \mathcal{A}_{\alpha,P}^{k,p_0;0} \vec{f}(x) > C^* \prod_{i=1}^m \langle f_i \rangle_{P,p_0}.$$

Thus we have

$$\begin{aligned}
|F(P)| &\le \Big|\Big\{x\in P : \mathcal{G}_{P,p_0}\vec f(x) + \mathcal{A}^{k,p_0;0}_{\alpha,P}\vec f(x) > C^*\prod_{i=1}^m \langle f_i\rangle_{P,p_0}\Big\}\Big| \\
&\le \frac{\|\mathcal{G}_{P,p_0} + \mathcal{A}^{k,p_0;0}_{\alpha,P}\|^{p_0/m}_{L^{p_0}(P)\times\cdots\times L^{p_0}(P)\to L^{p_0/m,\infty}(P)}}{(C^*\prod_{i=1}^m\langle f_i\rangle_{P,p_0})^{p_0/m}}\Big(\prod_{i=1}^m \|f_i\|_{L^{p_0}(P)}\Big)^{p_0/m} \\
&\le 2^{p_0/m}|P|\bigg(\frac{\|\mathcal{G}_{P,p_0}\|^{p_0/m}_{L^{p_0}(P)\times\cdots\times L^{p_0}(P)\to L^{p_0/m,\infty}(P)}}{(C^*)^{p_0/m}} \\
&\qquad\qquad + \frac{\|\mathcal{A}^{k,p_0;0}_{\alpha,P}\|^{p_0/m}_{L^{p_0}(P)\times\cdots\times L^{p_0}(P)\to L^{p_0/m,\infty}(P)}}{(C^*)^{p_0/m}}\bigg) \\
&\le \frac{|P|}{2}\bigg(\frac{\|\mathcal{G}_{P,p_0}\|^{p_0/m}_{L^{p_0}(P)\times\cdots\times L^{p_0}(P)\to L^{p_0/m,\infty}(P)}}{2} + \frac{1}{2}\bigg).
\end{aligned}$$

Finally, we observe that the operator $\mathcal{G}_{P,p_0}$ is bounded above by the multi-sublinear operator

$$\mathcal{P}_{P,p_0}\vec f = \sum_{R\in\mathcal{R}}\Big(\prod_{i=1}^m\langle f_i\rangle_{R,p_0}\Big)\chi_R,$$

which is contractive from $L^{p_0}(P)\times\cdots\times L^{p_0}(P)$ to $L^{p_0/m,\infty}(P)$. Therefore, the norm of $\mathcal{G}_{P,p_0}$ from $L^{p_0}(P)\times\cdots\times L^{p_0}(P)$ to $L^{p_0/m,\infty}(P)$ is bounded by 1. This is enough to obtain the assertion.

Step 4. *Pointwise bound.* Following the proof of [CR, Lemma 2.3], one gets the pointwise bound

$$\mathcal{A}^{k,p_0;0}_{\alpha,P_0}\vec f(x) \lesssim_{n,m} \mathrm{W}(p_0,k)\sum_{Q\in\mathcal{S}}\Big(\prod_{i=1}^m\langle f_i\rangle_{Q,p_0}\Big)\chi_Q(x).$$

Therefore, we only need to prove $\mathrm{W}(p_0,k)\lesssim_{p_0,n,m} 1$.

Step 5. *Weak type estimate for $\mathcal{A}^{k,p_0}_{\alpha,P}$.* Fix some $P\in\mathscr{D}$ and some normalized Carleson sequence α such that $\alpha_Q\neq 0$ only if $Q\in\mathscr{D}(P)$. We need to show that

$$\|\mathcal{A}^{k,p_0}_{\alpha,P}\|_{L^{p_0}\times\cdots\times L^{p_0}\to L^{p_0/m,\infty}} \lesssim_{n,m,p_0} 1.$$

To prove it, we first establish an L^{2p_0m} estimate. We will use the estimate of [CD], which reads

$$\Big(\sum_{Q\in\mathscr{D}(P)}\alpha_Q\Big(\prod_{i=1}^m\frac{1}{|Q|}\int_Q f_i\Big)^q|Q|\Big)^{1/q} \le \prod_{i=1}^m p_i'\|f_i\|_{L^{p_i}(P)} \tag{3.1}$$

whenever $1/q = 1/p_1 + \cdots + 1/p_m$ and α is Carleson and normalized. We will show that

$$\|\mathcal{A}^{k,p_0}_{\alpha,P}\vec{f}\|_{L^{2p_0}} \lesssim \prod_{i=1}^{m} \|f_i\|_{L^{2p_0 m}}.$$

Indeed, by using duality we reduce to showing

$$\int_P g(x)\mathcal{A}^{k,p_0}_{\alpha,P}\vec{f}(x)\,dx \lesssim 1$$

assuming that $\|f_i\|_{L^{2p_0 m}} = \|g\|_{L^{(2p_0)'}} = 1$ for all $1 \le i \le m$ and $g \ge 0$. By definition and Hölder's inequality, it is enough to show

$$\Big(\sum_{Q\in\mathscr{D}_{\ge m}(P_0)} \alpha_Q\Big(\prod_{i=1}^{m}\langle f_i\rangle_{Q^{(k)},p_0}\Big)^{2p_0}|Q|\Big)^{1/(2p_0)} \times\Big(\sum_{Q\in\mathscr{D}_{\ge k}(P_0)} \alpha_Q\Big(\frac{1}{|Q|}\int_Q g\Big)^{(2p_0)'}|Q|\Big)^{1/(2p_0)'} \lesssim 1.$$

The second term can be estimated, using (3.1) in the linear case, by an absolute constant. For the first term observe that the sequence β_Q defined by

$$\beta_Q = \frac{1}{2^{nk}}\sum_{R\in\mathscr{D}_k(Q)} \alpha_R$$

is a Carleson sequence adapted to P of constant 1. Indeed, for any $Q \in \mathscr{D}(P)$,

$$\begin{aligned}\frac{1}{|Q|}\sum_{R\in\mathscr{D}(Q)}\beta_R|R| &= \frac{1}{|Q|}\sum_{R\in\mathscr{D}(Q)}|R|\frac{1}{2^{nk}}\sum_{T\in\mathscr{D}_k(R)}\alpha_T\\ &= \frac{1}{|Q|}\sum_{R\in\mathscr{D}(Q)}\sum_{T\in\mathscr{D}_k(R)}\alpha_T|T| = \frac{1}{|Q|}\sum_{R\in\mathscr{D}_{\ge k}(Q)}\alpha_R|R| \le 1.\end{aligned}$$

Therefore, we can write the first term as

$$\Big(\sum_{Q\in\mathscr{D}(P)}\beta_Q\Big(\sum_{i=1}^{m}\langle f_i\rangle_{Q,p_0}\Big)^{2p_0}|Q|\Big)^{1/(2p_0)},$$

which can also be estimated by (3.1), with $p_1 = \cdots = p_m = 2m$, $q = 2$:

$$\begin{aligned}\Big(\sum_{Q\in\mathscr{D}(P)}\beta_Q\Big(\prod_{i=1}^{m}\langle f_i\rangle_{Q,p_0}\Big)^{2p_0}\Big)^{1/(2p_0)} &= \Big(\sum_{Q\in\mathscr{D}(P)}\beta_Q\Big(\prod_{i=1}^{m}\langle |f_i|^{p_0}\rangle_{Q,1}\Big)^{2}|Q|\Big)^{1/(2p_0)}\\ &\lesssim_{p_0,m}\prod_{i=1}^{m}\|\,|f_i|^{p_0}\|_{2m}^{1/p_0} \lesssim 1.\end{aligned}$$

Combining both terms we arrive at the strong type result we want.

