FUNDAMENTA
MATHEMATICAE
239 (2017)

On $0,1$-laws and asymptotics of definable sets in geometric Fraïssé classes

by

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Abstract. We examine one consequence for the generic theory $T_{\mathbf{C}}$ of a geometric Fraïssé class $\mathbf{C}$ when $\mathbf{C}$ has the $0,1$-law for first-order logic with convergence to $T_{\mathbf{C}}$ itself. We show that in this scenario, if the asymptotic probability measure in play is not terribly exotic, then $\mathbf{C}$ is "very close" to being a 1-dimensional asymptotic class—so that $T_{\mathbf{C}}$ is supersimple of finite SU-rank.

Introduction. In much of the existing work on $0,1$-laws for Fraïssé classes $\mathbf{C}$, researchers have focused almost entirely on the most ordinary of asymptotic probability measures $\mu = (\mu_N)_N$—namely, μ_N is the uniform probability measure on members of $\mathbf{C}$ with universe $N = \{0, 1, \ldots, N-1\}$. When there is a $0,1$-law relative to such μ with $\mathrm{Th}(\mu) = T_{\mathbf{C}}$ (1), one very often finds that $T_{\mathbf{C}}$ was already a very special sort of theory. For example, if $\mathbf{G}$ is the class of all finite graphs, then $T_{\mathbf{G}}$, the theory of the random graph, is supersimple of SU-rank 1, and one finds something similar when $\mathbf{C}$ is the class of finite bipartite graphs, the class of finite partial orders of rank 2, finite directed trees of height 2, and so on. There is a sense that if a class $\mathbf{C}$ has the $0,1$-law in a way similar to $\mathbf{G}$, then geometrically speaking, $T_{\mathbf{C}}$ is very much like $T_{\mathbf{G}}$.

This discussion requires an answer to the question, "What does it mean for $\mathbf{C}$ to have the $0,1$-law in a way similar to $\mathbf{G}$?" In this paper, we answer this question for geometric Fraïssé classes (i.e. $T_{\mathbf{C}}$ is a geometric theory) by focusing on (i) the conditional independence properties of the asymptotic probability measure μ, and (ii) the requirement that $\mathrm{Th}(\mu) = T_{\mathbf{C}}$. We find

2010 *Mathematics Subject Classification*: 03C13, 03C15, 03C45, 05A16.
Key words and phrases: Fraïssé classes, asymptotic classes, $0,1$-laws, simple theories, geometric theories.
Received 12 August 2015; revised 11 November 2016 and 31 January 2017.
Published online 23 June 2017.

(1) $T_{\mathbf{C}}$ being the complete theory of the generic model, or Fraïssé limit, of $\mathbf{C}$.

DOI: 10.4064/fm122-1-2017

that if μ is not too weird, then not only is $T_{\mathbf{C}}$ supersimple of finite rank (as is $T_{\mathbf{G}}$), but up to excluding a neglible part of $\mathbf{C}$, the sizes of definable sets in members of $\mathbf{C}$ are under very tight, uniform control in a sense proposed in [14, 6, 7], generalizing results of [5] for finite fields (which in turn extend the better known Lang–Weil estimates [13]). Speaking formally, we prove the following theorem:

THEOREM 2.1. *Let* $\mathbf{C}$ *be a geometric Fraïssé class, and suppose that* $\mathbf{C}$ *has the* $0,1$*-law for first-order logic with* $\mathrm{Th}(\mu) = T_{\mathbf{C}}$ *relative to an asymptotic probability measure* μ *that has almost independent sampling. Then* $\mathbf{C}$ *contains a cofinal superrobust subclass* $\mathbf{D}$ *such that* $\lim_{N\to\infty} \mu_N[\mathbf{D}] = 1$ *and which is a* 1*-dimensional asymptotic class.*

Using theorems of [14, 7, 9], we derive Theorem 2.2 below as a corollary. This theorem can be viewed as the companion of the result of [1], which states that, for strictly binary languages, for any algebraically trivial $\aleph_0$-categorical supersimple theory T of SU-rank 1, one can define an asymptotic probability measure for the age of its countable model whose almost sure theory is T.

THEOREM 2.2. *Let* $\mathbf{C}$ *be a geometric Fraïssé class, and suppose that* $\mathbf{C}$ *has the* $0,1$*-law for first-order logic with* $\mathrm{Th}(\mu) = T_{\mathbf{C}}$ *relative to an asymptotic probability measure* μ *that has almost independent sampling. Then* $T_{\mathbf{C}}$ *is supersimple, and for every definable set* X *of the generic model* $\mathcal{M}$ *of* $\mathbf{C}$*,* $D(X)$ *is bounded by the algebraic dimension of* X*.*

Regarding certain open questions of the form, “Does a certain geometric Fraïssé class $\mathbf{C}$ have the $0,1$-law relative to some unknown measure?” there is another moral in this story: An asymptotic probability measure that yields a $0,1$-law with convergence to an *un*simple $\aleph_0$-categorical theory would have to be profoundly exotic.

Several classes that fall under our rubric here have, for some time, been known to have subclasses that are superrobust and 1-dimensional asymptotic. Among these are the class $\mathbf{G}$ of all finite graphs and, more generally, the class $\mathbf{H}_r$ of all finite r-regular hypergraphs (r-hypergraphs). In [14], it is noted that since the class of Paley graphs is uniformly definable in the class of finite fields, and since the class of finite fields is a 1-dimensional asymptotic class (by way of [5, 13], the motivating example of the definition), the class of Paley graphs is asymptotic. Moreover, it is known from [3] that the “eventual theory” of Paley graphs is precisely the theory of the random graph, and this class is superrobust. However, the class of Paley graphs is of asymptotic measure 0 within the class of all finite graphs, so Theorem 2.1 is stronger in that it guarantees that $\mathbf{G}$ is 1-dimensional asymptotic up to removing a “null set.” Our proof of Theorem 2.1 also obviates the appeal to algebra/number theory implicit in focusing on Paley graphs. In [2], it is shown that for every r, the generic r-hypergraph is interpretable in any infinite pseudo-finite field in a uniform way, and again, appealing to [5, 13], it follows that $\mathbf{H}_r$ contains a cofinal superrobust subclass that is 1-dimensional asymptotic. As in the Paley graph construction, the members of this subclass arise from solution sets of a well-chosen symmetric polynomial, so again, this subclass is "very small" compared to $\mathbf{H}_r$, while Theorem 2.1 guarantees that $\mathbf{H}_r$ is 1-dimensional asymptotic up to removing a "null set" and the proof is probabilistic instead of algebraic in nature.

Finally, we remark that there seems to be an important connection between our result here and the work of [12] on pseudo-finite countably categorical theories. The machinery of disjoint n-amalgamation in [12] appears to provide a means for generating asymptotic probability measures that have almost independent sampling, so it is not surprising that both approaches allow one to conclude that $T_{\mathbf{C}}$ is supersimple in many cases. An important divergence between [12] and our work here is the "filtered" case that allows for certain unsimple but still pseudo-finite generic theories of equivalence relations.

Outline of the paper. This paper consists of just two main sections (plus a few concluding thoughts). In the first section, we collect all of the relevant definitions, recalling the definitions of Fraïssé classes, geometric theories, superrobust classes, and 1-dimensional asymptotic classes. We also review the definitions of asymptotic probability measures and 0, 1-laws relative to them. Finally, we identify what is meant by a non-exotic measure: an asymptotic probability measure with almost independent sampling. In the second section, we prove our two main theorems. In the third section, we note a few open questions related to the results presented here.

1. Definitions: Asymptotic probability measures, almost independent sampling, and superrobustness. As already noted, in this section, we will introduce all of the definitions that are relevant throughout the paper. This includes Fraïssé classes, superrobustness, asymptotic classes, asymptotic probability measures, 0, 1-laws, and almost independent sampling. Before these definitions, however, we establish a few notational and terminological conventions as follows.

1.0.1. *Notational and terminological conventions.* Throughout this paper, we make certain demands on our languages, and we use a somewhat eccentric notation for finite structures. These are accounted for in the following two conventions ([2]).

([2]) The presentation here is quite similar to that in [9], and indeed there is some overlap. There are only so many ways one can review the basic material on Fraïssé classes.

