

# The variety of subadditive functions for finite group schemes

by

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**Abstract.** For a finite group scheme, the subadditive functions on finite-dimensional representations are studied. It is shown that the projective variety of the cohomology ring can be recovered from the equivalence classes of subadditive functions. Using Crawley-Boevey's correspondence between subadditive functions and endofinite modules, we obtain an equivalence relation on the set of point modules introduced in our joint work with Iyengar and Pevtsova. This corresponds to the equivalence relation on  $\pi$ -points introduced by Friedlander and Pevtsova.

**1. Introduction.** A theorem of Crawley-Boevey [5] gives a correspondence between endofinite modules for a ring  $A$  and subadditive functions on the finitely presented  $A$ -modules.

We examine this correspondence in the context of finite group schemes, for certain endofinite 'point modules' for a finite group scheme  $G$  introduced in our joint work with Iyengar and Pevtsova [3]. These point modules come from the  $\pi$ -points introduced by Friedlander and Pevtsova [6]. There is a natural equivalence relation on  $\pi$ -points, and it is proved in [6] that the equivalence classes of  $\pi$ -points can be used to reconstruct the variety  $\text{Proj } H^*(G, k)$ .

We translate the equivalence relation of Friedlander and Pevtsova into a corresponding equivalence relation on subadditive functions. This enables us to prove that one can recover  $\text{Proj } H^*(G, k)$  from the subadditive functions on finite-dimensional  $G$ -modules in a natural way (Theorem 4.1).

**2. Subadditive functions and endofinite modules.** We briefly review Crawley-Boevey's correspondence between subadditive functions and endofinite modules.

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We fix a ring  $A$  and consider the category of (right)  $A$ -modules. Let  $\text{mod } A$  denote the full subcategory of finitely presented  $A$ -modules. For an  $A$ -module  $M$  let  $\ell_A(M)$  denote its composition length.

A *subadditive function*  $\chi: \text{mod } A \rightarrow \mathbb{N}$  assigns to each finitely presented  $A$ -module a non-negative integer such that

- (1)  $\chi(X \oplus Y) = \chi(X) + \chi(Y)$  for all  $X, Y \in \text{mod } A$ , and
- (2)  $\chi(X) + \chi(Z) \geq \chi(Y)$  for each exact sequence  $X \rightarrow Y \rightarrow Z \rightarrow 0$  in  $\text{mod } A$ .

A subadditive function  $\chi \neq 0$  is *irreducible* if  $\chi$  cannot be written as a sum of two non-zero subadditive functions.

An  $A$ -module  $M$  is called *endofinite* if it has finite composition length when viewed as a left module over its endomorphism ring  $\text{End}_A(M)$ . An endofinite  $A$ -module  $M$  gives rise to a subadditive function  $\chi_M$  by setting

$$\chi_M(X) := \ell_{\text{End}_A(M)}(\text{Hom}_A(X, M)) \quad \text{for } X \in \text{mod } A.$$

The following theorem of Crawley-Boevey [5, §5] provides the context for our study of subadditive functions.

**THEOREM 2.1.** *Any subadditive function  $\text{mod } A \rightarrow \mathbb{N}$  can be written uniquely as a finite sum of irreducible subadditive functions. Sending an endofinite  $A$ -module  $M$  to  $\chi_M$  induces a bijection between the isomorphism classes of indecomposable endofinite  $A$ -modules and the irreducible subadditive functions  $\text{mod } A \rightarrow \mathbb{N}$ . ■*

Note that every endofinite module decomposes uniquely into indecomposable endofinite modules. We have  $\chi_{M \oplus M'} = \chi_M + \chi_{M'}$  when  $M$  and  $M'$  have no common indecomposable summand, while  $\chi_{M \oplus M} = \chi_M$  [5, §5].

**3. The additive locus of a subadditive function.** We fix a ring  $A$  such that  $\text{mod } A$  is an abelian category. For a subadditive function  $\chi: \text{mod } A \rightarrow \mathbb{N}$  we define the *additive locus*  $\text{Adloc}(\chi)$  as the full subcategory of objects  $Z \in \text{mod } A$  such that for every exact sequence  $0 \rightarrow X \rightarrow Y \rightarrow Z \rightarrow 0$  in  $\text{mod } A$ ,

$$\chi(X) - \chi(Y) + \chi(Z) = 0.$$

We collect some basic properties of the additive locus.