Now we can prove our weak type estimate

$$\sup_{\lambda>0} \lambda|\{x : \mathcal{A}^{k,p_0}_{\alpha,P} \vec{f}(x) > \lambda\}|^{m/p_0} \lesssim_{n,m,p_0} \prod_{i=1}^{m} \|f_i\|_{L^{p_0}}.$$

By homogeneity we can assume $\|f_i\|_{L^{p_0}} = 1$ and $f_i \geq 0$ for $1 \leq i \leq m$. We will use the previous strong bound and a standard Calderón–Zygmund decomposition of the positive tuple $(f_1^{p_0}, \dots, f_m^{p_0})$, which we explain now.

We need the following version of the dyadic maximal operator:

$$\mathcal{M}^{\mathscr{D}}_{p_0} g(x) = \sup_{x \in Q \in \mathscr{D}} \langle g \rangle_{Q,p_0}.$$

For $1 \leq i \leq m$, denote

$$\Omega_i = \{x \in P : \mathcal{M}^{\mathscr{D}}_{p_0} f_i(x) > \lambda^{1/m}\}.$$

If $\langle f_i \rangle_{P,p_0} > \lambda^{1/m}$ then

$$|P| \lambda^{p_0/m} < \|f_i\|^{p_0}_{L^{p_0}},$$

and the estimate follows by the homogeneity assumption. Therefore, we can assume $\langle f_i \rangle_{P,p_0} \leq \lambda^{1/m}$ for all $1 \leq i \leq m$. But then we can write Ω_i as a union of cubes in a collection $\mathcal{R}_i$ consisting of pairwise disjoint dyadic (strict) subcubes R of P with the properties

$$\langle f_i \rangle_{R,p_0} > \lambda^{1/m} \quad \text{and} \quad \langle f_i \rangle_{R^{(1)},p_0} \leq \lambda^{1/m}, \ R \in \mathcal{R}_i.$$

For each $1 \leq i \leq m$ let $b_i = \sum_{R \in \mathcal{R}_i} b_i^R$, where

$$b_i^R(x) := \big(f_i^{p_0}(x) - \langle f_i \rangle^{p_0}_{R,p_0}\big) \chi_R(x).$$

We now let $g_i = f_i^{p_0} - b_i$. Observe that

$$|g_i(x)| \lesssim \lambda^{p_0/m}, \qquad \|g_i\|_{L^1} \lesssim \|f_i\|_{L^{p_0}} = 1$$

as well as

$$|\Omega_i| = \sum_{R \in \mathcal{R}_i} |R| \leq \frac{1}{\lambda^{p_0/m}}.$$

Set $\Omega = \bigcup_i \Omega_i$. Now we have

$$\begin{aligned} |\{x : \mathcal{A}^{k,p_0}_{\alpha,P} \vec{f}(x) > \lambda\}| &\leq |\Omega| + |\{x \in \mathbb{R}^n \setminus \Omega : \mathcal{A}^{k,p_0}_{\alpha,P} \vec{f}(x) > \lambda\}| \\ &\leq \frac{m}{\lambda^{p_0/m}} + |\{x \in \mathbb{R}^n \setminus \Omega : \mathcal{A}^{k,p_0}_{\alpha,P} \vec{f}(x) > \lambda\}|. \end{aligned} \tag{3.2}$$

To estimate the second term above observe that

$$\langle f_i \rangle^{p_0}_{Q,p_0} \leq \left| \frac{1}{|Q|} \int_Q g_i \right| + \left| \frac{1}{|Q|} \int_Q b_i \right|.$$

Therefore, by the concavity of the function $x \mapsto |x|^{1/p_0}$ we obtain

$$\mathcal{A}^{k,p_0}_{\alpha,P}\vec{f}(x) \le |\mathcal{A}|^{k,p_0}_{\alpha,P}\vec{g}(x) + \sum_{j=1}^{2^m-1} |\mathcal{A}|^{k,p_0}_{\alpha,P}(h_1^j,\dots,h_m^j)(x),$$

where we have denoted $\vec{g} = (g_i)_{1\le i\le k}$, h_i^j is either g_i or b_i, and for each j, there is at least one $1 \le i \le m$ such that $h_i^j = b_i$. Also, we have used the notation

$$|\mathcal{A}|^{k,p_0}_{\alpha,P}\vec{h}(x) = \sum_{Q\in\mathscr{D}(P),\, Q^k\subset P} \alpha_Q \prod_{i=1}^m \bigg|\frac{1}{|Q^{(k)}|}\int_{Q^{(k)}} h_i\bigg|^{1/p_0} \chi_Q(x).$$

If $h_i^j = b_i$, then for all $x \notin \Omega_i$ we can see that $|\mathcal{A}|^{k,p_0}_{\alpha,P}(h_1^j,\dots,h_m^j)(x) = 0$ because each b_i^R has zero average. With this fact we can see that the second term in (3.2) is actually identical to

$$|\{x\in\mathbb{R}^n\setminus\Omega : |\mathcal{A}|^{k,p_0}_{\alpha,P}\vec{g}(x) > \lambda\}|.$$

Now we can use the L^{2p_0} bound. Writing $|\vec{g}|^{1/p_0} = (|g_1|^{1/p_0},\dots,|g_k|^{1/p_0})$, we have

$$\begin{aligned}
&|\{x\in\mathbb{R}^n\setminus\Omega : |\mathcal{A}|^{k,p_0}_{\alpha,P}\vec{g}(x) > \lambda\}| \\
&\qquad\le \frac{1}{\lambda^{2p_0}}\|\,|\mathcal{A}|^{k,p_0}_{\alpha,P}\vec{g}\|^{2p_0}_{L^{2p_0}} \le \frac{1}{\lambda^{2p_0}}\|\mathcal{A}^{k,p_0}_{\alpha,P}|\vec{g}|^{1/p_0}\|^{2p_0}_{L^{2p_0}} \\
&\qquad\lesssim \frac{1}{\lambda^{2p_0}}\prod_{i=1}^m \|\,|g_i|^{1/p_0}\|^{2p_0}_{L^{2p_0m}} \lesssim \frac{1}{\lambda^{p_0/m}}\prod_{i=1}^m \|g_i\|^{1/m}_{L^1} \lesssim \frac{1}{\lambda^{p_0/m}}.
\end{aligned}$$

Putting both estimates together we arrive at the desired result. This completes the proof of Theorem 1.1.