Convention. Throughout this article, any language $\mathscr{L}$ in question is the first-order language built over a countable signature $\mathrm{sig}(\mathscr{L})$ that has *no function symbols and only finitely many constant symbols.*

Given $\mathscr{L}$, $\mathscr{L}^p$ is the set of partitioned $\mathscr{L}$-formulas, usually written $\varphi(\overline{x};\overline{y})$—an ordinary $\mathscr{L}$-formula with a partition of the free variables understood (allowing that the tuple of parameter variables $\overline{y}$ may be empty). We write $\mathscr{L}^{\mathsf{qf}}$ for the set of quantifier-free $\mathscr{L}$-formulas. Also, if φ is a formula, then $\varphi^1 = \varphi$ and $\varphi^0 = \neg\varphi$.

Convention. Fix a language $\mathscr{L}$.

- $\mathscr{L}$-structures that *might be* infinite are rendered as uppercase calligraphic letters, $\mathcal{M}, \mathcal{N}, \ldots$, and their universes are the respective uppercase italic characters, $M, N, \ldots$, with cardinalities $|M|, |N|, \ldots$.
- If $\mathcal{M}$ is an $\mathscr{L}$-structure and $A \subseteq M$, then $\mathcal{M}[A]$ denotes the induced substructure of $\mathcal{M}$ with universe $A \cup \{c^{\mathcal{M}} : c \text{ a constant of } \mathrm{sig}(\mathscr{L})\}$. We write $\mathcal{M} \leq \mathcal{N}$ to mean that $\mathcal{M}$ is an induced substructure of $\mathcal{N}$: $M \subseteq N$ and $\mathcal{N}[M] = \mathcal{M}$. Extending this notation somewhat, we write $\mathcal{M} \leq^* \mathcal{N}$ to mean that there is an embedding $\mathcal{M} \to \mathcal{N}$.
- We use lowercase gothic letters, $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \ldots$, to denote $\mathscr{L}$-structures that *are certainly finite.* The universe of $\mathfrak{a}$ is $\|\mathfrak{a}\|$, and $|\mathfrak{a}|$ is the cardinality of $\mathfrak{a}$, i.e. $|\mathfrak{a}| = \big|\|\mathfrak{a}\|\big|$. The notations already mentioned pertain to finite structures as well.

In order to reduce the amount of qualifying verbiage in our discussions of classes of finite structures, we establish the following basic qualifications once and for all.

Convention. To say, "$\mathbf{C}$ is a class of finite structures," we require:

- All members of $\mathbf{C}$ are finite structures for the same language $\mathscr{L}$.
- $\mathbf{C}$ is closed under isomorphism: If $\mathfrak{a} \in \mathbf{C}$ and $\mathfrak{b} \cong \mathfrak{a}$, then $\mathfrak{b} \in \mathbf{C}$.
- $\mathbf{C}$ is infinite *modulo* isomorphisms.
- For every $0 < n < \omega$, the set $S_n^{\mathsf{qf}}(\mathbf{C}) = \{\mathrm{qftp}^{\mathfrak{a}}(\overline{b}) : \mathfrak{a} \in \mathbf{C}, \overline{b} \in \|\mathfrak{a}\|^n\}$ is finite. Here, $\mathrm{qftp}^{\mathfrak{a}}(\overline{b})$ denotes the quantifier-free-complete type of $\overline{b}$ in the sense of $\mathfrak{a}$.

Given some structure $\mathcal{N}$ (possibly finite), we write $\mathbf{C}(\mathcal{N})$ for the set $\{\mathfrak{a} \in \mathbf{C} : \mathfrak{a} \leq \mathcal{N}\}$.

Associated with any class $\mathbf{C}$ of finite structures, there is a *foundation rank* $\mathrm{rk}^{\mathbf{C}}$, which may or may not be useful. (For superrobust classes, the foundation rank is fundamental; for asymptotic classes in general, the foundation rank does not play any role to speak of.)

Definition 1.1. Let $\mathbf{C}$ be a class of finite structures. Then we define its *rank function* $\mathrm{rk}^{\mathbf{C}} : \mathbf{C} \to \omega$ as follows:

- $\mathrm{rk}^{\mathbf{C}}(\mathfrak{a}) \geq 0$ for all $\mathfrak{a} \in \mathbf{C}$.
- $\mathrm{rk}^{\mathbf{C}}(\mathfrak{a}) \geq n+1$ if there is an $\mathfrak{a}_0 \in \mathbf{C}(\mathfrak{a}) \setminus \{\mathfrak{a}\}$ and $\mathrm{rk}^{\mathbf{C}}(\mathfrak{a}) \geq n$.

1.1. Fraïssé and superrobust classes, cofinality of classes of structures. Our next batch of definitions concerns classes of finite structures with additional properties that allow them to be canonically pieced together into a single countably infinite structure. The first examples of such classes noted in the literature (e.g. [8]) were Fraïssé classes, although related formulations are now widely studied under various names.

1.1.1. *Fraïssé classes and cofinality*

Definition 1.2. Let $\mathbf{C}$ be a class of finite structures. We say that $\mathbf{C}$ is a *Fraïssé class* if it has the following three properties:

Joint-embedding **(JEP)** For any $\mathfrak{a}_0, \mathfrak{a}_1 \in \mathbf{C}$, there are $\mathfrak{b} \in \mathbf{C}$ and embeddings $\mathfrak{a}_0 \to \mathfrak{b}$ and $\mathfrak{a}_1 \to \mathfrak{b}$.

Amalgamation **(AP)** For $\mathfrak{a}, \mathfrak{b}_0, \mathfrak{b}_1 \in \mathbf{C}$ and embeddings $f_i : \mathfrak{a} \to \mathfrak{b}_i$ $(i < 2)$, there are $\mathfrak{c} \in \mathbf{C}$ and embeddings $f_i' : \mathfrak{b}_i \to \mathfrak{c}$ $(i < 2)$ such that $f_0' \circ f_0 = f_1' \circ f_1$.

Heredity **(HP)** $\mathfrak{a}[B] \in \mathbf{C}$ whenever $\mathfrak{a} \in \mathbf{C}$ and $B \subseteq \|\mathfrak{a}\|$.

Following the terminology of [9], a class that has JEP and AP is a *semicoherent class.*

Fraïssé classes, and semicoherent classes more generally, are attractive precisely because AP and JEP generate the following proposition.

Proposition 1.3 ([10, Chapter 7], essentially). *Let* $\mathbf{C}$ *be a semicoherent class in a language* $\mathscr{L}$. *There is a countably infinite* $\mathscr{L}$*-structure* $\mathcal{M}$ *satisfying the following:*

1. ($\mathbf{C}$-Universality) *For every* $\mathfrak{a} \in \mathbf{C}$*, there is an embedding* $\mathfrak{a} \to \mathcal{M}$.
2. ($\mathbf{C}$-Homogeneity) *For every* $\mathfrak{a} \in \mathbf{C}$ *such that* $\mathfrak{a} \leq \mathcal{M}$*, and every embedding* $f : \mathfrak{a} \to \mathcal{M}$*, there is an automorphism* $g \in \mathrm{Aut}(\mathcal{M})$ *such that* $f \subseteq g$.
3. ($\mathbf{C}$-Closedness) *For every* $A \subset_{\mathrm{fin}} M$*, there is a finite substructure* $\mathfrak{b} \leq \mathcal{M}$ *such that* $\mathfrak{b} \in \mathbf{C}$ *and* $A \subseteq \|\mathfrak{b}\|$.
4. $\mathcal{M}$ *is the prime model of its own theory.*

Furthermore, up to isomorphism, $\mathcal{M}$ is the unique countable structure that satisfies items (1)–(3). Thus, we may speak of *the* generic model of a semicoherent class $\mathbf{C}$, and the assignment $T_{\mathbf{C}} := \mathrm{Th}(\mathcal{M})$ is well-defined. (Primeness of $\mathcal{M}$ follows from the fact that for $A \subset_{\mathrm{fin}} M$, if $\mathcal{M}[A] \in \mathbf{C}$, then any partial isomorphism $A \to \mathcal{M}$ extends to an automorphism of $\mathcal{M}$.)

Obviously every Fraïssé class is a semicoherent class, so one naturally asks if there is a characterization of Fraïsséness in terms of generic models and generic theories. The following proposition provides such a characterization.

