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# Badly approximable points in twisted Diophantine approximation and Hausdorff dimension

by

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**1. Introduction.** The classical result due to Dirichlet: for any real number $\theta$ there exist infinitely many natural numbers $q$ such that

$$\|q\theta\| \le q^{-1}, \tag{1}$$

where $\|\cdot\|$ denotes the distance to the nearest integer, has higher dimension generalisations. Consider any $n$-tuple $(j_1, \ldots, j_n)$ of real numbers such that

$$j_1, \ldots, j_n > 0 \quad \text{and} \quad \sum_{i=1}^{n} j_i = 1. \tag{2}$$

Then, for any vector $\theta = (\theta_1, \ldots, \theta_n) \in \mathbb{R}^n$, there exist infinitely many natural numbers $q$ such that

$$\max_{1\le i\le n} \|q\theta_i\|^{1/j_i} \le q^{-1}. \tag{3}$$

The two results above motivate the study of real numbers and real vectors $\theta \in \mathbb{R}^n$ for which the right hand sides of (1) and (3) respectively cannot be improved by an arbitrary constant. Those numbers and vectors respectively constitute the sets Bad of badly approximable numbers and $\mathrm{Bad}(j_1, \ldots, j_n)$ of $(j_1, \ldots, j_n)$-badly approximable tuples. Hence

$$\mathrm{Bad}(j_1, \ldots, j_n) := \Big\{(\theta_1, \ldots, \theta_n) \in \mathbb{R}^n : \inf_{q\in\mathbb{N}} \max_{1\le i\le n} q^{j_i}\|q\theta_i\| > 0\Big\}.$$

In the 1-dimensional case, it is well known that the set of badly approximable numbers has Lebesgue measure zero but maximal Hausdorff dimension. In the $n$-dimensional case, it is also a classical result that $\mathrm{Bad}(j_1, \ldots, j_n)$ has Lebesgue measure zero, and Schmidt proved in 1966 that the particular

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set Bad$(1/2, 1/2)$ has full Hausdorff dimension. But the result of maximal dimension in the weigthed setting has not been proved until almost 40 years later, by Pollington and Velani [21]. In the 2-dimensional case, An [1] showed that Bad$(j_1, j_2)$ is in fact winning for the now famous Schmidt games (see [22]). Thus he provided a direct proof of a conjecture of Schmidt stating that any countable intersection of sets Bad$(j_1, j_2)$ is non-empty (see also [2]).

Recently, interest in the size of related sets, usually referred to as 'twists' of the sets Bad$(j_1, \ldots, j_n)$, has developed. The study of these new sets started in the 1-dimensional setting: we fix $\theta \in \mathbb{R}$ and consider the twist of Bad:

$$\mathrm{Bad}_\theta := \Big\{\eta \in \mathbb{R} : \inf_{q\in\mathbb{N}} q\|q\theta - \eta\| > 0\Big\}.$$

The set $\mathrm{Bad}_\theta$ has a palpable interpretation in terms of rotations of the unit circle. If we identify the circle with the unit interval $[0, 1)$, the value $q\theta$ (modulo 1) may be thought of as the position of the origin after $q$ rotations by the angle $\theta$. If $\theta$ is rational, the rotation is periodic. If $\theta$ is irrational, a classical result of Weyl [25] implies that $q\theta$ (modulo 1) is equidistributed, so $q\theta$ visits any fixed subinterval of $[0, 1)$ infinitely often. The natural question of what happens if the subinterval is allowed to shrink with time arises.

Shrinking a subinterval corresponds to making its length decay according to some specified function. The set $\mathrm{Bad}_\theta$ corresponds to considering, for any $\epsilon > 0$, the shrinking interval $(\eta - \epsilon/q, \eta + \epsilon/q)$ centred at the point $\eta$ and where the specified function is $\epsilon/q$. Khinchin [14] showed that

$$\|q\theta - \eta\| < \frac{1+\delta}{\sqrt{5}\,q} \quad (\delta > 0) \tag{4}$$

is satisfied for infinitely many integers $q$, and Theorem III in Chapter III of Cassels' book [5] shows that the right hand side of (4) cannot be improved by an arbitrary constant for every irrational $\theta$ and every real $\eta$. This motivates the study of the set $\mathrm{Bad}_\theta$. Kim [16] proved in 2007 that it has Lebesgue measure zero, and later it was shown by Tseng [23] that it has full Hausdorff dimension (actually Tseng proved that $\mathrm{Bad}_\theta$ has the stronger property of being winning for any $\theta \in \mathbb{R}$).