**LEMMA 3.1.** *Let  $M$  be an endofinite  $A$ -module and  $Z \in \text{mod } A$ . Then  $Z$  belongs to  $\text{Adloc}(\chi_M)$  if and only if  $\text{Ext}_A^1(Z, M) = 0$ .*

*Proof.* Applying  $\text{Hom}_A(-, M)$  to an exact sequence  $0 \rightarrow X \rightarrow Y \rightarrow Z \rightarrow 0$  in  $\text{mod } A$  induces a long exact sequence

$$0 \rightarrow \text{Hom}_A(Z, M) \rightarrow \text{Hom}_A(Y, M) \rightarrow \text{Hom}_A(X, M) \rightarrow \text{Ext}_A^1(Z, M) \rightarrow \dots$$

of  $\text{End}_A(M)$ -modules. Clearly,  $\text{Ext}_A^1(Z, M) = 0$  implies

$$\chi_M(X) - \chi_M(Y) + \chi_M(Z) = 0.$$

The converse follows by choosing for  $Y$  a projective  $A$ -module. ■

LEMMA 3.2. *Let  $\chi, \chi' : \text{mod } A \rightarrow \mathbb{N}$  be subadditive functions. Then*

$$\text{Adloc}(\chi + \chi') = \text{Adloc}(\chi) \cap \text{Adloc}(\chi').$$

*Proof.* Let  $\eta : 0 \rightarrow X \rightarrow Y \rightarrow Z \rightarrow 0$  be an exact sequence in  $\text{mod } A$ . Then  $\chi(\eta) := \chi(X) - \chi(Y) + \chi(Z)$  is a non-negative integer. Thus  $(\chi + \chi')(\eta) = 0$  if and only if  $\chi(\eta) = 0 = \chi'(\eta)$ . From this observation the assertion of the lemma follows. ■

LEMMA 3.3. *Let  $\chi : \text{mod } A \rightarrow \mathbb{N}$  be a subadditive function. Then there exists an endofinite  $A$ -module  $M$  such that  $\text{Adloc}(\chi) = \text{Adloc}(\chi_M)$ .*

*Proof.* We apply Theorem 2.1. Write  $\chi = \sum_i n_i \chi_i$  with  $\chi_i$  irreducible and  $\chi_i \neq \chi_j$  for all  $i \neq j$ . There are indecomposable endofinite  $A$ -modules  $M_i$  such that  $\chi_i = \chi_{M_i}$  for all  $i$ , and  $\sum_i \chi_i = \chi_M$  for  $M = \bigoplus_i M_i$ . Then  $\text{Adloc}(\chi) = \text{Adloc}(\chi_M)$  by Lemma 3.2. ■

Let  $\chi, \chi' : \text{mod } A \rightarrow \mathbb{N}$  be subadditive functions. We set

$$\chi \geq \chi' :\Leftrightarrow \text{Adloc}(\chi) \subseteq \text{Adloc}(\chi')$$

and call  $\chi$  and  $\chi'$  *equivalent* if  $\text{Adloc}(\chi) = \text{Adloc}(\chi')$ . Thus the equivalence classes of subadditive functions form a partially ordered set.

**4. Subadditive functions for finite group schemes.** Let  $G$  be a finite group scheme over a field  $k$ . Thus  $G$  is an affine group scheme such that its coordinate algebra  $k[G]$  is finite-dimensional as a  $k$ -vector space. The  $k$ -linear dual of  $k[G]$  is a cocommutative Hopf algebra, called the *group algebra* of  $G$ , and denoted  $kG$ . We identify  $G$ -modules with modules over the group algebra  $kG$ . The category of finite-dimensional  $G$ -modules is denoted by  $\text{mod } G$ .

The tensor product over  $k$  induces one for  $\text{mod } G$  via the diagonal action of  $G$ . A subcategory  $\mathcal{C}$  of  $\text{mod } G$  is called *tensor closed* if  $M \in \mathcal{C}$  and  $N \in \text{mod } G$  imply that  $M \otimes_k N$  is in  $\mathcal{C}$ .

We write  $H^*(G, k)$  for the cohomology algebra of  $G$ , and  $\text{Proj } H^*(G, k)$  for the set of its homogeneous prime ideals not containing the unique maximal ideal of positive degree elements. Note that  $H^*(G, k)$  acts on  $\text{Ext}_G^*(M, N)$  for all  $G$ -modules  $M, N$ . The *support* of a finite-dimensional  $G$ -module  $M$  is

$$\text{supp}_G(M) := \{\mathfrak{p} \in \text{Proj } H^*(G, k) \mid \text{Ext}_G^*(M, M)_{\mathfrak{p}} \neq 0\}.$$

The following theorem says that  $\text{Proj } H^*(G, k)$  can be recovered from the equivalence classes of subadditive functions on  $\text{mod } G$ .