PROPOSITION 1.4. *Let* $\mathbf{C}$ *be a semicoherent class in a language* $\mathscr{L}$, *and let* $\mathcal{M}$ *be its generic model. The following are equivalent:*

(1) $T_{\mathbf{C}}$ *is* $\aleph_0$*-categorical and eliminates quantifiers.*
(2) $\mathrm{Age}(\mathcal{M}) = \{\mathfrak{a} : \mathfrak{a} \leq^* \mathcal{M}\} \supseteq \mathbf{C}$ (*up to isomorphism*) *is a Fraïssé class.*

A Fraïssé class may contain finite structures that are pathological relative to the other structures in the class. For example, the class $\mathbf{G}$ of all finite graphs contains all of the cycles, but a large "typical" graph—a typical member of $\mathbf{G}$—is not remotely like a cycle. The following definition of cofinality for semicoherent classes just gives us a framework for eliminating pathological members of a class without changing the class in any essential way.

DEFINITION 1.5. Let $\mathbf{C}, \mathbf{D}$ be semicoherent classes. We say that $\mathbf{C}$ and $\mathbf{D}$ are *cofinal* if the following are both true:

- For every $\mathfrak{c} \in \mathbf{C}$, there is a $\mathfrak{d} \in \mathbf{D}$ such that $\mathfrak{c} \leq \mathfrak{d}$.
- For every $\mathfrak{d} \in \mathbf{D}$, there is a $\mathfrak{c} \in \mathbf{C}$ such that $\mathfrak{d} \leq \mathfrak{c}$.

(Obviously, a cofinal subclass of $\mathbf{C}$ is a subclass $\mathbf{C}_0 \subseteq \mathbf{C}$ such that $\mathbf{C}$ and $\mathbf{C}_0$ are cofinal.) We note that if $\mathbf{C}$ and $\mathbf{D}$ are cofinal semicoherent classes, then $T_{\mathbf{C}} = T_{\mathbf{D}}$, and the converse holds as well.

1.1.2. *Superrobustness.* In general, first-order properties and first-order definable sets of the generic model and those of members of a semicoherent class $\mathbf{C}$ need not coincide. (Think of $\mathbb{Q}$ and the class of finite linear orders, say.) In [15], and then in [9], the notion of *robustness* of a chain or class of finite structures was introduced; in both cases, one introduces a graded, approximate form of the elementary substructure relation.

DEFINITION 1.6. Let $\mathbf{C}$ be a semicoherent class of finite $\mathscr{L}$-structures. We say that $\mathbf{C}$ is *superrobust* if there is a function $\nu : \mathscr{L} \to \omega$ such that for all $\mathfrak{a}, \mathfrak{b} \in \mathbf{C}$, all $\varphi(x_0, \ldots, x_{n-1}) \in \mathscr{L}$, and all $\overline{a} \in \|\mathfrak{a}\|^n$, if $\mathrm{rk}^{\mathbf{C}}(\mathfrak{a}) \geq \nu(\varphi)$ and $\mathfrak{a} \leq \mathfrak{b}$, then $\mathfrak{a} \models \varphi(\overline{a}) \Leftrightarrow \mathfrak{b} \models \varphi(\overline{a})$.

In [9], a large number of equivalents of superrobustness are examined. In this paper, we will have need of one of these. (Eventually, we will use it to transfer the supersimplicity of ultraproducts over a derived 1-dimensional asymptotic class back to the generic theory of a given Fraïssé class.)

THEOREM 1.7. *Let* $\mathbf{C}$ *be a Fraïssé class with generic model* $\mathcal{M}$, *and let* $\mathbf{D}$ *be a superrobust cofinal subclass of* $\mathbf{C}$. *Let* $\mathfrak{d}_0, \mathfrak{d}_1, \ldots$ *be any enumeration of* (*representatives of*) *isomorphism types in* $\mathbf{D}$, *and let* $\mathscr{U}$ *be any non-principal ultrafilter on* ω. *Then* $\prod_n \mathfrak{d}_n/\mathscr{U} \models T_{\mathbf{C}}$.

1.2. Geometric theories. The formulation of asymptotic classes (below) asks for both dimension and measure of definable sets in finite structures. For a general Fraïssé class, it is not at all clear where to find a "canonical" dimension theory. (We note that pseudo-finite dimensions, as in [11], can vary wildly depending on the ultrafilter chosen.) In this paper, we dispense with this issue by focusing attention on Fraïssé classes in which *there is* a natural dimension theory—those classes for which the generic theory is geometric in the following sense.

Definition 1.8. Let T be a complete theory with infinite models. We say that T is a *geometric theory* if both of the following hold:

- T eliminates $\exists^\infty$: For every $\varphi(\overline{x};\overline{y})$, there is a number n_φ such that for all $\mathcal{M} \models T$ and $\overline{b} \in M^{|\overline{y}|}$, if $\mathcal{M} \models (\exists_{\geq n_\varphi}\overline{x})\ \varphi(\overline{x},\overline{b})$, then $\varphi(\mathcal{M},\overline{b})$ is infinite. (This condition is, of course, immediate for all $\aleph_0$-categorical theories in countable languages.)
- For every $\mathcal{M} \models T$, $\mathrm{acl}^{\mathcal{M}}$ is the closure operator of a pre-geometry on M—meaning that for all $C \subseteq M$ and $a, b \in M$, if $a \in \mathrm{acl}^{\mathcal{M}}(bC) \setminus \mathrm{acl}^{\mathcal{M}}(C)$, then $b \in \mathrm{acl}^{\mathcal{M}}(aC)$.

For our purposes, then, a Fraïssé class $\mathbf{C}$ is *geometric* just in case $T_{\mathbf{C}}$ is a geometric theory, and if $\mathbf{C}$ is a geometric Fraïssé class, then we write $\mathbf{C}^{\mathrm{alg}}$ for the subclass consisting of those $\mathfrak{a} \in \mathbf{C}$ for which there is an embedding $f : \mathfrak{a} \to \mathcal{M}$, where $\mathcal{M} \models T_{\mathbf{C}}$, such that $\mathrm{acl}^{\mathcal{M}}(\|f\mathfrak{a}\|) = \|f\mathfrak{a}\|$. (By quantifier-elimination in $T_{\mathbf{C}}$, algebraic closedness is an invariant of the isomorphism type of a member of $\mathbf{C}$.)

Once a theory is known to be geometric, we also know, as with strongly minimal theories, that cardinalities of algebraically independent subsets are invariants, and this provides us with a rudimentary dimension theory that extends easily to a dimension theory for definable sets as follows.

Definition 1.9. If T is a geometric theory, then every model $\mathcal{M} \models T$ admits a dimension function of the form

$$\dim_{\mathbf{a}}(\overline{a}/B) = \max\left\{|S| : \begin{array}{c} S \subseteq n,\ |\{a_i\}_{i\in S}| = |S|, \\ \{a_i\}_{i\in S} \text{ is alg. ind. over } B \end{array}\right\}$$

for $0 < n < \omega$, $\overline{a} \in M^n$, and $B \subseteq M$. One extends this rudimentary dimension function to a dimension function on definable sets: for a definable set $X \in \mathsf{Def}(\mathcal{M})$,

$$\dim_{\mathbf{a}}(X) = \min\left\{\max\{\dim_{\mathbf{a}}(\overline{a}/B) : \overline{a} \in X\} : \begin{array}{c} B \subset_{\mathrm{fin}} M, \\ X \text{ is over } B \end{array}\right\}.$$

Everything in what follows is applied more or less exclusively to geometric Fraïssé classes, so it will be convenient (for succinctness) to establish the following blanket assumption.

From now on, unless stated otherwise, $\mathbf{C}$ *denotes a* geometric *Fraïssé class, and* $\mathcal{M}$ *is its generic model.*

1.3. Some probability theory. It may be convenient for the reader if we pause now to recall some notation and ideas from probability theory. We use conditional probabilities in several places, so we recall that if $E_0, E_1, \ldots, E_n$ are events in some probability space, then

$$\mathbb{P}(E_0 \wedge E_1 \wedge \cdots \wedge E_{n-1}) \\ = \mathbb{P}(E_0 \mid E_1 \wedge \cdots \wedge E_n) \cdot \mathbb{P}(E_1 \mid E_2 \wedge \cdots \wedge E_n) \cdots \mathbb{P}(E_{n-1} \mid E_n).$$

When we make calculations using this equation, we will sometimes refer to the "calculus of conditional probabilities." Another notation we will use is "$X \sim \mathsf{Binomial}(N, p)$," which means that X is a random variable that follows the binomial distribution with parameters $N < \omega$ and $p \in [0, 1]$. More precisely, the range of values of X is $\{0, 1, \ldots, N\}$, and for each k,

$$\mathbb{P}(X = k) = \binom{N}{k} p^k (1-p)^{N-k}$$

is the probability of getting exactly k successes in N independent trials, each trial with probability p of success.