By generalising circle rotations to rotations on tori of higher dimensions, i.e. by considering the sequence $q\theta$ (modulo 1) in $[0, 1)^n$ where $\theta = (\theta_1, \ldots, \theta_n) \in \mathbb{R}^n$, we obtain 'twists' of the sets Bad$(j_1, \ldots, j_n)$:

$$\mathrm{Bad}_\theta(j_1, \ldots, j_n) = \Big\{(\eta_1, \ldots, \eta_n) \in \mathbb{R}^n : \inf_{q\in\mathbb{N}} \max_{1\le i\le n} q^{j_i}\|q\theta_i - \eta_i\| > 0\Big\}. \tag{5}$$

Bugeaud et al. [3] proved that the non-weighted set $\mathrm{Bad}_\theta(1/n, \ldots, 1/n)$ has full Hausdorff dimension. Recently, Einsiedler and Tseng [8] extended the results of [3, 23] by showing, among other results, that $\mathrm{Bad}_\theta(1/n, \ldots, 1/n)$ is also winning. It was shown in [18] that such results may be obtained by classical methods developed by Khinchin [15] and Jarník [12, 13], and discussed in Chapter V of Cassels' book [5]. Unfortunately, these methods cannot be directly extended to the weighted setting. For the latter, less has been known. Harrap [10] gave the first contribution in the 2-dimensional case, by proving that $\mathrm{Bad}_\theta(j_1, j_2)$ has full Hausdorff dimension provided that the fixed point $\theta \in \mathbb{R}^2$ belongs to $\mathrm{Bad}(j_1, j_2)$, which is a significantly restrictive condition. Recently, under the hypothesis $\theta \in \mathrm{Bad}(j_1, \ldots, j_n)$, Harrap and Moshchevitin [11] have extended to weighted linear forms in higher dimension and improved to winning the result of [10].

In this paper, we prove that the weighted set $\mathrm{Bad}_\theta(j_1, \ldots, j_n)$ has full Hausdorff dimension for any $\theta \in \mathbb{R}^n$. Moreover, the following theorem holds.

THEOREM 1.1. *For any $\theta \in \mathbb{R}^n$ and all $j_1, \ldots, j_n > 0$ with $\sum_{i=1}^n j_i = 1$,*

$$\dim\big(\mathrm{Bad}_\theta(j_1, \ldots, j_n) \cap \mathrm{Bad}(1, 0, \ldots, 0) \cap \cdots \cap \mathrm{Bad}(0, \ldots, 0, 1)\big) = n.$$

The same type of theorem holds in the classical non-twisted setting [21, Theorem 2].

Note that if $1, \theta_1, \ldots, \theta_n$ are linearly dependent over $\mathbb{Z}$, then Theorem 1.1 is obvious. Indeed, in this case $\{q\theta : q \in \mathbb{Z}\}$ is restricted to a hyperplane $H$ of $\mathbb{R}^n$, so $\mathrm{Bad}_\theta(j_1, \ldots, j_n) \supset \mathbb{R}^n \setminus H$ is winning. Hence $\mathrm{Bad}_\theta(j_1, \ldots, j_n) \cap \mathrm{Bad}(1, 0, \ldots, 0) \cap \cdots \cap \mathrm{Bad}(0, \ldots, 0, 1)$ is winning and in particular has full dimension ([1]). Therefore we suppose throughout that $1, \theta_1, \ldots, \theta_n$ are linearly independent over $\mathbb{Z}$.

The strategy of the proof of Theorem 1.1 is as follows. We start in Section 3 by defining a set $\mathcal{V} \subset \mathrm{Bad}_\theta(j_1, \ldots, j_n)$ related to the best approximations (Section 2) to the fixed point $\theta \in \mathbb{R}^n$. Then (Section 4) we construct a Cantor-type set $K(R)$ inside $\mathcal{V} \cap \mathrm{Bad}(1, 0, \ldots, 0) \cap \cdots \cap \mathrm{Bad}(0, \ldots, 0, 1)$. Finally (Section 5), we describe a probability measure supported on $K(R)$ to which we can apply the mass distribution principle and thus find a lower bound for the dimension of $K(R)$ (Section 6).

In the following, we let $n \in \mathbb{N}$, fix an $n$-tuple $(j_1, \ldots, j_n) \in \mathbb{R}^n$ satisfying (2) and a vector $\theta = (\theta_1, \ldots, \theta_n) \in \mathbb{R}^n$ such that $1, \theta_1, \ldots, \theta_n$ are linearly independent over $\mathbb{Z}$. We denote by $x \cdot y$ the scalar product of two vectors $x$ and $y$ in $\mathbb{R}^n$, and by $\|\cdot\|$ the distance to the nearest integer.