4. Proofs of Theorems 1.2 and 1.3

Proof of Theorem 1.2. We borrow some ideas from [LMS, Theorem 3.2], where the case $p_0 = 1$ is considered. Throughout the proof, we set $a = p/p_0$ and $a_i = p_i/p_0$ for $i = 1,\dots,m$. Let $\sigma_i = w_i^{1-a_i'}$, $\vec{f}_{\sigma,p_0} = (f_1\sigma_1^{1/p_0},\dots,f_m\sigma_m^{1/p_0})$ and $f_i \ge 0$. We have $\sigma_i, \nu_{\vec{w}} \in A_\infty$ (see [LO+, Theorem 3.6]). It suffices to prove that

$$\|\mathcal{A}^{p_0}_{\mathscr{D},S}(\vec{f}_{\sigma,p_0})\|_{L^p(\nu_{\vec{w}})} \lesssim [\vec{w}]^{\max\{1,a_1'/(p_0a),\dots,a_m'/(p_0a)\}}_{A_{\vec{P}/p_0}} \prod_{i=1}^m \|f_i\|_{L^{p_i}(\sigma_i)}. \tag{4.1}$$

By definition, for any cube $Q\subset\mathbb{R}^n$, we have

$$[\vec{w}]_{A_{\vec{P}/p_0}} \ge \bigg(\frac{1}{|Q|}\int_Q \nu_{\vec{w}}\bigg)\prod_{j=1}^m\bigg(\frac{1}{|Q|}\int_Q w_j^{1-a_j'}\bigg)^{a/a_j'}. \tag{4.2}$$

Denote $\beta = \max\{1, a_1'/(p_0 a), \ldots, a_m'/(p_0 a)\}$, and assume that $0 \le g \in L^{p'}(\nu_{\vec{w}})$. We have

$$\int_{\mathbb{R}^n} \mathcal{A}^{p_0}_{\mathscr{D},S}(\vec{f}_{\sigma,p_0}) g\nu_{\vec{w}} = \sum_{Q\in\mathcal{S}} \int_Q g\nu_{\vec{w}} \times \biggl(\prod_{i=1}^m \Bigl(\frac{1}{|Q|}\int_Q |f_i|^{p_0}\sigma_i\Bigr)^{1/p_0}\biggr).$$

From this and the definition of $[\vec{w}]_{A_{\vec{P}/p_0}}$, we obtain

$$\begin{aligned}
&\sum_{Q\in\mathcal{S}} \int_Q g\nu_{\vec{w}} \times \Bigl(\prod_{i=1}^m \frac{1}{|Q|}\int_Q |f_i|^{p_0}\sigma_i\Bigr)^{1/p_0} \\
&\quad\le [\vec{w}]^\beta_{A_{\vec{P}}} \sum_{Q\in\mathcal{S}} \frac{|Q|^{m(\beta a-1/p_0)}}{\nu_{\vec{w}}(Q)^{\beta-1}\prod_{i=1}^m \sigma_i(Q)^{\beta a/a_i'-1/p_0}} \\
&\qquad\times \Bigl(\frac{1}{\nu_{\vec{w}}(Q)}\int_Q g\nu_{\vec{w}}\Bigr) \times \Bigl(\prod_{i=1}^m \frac{1}{\sigma_i(Q)}\int_Q |f_i|^{p_0}\sigma_i\Bigr)^{1/p_0} \\
&\quad\le 2^{m(\beta a-1/p_0)}[\vec{w}]^\beta_{A_{\vec{P}}} \sum_{Q\in\mathcal{S}} \frac{|E_Q|^{m(\beta a-1/p_0)}}{\nu_{\vec{w}}(Q)^{\beta-1}\prod_{i=1}^m \sigma_i(Q)^{\beta a/a_i'-1/p_0}} \\
&\qquad\times \Bigl(\frac{1}{\nu_{\vec{w}}(Q)}\int_Q g\nu_{\vec{w}}\Bigr) \times \Bigl(\prod_{i=1}^m \frac{1}{\sigma_i(Q)}\int_Q |f_i|^{p_0}\sigma_i\Bigr)^{1/p_0} \\
&\quad\le 2^{m(\beta a-1/p_0)}[\vec{w}]^\beta_{A_{\vec{P}}} \sum_{Q\in\mathcal{S}} \frac{|E_Q|^{m(\beta a-1/p_0)}}{\nu_{\vec{w}}(E_Q)^{\beta-1}\prod_{i=1}^m \sigma_i(E_Q)^{\beta a/a_i'-1/p_0}} \\
&\qquad\times \Bigl(\frac{1}{\nu_{\vec{w}}(Q)}\int_Q g\nu_{\vec{w}}\Bigr) \times \Bigl(\prod_{i=1}^m \frac{1}{\sigma_i(Q)}\int_Q |f_i|^{p_0}\sigma_i\Bigr)^{1/p_0},
\end{aligned}$$

where in the last inequality we have used the inequalities $\nu_{\vec{w}}(Q) \ge \nu_{\vec{w}}(E_Q)$, $\sigma_i(Q) \ge \sigma_i(E_Q)$ and the positivity of the exponents. On the other hand, by Hölder's inequality, we have

$$|E_Q| = \int_{E_Q} \nu_{\vec{w}}^{1/(ma)} \prod_{i=1}^m \sigma_i^{1/(ma_i')} \le \nu_{\vec{w}}(E_Q)^{1/(ma)} \prod_{i=1}^m \sigma_i(E_Q)^{1/(ma_i')}. \tag{4.3}$$