CONVENTION. For $a, b \in [0, \infty)$ and $0 < \varepsilon < 1$, we write $a \in (1 \pm \varepsilon)b$ as shorthand for

$$(1 - \varepsilon)b < a < (1 + \varepsilon)b.$$

Also, if X is some set and $0 < n < \omega$, then $X^{(n)}$ is the set of $\overline{x} \in X^n$ such that $x_i \neq x_j$ for all $i < j < n$.

1.4. Asymptotic probability measures, 0, 1-laws, and almost independent sampling. In this subsection, we finally formalize the notions of asymptotic probability measures and 0, 1-laws relative to them for geometric Fraïssé classes. The first definition below accounts for asymptotic probability measures for geometric Fraïssé classes, and in the subsequent definition, we identify a few notational tricks that are needed for our formulation of 0, 1-laws for first-order logic which is given in Definition 1.13. We note that our definitions do not require that $\mathscr{L}$'s signature is finite, only that $\mathbf{C}_N$ is finite for every N.

DEFINITION 1.10. For every $0 < N < \omega$, $\mathbf{C}_N$ is the set $\{\mathfrak{a} \in \mathbf{C} : \|\mathfrak{a}\| = N\}$. Now, an *asymptotic probability measure for* $\mathbf{C}$ is a sequence $\mu = (\mu_N)_N$ such that:

- For every $0 < N < \omega$, $\mu_N : \mathbf{C}_N \to [0, 1]$ is an isomorphism-invariant probability mass function.
- For all $N < N_1 < \omega$ and all $\mathfrak{a} \in \mathbf{C}_N \cap \mathbf{C}^{\mathrm{alg}}$, $\mu_N(\mathfrak{a}) = \sum_{\mathfrak{b} \in \mathbf{C}_{N_1} : \mathfrak{a} \leq \mathfrak{b}} \mu_{N_1}(\mathfrak{b})$.

For $X \subseteq \mathbf{C}_N$, we write $\mu_N X$ or $\mu_N[X]$ as shorthand for $\sum_{\mathfrak{a} \in X} \mu_N(\mathfrak{a})$.

DEFINITION 1.11. Let μ be an asymptotic probability measure for $\mathbf{C}$, and let $0 < N < \omega$.

- For a sentence $\varphi \in \mathrm{Sent}(\mathscr{L}(N))$, where $\mathscr{L}(N)$ is the expansion of $\mathscr{L}$ by new constants $0, 1, \ldots, N-1$, we define
$$\mu_N[\varphi] = \sum_{\mathfrak{a} \in \mathbf{C}_N : \mathfrak{a} \models \varphi} \mu_N(\mathfrak{a}) = \mu_N\{\mathfrak{a} \in \mathbf{C} : \mathfrak{a} \models \varphi\}$$
where "$\mathfrak{a} \models \varphi$" is given its natural meaning.
- Suppose $\theta = \theta(\bar{b})$ and $\varphi = \varphi(\bar{a}, \bar{b})$ are sentences in $\mathscr{L}(N)$, and suppose $\mu_N[\theta] > 0$. Then
$$\mu_N[\varphi|\theta] = \frac{\mu_N[\varphi \wedge \theta]}{\mu_N[\theta]}.$$

Furthermore, we define a partial function $\mu_\infty : \mathrm{Sent}(\mathscr{L}(\omega)) \rightharpoonup [0, 1]$ by
$$\mu_\infty[\varphi] = \begin{cases} \lim_{N\to\infty} \mu_N[\varphi] & \text{if the limit exists,} \\ \uparrow & \text{otherwise.} \end{cases}$$

OBSERVATION 1.12. Let μ be an asymptotic probability measure for $\mathbf{C}$.

- If θ is a *quantifier-free* sentence in $\mathscr{L}(\omega)$, then $\mu_\infty[\theta]$ exists.
- Suppose $\theta = \theta(\bar{b})$ and $\varphi = \varphi(\bar{a}, \bar{b})$ are quantifier-free sentences in $\mathscr{L}(\omega)$, and $\mu_\infty[\theta] > 0$. Then the limit $\mu_\infty[\varphi|\theta] = \lim_{N\to\infty} \mu_N[\varphi|\theta]$ exists, and $\mu_\infty[\varphi|\theta] = \mu_\infty[\varphi \wedge \theta]/\mu_\infty[\theta]$.

The second item is explained as follows: In the special case where φ and θ are quantifier-free *diagrams* (or at least, isolated single isomorphism types of structures modulo $\mathbf{C}$), the claim follows by the marginalization condition in the definition of an asymptotic probability measure; then, the general case follows by the calculus of marginal and conditional probabilities.

DEFINITION 1.13. Let μ be an asymptotic probability measure for $\mathbf{C}$. We define
$$\mathrm{Th}(\mu) = \{\varphi \in \mathrm{Sent}(\mathscr{L}) : \mu_\infty[\varphi]{\downarrow} = 1\}.$$

Ordinarily, $\mathbf{C}$ would be said to have the 0, 1-law for first-order logic relative to μ just in case $\mathrm{Th}(\mu)$ is a complete theory, but in this paper, it will be more convenient to fix a stronger definition. We say that $\mathbf{C}$ *has the* 0, 1-*law for first-order logic relative to* μ if $\mathrm{Th}(\mu) = T_{\mathbf{C}}$.

We must note that our last requirement in the specification of a 0, 1-law—that $\mathrm{Th}(\mu) = T_{\mathbf{C}}$—is not a triviality; the class of finite triangle-free graphs provides a good example.

REMARK 1.14. Let $\mathbf{C}$ be the class of finite triangle-free graphs in the language whose signature consists of a single binary relation symbol, and let $\mu = (\mu_N)_N$ be such that for every N, μ_N is the uniform distribution on $\mathbf{C}_N$.

Then $\mathrm{Th}(\mu)$ is a complete theory (the theory of the generic bipartite graph), but $\mathrm{Th}(\mu) \neq T_{\mathbf{C}}$.

Now, we turn to the idea of *almost independent sampling*, to which we will appeal in identifying measures of definable sets (relative to their algebraic dimensions). To motivate the definition, first recall that **G** denotes the class of all finite graphs. Then, we note that in generating a random member of $\mathbf{G}_N$ by independent unbiased coin flips, the event of any particular vertex $i \in N \setminus C$ satisfying the requirements of an extension axiom (over some clump C of vertices) is independent of the event of any *other* vertex $j \in N \setminus C$ satisfying the same extension axiom. In [17] (presented more tractably in [4]), this observation is mined to prove that typical finite graphs satisfy *strong* extension axioms, which are very similar to the conclusion established in our Proposition 2.3 below.

G is a geometric Fraïssé class, but the algebraic closure operation in $T_{\mathbf{G}}$ is trivial. To extend the idea of independent sampling to an arbitrary geometric Fraïssé class, we need to tailor the discussion a little more carefully. The basic objects, then, are d,n-extension problems, which are used both in the formulation of almost independent sampling and heavily in the rest of the paper.

Definition 1.15. We define a *d,n-extension problem of* **C** to be a pair (φ, θ) where $\theta(\overline{y}) \in S^{\mathsf{qf}}_m(T_{\mathbf{C}})$, $\varphi(\overline{x};\overline{y}) \in S^{\mathsf{qf}}_{d+m}(T_{\mathbf{C}})$, and together these satisfy:

- $\theta(\overline{y}) \vDash \bigwedge_{i<j<m} y_i \neq y_j$.
- $\varphi(\overline{x};\overline{y}) \vDash \theta(\overline{y}) \wedge \bigwedge_{i<j<d} x_i \neq x_j$.
- If $\mathcal{M} \vDash \varphi(\overline{a},\overline{b})$, where $\mathcal{M}$ is the generic model of **C**, then $\overline{a} \cap \mathrm{acl}(\overline{b}) = \emptyset$ and $\{a_0, \ldots, a_{d-1}\}$ is algebraically independent over $\overline{b}$.

As it is slightly easier to typeset (and slightly more illustrative), we often write φ/θ in place of (φ,θ) for a d,n-extension problem of **C**.