## 2. Best approximations

DEFINITION 2.1. An $n$-dimensional vector $m = (m_1, \ldots, m_n) \in \mathbb{Z}^n \setminus \{0\}$ is called a *best approximation* to $\theta$ if for all $v \in \mathbb{Z}^n \setminus \{0, -m, m\}$ the following

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([1]) We recall that winning sets in $\mathbb{R}^n$ have maximal Hausdorff dimension, and that countable intersections of winning sets are again winning. We refer the reader to [22] for all necessary definitions and results on winning sets.

implication holds:

$$\max_{1\le i\le n} |v_i|^{1/j_i} \le \max_{1\le i\le n} |m_i|^{1/j_i} \;\Rightarrow\; \|v\cdot\theta\| > \|m\cdot\theta\|.$$

Note that the condition that $1,\theta_1,\ldots,\theta_n$ are $\mathbb{Z}$-linearly independent allows us to demand a strict inequality on the right hand side of the implication above.

Note also that when $n=1$, the best approximations to a real number $x$ are, up to the sign, the denominators of the convergents to $x$.

Since $1,\theta_1,\ldots,\theta_n$ are $\mathbb{Z}$-linearly independent, we have an infinite number of best approximations to $\theta$. They can be arranged up to the sign—so that two vectors of opposite sign do not both appear—in an infinite sequence

$$m_\nu = (m_{\nu,1},\ldots,m_{\nu,n}),\qquad \nu\ge 1, \tag{6}$$

such that the values

$$M_\nu = \max_{1\le i\le n} |m_{\nu,i}|^{1/j_i} \tag{7}$$

form a strictly increasing sequence, and the values

$$\zeta_\nu = \|m_\nu\cdot\theta\| \tag{8}$$

form a strictly decreasing sequence. Hence each value $M_\nu$ corresponds to a single best approximation $m_\nu$. The quantity $M_\nu$ can be referred to as the 'height' of $m_\nu$.

Best approximation vectors have often been used in proofs, but not always explicitly. In particular, Voronoi [24] selected some points in a lattice that correspond exactly to the best approximation vectors (see also [7]). Similar constructions were introduced in [17] or [4, Section 2]. Some important properties of the best approximation vectors are discussed in [19, 20], and a recent survey on the topic is due to Chevallier [6].

For each $\nu\ge 1$, it is easy to see that the region

$$\Big\{(x_0,\ldots,x_n)\in\mathbb{R}^{n+1} : \max_{1\le i\le n}|x_i|^{1/j_i} < M_{\nu+1},\ \Big|x_0+\sum_{i=1}^n x_i\theta_i\Big| < \zeta_\nu\Big\}$$

does not contain any integer point different from 0. Since this region has volume $2^{n+1}M_{\nu+1}\zeta_\nu$ (see [5, Lemma 4 in Appendix B]), it follows from Minkowski's convex body theorem that

$$\zeta_\nu M_{\nu+1}\le 1. \tag{9}$$

The inequality above will be used later, together with the following lemma stating that the sequence of heights $M_\nu$ is lacunary.

Lemma 2.2. *For every $\nu\ge 1$,*

$$M_{\nu+2\cdot 3^n}\ge 2M_\nu.$$

*Proof.* Given $\nu \geq 1$, we show that at most $2 \cdot 3^n$ vectors $m_{\nu+r}$ satisfy $r \geq 0$ and $M_{\nu+r} < 2M_\nu$. The goal is to see that the 0-symmetric region

$$\Big\{(x_0, \ldots, x_n) \in \mathbb{R}^{n+1} : \max_{1 \leq i \leq n} |x_i|^{1/j_i} < 2M_\nu,\ \Big|x_0 + \sum_{i=1}^n x_i\theta_i\Big| \leq \zeta_\nu\Big\} \tag{10}$$

contains at most $4 \cdot 3^n$ integer points other than 0. The region (10) is covered by sets of the form

$$T(\xi) = \left\{\begin{array}{r} (x_0, \ldots, x_n) \in \mathbb{R}^{n+1} : \max_{1 \leq i \leq n} |x_i - \xi_i|^{1/j_i} \leq M_\nu \\ \text{and } |x_0 - \xi_0 + \sum_{i=1}^n (x_i - \xi_i)\theta_i| \leq \zeta_\nu \end{array}\right\},$$

with

$$\xi_i \in \{-2M_\nu^{j_i}, 0, 2M_\nu^{j_i}\}, \qquad \xi_0 = -\sum_{i=1}^n \xi_i\theta_i. \tag{11}$$