Inserting this into the estimate above we conclude that

$$\begin{aligned}
&\sum_{Q\in\mathcal{S}} \int_Q g\nu_{\vec{w}} \times \Bigl(\prod_{i=1}^m \frac{1}{|Q|}\int_Q |f_i|^{p_0}\sigma_i\Bigr)^{1/p_0} \\
&\quad\le 2^{m(\beta a-1/p_0)}[\vec{w}]^\beta_{A_{\vec{P}}} \sum_{Q\in\mathcal{S}} \frac{\nu_{\vec{w}}(E_Q)^{(\beta a-1/p_0)/a}\prod_{i=1}^m \sigma_i(E_Q)^{(\beta a-1/p_0)/a_i'}}{\nu_{\vec{w}}(E_Q)^{\beta-1}\prod_{i=1}^m \sigma_i(E_Q)^{\beta a/a_i'-1/p_0}} \\
&\qquad\times \Bigl(\frac{1}{\nu_{\vec{w}}(Q)}\int_Q g\nu_{\vec{w}}\Bigr) \times \Bigl(\prod_{i=1}^m \frac{1}{\sigma_i(Q)}\int_Q |f_i|^{p_0}\sigma_i\Bigr)^{1/p_0}
\end{aligned}$$

$$
\begin{aligned}
&\le 2^{m(\beta a-1/p_0)}[\vec{w}]^{\beta}_{A_{\vec{P}}} \sum_{Q\in\mathcal{S}} \nu_{\vec{w}}(E_Q)^{1-1/(ap_0)} \prod_{i=1}^m \sigma_i(E_Q)^{1/(p_0a_i)} \\
&\qquad\qquad \times \bigg(\frac{1}{\nu_{\vec{w}}(Q)}\int_Q g\nu_{\vec{w}}\bigg) \times \bigg(\prod_{i=1}^m \frac{1}{\sigma_i(Q)}\int_Q |f_i|^{p_0}\sigma_i\bigg)^{1/p_0} \\
&= 2^{mq(\beta a-1/p_0)}[\vec{w}]^{\beta}_{A_{\vec{P}}} \sum_{Q\in\mathcal{S}} \nu_{\vec{w}}(E_Q)^{1/p'} \prod_{i=1}^m \sigma_i(E_Q)^{1/p_i} \bigg(\frac{1}{\nu_{\vec{w}}(Q)}\int_Q g\nu_{\vec{w}}\bigg) \\
&\qquad\qquad \times \bigg(\prod_{i=1}^m \frac{1}{\sigma_i(Q)}\int_Q |f_i|^{p_0}\sigma_i\bigg)^{1/p_0} \\
&= 2^{mq(\beta a-1/p_0)}[\vec{w}]^{\beta}_{A_{\vec{P}}} \sum_{Q\in\mathcal{S}} \bigg[\bigg(\frac{1}{\nu_{\vec{w}}(Q)}\int_Q g\nu_{\vec{w}}\bigg)\nu_{\vec{w}}(E_Q)^{1/p'}\bigg] \\
&\qquad\qquad \times \bigg[\prod_{i=1}^m \bigg(\frac{1}{\sigma_i(Q)}\int_Q |f_i|^{p_0}\sigma_i\bigg)\sigma_i(E_Q)^{p_0/p_i}\bigg]^{1/p_0}.
\end{aligned}
$$

This, together with Hölder's inequality and the disjointness of the family $\{E_Q\}_{Q\in\mathcal{S}}$, yields

$$
\begin{aligned}
&\sum_{Q\in\mathcal{S}} \int_Q g\nu_{\vec{w}} \times \bigg(\prod_{i=1}^m \frac{1}{|Q|}\int_Q |f_i|^{p_0}\sigma_i\bigg)^q \\
&\quad\le 2^{mq(\beta a-1/p_0)}[\vec{w}]^{\beta}_{A_{\vec{P}}} \bigg[\sum_{Q\in\mathcal{S}} \bigg(\frac{1}{\nu_{\vec{w}}(Q)}\int_Q g\nu_{\vec{w}}\bigg)^{p'} \nu_{\vec{w}}(E_Q)\bigg]^{1/p'} \\
&\quad\quad \times \prod_{i=1}^m \bigg[\sum_{Q\in\mathcal{S}} \bigg(\frac{1}{\sigma_i(Q)}\int_Q |f_i|_i^{p_0}\sigma_i\bigg)^{p_i/p_0} \sigma_i(E_Q)\bigg]^{1/p_i} \\
&\quad\le 2^{m(\beta a-1/p_0)}[\vec{w}]^{\beta}_{A_{\vec{P}}} \|M^{\mathscr{D}}_{\nu_{\vec{w}}}(g)\|_{L^{p'}(\nu_{\vec{w}})} \times \prod_{i=1}^m \|M^{\mathscr{D}}_{\sigma_i}(|f_i|^{p_0})\|^{1/p_0}_{L^{p_i/p_0}(\sigma_i)} \\
&\quad\lesssim 2^{m(\beta a-1/p_0)}[\vec{w}]^{\beta}_{A_{\vec{P}}} \|g\|_{L^{p'}(\nu_{\vec{w}})} \times \prod_{i=1}^m \|f_i^{p_0}\|^{1/p_0}_{L^{p_i/p_0}(\sigma_i)} \\
&\quad= 2^{m(\beta a-1/p_0)}[\vec{w}]^{\beta}_{A_{\vec{P}}} \|g\|_{L^{p'}(\nu_{\vec{w}})} \times \prod_{i=1}^m \|f_i\|_{L^{p_i}(\sigma_i)},
\end{aligned}
$$

by applying (2.3) to get the last inequality. This proves (4.1). ∎

The following proposition plays an important role in proving Theorem 1.3.

PROPOSITION 4.1. *Let T satisfy* (H1) *and* (H2). *Then, for any cube $Q \subset \mathbb{R}^n$,*

$$\omega_\lambda(T\vec{f};Q) \le c(T,\lambda,m,n)\sum_{\ell=0}^{\infty} 2^{-\ell\delta_0}\bigg(\prod_{i=1}^{m}\frac{1}{|2^\ell Q|}\int_{2^\ell Q}|f_i(y)|^{p_0}\,dy\bigg)^{1/p_0},$$

where $\delta_0 = \delta - n/p_0$.

Proof. The proof is standard. For completeness, we sketch it here. For each $i = 1,\ldots,m$, we define $f_i^0 = f_i\chi_{Q^*}$ and $f_i^\infty = f_i - f_i^0$. Setting $\vec{f}^{\,0} = (f_1\chi_{4Q},\ldots,f_m\chi_{4Q})$, we have

$$T(\vec{f})(z) = T(\vec{f}^{\,0})(z) + \sum_{\vec{\alpha}\in\mathcal{I}_0} T(f_1^{\alpha_1},\ldots,f_m^{\alpha_m})(z), \tag{4.4}$$

where $\mathcal{I}_0 := \{\vec{\alpha} = (\alpha_1,\ldots,\alpha_m) : \alpha_i \in \{0,\infty\}$ and at least one $\alpha_i \ne 0\}$. We first observe that