Definition 1.16. Let μ be an asymptotic probability measure for **C**. We say that μ *has almost independent sampling* if for every $1,n$-extension problem φ/θ of **C**, there are numbers $0 \le \varepsilon < 1$ and $0 < \delta < 1$ such that for all but finitely many $N \in \omega$, for every $\overline{b} \in N^{(n)}$, for any $\{i_0 < \cdots < i_{t-1}\} \subseteq N \setminus \overline{b}$ where $t = \lceil \delta N \rceil$, for every $s : t \to 2$,

$$\mu_N\Big[\bigwedge_{j<t} \varphi(i_j;\overline{b})^{s(j)} \;\Big|\; \theta(\overline{b})\Big] \in (1 \pm \varepsilon) \prod_{j<t} \mu_N[\varphi(i_j;\overline{b})^{s(j)} \,|\, \theta(\overline{b})].$$

1.5. Asymptotic classes. The last definition required for this paper is that of an asymptotic class. These were introduced in [14, 6, 7], generalizing the situation for finite fields as proved in [5].

Definition 1.17. Let **D** be a class of finite structures in a countable language $\mathscr{L}$. We say that **D** is a 1-*dimensional asymptotic class* if there is a function

$$\mathbb{A} : \mathscr{L}^p \to \mathcal{P}_{\text{fin}}(\mathscr{L} \times \omega \times [0, \infty)) : \varphi(\overline{x}; \overline{y}) \mapsto \mathbb{A}(\varphi)$$

such that for all $\varphi(\overline{x}; \overline{y}) \in \mathscr{L}^p$:

- $\mathbb{A}(\varphi) \neq \emptyset$, and for every $(\theta, d, m) \in \mathbb{A}(\varphi)$, θ is of the form $\theta(\overline{y})$.
- $\bigvee_{(\theta,d,m)\in\mathbb{A}(\varphi)} \theta(\overline{y}) \equiv \mathsf{True}$ modulo $\mathbf{D}$.
- For (θ_1, d_1, m_1) and (θ_2, d_2, m_2) in $\mathbb{A}(\varphi)$, if $\theta_1 \wedge \theta_2 \not\equiv \mathsf{False}$ modulo $\mathbf{D}$, then $d_1 = d_2$ and $m_1 = m_2$.
- For every $\varepsilon > 0$, there is an $N_{\varphi,\varepsilon} < \omega$ such that for all $\mathfrak{a} \in \mathbf{D}$, all $(\theta, d, m) \in \mathbb{A}(\varphi)$, and all $\overline{b} \in \|\mathfrak{a}\|^{|\overline{y}|}$, if $\mathfrak{a} \vDash \theta(\overline{b})$ and $|\mathfrak{a}| \geq N_{\varphi,\varepsilon}$, then
$$\left||\varphi(\overline{a}; \overline{b})| - m|\mathfrak{a}|^d\right| < \varepsilon|\mathfrak{a}|^d.$$

(Clearly, it follows that if $(\theta, d, m) \in \mathbb{A}(\varphi(x_0, \ldots, x_{r-1}; \overline{y}))$, then $d \leq r$.)

For any number $0 < K < \omega$, we recover the notion of a *K-dimensional asymptotic class* by simply replacing ω in the range of $\mathbb{A}$ by $\{k/K : k < \omega\} \subseteq \mathbb{Q} \cap [0, \infty)$. Thus, for a K-dimensional asymptotic class, if $(\theta, k/K, m) \in \mathbb{A}(\varphi(x_0, \ldots, x_{r-1}; \overline{y}))$, then $k \leq Kr$.

We conclude this section by noting an important theorem showing that the infinitary model theory of ultraproducts over an asymptotic class is quite tame. We will also use this theorem to derive Theorem 2.2 from Theorem 2.1.

THEOREM 1.18 ([6]). *Let $\mathbf{D}$ be a class of finite structures in a countable language $\mathscr{L}$. If $\mathbf{D}$ is a K-dimensional asymptotic class, then the theory of any infinite ultraproduct $\mathcal{M}$ of members of $\mathbf{D}$ is supersimple, and if $\mathcal{M}$ is geometric, then $D(X) \leq K \cdot \mathrm{ari}(X)$ for any definable set $X \in \mathsf{Def}(\mathcal{M})$.*

2. Demonstrations. In this section, of course, we prove the two main theorems of this paper. In the first, we show that a geometric Fraïssé class that has the 0, 1-law for first-order logic relative to a "non-exotic" asymptotic probability measure is necessarily very close to being an asymptotic class. An important step in the proof of Theorem 2.1 is the "tail inequality" encoded as Proposition 2.3; we use the lower bounds resulting from Proposition 2.3 to identify measures of definable sets relative to their dimensions. Using Theorem 2.1, it is then a fairly simple matter to prove Theorem 2.2 using Theorems 1.18 and 1.7.

THEOREM 2.1. *Let $\mathbf{C}$ be a geometric Fraïssé class, and suppose that $\mathbf{C}$ has the* 0, 1*-law for first-order logic relative to an asymptotic probability measure μ that has almost independent sampling. Then $\mathbf{C}$ contains a cofinal superrobust subclass $\mathbf{D}$, which is a* 1*-dimensional asymptotic class via $\mathbb{A}$, with*

$$\mathbb{A}(\varphi(\overline{x}; \overline{y})) \subset_{\text{fin}} \left\{\left(\mathrm{qftp}(\overline{a}), \dim_{\mathbf{a}}(\varphi(\mathcal{M}; \overline{a})), m\right) : \overline{a} \in M^{|\overline{y}|},\ m \in [0, \infty)\right\}.$$

Theorem 2.2. *Let* $\mathbf{C}$ *be a geometric Fraïssé class, and suppose that* $\mathbf{C}$ *has the* $0,1$*-law for first-order logic relative to an asymptotic probability measure* μ *that has almost independent sampling. Then* $T_{\mathbf{C}}$ *is supersimple, and* $D(X) \leq \dim_{\mathbf{a}}(X)$ *for every definable set* X *of the generic model* $\mathcal{M}$ *of* $\mathbf{C}$*.*

2.1. A tail inequality. The focus of this subsection is to prove Proposition 2.3 below. Our proof draws heavily on a related argument given in [4]. The starting point for our proof (and theirs) is the following standard result from discrete probability theory.

Fact (Chernoff bound, see [16, Chapter 4]). *For any* $0 < p, \alpha < 1$, *there exists* $0 < c < 1$ *such that for any* $N < \omega$, *if* $X \sim \mathsf{Binomial}(N, p)$, *then* $\mathbb{P}[X \leq \alpha p N] \leq c^N$.

An alternative statement, which is a little more convenient for us, is the following: *There is a function* $\xi : (0,1)\times(0,1) \to (0,1)$ *such that for any* $0 < p, \alpha < 1$ *and any* $N < \omega$, *if* $X \sim \mathsf{Binomial}(N, p)$, *then* $\mathbb{P}[X \leq \alpha p N] \leq \xi(p, \alpha)^N$.

Proposition 2.3. *Let* $\mathbf{C}$ *be a geometric Fraïssé class, and let* μ *be an asymptotic probability measure for* $\mathbf{C}$ *that has almost independent sampling. Then, for every* d, n*-extension problem* φ/θ *of* $\mathbf{C}$*, there is a number* $0 < \beta < 1$ *such that*

$$\mu_N\{\mathfrak{a} : (\exists \overline{b} \in \theta(\mathfrak{a}))\ |\varphi(\mathfrak{a}; \overline{b})| \leq \beta |\mathfrak{a}|^d\} \in O(N^k c^N)$$

for some $0 < c < 1$ *and* $k \geq 0$.

Proof. First, we will observe that the full proposition for d, n-extension problems reduces to the proposition for just $1, n$-extension problems (n arbitrary). Let $\varphi(\overline{x}; \overline{y})/\theta(\overline{y})$ be a d, n-extension problem where $d > 1$. If $\mathcal{M}$ is the generic model of $\mathbf{C}$, let $\overline{a} \in M^d$ and $\overline{b} \in M^n$ be such that $\mathcal{M} \vDash \varphi(\overline{a}; \overline{b})$—so $\mathcal{M} \vDash \theta(\overline{b})$. For $k = 0, 1, \ldots, d-2$, let $\theta_k(x_{k+1}, \ldots, x_{d-1}, \overline{y}) = \mathrm{qftp}^{\mathcal{M}}(a_{k+1}, \ldots, a_{d-1}, \overline{b})$, and let $\varphi_k(x_k; x_{k+1}, \ldots, x_{d-1}, \overline{y})$ be the natural partitioning of the type $\mathrm{qftp}^{\mathcal{M}}(a_k, a_{k+1}, \ldots, a_{d-1}, \overline{b})$. Also, let $\theta_{d-1} = \theta$ and $\varphi_{d-1}(x_{d-1}; \overline{y}) = \mathrm{qftp}^{\mathcal{M}}(a_{d-1}, \overline{b})$. For each k, φ_k/θ_k is a $1, n{+}(d-k-1)$-extension problem. Now, if the proposition holds for $1, n$-extension problems for all n, then by the calculus of conditional probabilities, the proposition holds for d, n-extension problems for all d, n.