Each region $T(\xi)$ is the translate by $(\xi_0, \ldots, \xi_n)$ of the set

$$\Big\{(x_0, \ldots, x_n) \in \mathbb{R}^{n+1} : \max_{1 \leq i \leq n} |x_i|^{1/j_i} \leq M_\nu,\ \Big|x_0 + \sum_{i=1}^n x_i\theta_i\Big| \leq \zeta_\nu\Big\},$$

which contains exactly three integer points: 0 and two best approximations with opposite sign. Hence each $T(\xi)$ contains at most four integer points. Since there are $3^n$ possible choices for $(\xi_0, \ldots, \xi_n)$ satisfying (11), the set (10) contains at most $4 \cdot 3^n$ integer points. ∎

**3. The set $\mathcal{V}$ included in** $\mathrm{Bad}_\theta(j_1, \ldots, j_n)$**.** The following proposition allows us to work with a set defined by the best approximations to $\theta$ instead of working directly with $\mathrm{Bad}_\theta(j_1, \ldots, j_n)$.

PROPOSITION 3.1. *If $\eta \in \mathbb{R}^n$ satisfies*

$$\inf_\nu \|m_\nu \cdot \eta\| > 0, \tag{12}$$

*then $\eta \in \mathrm{Bad}_\theta(j_1, \ldots, j_n)$.*

*Proof.* Let $\eta = (\eta_1, \ldots, \eta_n) \in \mathbb{R}^n$ satisfy

$$\|m_\nu \cdot \eta\| > \gamma \qquad \forall \nu \geq 1$$

for some $\gamma > 0$. For all $q \in \mathbb{N}$ and $\nu \geq 1$, we have the identity

$$m_\nu \cdot \eta = m_\nu \cdot (\eta - q\theta) + q m_\nu \cdot \theta,$$

from which we obtain the inequalities

$$\gamma < \|m_\nu \cdot \eta\| \leq n \max_{1 \leq i \leq n} |m_{\nu,i}| \cdot \|\eta_i - q\theta_i\| + q\zeta_\nu. \tag{13}$$

Since $\zeta_\nu$ is strictly decreasing and $\zeta_\nu \to 0$ as $\nu \to \infty$, there exists $\nu \geq 1$ such that

$$\frac{\gamma}{2\zeta_\nu} \leq q \leq \frac{\gamma}{2\zeta_{\nu+1}}. \tag{14}$$

On the one hand, from (13) and the upper bound in (14), we deduce that

$$\max_{1\leq i\leq n} \|\eta_i - q\theta_i\| \cdot |m_{\nu+1,i}| > \frac{\gamma}{2n}. \tag{15}$$

On the other hand, from the lower bound in (14) and (9),

$$q \geq \frac{\gamma}{2} M_{\nu+1}.$$

Therefore

$$q^{j_i} \geq c|m_{\nu+1,i}| \quad \forall i = 1, \ldots, n, \tag{16}$$

where

$$c = \min_{1\leq i\leq n} \left(\frac{\gamma}{2}\right)^{j_i}.$$

Finally, by combining (15) and (16), we get

$$\max_{1\leq i\leq n} \|\eta_i - q\theta_i\| q^{j_i} > \frac{\gamma c}{2n}. \ \blacksquare$$

We define

$$\mathcal{V} := \Big\{\eta \in \mathbb{R}^n : \inf_{\nu\geq 1} \|m_\nu \cdot \eta\| > 0\Big\}.$$

Clearly

$$\mathcal{V} \subset \mathrm{Bad}_\theta(j_1, \ldots, j_n). \tag{17}$$

**4. The Cantor-type set $K(R)$.** In this section we construct the Cantor-type set $K(R)$ inside

$$\mathrm{Bad}_\theta(j_1, \ldots, j_n) \cap \mathrm{Bad}(1, 0, \ldots, 0) \cap \cdots \cap \mathrm{Bad}(0, \ldots, 0, 1).$$

To lighten notation, throughout this section we denote by $\mathcal{M}$ the set of best approximations in the sequence (6), and for each $m \in \mathcal{M}$, by $M_m$ the quantity defined by (7), i.e.

$$M_m = \max_{1\leq i\leq n} |m_i|^{1/j_i}.$$

Hence

$$\mathcal{V} = \Big\{\eta \in \mathbb{R}^n : \inf_{m\in\mathcal{M}} \|m \cdot \eta\| > 0\Big\}.$$

We define the following partition of $\mathcal{M}$:

$$\mathcal{M}_k := \{m \in \mathcal{M} : R^{k-1} \leq M_m < R^k\} \quad (k \geq 0). \tag{18}$$

Note that $\mathcal{M}_0 = \emptyset$. We have $\mathcal{M} = \bigcup_{k=0}^\infty \mathcal{M}_k$.