$$\begin{aligned}\Big[\Big(T(\vec{f}) - \sum_{\vec{\alpha}\in\mathcal{I}_0} T(f_1^{\alpha_1},\ldots,f_m^{\alpha_m})(x_0)\Big)\chi_Q\Big]^*(\lambda|Q|)&\\ \le 2(T(\vec{f}^{\,0})\chi_Q)^*(\lambda|Q|/2)&\\ + 2\Big\|\sum_{\vec{\alpha}\in\mathcal{I}_0} T(f_1^{\alpha_1},\ldots,f_m^{\alpha_m})(\cdot) - \sum_{\vec{\alpha}\in\mathcal{I}_0} T(f_1^{\alpha_1},\ldots,f_m^{\alpha_m})(x_0)\Big\|_{L^\infty(Q)},&\end{aligned}$$

where x_0 is the center of Q. Since T maps $L^{p_0}\times\cdots\times L^{p_0}$ into $L^{p_0/m,\infty}$, we have

$$\begin{aligned}(T\vec{f}^{\,0})^*(\lambda|Q|) &\le C_{n,T,\lambda}\|T\vec{f}^{\,0}\|_{L^{p_0/m,\infty}(Q,dx/|Q|)}\\ &\le C_{n,T,\lambda}\bigg(\prod_{i=1}^{m}\frac{1}{|4Q|}\int_{4Q}|f_i(y)|^{p_0}\,dy\bigg)^{1/p_0}.\end{aligned}$$

On the other hand, for $x \in Q$, the argument in [BD, Theorem 3.1] yields

$$\begin{aligned}\Big|\sum_{\vec{\alpha}\in\mathcal{I}_0} T(f_1^{\alpha_1},\ldots,f_m^{\alpha_m})(x) - \sum_{\vec{\alpha}\in\mathcal{I}_0} T(f_1^{\alpha_1},\ldots,f_m^{\alpha_m})(x_0)\Big|&\\ \le C_{n,m,T}\sum_{\ell=0}^{\infty}2^{-\ell\delta_0}\bigg(\prod_{i=1}^{m}\frac{1}{|2^\ell Q|}\int_{2^\ell Q}|f_i(y)|^{p_0}dy\bigg)^{1/p_0}&\end{aligned}\tag{4.5}$$

with $\delta_0 = \delta - n/p_0$. Taking the last two estimates into account we obtain

$$\begin{aligned}\Big[\Big(T(\vec{f}) - \sum_{\vec{\alpha}\in\mathcal{I}_0} T(f_1^{\alpha_1},\ldots,f_m^{\alpha_m})(x_0)\Big)\chi_Q\Big]^*(\lambda|Q|)&\\ \le C_{n,m,T,\lambda}\sum_{\ell=0}^{\infty}2^{-\ell\delta_0}\bigg(\prod_{i=1}^{m}\frac{1}{|2^\ell Q|}\int_{2^\ell Q}|f_i(y)|^{p_0}dy\bigg)^{1/p_0}.&\end{aligned} \qquad \blacksquare$$

At this stage, Theorem 1.3 follows immediately from Theorem 1.2 via the following short argument:

Proof of Theorem 1.3. From Theorem 2.1 and Proposition 4.1, for $Q_0 \in \mathscr{D}$, we can pick a sparse family $\mathcal{S}(Q_0) \subset \mathscr{D}(Q_0)$ so that

$$|T(\vec{f})(x) - m_{T(\vec{f})}(Q_0)| \\ \le c(T,n,m) \sum_{Q\in\mathcal{S}(Q_0)} \sum_{\ell=0}^{\infty} 2^{-\ell\delta_0} \Big(\prod_{i=1}^{m} \frac{1}{|2^\ell Q|} \int_{2^\ell Q} |f_i(y)|^{p_0} dy\Big)^{1/p_0} \chi_Q(x)$$

for a.e. $x \in Q_0$. Since T maps $L^{p_0} \times \cdots \times L^{p_0}$ into $L^{p_0/m,\infty}$, we can write

$$|m_{T(\vec{f})}(Q_0)| \le (T(\vec{f})\chi_{Q_0})^*(|Q_0|/2) \le (2/|Q_0|)^{m/p_0} \|T(\vec{f})\chi_{Q_0}\|_{L^{p_0/m,\infty}} \\ \lesssim \|T\|_{L^{p_0}\times\cdots\times L^{p_0}\to L^{p_0/m,\infty}} \prod_{i=1}^{m} \langle |f_i|\rangle_{Q_0,p_0}.$$

Therefore, after adding the median term to the right hand side and relabelling we are left with the estimate

$$|T(\vec{f})(x)| \lesssim \sum_{\ell=0}^{\infty} 2^{-\ell\delta_0} \mathcal{T}^{\ell,p_0}_{\mathcal{S}(Q_0)}(\vec{f})(x) \quad \text{for a.e. } x \in Q_0.$$

We now argue as in [CR, Corollary A.1] (see also [C]) and apply Theorem 1.1 to obtain the pointwise estimate

$$|T(\vec{f})(x)| \lesssim \sum_{j=1}^{c_n} \mathcal{A}^{0,p_0}_{\mathscr{D}^j,\mathcal{S}_j}(\vec{f})(x) \quad \text{for a.e. } x \in Q_0, \tag{4.6}$$

which is the first assertion of Theorem 1.3. Finally, taking a sequence of cubes that grow to fill the space and by a limit procedure we obtain the second assertion:

$$\|T(\vec{f})\|_{\mathbb{X}} \lesssim_{T,m,n} \sup_{\mathscr{D},\mathcal{S}} \|\mathcal{A}^{p_0}_{\mathscr{D},\mathcal{S}}(\vec{f})\|_{\mathbb{X}}. \ \blacksquare$$

REMARK 4.2. Our local pointwise estimate (4.6) holds only for points inside a compact set. This is irrelevant in applications, since the estimate is powerful enough to deduce full norm estimates from it. In the particular case of ω-Calderón–Zygmund operators, the global pointwise estimate can be deduced from the local one by taking an appropriate partition of $\mathbb{R}^n$ (see for example [Ler3]). However, this technique does not seem feasible in our present setting. The reason is that for a given cube Q, Lerner [Ler3] can bound $T(f1_{3Q})$ by a sparse family of cubes inside Q, whereas we are able to bound $T(f1_{3Q})$ by a sparse family of cubes inside $3Q$.

5. Applications to certain singular integral operators with non-smooth kernels

5.1. Linear Fourier multipliers. Let m be a bounded function on $\mathbb{R}^n$. We define the multiplier operator T_m by setting

$$(T_m f)^\wedge(x) = m(x)\hat{f}(x)$$

where $\hat{f}$ is the Fourier transform of f.