We call a subadditive function on  $\text{mod } G$  *tensor closed* if its additive locus is a tensor closed subcategory of  $\overline{\text{mod } G}$ .

An element  $x$  of a poset is *join irreducible* if it is not the supremum of elements that are strictly smaller than  $x$ .

**THEOREM 4.1.** *Let  $G$  be a finite group scheme over a field  $k$  and let  $P(G, k)$  denote the partially ordered set of equivalence classes of tensor closed subadditive functions  $\chi: \text{mod } G \rightarrow \mathbb{N}$ . If the class of  $\chi$  is join irreducible, then there exists a unique  $\mathfrak{p} \in \text{Proj } H^*(G, k)$  such that*

$$\text{Adloc}(\chi) = \{M \in \text{mod } G \mid \mathfrak{p} \notin \text{supp}_G(M)\}.$$

*Sending  $\chi$  to  $\mathfrak{p}$  induces an order isomorphism between the set of join irreducible elements of  $P(G, k)$  and  $\text{Proj } H^*(G, k)$ .*

The proof will be given in §7.

**5. The additive locus of a tensor closed function.** In this section we study the additive locus of a subadditive function that is tensor closed. We need the following result, which essentially comes from [1].

For  $G$ -modules  $M, N$  and  $n \in \mathbb{Z}$ , we write  $\widehat{\text{Ext}}_G^n(M, N)$  for the  $n$ th Tate extension group; it equals  $H^n(\text{Hom}_G(\mathfrak{t}M, N))$  where  $\mathfrak{t}M$  denotes a Tate resolution of  $M$ . Note that

$$\widehat{\text{Ext}}_G^n(M, N) \cong \text{Ext}_G^n(M, N) \quad \text{for } n > 0.$$

**THEOREM 5.1.** *Given a finite group scheme  $G$  over a field  $k$ , there exists a positive integer  $r$  such that for any  $G$ -modules  $M$  and  $N$ , if  $\widehat{\text{Ext}}_G^n(M, N) = 0$  for  $r$  consecutive values of  $n$  then  $\widehat{\text{Ext}}_G^n(M, N) = 0$  for all  $n$ , positive and negative.*

*Proof.* The proof for finite groups given in [1] works just as well for finite group schemes. The input is the finite generation of cohomology, which for finite group schemes was proved by Friedlander and Suslin [7]. ■

A full subcategory  $\mathcal{C}$  of  $\text{mod } G$  is said to be *thick* if any direct summand of a module in  $\mathcal{C}$  is also in  $\mathcal{C}$  and for every exact sequence  $0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$  in  $\text{mod } G$  with two of  $M, M', M''$  in  $\mathcal{C}$  also the third is in  $\mathcal{C}$ .

**COROLLARY 5.2.** *Let  $\chi: \text{mod } G \rightarrow \mathbb{N}$  be a tensor closed subadditive function. Then  $\text{Adloc}(\chi)$  is a thick subcategory of  $\text{mod } G$ . If  $\chi = \chi_M$  for some endofinite  $G$ -module  $M$ , then*

$$\text{Adloc}(\chi_M) = \{X \in \text{mod } G \mid \widehat{\text{Ext}}_G^*(X, M) = 0\}.$$

*Proof.* We may assume that  $\chi = \chi_M$  for some endofinite  $G$ -module  $M$  by Lemma 3.3. Fix  $X \in \text{mod } G$ . From Lemma 3.1 it follows that  $X$  is in  $\text{Adloc}(\chi)$  if and only if  $\text{Ext}_G^1(X, M) = 0$ . Let  $\Omega(X)$  denote the kernel of a projective cover of  $X$ , and observe that  $\Omega(X) \cong X \otimes_k \Omega(k)$  up to

projective direct summands. Using dimension shift and the fact that  $\chi$  is tensor closed, we find that  $X$  is in  $\text{Adloc}(\chi)$  if and only if  $\text{Ext}_G^n(X, M) = 0$  for every  $n > 0$ . Now Theorem 5.1 implies that  $X$  is in  $\text{Adloc}(\chi)$  if and only if  $\widehat{\text{Ext}}_G^*(X, M) = 0$ . From this description of  $\text{Adloc}(\chi)$  and the long exact sequence for  $\widehat{\text{Ext}}_G^*(-, M)$  it follows that  $\text{Adloc}(\chi)$  is a thick subcategory. ■

REMARK 5.3. Given any subadditive function  $\chi: \text{mod } G \rightarrow \mathbb{N}$ , we see that

$$\chi' := \sum_{S \text{ simple}} \chi(- \otimes_k S)$$

is up to equivalence the unique minimal tensor closed subadditive function such that  $\chi' \geq \chi$ . In fact, Lemma 3.2 shows that

$$\text{Adloc}(\chi') = \{X \in \text{mod } G \mid X \otimes_k Y \in \text{Adloc}(\chi) \text{ for all } Y \in \text{mod } G\}.$$

**6. Tensor closed thick subcategories and  $\pi$ -points.** In this section we recall some of the results from our joint work with Iyengar and Pevtsova [2, 3]. The first result is a classification of tensor closed thick subcategories of  $\text{mod } G$ , which has been anticipated in [6].