We also need, for each $1 \le i \le n$, the following partitions of $\mathbb{N}$:

$$\mathcal{Q}_k^{(i)} := \{q \in \mathbb{N} : R^{(k-1)j_i/2} \le q < R^{kj_i/2}\} \quad (k \ge 0). \tag{19}$$

Note that $\mathcal{Q}_0^{(i)} = \emptyset$, and that for each $1 \le i \le n$, we have $\mathbb{N} = \bigcup_{k=0}^{\infty} \mathcal{Q}_k^{(i)}$.

At the heart of the construction of $K(R)$ is constructing a collection $\mathcal{F}_k$ of hyperrectangles $H_k$ inside the hypercube $[0,1]^n$ that satisfy the following $n+1$ conditions, for some $\epsilon > 0$:

(0) $|m \cdot \eta + p| \ge \epsilon \quad \forall \eta \in H_k, \forall m \in \mathcal{M}_{k-1}, \forall p \in \mathbb{Z}$;
(1) $q|q\eta_1 - p| \ge \epsilon \quad \forall \eta \in H_k, \forall q \in \mathcal{Q}_{k-1}^{(1)}, \forall p \in \mathbb{Z}$;
...
($n$) $q|q\eta_n - p| \ge \epsilon \quad \forall \eta \in H_k, \forall q \in \mathcal{Q}_{k-1}^{(n)}, \forall p \in \mathbb{Z}$.

We start by inductively constructing a collection $(\mathcal{G}_k^{(0)})_{k \ge 0}$ of hyperrectangles satisfying condition (0). Then we define a subcollection $\mathcal{G}_k^{(1)} \subset \mathcal{G}_k^{(0)}$ of hyperrectangles that also satisfy (1), a subcollection $\mathcal{G}_k^{(2)} \subset \mathcal{G}_k^{(1)}$ that also satisfies (2), etc. This process ends with a subcollection $\mathcal{G}_k^{(n)}$ that satisfies the $n+1$ conditions above. We would like to quantify $\#\mathcal{G}_k^{(n)}$. We can give a lower bound, but not the exact cardinality. So we refine the collection $\mathcal{G}_k^{(n)}$ by choosing a right and final subcollection $\mathcal{F}_k$ that we can quantify.

Let

$$j_{\min} = \min_{1 \le i \le n} j_i, \qquad j_{\max} = \max_{1 \le i \le n} j_i.$$

Let $R > 4^{1/j_{\min}}$ and $\epsilon > 0$ be such that

$$\epsilon < \frac{1}{2R^{2j_{\max}}}. \tag{20}$$

The parameter $R$ will be chosen later to be sufficiently large in order to satisfy various conditions.

**4.1. The collection $\mathcal{G}_k^{(0)}$.** For each $m \in \mathcal{M}$ and $p \in \mathbb{Z}$, let

$$\Delta(m,p) := \{x \in \mathbb{R}^n : |m \cdot x + p| < \epsilon\}.$$

Geometrically, $\Delta(m,p)$ is the thickening of the hyperplane

$$\mathcal{L}(m,p) := \{x \in \mathbb{R}^n : m \cdot x + p = 0\} \tag{21}$$

with width $2\epsilon/m_i$ in all the $x_i$-coordinate directions.

Next we describe the induction procedure in order to define the collection $(\mathcal{G}_k^{(0)})_{k \ge 0}$. We work within the closed hypercube $H_0 = [0,1]^n$ and set $\mathcal{G}_0^{(0)} = \{H_0\}$. For $k \ge 0$, we divide each $H_k \in \mathcal{G}_k^{(0)}$ into new hyperrectangles $H_{k+1}$ of size

$$R^{-(k+1)j_1} \times \cdots \times R^{-(k+1)j_n}.$$

Note that if $R^{j_i} \notin \mathbb{Z}$ for some $1 \le i \le n$, the division will not be exact, in the sense that the new hyperrectangles will not cover $H_k$. This division gives at least $\prod_{i=1}^n [R^{j_i}] > R - \sum_{i=1}^n R^{j_i}$ new hyperrectangles. Among these new hyperrectangles, we denote by $\mathcal{G}^{(0)}(H_k)$ the collection of hyperrectangles $H_{k+1} \subset H_k$ satisfying

$$H_{k+1} \cap \Delta(m,p) = \emptyset \quad \forall m \in \mathcal{M}_k,\ \forall p \in \mathbb{Z}.$$