Next, we prove the statement when $d = 1$. That is, we prove just that *for every* $1, n$*-extension problem* φ/θ *of* $\mathbf{C}$*, there are numbers* $0 < \beta \leq 1$, $0 < c < 1$, *and* $k \geq 0$ *such that* $\mu_N\{\mathfrak{a} : (\exists \overline{b} \in \theta(\mathfrak{a}))\ |\varphi(\mathfrak{a}; \overline{b})| \leq \beta |\mathfrak{a}|\} \in O(N^k c^N)$.

Claim. *Let* φ/θ *be a* $1, m$*-extension problem of* $\mathbf{C}$*. There is a constant* $0 < c < 1$ *such that for sufficiently large* N*, and each* $\overline{b} \in N^{(n)}$,

$$\mu_N\big[|\varphi(*;\overline{b})| \leq \tfrac{1}{2}p\,\delta N \;\big|\; \theta(\overline{b})\big] \leq c^N$$

where $p = (1-\varepsilon)\mu_N[\varphi(n;0,\ldots,n-1) \mid \theta(0,\ldots,n-1)]$ *for* ε *and* δ *as in Definition* 1.16*, and* $\mu_N\big[|\varphi(*;\overline{b})| \leq \frac{1}{2}p\delta N \;\big|\; \theta(\overline{b})\big]$ *is shorthand for the conditional probability*

$$\frac{\mu_N\big\{\mathfrak{a} : \mathfrak{a} \vDash \theta(\overline{b}) \wedge |\varphi(\mathfrak{a},\overline{b})| \leq \frac{1}{2}p\,\delta N\big\}}{\mu_N[\theta(\overline{b})]}.$$

To prove the claim, it is helpful to isolate one straightforward observation on ensembles of $\{0,1\}$-valued random variables.

Observation. Suppose $0 < p < 1$ and $0 < \alpha < 1$, and for some $K_0 < \omega$, let $Y_0 \sim \mathsf{Binomial}(K_0, p)$. Let $K_0 \leq K < \omega$, and let $q : \{0,1\}^K \to [0,1]$ be a probability mass function that is invariant under permutations of coordinates and such that for all $\ell \leq K_0$ and $\{i_0 < \cdots < i_{\ell-1}\} \subseteq K$,

$$\mathbb{P}_q\Big\{\overline{x} \in \{0,1\}^K : \sum_{j<\ell} x_{i_j} \geq \ell\Big\} \geq \binom{K_0}{\ell} p^{\ell}(1-p)^{K_0-\ell}.$$

Then

$$\mathbb{P}_q\Big\{\overline{x} \in \{0,1\}^K : \sum_{i<K_0} x_i \leq \alpha p K_0\Big\} \leq \mathbb{P}[Y_0 \leq \alpha p K_0].$$

Proof of Claim. Let $K = N - n$ and $K_0 = \lceil \delta N \rceil$ (so for large enough N, $K_0 < K$), and define $q : \{0,1\}^K \to [0,1]$ by

$$q(\overline{x}) = \mu_N\Big[\bigwedge_{i<K} \varphi(n{+}i;0,\ldots,n-1)^{x_i} \;\Big|\; \theta(0,\ldots,n-1)\Big].$$

Also, let $X \sim \mathsf{Binomial}(K_0, p)$. Then, for each $\overline{b} \in N^{(n)}$ individually,

$$\begin{aligned}\mu_N\big[|\varphi(*;\overline{b})| \leq \tfrac{1}{2}p\,\delta N \;\big|\; \theta(\overline{b})\big] &= \mathbb{P}_q\Big\{\overline{x} \in \{0,1\}^K : \sum_{j<K_0} x_i \leq \tfrac{1}{2}pK_0\Big\}\\ &\leq \mathbb{P}\big[X \leq \tfrac{1}{2}pK_0\big] \leq \xi(p,\delta/2)^{\delta N},\end{aligned}$$

so we set $c = \xi(p,\delta/2)^{\delta}$. ■

The proof of the proposition now concludes with just a little more calculation:

$$\begin{aligned}\mu_N\Big[\bigvee_{\overline{b}\in N^{(n)}} \theta(\overline{b}) \wedge |\varphi(*;\overline{b})| \leq \tfrac{1}{2}p\delta N\Big] &\leq \sum_{\overline{b}\in N^{(n)}} \mu_N\big[\theta(\overline{b}) \wedge |\varphi(*;\overline{b})| \leq \tfrac{1}{2}p\,\delta N\big]\\ &= \sum_{\overline{b}\in N^{(n)}} \mu_N\big[|\varphi(*;\overline{b})| \leq \tfrac{1}{2}p\,\delta N \;\big|\; \theta(\overline{b})\big]\mu_N[\theta(\overline{b})]\end{aligned}$$

$$= \mu_N[\theta(0,\ldots,n-1)] \sum_{\overline{b}\in N^{(n)}} \mu_N\left[|\varphi(*;\overline{b})| \le \tfrac{1}{2}p\,\delta N \;\middle|\; \theta(\overline{b})\right]$$

$$\le \left(\mu_N[\theta(0,\ldots,n-1)]n!\binom{N}{n}\right)c^N. \blacksquare$$

We observe that Proposition 2.3 does not actually require the 0, 1-law as a hypothesis. As it directly pertains only to quantifier-free types, the assumption that μ has almost independent sampling is sufficient. Its pertinence to 0, 1-laws for a geometric Fraïssé class $\mathbf{C}$ arises from the fact that $T_{\mathbf{C}}$ eliminates quantifiers, so that with probability tending to 1, each definable set in a member of $\mathbf{C}$ is (uniformly) quantifier-free definable.

2.2. Proof of Theorems 2.1 and 2.2. From this point, the remaining work in proving Theorem 2.1 consists in the following:

- Identify what the dimension-relative measures of definable sets *should* be in the 1-dimensional asymptotic subclass that we eventually extract.
- Extract that subclass.

In Definition 2.4 and Lemma 2.5, we use Proposition 2.3 to recover measures for d,n-extension problems, and in Definition 2.6, we state how one can form measures for arbitrary definable sets using those of extension problems (we take it as clear that this method works as advertised), completing the specification $\mathbb{A}$ for the 1-dimensional asymptotic class.

Definition 2.4. Let φ/θ be a d,n-extension problem of $\mathbf{C}$ for some $0 < d < \omega$ and $n < \omega$. Then we define

$$U(\varphi/\theta) = \big\{\beta \in [0,1] : (\exists\alpha \in (0,1))(\text{a.e. } N)(\forall\overline{b} \in N^{(n)}) \\ \mu_N[|\varphi(*;\overline{b})| \ge \beta N^d \mid \theta(\overline{b})] \le \alpha^N\big\},$$

$$\beta(\varphi/\theta) = \sup U(\varphi/\theta).$$

Now, for $n < \omega$, $0 < d < \omega$, and $\theta(\overline{y}) \in S_n^{\mathsf{qf}}(T_{\mathbf{C}})$, let $J_d(\theta)$ be the set of all d,n-extension problems of $\mathbf{C}$ of the form φ/θ. Then for each $\varphi/\theta \in J_d(\theta)$, we define

$$\gamma(\varphi/\theta) = \frac{1}{2}\Big(\beta(\varphi/\theta) + \Big(1 - \sum_{\psi/\theta\in J_d(\theta):\,\psi\not\equiv\varphi} \beta(\psi/\theta)\Big)\Big).$$

Lemma 2.5. *Let φ/θ be a d,n-extension problem of $\mathbf{C}$ for some $0 < d < \omega$ and $n < \omega$. Then for every $0 < \varepsilon < 1$, there is a number $0 \le c(\theta,\varepsilon) < 1$ such that for every large enough N and every $\overline{b} \in N^{(n)}$,*

$$\mu_N\left[\big||\varphi(*;\overline{b})| - \gamma(\varphi/\theta)N^d\big| > \varepsilon N^d \;\middle|\; \theta(\overline{b})\right] \le c(\theta,\varepsilon)^N.$$

Proof. In the first place, we note that if $J_d(\theta) = \{\varphi/\theta\}$ is a singleton, then $\beta(\varphi/\theta) = 1$, so $\gamma(\varphi/\theta) = 1$, and the claim of the lemma is immediate.