For a subcategory  $\mathcal{C}$  of  $\text{mod } G$  we set

$$\text{supp}_G(\mathcal{C}) := \bigcup_{M \in \mathcal{C}} \text{supp}_G(M).$$

A subset  $\mathcal{U}$  of  $\text{Proj } H^*(G, k)$  is called *specialisation closed* if whenever  $\mathfrak{p}$  is in  $\mathcal{U}$  so is any prime  $\mathfrak{q}$  containing  $\mathfrak{p}$ .

THEOREM 6.1. *Let  $G$  be a finite group scheme over a field  $k$ . Then the assignments*

$$\mathcal{C} \mapsto \text{supp}_G(\mathcal{C}) \quad \text{and} \quad \mathcal{U} \mapsto \{M \in \text{mod } G \mid \text{supp}_G(M) \subseteq \mathcal{U}\}$$

*give mutually inverse and inclusion preserving bijections between the non-zero tensor closed thick subcategories of  $\text{mod } G$  and the specialisation closed subsets of  $\text{Proj } H^*(G, k)$ .*

*Proof.* See [2, Theorem 10.3]. ■

In [6], Friedlander and Pevtsova introduced, for a finite group scheme  $G$  over a field  $k$  of characteristic  $p > 0$ , the notion of a  $\pi$ -point. This is by definition a flat algebra homomorphism  $\alpha: K[t]/(t^p) \rightarrow KG$  for some field extension  $K/k$  such that  $\alpha$  factors through the group algebra of a unipotent abelian subgroup scheme of  $G_K$ , where  $G_K$  denotes the group scheme over  $K$  with group algebra  $KG := kG \otimes_k K$ . Let  $\alpha^*: \text{mod } G_K \rightarrow \text{mod } K[t]/(t^p)$  denote restriction along  $\alpha$ . We set

$$\text{Thick}(\alpha) := \{M \in \text{mod } G \mid \alpha^*(M \otimes_k K) \text{ is projective}\}$$

and observe that  $\text{Thick}(\alpha)$  is a tensor closed thick subcategory of  $\text{mod } G$ .

Two  $\pi$ -points  $\alpha$  and  $\beta$  are *equivalent* if  $\text{Thick}(\alpha) = \text{Thick}(\beta)$ , and the equivalence classes are in natural bijection with the points of  $\text{Proj } H^*(G, k)$  (see [6, Theorem 3.6]). The following theorem makes this correspondence explicit.

**THEOREM 6.2.** *Let  $\mathfrak{p} \in \text{Proj } H^*(G, k)$ . Then there exists a  $\pi$ -point  $\alpha$  such that*

$$\text{Thick}(\alpha) = \{M \in \text{mod } G \mid \mathfrak{p} \notin \text{supp}_G(M)\}.$$

*Proof.* See [2, Theorem 6.1]. ■

**7. Proof of the main theorem.** This section provides the proof of Theorem 4.1, and we begin with some preparation.

Let  $\alpha: K[t]/(t^p) \rightarrow KG$  be a  $\pi$ -point for  $G$ . This gives rise to a subadditive function  $\chi_\alpha$  on  $\text{mod } G$  by setting

$$\chi_\alpha := \dim_K(\text{Hom}_{K[t]/(t^p)}(\alpha^*(- \otimes_k K), K)).$$

We may think of  $\chi_\alpha$  as the composite

$$\text{mod } G \xrightarrow{- \otimes_k K} \text{mod } G_K \xrightarrow{\alpha^*} \text{mod } K[t]/(t^p) \xrightarrow{\chi_K} \mathbb{N}.$$