We define

$$\mathcal{G}_{k+1}^{(0)} := \bigcup_{H_k \in \mathcal{G}_k^{(0)}} \mathcal{G}^{(0)}(H_k).$$

Hence $\mathcal{G}_{k+1}^{(0)}$ is nested in $\mathcal{G}_k^{(0)}$, and it is a collection of 'good' hyperrectangles with respect to all the best approximations $m$ satisfying $M_m < R^k$ and all the integers $p$. The collection $\mathcal{G}^{(0)}(H_k)$ is the collection of 'good' hyperrectangles that we obtain from the division of $H_k$.

Next we give a lower bound for $\#\mathcal{G}_k^{(0)}$. Actually, for a fixed hyperrectangle $H_k \in \mathcal{G}_k^{(0)}$, we give a lower bound for the number of hyperrectangles $H_{k+1} \in \mathcal{G}^{(0)}(H_k)$. Alternatively, we give an upper bound for the number of 'bad' hyperrectangles in $H_k$; these are the hyperrectangles $H_{k+1} \subset H_k$ that intersect the thickening $\Delta(m,p)$ of some hyperplane $\mathcal{L}(m,p)$ with $m \in \mathcal{M}_k$. Facts 1 and 2 bound the number of thickenings $\Delta(m,p)$ with $m \in \mathcal{M}_k$ and $p \in \mathbb{Z}$ that intersect $H_k$. Fact 3 bounds the number of hyperrectangles $H_{k+1} \subset H_k$ that are intersected by a thickening $\Delta(m,p)$ with $m \in \mathcal{M}_k$ and $p \in \mathbb{Z}$.

FACT 1. We show that for each $k \ge 1$, the set $\mathcal{M}_k$ contains at most $2 \cdot 3^n(1 + \log_2(R))$ best approximations. Indeed, Lemma 2.2 implies that

$$M_{\nu+2\cdot 3^n(1+\log_2(R))} \ge 2^{1+\log_2(R)} M_\nu \overset{(18)}{\ge} 2^{1+\log_2(R)} R^{k-1} > R^k.$$

Therefore, there are at most $2 \cdot 3^n(1+\log_2(R))$ best approximations in $\mathcal{M}_k$.

FACT 2. Fix $m \in \mathcal{M}_k$. We show that there are at most $2^n n$ thickenings $\Delta(m,p)$ that intersect $H_k$. Indeed, suppose that two different thickenings $\Delta(m,p)$ and $\Delta(m,p')$ intersect the same edge of $H_k$. This edge of $H_k$ is a segment of a line which is parallel to an $x_l$-axis. Let $P = (y_1, \dots, y_n)$ and $P' = (y_1', \dots, y_n')$ denote the points of intersection of this line parallel to the $x_l$-axis with $\mathcal{L}(m,p)$ and $\mathcal{L}(m,p')$ respectively. The fact that $P$ and $P'$ respectively belong to $\mathcal{L}(m,p)$ and $\mathcal{L}(m,p')$ is described by the equations

$$m \cdot y + p = 0, \qquad m \cdot y' + p' = 0. \tag{22}$$

The fact that $P$ and $P'$ both belong to a line parallel to the $x_l$-axis implies that $y_i = y_i'$ for all $i \ne l$. Hence, by subtracting the second equation in (22) from the first one, we get

$$|y_l - y_l'| - \frac{2\epsilon}{|m_l|} \geq \frac{|p-p'|}{|m_l|} - \frac{2\epsilon}{|m_l|} > \frac{1}{R^{kj_l}} - \frac{1}{2R^{kj_l}} = \frac{1}{2}R^{-kj_l}. \tag{23}$$

Since the length of $H_k$ in the $x_l$-direction is $R^{-kj_l}$, (23) implies that there are no more than two thickenings intersecting the same edge of $H_k$. Thus the number of thickenings $\Delta(m,p)$ that intersect $H_k$ is at most twice the number of edges of $H_k$, and this is $2^n n$.