So, let us assume that

$$J_d(\theta) = \{\varphi_0/\theta{=}\varphi/\theta, \varphi_1/\theta, \ldots, \varphi_{r-1}/\theta\}$$

for some $r > 1$.

Let $0 < \varepsilon < 1$ be given. Towards a contradiction, suppose that for every $0 < c < 1$, there are infinitely many N such that

$$\mu_N[|\varphi_0(*;\overline{b})| - \gamma(\varphi_0/\theta)N^d > \varepsilon N^d \,|\, \theta(\overline{b})] > c^N$$

where $\overline{b} \in \omega^{(n)}$ is fixed arbitrarily. (Since we are thinking about infinitely many N's, and $\theta(\overline{b})$ implies that $\overline{b}$ is algebraically closed, the definition of an asymptotic probability measure ensures both that we can choose such $\overline{b}$ and that it does not matter which $\overline{b}$ we use.) Hence, for some $0 < \varepsilon' < 1$ and every k,

$$\mu_N\Big[\Big|\bigvee_{0<i<r} \varphi_i(*;\overline{b})\Big| < \varepsilon'\Big(\sum_{0<i<r} \beta(\varphi_i/\theta)\Big)N^d \,\Big|\, \theta(\overline{b})\Big] > (1-2^{-k})^N$$

for infinitely many N. By the Pigeonhole Principle, we may assume that there is a single $0 < i < r$ such that for all k,

$$\mu_N[|\varphi_i(*;\overline{b})| < \varepsilon'\beta(\varphi_i/\theta)N^d \,|\, \theta(\overline{b})] > (1-2^{-k})^N$$

for infinitely many N. Since $\varepsilon'\beta(\varphi_i/\theta) < \beta(\varphi_i/\theta)$, this contradicts the definition of $\beta(\varphi_i/\theta)$. Thus, there is a number $0 < c^+(\theta,\varepsilon) < 1$ such that

$$\mu_N[|\varphi_0(*;\overline{b})| - \gamma(\varphi_0/\theta)N^d \le \varepsilon N^d \,|\, \theta(\overline{b})] \le c^+(\theta/\varepsilon)^N$$

for almost all N. Arguing in largely the same way (in this case it is $\varphi_i(*;\overline{b})$ for some $i > 0$ that is carrying excess "mass" instead of $\varphi_0(*;\overline{b})$), we also recover a number $0 < c^-(\theta,\varepsilon) < 1$ such that

$$\mu_N[\gamma(\varphi_0/\theta)N^d - |\varphi_0(*;\overline{b})| \le \varepsilon N^d \,|\, \theta(\overline{b})] \le c^-(\theta/\varepsilon)^N$$

for all but finitely many N, and we set $c(\theta,\varepsilon) = \max\{c^+(\theta,\varepsilon), c^-(\theta,\varepsilon)\}$ to complete the proof. ∎

Definition 2.6. Let $\psi(x_0,\ldots,x_{k-1};y_0,\ldots,y_{n-1}) \in \mathscr{L}^p$.

- Let $\psi^{\mathsf{qf}}(\overline{x};\overline{y})$ be the disjunction of all quantifier-free-complete types $q(\overline{x};\overline{y}) \in S^{\mathsf{qf}}_{k+n}(T_{\mathbf{C}})$ such that $T_{\mathbf{C}} \vDash q(\overline{x};\overline{y}) \to \psi(\overline{x};\overline{y})$.
- Let $d \le k$, and let $\theta(\overline{y}) \in S^{\mathsf{qf}}_n(T_{\mathbf{C}})$ be such that $\psi(\overline{x};\overline{y}) \wedge \theta(\overline{y})$ is consistent with $T_{\mathbf{C}}$. Then $K^+_d(\psi/\theta)$ is the set of pairs $(\varphi'(\overline{x};\overline{y}), \varphi/\theta)$, where $\varphi = \varphi(x_{i_0},\ldots,x_{i_{d-1}};\overline{y})$, such that if $\mathcal{M} \vDash \varphi(a_{i_0},\ldots,a_{i_{d-1}};\overline{b})$, then:
  - φ' is a quantifier-free-complete type, and $T_{\mathbf{C}} \vDash \varphi'(\overline{x};\overline{y}) \to \psi(\overline{x};\overline{y}) \wedge \varphi(x_{i_0},\ldots,x_{i_{d-1}};\overline{y})$.
  - φ/θ is a d,n-extension problem.
  - If $\mathcal{M} \vDash \varphi'(\overline{a};\overline{b})$, then $i_0 = \min\{t < k : a_t \notin \operatorname{acl}(\overline{b})\}$, and for $0 < \ell < d$, $i_\ell = \min\big\{t : i_{\ell-1} < t < k \wedge a_t \notin \operatorname{acl}(\overline{b} \cup \{a_{i_0},\ldots,a_{i_{\ell-1}}\})\big\}$.

We then define $K_d(\psi/\theta) = \{\varphi/\theta : (\varphi', \varphi/\theta) \in K^+_d(\psi/\theta)\}$.

- Let $\theta(\overline{y}) \in S_n^{\mathsf{qf}}(T_{\mathbf{C}})$ be such that $\psi(\overline{x};\overline{y}) \wedge \theta(\overline{y})$ is consistent with $T_{\mathbf{C}}$. Let $d = \dim_{\mathbf{a}}(\psi(\mathcal{M};\overline{b}))$ where $\overline{b} \in \theta(\mathcal{M})$. If $d > 0$, then we define
$$\gamma(\psi/\theta) = \sum_{\varphi \in K_d(\psi/\theta)} \gamma(\varphi/\theta),$$
and if $d = 0$—so that $\psi(\mathcal{M},\overline{b})$ is finite—then we set $\gamma(\psi/\theta) = |\psi(\mathcal{M};\overline{b})|$.
- We may then define
$$\mathbb{A}(\psi) = \left\{(\theta, d, \gamma(\psi/\theta)) : \begin{array}{c} \overline{b} \in M^n,\ \theta = \mathrm{qftp}(\overline{b}), \\ d = \dim_{\mathbf{a}}(\psi(\mathcal{M};\overline{b})) \end{array}\right\}.$$

We now have the data $\mathbb{A}$ for a 1-dimensional asymptotic class, and our next (and last) task in proving Theorem 2.1 is to pair down the original class to a cofinal subclass $\mathbf{D}$ that is superrobust and 1-dimensional asymptotic via $\mathbb{A}$. The construction is in stages in which we reduce $\mathbf{C}$ to more and more $\mathbb{A}$-asymptotic subclasses. The definitions needed for this process are listed in Definition 2.7, and the process itself immediately follows Lemma 2.8.

Definition 2.7. Let $F_m \subset_{\mathrm{fin}} T_{\mathbf{C}} \cup \mathscr{L}^p$ for each $m < \omega$ such that $\bigcup_m F_m = T_{\mathbf{C}} \cup \mathscr{L}^p$. Then for each m, we define $\mathbf{C}(m)$ to be the subclass consisting of all $\overline{a} \in \mathbf{C}$ such that:

- $\mathrm{Th}(\mathfrak{a}) \cap T_{\mathbf{C}} \vDash \psi \leftrightarrow \psi^{\mathsf{qf}}$ if $\psi \in F_m \cap \mathscr{L}^p$, where ψ^{qf} is as in Definition 2.6;
- $\mathfrak{a} \vDash F_m \cap T_{\mathbf{C}}$.