Let $s \geq 1$, l be a positive integer and $\alpha = (\alpha_1, \ldots, \alpha_n)$ be a multi-index of nonnegative integers α_j with length $|\alpha| = |\alpha_1| + \cdots + |\alpha_n|$. Following [KW], we say that the function m is in $M(s,l)$ if

$$\sup_{R>0}\Big(R^{s|\alpha|-n} \int\limits_{R<|x|<2R} |\partial^\alpha m(x)|^s\,dx\Big)^{1/s} < \infty \quad \text{for all } |\alpha| \leq l. \tag{5.1}$$

Hörmander [H] showed that if $m \in M(2,l)$ and $l > n/2$, then the associated operators T_m are bounded on L^p for $1 < p < \infty$. The condition $m \in M(s,l)$ with $s \geq 2$ and $l > n/s$ was considered in [CT]. In [KW], the authors consider the class of $m \in M(s,l)$ with $s \leq 2$ and $l > n/s$. The following result is a direct consequence of [KW, Theorem 1].

Theorem 5.1. *Let $1 < s \leq 2$ and $m \in M(s,l)$ with $n/s < l < n$. Then T_m is bounded on L^p for $1 < p < \infty$.*

Moreover, the following estimate follows from [KW, Lemma 1]:

Lemma 5.2. *Let $1 < s \leq 2$ and $m \in M(s,l)$ with $n/s < l < n$. Then for any $p_0 > n/l$ there exists $\epsilon > 0$ such that for any ball B and $x, \overline{x} \in B$,*

$$\Big(\int\limits_{S_k(B)} |K(x,y) - K(\overline{x},y)|^{p_0'}\,dy\Big)^{1/p_0'} \leq C\frac{2^{-k\epsilon}}{(2^k r_B)^{n/p_0}}$$

for all $k \geq 2$, where $K(x,y)$ is the kernel of T_m.

Therefore, T_m satisfies conditions (H1) and (H2). Thus, as a consequence of Theorem 1.3, we obtain the following result.

Theorem 5.3. *Let $1 < s \leq 2$ and $m \in M(s,l)$ with $n/s < l < n$. For any $n/l < p_0 < \infty$:*

(a) *For $p_0 < p < \infty$ and $w \in A_{p/p_0}$, we have*

$$\|T_m f\|_{L^p(w)} \leq C_{T_m,p,p_0}[w]_{A_{p/p_0}}^{\max\{1,1/(p-p_0)\}}\|f\|_{L^p(w)}.$$

(b) *For $1 < p < p_0'$ and $w \in A_p \cap RH_{(p_0'/p)'}$, we have*

$$\|T_m f\|_{L^p(w)} \leq C_{T_m,p,p_0}[w]_{RH_{(p_0'/p)'}}^{\max\{p'-1,\frac{p'-1}{p'-p_0}\}}[w]_{A_p}^{\max\{p'-1,\frac{p'-1}{p'-p_0}\}}\|f\|_{L^p(w)}.$$

Proof. (a) From Theorem 5.1 and Lemma 5.2, T_m satisfies (H1) and (H2) for p_0. As a consequence of Theorem 1.3 we get

$$\begin{aligned}\|T_m f\|_{L^p(w)} &\le C_{T_m,p,p_0}[w]_{A_{p/p_0}}^{\max\{1,\frac{(p/p_0)'}{p}\}}\|f\|_{L^p(w)}\\ &= C_{T_m,p,p_0}[w]_{A_{p/p_0}}^{\max\{1,\frac{1}{p-p_0}\}}\|f\|_{L^p(w)}.\end{aligned}$$

(b) For $w \in A_p \cap RH_{(p_0'/p)'}$, we claim that

$$[\sigma]_{A_{p'/p_0}} \le [w]_{RH_{(p_0'/p)'}}^{p'-1}[w]_{A_p}^{p'-1} := [w]_{RH_{(p_0'/p)'}}^{1/(p-1)}[w]_{A_p}^{1/(p-1)}, \tag{5.2}$$

where $\sigma = w^{1-p'}$. Once (5.2) is proved, (b) follows immediately by duality.

To prove (5.2), we write

$$[\sigma]_{A_{p'/p_0}} = \sup_Q\Big(\fint_Q \sigma\Big)\Big(\fint_Q \sigma^{1-(p'/p_0)'}\Big)^{p'/p_0-1}.$$

This along with the fact that

$$(1-p')\Big[1-\Big(\frac{p'}{p_0}\Big)'\Big] = \frac{p_0}{p-p_0(p-1)} = \Big(\frac{p_0'}{p}\Big)'$$

implies that

$$[\sigma]_{A_{p'/p_0}} = \sup_Q\Big(\fint_Q w^{1-p'}\Big)\Big(\fint_Q w^{(p_0'/p)'}\Big)^{p'/p_0-1}.$$

Using the facts that $w \in RH_{(p_0'/p)'}$ and

$$\Big[\frac{p'}{p_0}-1\Big]\Big(\frac{p_0'}{p}\Big)' = \frac{1}{p-1},$$

we obtain

$$\begin{aligned}[\sigma]_{A_{p'/p_0}} &\le [w]_{RH_{(p_0'/p)'}}^{1/(p-1)}\sup_Q\Big(\fint_Q w^{1-p'}\Big)\Big(\fint_Q w\Big)^{1/(p-1)}\\ &\le [w]_{RH_{(p_0'/p)'}}^{1/(p-1)}[w]_{A_p}^{1/(p-1)}.\end{aligned}$$

This proves (5.2). ∎

Remark 5.4. Note that it was proved in [KW, Theorem 1] that under the assumptions of Theorem 5.3, T_m is bounded on $L^p(w)$ for $n/l < p < \infty$ and $w \in A_{pl/n}$ and hence by duality T_m is bounded on $L^p(w)$ for $1 < p < (n/l)'$ and $w \in A_p \cap RH_{((n/l)'/p)'}$. Hence, it is reasonable to expect that the weighted bounds in Theorem 5.3 still hold for $p_0 = n/l$. We leave it as an open problem.

5.2. Riesz transforms related to Schrödinger operators. Let $L = -\Delta + V$ be a Schrödinger operator on $\mathbb{R}^n$ with $n \geq 3$ where the potential V is in the reverse Hölder class RH_q for some $q > n/2$. Note that if $V \in RH_q$, $q \geq n$, then the Riesz transforms $\nabla L^{-1/2}$ and $L^{-1/2}\nabla$ turn out to be Calderón–Zygmund operators (see e.g. [S]). That is why we restrict ourselves to the case $V \in RH_q$ with $n/2 < q < n$.

We now recall the following result of [S] concerning the boundedness of $\nabla L^{-1/2}$ and $L^{-1/2}\nabla$.

Theorem 5.5. *Let $L = -\Delta + V$ be a Schrödinger operator on $\mathbb{R}^n$ with $n \geq 3$. Assume that $V \in RH_q, n/2 < q < n$. Let $p_0 = qn/(n-q)$. Then:*

(a) *$L^{-1/2}\nabla$ is bounded on L^p for $p_0' \leq p < \infty$.*
(b) *$\nabla L^{-1/2}$ is bounded on L^p for $1 < p \leq p_0$.*

We now apply Theorem 1.3 to get weighted bounds for these operators.