Fact 3. For a thickening $\Delta(m,p)$, we give an upper bound for the number of hyperrectangles $H_{k+1} \subset H_k$ that intersect $\Delta(m,p)$. Fix $m \in \mathcal{M}_k$ and $p \in \mathbb{Z}$. Let $l$ be such that $M_m = |m_l|^{1/j_l}$. Consider the projection of $\Delta(m,p) \cap H_k$ onto one of the faces of $H_k$ parallel to the plane given by the $x_l$-axis and an $x_i$-axis. We split this projection of $\Delta(m,p) \cap H_k$ into right triangles with perpendicular sides of length $2\epsilon/|m_l|$ and $2\epsilon/|m_i|$ respectively. From this splitting and the inequality

$$\frac{2\epsilon}{|m_l|} < \frac{1}{2R^{j_l(k+1)}},$$

we deduce that $\Delta(m,p)$ intersects at most $2[R^{1-j_{\min}}]$ hyperrectangles $H_{k+1} \subset H_k$.

Conclusion. There are at most $[2^{n+2}3^n n(1+\log_2(R))R^{1-j_{\min}}]$ hyperrectangles $H_{k+1} \subset H_k$ that intersect some $\Delta(m,p)$ with $m \in \mathcal{M}_k$, $p \in \mathbb{Z}$. Hence

$$\#\mathcal{G}^{(0)}(H_k) \geq R - \sum_{i=1}^{n} R^{j_i} - [2^{n+2}3^n n(1+\log_2(R))R^{1-j_{\min}}].$$

**4.2. The subcollections $\mathcal{G}_k^{(i)}$.** For each $q \in \mathbb{N}$ and $p \in \mathbb{Z}$, consider the sets

$$\Gamma_i(q,p) := \{x \in \mathbb{R}^n : q|qx_i - p| < \epsilon\} \quad (1 \leq i \leq n). \tag{24}$$

Geometrically, each $\Gamma_i(q,p)$ is a thickening of the hyperplane described by the equation $x_i = p/q$ with width $2\epsilon/q^2$ in the $x_i$-coordinate direction.

We construct a tower of subcollections

$$\mathcal{G}_k^{(n)} \subset \mathcal{G}_k^{(n-1)} \subset \cdots \subset \mathcal{G}_k^{(0)},$$

where each $\mathcal{G}_k^{(i)}$ consists of hyperrectangles in $\mathcal{G}_k^{(i-1)}$ whose points avoid each thickening $\Gamma_i(q,p)$ for $q \in \mathcal{Q}_{k-1}^{(i)}$. More precisely, for $1 \leq i \leq n$, we form $\mathcal{G}_k^{(i)}$ by letting

$$\mathcal{G}^{(i)}(H_k) := \{H_{k+1} \in \mathcal{G}^{(i-1)}(H_k) : H_{k+1} \cap \Gamma_i(q,p) = \emptyset \ \forall q \in \mathcal{Q}_k^{(i)}\}$$

and

$$\mathcal{G}_{k+1}^{(i)} := \bigcup_{H_k \in \mathcal{G}_k^{(i-1)}} \mathcal{G}^{(i)}(H_k).$$

Clearly the hyperrectangles in $\mathcal{G}_{k+1}^{(i)}$ satisfy conditions $(0), (1), \ldots, (i)$, so the collection $\mathcal{G}_k^{(n)}$ satisfies the $n+1$ conditions $(0), (1), \ldots, (n)$.

Next, for each $1 \le i \le n$ and $H_k \in \mathcal{G}_k^{(i-1)}$, we give a lower bound of $\#\mathcal{G}^{(i)}(H_k)$. Suppose that there are $(q,p), (q',p') \in \mathcal{Q}_k^{(i)} \times \mathbb{Z}$ such that

$$H_k \cap \Gamma_i(q,p) \neq \emptyset, \qquad H_k \cap \Gamma_i(q',p') \neq \emptyset.$$

In other words, suppose there exist $\eta, \eta'$ in $H_k$ such that

$$q|q\eta_i - p| < \epsilon, \qquad q'|q'\eta_i' - p'| < \epsilon. \tag{25}$$

Then, by (19) and (20), we have

$$\left|\frac{p}{q} - \frac{p'}{q'}\right| - \frac{\epsilon}{q^2} - \frac{\epsilon}{q'^2} \ge \frac{1}{qq'} - \frac{\epsilon}{q^2} - \frac{\epsilon}{q'^2} > \frac{1}{R^{kj_i}} - \frac{1}{2R^{kj_i}} = \frac{1}{2}R^{-kj_i}. \tag{26}$$

Since the length of $H_k$ in the $x_i$-direction is $R^{-kj_i}$, (26) implies that at most two thickenings of the form (24) can intersect $H_k$.