**LEMMA 7.1.** *We have  $\text{Adloc}(\chi_\alpha) = \text{Thick}(\alpha)$ .*

*Proof.* Let  $Z \in \text{mod } G$ . If  $\alpha^*(Z \otimes_k K)$  is a projective  $K[t]/(t^p)$ -module, then for any exact sequence  $0 \rightarrow X \rightarrow Y \rightarrow Z \rightarrow 0$  in  $\text{mod } G$  we have

$$\chi_\alpha(X) - \chi_\alpha(Y) + \chi_\alpha(Z) = 0.$$

For the converse, choose an exact sequence  $0 \rightarrow X \rightarrow Y \rightarrow Z \rightarrow 0$  in  $\text{mod } G$  with  $Y$  projective. Thus  $\alpha^*(Y \otimes_k K)$  is projective. If

$$\chi_\alpha(X) - \chi_\alpha(Y) + \chi_\alpha(Z) = 0,$$

then the sequence

$$0 \rightarrow \alpha^*(X \otimes_k K) \rightarrow \alpha^*(Y \otimes_k K) \rightarrow \alpha^*(Z \otimes_k K) \rightarrow 0$$

splits, and therefore  $\alpha^*(Z \otimes_k K)$  is projective. ■

**LEMMA 7.2.** *Let  $X$  be a  $T_0$  topological space. Fix a set  $P$  of closed subsets of  $X$  that contains the closure  $\{x\}$  for each  $x \in X$ , and view  $P$  as a poset via the inclusion order. Then the assignment  $x \mapsto \overline{\{x\}}$  identifies  $X$  with the set of join irreducible elements of  $P$ .*

*Proof.* Straightforward. ■

*Proof of Theorem 4.1.* We consider  $\text{Proj } H^*(G, k)$  with the Hochster dual of the Zariski topology. Thus the open subsets are precisely the specialisation closed subsets [8]. The assignment

$$\chi \mapsto \text{Proj } H^*(G, k) \setminus \text{supp}_G(\text{Adloc}(\chi))$$

identifies the equivalence classes of subadditive functions  $\text{mod } G \rightarrow \mathbb{N}$  that are tensor closed with certain closed subsets of  $\text{Proj } H^*(G, k)$ . This follows from Corollary 5.2 and Theorem 6.1. On the other hand, Theorem 6.2 provides for each  $\mathfrak{p} \in \text{Proj } H^*(G, k)$  a  $\pi$ -point  $\alpha$  such that  $\chi_\alpha$  is a tensor closed subadditive function satisfying

$$\text{Adloc}(\chi_\alpha) = \{M \in \text{mod } G \mid \mathfrak{p} \notin \text{supp}_G(M)\}$$

by Lemma 7.1. Thus the function  $\chi_\alpha$  is sent to the closure  $\overline{\{\mathfrak{p}\}} = \{\mathfrak{q} \mid \mathfrak{q} \subseteq \mathfrak{p}\}$ . Moreover, given a  $\pi$ -point  $\beta$  corresponding to  $\mathfrak{q} \in \text{Proj } H^*(G, k)$  we have

$$\chi_\alpha \geq \chi_\beta \Leftrightarrow \text{Adloc}(\chi_\alpha) \subseteq \text{Adloc}(\chi_\beta) \Leftrightarrow \mathfrak{p} \supseteq \mathfrak{q}.$$

Now the assertion of the theorem follows from Lemma 7.2. ■

**8.  $\pi$ -points and point modules.** Let  $G$  be a finite group scheme over a field  $k$  of characteristic  $p > 0$ . To each  $\pi$ -point  $\alpha: K[t]/(t^p) \rightarrow KG$  corresponds a *point module*

$$\Delta_G(\alpha) := \text{res}_k^K(\text{Hom}_{K[t]/(t^p)}(KG, K)).$$

This is an endofinite  $G$ -module and plays a prominent role in our recent work with Iyengar and Pevtsova [3].

LEMMA 8.1. *We have  $\chi_{\Delta_G(\alpha)} = \chi_\alpha$ .*

*Proof.* Adjunction gives for each  $M \in \text{mod } G$  a natural isomorphism

$$\begin{aligned} \text{Hom}_G(M, \Delta_G(\alpha)) &\cong \text{Hom}_{G_K}(M \otimes_k K, \text{Hom}_{K[t]/(t^p)}(KG, K)) \\ &\cong \text{Hom}_{K[t]/(t^p)}(\alpha^*(M \otimes_k K), K), \end{aligned}$$

which restricts to submodules over the endomorphism rings of  $\Delta_G(\alpha)$  and  $K$  respectively. ■

COROLLARY 8.2. *Let  $G$  be a finite group scheme over a field  $k$ . Given  $\pi$ -points  $\alpha$  and  $\beta$  of  $G$ , the following conditions are equivalent:*

- (1) *The  $\pi$ -points  $\alpha$  and  $\beta$  are equivalent.*
- (2) *The subadditive functions  $\chi_{\Delta_G(\alpha)}$  and  $\chi_{\Delta_G(\beta)}$  are equivalent.*

*Proof.* Combine Lemmas 7.1 and 8.1. ■

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