For m and $0 < \delta < 1$, we define $\mathbf{C}^{\delta}(m)$ to be the subclass consisting of those $\mathfrak{a} \in \mathbf{C}(m)$ such that for each d, n-extension problem φ/θ, if $\varphi \in F_m$, then
$$\big||\varphi(\mathfrak{a};\overline{b})| - \gamma(\varphi/\theta)|\mathfrak{a}|^d\big| \le \delta|\mathfrak{a}|^d \quad \text{for every } \overline{b} \in \theta(\mathfrak{a}).$$

Lemma 2.8. *For all $0 < m < \omega$ and $0 < \delta < 1$,*
$$\lim_{N\to\infty} \mu_N[\mathbf{C}^{\delta}(m) \cap \mathbf{C}_N] = 1.$$

Proof. Let $0 < m < \omega$ and $0 < \delta < 1$ be given. Then let $F'_m \subset_{\mathrm{fin}} T_{\mathbf{C}}$ be the finite set of sentences
$$(F_m \cap T_{\mathbf{C}}) \cup \{(\forall\,\overline{x})(\forall\,\overline{y})\ (\psi(\overline{x};\overline{y}) \leftrightarrow \psi^{\mathsf{qf}}(\overline{x};\overline{y})) : \psi(\overline{x};\overline{y}) \in F_m \cap \mathscr{L}^p\}.$$
Since by hypothesis $\mathbf{C}$ has the 0,1-law with respect to μ and $\mathrm{Th}(\mu) = T_{\mathbf{C}}$, we know that $\lim_{N\to\infty} \mu_N[\bigwedge F'_m] = 1$. Since $\mathbb{P}(A \cap B) \ge (1 - \mathbb{P}(\neg A) - \mathbb{P}(\neg B))$ in general, Lemma 2.5 implies that $\lim_{N\to\infty} \mu_N[\mathbf{C}^{\delta}(m) \cap \mathbf{C}_N] = 1$. ∎

Now, we choose a fast growing, strictly increasing function $f : \omega \to \omega$ and a decreasing function $\delta : \omega \to (0,1) : k \mapsto \delta_k$ such that $\lim_{k\to\infty} \delta_k = 0$. Then, we proceed as follows:

- $\mathbf{D}_0 = \mathbf{C}$.
- If $k > 0$, then $\mathbf{D}_k = \{\mathfrak{a} \in \mathbf{D}_{k-1} : |\mathfrak{a}| \ge f(k) \Rightarrow \mathfrak{a} \in \mathbf{C}^{\delta_k}(k)\}$.

In the "end," we set $\mathbf{D} = \bigcap_{k<\omega} \mathbf{D}_k$.

Proof of Theorem 2.1. At this point, the proof of the theorem amounts to three very straightforward observations as follows.

Observation. By construction, $\lim_{N\to\infty} \mu_N[\mathbf{D} \cap \mathbf{C}_N] = 1$.

Observation. For each $\mathfrak{a} \in \mathbf{C}$, there is a number $m_{\mathfrak{a}}$ such that $\mathfrak{a} \leq \mathfrak{b}^*$ whenever $m \geq m_{\mathfrak{a}}$ and $\mathfrak{b} \in \mathbf{C}(m)$. From this, it follows that $\mathbf{D}$ is a cofinal subclass of $\mathbf{C}$.

Observation. $\mathbf{D}$ is a 1-dimensional asymptotic class via $\mathbb{A}$ as defined above.

To see this, let $\psi(\overline{x};\overline{y}) \in \mathscr{L}^p$, $0 < \varepsilon < 1$, and $(\theta(\overline{y}), d, \gamma(\psi/\theta)) \in \mathbb{A}(\psi)$. Then we may choose $k < \omega$ such that $\delta_k \leq \varepsilon$. By construction, for any $\mathfrak{a} \in \mathbf{D}$, if $|\mathfrak{a}| \geq f(k)$, then

$$\big||\psi(\mathfrak{a};\overline{b})| - \gamma(\psi/\theta)|\mathfrak{a}|^d\big| \leq \delta_k|\mathfrak{a}|^d \leq \varepsilon|\mathfrak{a}|^d.$$

This completes the proof of the theorem. ∎

The last task of this paper is to prove Theorem 2.2. As already noted, it follows quickly as a corollary of Theorems 1.7, 1.18, and 2.1.

Proof of Theorem 2.2. By Theorem 2.1, let $\mathbf{D}$ be a superrobust cofinal subclass of $\mathbf{C}$ that is 1-dimensional asymptotic. Let $\mathfrak{d}_0, \mathfrak{d}_1, \ldots$ be pairwise non-isomorphic members of $\mathbf{D}$ that exhaust all isomorphism types in $\mathbf{D}$, and let $\mathscr{U}$ be a non-principal ultrafilter on ω. Since $\mathbf{D}$ is a cofinal superrobust subclass of $\mathbf{C}$, by Theorem 1.7 we have $\mathrm{Th}(\prod_n \mathfrak{d}_n/\mathscr{U}) = T_{\mathbf{D}} = T_{\mathbf{C}}$. Since $\mathbf{D}$ is 1-dimensional asymptotic, by Theorem 1.18, $T_{\mathbf{C}} = \mathrm{Th}(\prod_n \mathfrak{d}_n/\mathscr{U})$ is supersimple, and $D(X) \leq \dim_{\mathbf{a}}(X)$ for every definable set X of $\prod_n \mathfrak{d}_n/\mathscr{U}$ (which elementarily embeds $\mathcal{M}$)—and this completes the proof of the theorem. ∎

3. Concluding remarks. To conclude this paper, we make two conjectures which may help to motivate some future work in this area. For the first conjecture, we observe that the 1-dimensionality of the 1-dimensional asymptotic class recovered in Theorem 2.1 seems to come from the role of $\lceil \delta N \rceil$ in the formulation of almost independent sampling. To account for K-dimensional asymptotic classes with $K > 1$, it seems that we would perhaps need a yet more relaxed version of independent samplings as follows.

Definition 3.1. Let μ be an asymptotic probability measure for $\mathbf{C}$. We say that μ *has broadly almost independent sampling* if there is an integer $0 < K < \omega$ such that for every $1, n$-extension problem φ/θ of $\mathbf{C}$, there are numbers $0 \leq \varepsilon < 1$, $0 < \delta \leq 1$, and $k \leq K$ such that for all but finitely many N, for every $\overline{b} \in N^{(n)}$, for any $\{i_0 < \cdots < i_{t-1}\} \subseteq N \setminus \overline{b}$ where

$t = \lceil \delta N^{k/K} \rceil$, for every $s : t \to 2$,

$$\mu_N\Big[\bigwedge_{j<t} \varphi(i_j;\bar{b})^{s(j)} \;\Big|\; \theta(\bar{b}) \Big] \in (1 \pm \varepsilon) \prod_{j<t} \mu_N[\varphi(i_j;\bar{b})^{s(j)} \,|\, \theta(\bar{b})].$$

CONJECTURE 3.2. *Let* $\mathbf{C}$ *be a geometric Fraïssé class with generic model* $\mathcal{M}$*, and suppose that* $\mathbf{C}$ *has the* $0,1$*-law for first-order logic relative to an asymptotic probability measure* μ *that has broadly almost independent sampling. Then for some* K*,* $\mathbf{C}$ *contains a cofinal superrobust subclass* $\mathbf{D}$*, which is a* K*-dimensional asymptotic class via* $\mathbb{A}$*, with*

$$\mathbb{A}(\varphi(\overline{x};\overline{y})) \subset_{\text{fin}} \{(\text{qftp}(\overline{a}), k/K, m) : \overline{a} \in M^{|\overline{y}|}, \\ k \le K\dim_{\mathbf{a}}(\varphi(\mathcal{M};\overline{a})),\ m \in [0,\infty)\}.$$

Consequently, $T_{\mathbf{C}}$ *is supersimple, and* $D(X) \le K\dim_{\mathbf{a}}(X)$ *for any definable set* X *in* $\mathcal{M} \vDash T$.

Unfortunately, there is a new asymmetry in broadly almost independent sampling: while $t = \lfloor \delta N^{k/K} \rfloor$ is adequate, this does not always preclude, say, $t' = \lfloor \delta' N^{(2k+1)/(2K)} \rfloor$ from also being adequate. Thus, the techniques used in this paper are not alone sufficient to prove something like Conjecture 3.2.

On the other hand, it is not easy (if at all possible) to come up with a geometric Fraïssé class that has the $0,1$-law for first-order logic, is supersimple, and so forth, without building a 1-dimensional asymptotic class along the way. It would be quite interesting, if not totally unexpected (especially given the results of [1]), if the following were found to be true.

CONJECTURE 3.3. *Suppose* $\mathbf{C}$ *has the* $0,1$*-law for first-order logic relative to some asymptotic probability measure* μ*. Further, suppose that* $T_{\mathbf{C}}$ *is geometric, supersimple, and for every definable set* X *of the generic model* $\mathcal{M}$ *of* $\mathbf{C}$*,* $D(X)$ *is bounded by the algebraic dimension of* X*. Then there is an asymptotic probability measure* μ' *with almost independent sampling, relative to which* $\mathbf{C}$ *has the* $0,1$*-law for first-order logic.*

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