Theorem 5.6. *Let $L = -\Delta + V$ be a Schrödinger operator on $\mathbb{R}^n$ with $n \geq 3$. Assume that $V \in RH_q$, $n/2 < q < n$. Let $p_0 = qn/(n-q)$. Then:*

(a) *For $p_0' < p < \infty$ and $w \in A_{p/p_0'}$, we have*

$$\|L^{-1/2}\nabla f\|_{L^p(w)} \leq C_{L,p,q}[w]_{A_{p/p_0'}}^{\max\{1,1/(p-p_0')\}}\|f\|_{L^p(w)}.$$

(b) *For $1 < p < p_0$ and $w \in A_p \cap RH_{(p_0/p)'}$, we have*

$$\|\nabla L^{-1/2} f\|_{L^p(w)} \leq C_{L,p,q}[w]_{RH_{(p_0/p)'}}^{\max\{p'-1,\frac{p'-1}{p'-p_0'}\}}[w]_{A_p}^{\max\{p'-1,\frac{p'-1}{p'-p_0'}\}}\|f\|_{L^p(w)}.$$

Proof. (a) Let $K(x,y)$ be the kernel of $L^{-1/2}\nabla$. According to [GLP, proof of Theorem 1.6(iii)] there exists $\epsilon > 0$ such that for any ball B and $x, \overline{x} \in B$,

$$\Big(\int_{S_k(B)} |K(x,y) - K(\overline{x},y)|^{p_0}\,dy\Big)^{1/p_0} \leq C\frac{2^{-k\epsilon}}{(2^k r_B)^{n/p_0'}}$$

for all $k \geq 2$. Hence, (a) follows immediately from Theorem 1.3.

(b) This follows from (a) and the duality argument used in Theorem 5.3. ∎

Remark 5.7. It is worth noting that our approach also yields weighted bounds for other Riesz transforms, like $V^{1/2}L^{-1/2}, L^{-1/2}V^{1/2}$, VL^{-1} and $L^{-1}V$.

5.3. Multilinear Fourier multipliers. Another application of Theorem 1.3 is to obtain weighted bounds for multilinear Fourier multiplier operators.

For simplicity, we only consider the bilinear case. Let $m \in C^s(\mathbb{R}^{2n}\setminus\{0\})$, for some integer s, satisfy

$$|\partial_\xi^\alpha \partial_\eta^\beta m(\xi,\eta)| \le C_{\alpha,\beta}(|\xi|+|\eta|)^{-(|\alpha|+|\beta|)} \tag{5.3}$$

for all $|\alpha|+|\beta| \le s$ and $(\xi,\eta) \in \mathbb{R}^{2n}\setminus\{0\}$. The bilinear Fourier multiplier operator T_m is defined by

$$T_m(f,g)(x) = \frac{1}{(2\pi)^{2n}} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{ix\cdot(\xi+\eta)} m(\xi,\eta)\hat{f}(\xi)\hat{g}(\eta)\,d\xi\,d\eta$$

for all $f,g \in \mathcal{S}(\mathbb{R}^n)$. The associated kernel $K(x,y_1,y_2)$ is given by

$$K(x,y_1,y_2) = \check{m}(x-y_1,x-y_2) \tag{5.4}$$

where $\check{m}$ is the inverse Fourier transform of m. It is proved in [BD] that K satisfies (H2).

Proposition 5.8. *For any $p > 2n/s$, we have*

$$\Big(\int_{S_j(Q)}\int_{S_k(Q)} |K(x,y_1,y_2)-K(\overline{x},y_1,y_2)|^{p'}\,dy_1\,dy_2\Big)^{1/p'} \le C\frac{|x-\overline{x}|^{s-2n/p}}{|Q|^{s/n}}2^{-s\max\{j,k\}} \tag{5.5}$$

for all balls Q, all $x,\overline{x} \in \frac{1}{2}Q$ and $(j,k) \neq (0,0)$.

It was shown in [CM] that if (5.3) holds for $s > 4n$ then T_m maps $L^{p_1} \times L^{p_2}$ into L^p for all $1 < p_1,p_2,p < \infty$ with $1/p_1+1/p_2 = 1/p$. It was proved in [GT] that T_m maps boundedly $L^{p_1}\times L^{p_2}$ into L^p for all $1 < p_1,p_2 < \infty$ such that $1/p_1+1/p_2 = 1/p$ provided that (5.3) holds for $s \ge 2n+1$. However, in view of the linear case, the number of derivatives $s \ge 2n+1$ is not optimal and it is natural to expect that we only need $s \ge n+1$. The first positive answer is due to Tomita [T] who proved that if (5.3) holds for $s \ge n+1$, then T_m maps $L^{p_1}\times L^{p_2}$ into L^p for all $2 \le p_1,p_2,p < \infty$ such that $1/p_1+1/p_2 = 1/p$, and then by using multilinear interpolation and duality arguments he proved that T_m maps $L^{p_1}\times L^{p_2}$ into L^p for all $1 < p_1,p_2,p < \infty$ such that $1/p_1+1/p_2 = 1/p$. This result was then improved in [GS] to $p \le 1$ by using the L^r-based Sobolev space, $1 < r \le 2$. A particular case of [GS, Theorem 1.1] is the following theorem.

Theorem 5.9. *Assume that* (5.3) *holds for some $n+1 \le s \le 2n$. Then for any p_1,p_2 and p such that $2n/s < p_1,p_2 < \infty$ and $1/p_1+1/p_2 = 1/p$, the operator T_m maps $L^{p_1}\times L^{p_2}$ into L^p.*

We remark that the number $2n/s$ in Theorem 5.9 is contained implicitly in the proof of [GS, Theorem 1.1].

For any $2n/s < p_0$, by Theorem 5.9 and Proposition 5.8, T_m satisfies (H1) and (H2) for p_0. Then applying Theorem 1.3 we obtain

THEOREM 5.10. *Assume that* (5.3) *holds for some* $n+1 \leq s \leq 2n$. *Let* $2n/s < p_0$. *Then for any* p_1, p_2, p *such that* $p_0 < p_1, p_2 < \infty$, $1/p_1 + 1/p_2 = 1/p$, *and* $\vec{\omega} = (w_1, w_2) \in A_{\vec{P}/p_0}$ *with* $\vec{P} = (p_1, p_2)$, *we have*

$$\|T_m(f_1, f_2)\|_{L^p(v_{\vec{\omega}})} \leq C[\vec{w}]_{A_{\vec{P}/p_0}}^{\max\{1,(p_1/p_0)'/p,(p_2/p_0)'/p\}} \|f_1\|_{L^{p_1}(w_1)} \|f_2\|_{L^{p_2}(w_2)}.$$

REMARK 5.11. Similarly to the linear case in Theorem 5.3, it is natural to ask whether the weighted bound in Theorem 5.10 holds true for $p_0 = 2n/s$. This will be a subject of our future research.

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References

[Bu] S. M. Buckley, *Estimates for operator norms on weighted spaces and reverse Jensen inequalities*, Trans. Amer. Math. Soc. 340 (1993), 253–272.

[BD] T. A. Bui and X. T. Duong, *Weighted norm inequalities for multilinear operators and applications to multilinear Fourier multipliers*, Bull. Sci. Math. 137 (2013), 63–75.

[CT] A. P. Calderón and A. Torchinsky, *Parabolic maximal functions associated with a distribution. II*, Adv. Math. 24 (1977), 101–171.

[CD] W. Chen and W. Damian, *Weighted estimates for the multisublinear maximal function*, Rend. Circ. Mat. Palermo 62 (2015), 379–391.

[CM] R. Coifman et Y. Meyer, *Au delà des opérateurs pseudo-différentiels*, Astérisque 57 (1978).

[C] J. M. Conde-Alonso, *Geometric and probabilistic methods in Calderón–Zygmund theory*, PhD thesis, 2015.

[CR] J. M. Conde-Alonso and G. Rey, *A pointwise estimate for positive dyadic shifts and some applications*, Math. Ann. 365 (2016), 1111–1135.

[DLP] W. Damián, A. K. Lerner and C. Pérez, *Sharp weighted bounds for multilinear maximal functions and Calderón–Zygmund operators*, J. Fourier Anal. Appl. 21 (2015), 161–181.

[DGPP] O. Dragičević, L. Grafakos, M.C. Pereyra and S. Petermichl, *Extrapolation and sharp norm estimates for classical operators on weighted Lebesgue spaces*, Publ. Mat. 49 (2005), 73–91.

[G] L. Grafakos, *Modern Fourier Analysis*, 3rd ed., Grad. Texts Math. 250, Springer, New York, 2014.

[GS] L. Grafakos and Z. Si, *The Hörmander type multiplier theorem for multilinear operators*, J. Reine Angew. Math. 668 (2012), 133–147.

[GT] L. Grafakos and R. H. Torres, *Multilinear Calderón–Zygmund theory*, Adv. Math. 165 (2002), 124–164.

[GLP] Z. Guo, P. Li and L. Peng, *L^p boundedness of commutators of Riesz transforms associated to Schrödinger operators*, J. Math. Anal. Appl. 341 (2008), 421–432.

[H] L. Hörmander, *Estimates for translation invariant operators in L^p spaces*, Acta Math. 104 (1960), 93–139.

[Hyt1] T. Hytönen, *The sharp weighted bound for general Calderón–Zygmund operators*, Ann. of Math. (2) 175 (2012), 1473–1506.

[Hyt2] T. Hytönen, *The A_2 theorem: Remarks and complements*, in: Contemp. Math. 612, Amer. Math. Soc., Providence, RI, 2014, 91–106.

[HLP] T. Hytönen, M. Lacey and C. Pérez, *Sharp weighted bounds for the q-variation of singular integrals*, Bull. London Math. Soc. 45 (2013), 529–540.

[KW] D. S. Kurtz and R. L. Wheeden, *Results on weighted norm inequalities for multipliers*, Trans. Amer. Math. Soc. 255 (1979), 343–362.

[Ler1] A. K. Lerner, *A pointwise estimate for the local sharp maximal function with applications to singular integrals*, Bull. London Math. Soc. 42 (2010), 843–856.

[Ler2] A. K. Lerner, *On an estimate of Calderón–Zygmund operators by dyadic positive operators*, J. Anal. Math. 121 (2013), 141–161.

[Ler3] A. K. Lerner, *On pointwise estimates involving sparse operators*, New York J. Math. 22 (2016), 341–349.

[Ler4] A. K. Lerner, *A simple proof of the A_2 conjecture*, Int. Math. Res. Notices 2013, 3159–3170.

[LO$^+$] A. K. Lerner, S. Ombrosi, C. Pérez, R. H. Torres and R. Trujillo-González, *New maximal functions and multiple weights for the multilinear Calderón–Zygmund theory*, Adv. Math. 220 (2009), 1222–1264.

[LMS] K. Li, K. Moen and W. Sun, *The sharp weighted bound for multilinear maximal functions and Calderón–Zygmund operators*, J. Fourier Anal. Appl. 20 (2014), 751–765.

[LMPR] M. Lorente, J. M. Martell, C. Pérez and M. S. Riveros, *Generalized Hörmander conditions and weighted endpoint estimates*, Studia Math. 195 (2009), 157–192.

[LMRT] M. Lorente, J. M. Martell, M. S. Riveros and A. de la Torre, *Generalized Hörmander's condition, commutators and weights*, J. Math. Anal. Appl. 342 (2008), 1399–1425.

[LRT] M. Lorente, M. S. Riveros and A. de la Torre, *Weighted estimates for singular integral operators satisfying Hörmander's conditions of Young type*, J. Fourier Anal. Apl. 11 (2005), 497–509.

[Mo] K. Moen, *Sharp weighted bounds without testing or extrapolation*, Arch. Math. (Basel) 99 (2012), 457–466.

[Mu] B. Muckenhoupt, *Weighted norm inequalities for the Hardy maximal function*, Trans. Amer. Math. Soc. 165 (1972), 207–226.

[P1] S. Petermichl, *The sharp bound for the Hilbert transform on weighted Lebesgue spaces in terms of the classical A_p characteristic*, Amer. J. Math. 129 (2007), 1355–1375.

[P2] S. Petermichl, *The sharp weighted bound for the Riesz transforms*, Proc. Amer. Math. Soc. 136 (2008), 1237–1249.

[S] Z. Shen, *L^p estimates for Schrödinger operators with certain potentials*, Ann. Inst. Fourier (Grenoble) 45 (1995), 513–546.

[T] N. Tomita, *A Hörmander type multiplier theorem for multilinear operators*, J. Funct. Anal. 259 (2010), 2028–2044.

The Anh Bui, Xuan Thinh Duong
Department of Mathematics
Macquarie University
North Ryde, NSW 2109, Australia
E-mail: the.bui@mq.edu.au
bt_anh80@yahoo.com
xuan.duong@mq.edu.au

José M. Conde-Alonso
Instituto de Ciencias Matemáticas
Consejo Superior de Investigaciones Científicas
C/ Nicolás Cabrera 13-15
28049 Madrid, Spain
E-mail: jose.conde@icmat.es

Mahdi Hormozi
Department of Mathematical Sciences
Division of Mathematics
University of Gothenburg
41296 Gothenburg, Sweden
and
Department of Mathematics
Shiraz University
Shiraz 71454, Iran
E-mail: me.hormozi@gmail.com