Now, from (19) and (20), it follows that if $\eta \in \Gamma_i(q,p)$, then

$$\left|\eta_i - \frac{p}{q}\right| < \frac{\epsilon}{q^2} < \frac{1}{2}R^{-kj_i},$$

which implies that each thickening $\Gamma_i(q,p)$ intersects at most

$$2[R^{j_1}] \times \cdots \times \widehat{[R^{j_i}]} \times \cdots \times [R^{j_n}] \le 2[R^{1-j_i}]$$

hyperrectangles $H_{k+1} \subset H_k$.

Therefore, there are at most $4[R^{1-j_{\min}}]$ hyperrectangles $H_{k+1} \subset H_k$ that do not satisfy condition $(i)$ (p. 307). Hence

$$\#\mathcal{G}^{(i)}(H_k) \ge R - \sum_{l=1}^{n} R^{j_l} - [2^{n+2}3^n n(1+\log_2(R))R^{1-j_{\min}}] - 4i[R^{1-j_{\min}}]. \tag{27}$$

**4.3. The right subcollection $\mathcal{F}_k$.** We choose a subcollection of $\mathcal{G}_k^{(n)}$ that we can exactly quantify in the following way. Let $\mathcal{F}_0 := \mathcal{G}_0^{(0)}$. Choose $R$ sufficiently large so that

$$n_R := \left[R - \sum_{i=1}^{n} R^{j_i} - 2^{n+2}3^n n(1+\log_2(R)) \cdot R^{1-j_{\min}} - 4nR^{1-j_{\min}}\right] > 1.$$

For $k \ge 0$, for each $H_k \in \mathcal{F}_k$, we choose exactly $n_R$ hyperrectangles from the collection $\mathcal{G}^{(n)}(H_k)$ and denote this new collection by $\mathcal{F}(H_k)$. Trivially,

$$\#\mathcal{F}(H_k) = n_R > 1, \tag{28}$$

so each hyperrectangle $H_k \in \mathcal{F}_k$ gives rise to exactly the same number of hyperrectangles $H_{k+1}$ in $\mathcal{F}(H_k)$. Finally, define

$$\mathcal{F}_{k+1} := \bigcup_{H_k \in \mathcal{F}_k} \mathcal{F}(H_k).$$

This completes the construction of the Cantor-type set

$$K(R) := \bigcap_{k=0}^{\infty} \mathcal{F}_k.$$

By construction, $K(R) \subset \mathcal{V} \cap \mathrm{Bad}(1,0,\ldots,0) \cap \cdots \cap \mathrm{Bad}(0,\ldots,0,1)$. Moreover, in view of (28),

$$\#\mathcal{F}_{k+1} = \#\mathcal{F}_k \, \#\mathcal{F}(H_k) = n_R^{k+1}. \tag{29}$$

**5. The measure $\mu$ on $K(R)$.** We now describe a probability measure $\mu$ supported on the Cantor-type set $K(R)$ constructed in the previous section. The measure we define is analogous to the probability measure used in [21] and [2] on a Cantor-type set of $\mathbb{R}^2$. For any hyperrectangle $H_k \in \mathcal{F}_k$ we attach a weight $\mu(H_k)$ which is defined recursively as follows: for $k = 0$,

$$\mu(H_0) = \frac{1}{\#\mathcal{F}_0} = 1,$$

and for $k \geq 1$,

$$\mu(H_k) = \frac{1}{\#\mathcal{F}(H_{k-1})} \mu(H_{k-1}) \quad (H_k \in \mathcal{F}(H_{k-1})).$$

This procedure defines inductively a mass on any hyperrectangle used in the construction of $K(R)$. Moreover, $\mu$ can be further extended to all Borel subsets $X$ of $\mathbb{R}^n$, so that $\mu$ actually defines a measure supported on $K(R)$, by letting

$$\mu(X) = \inf \sum_{H \in \mathcal{C}} \mu(H)$$

where the infimum is taken over all coverings $\mathcal{C}$ of $X$ by rectangles $H \in \{\mathcal{F}_k : k \geq 0\}$. For further details, see [9, Proposition 1.7].

Notice that, in view of (29), we have

$$\mu(H_k) = 1/\#\mathcal{F}_k \quad (k \geq 0).$$

A classical method for obtaining a lower bound for the Hausdorff dimension of an arbitrary set is the following mass distribution principle (see [9, p. 55]).

LEMMA 5.1 (Mass distribution principle). *Let $\delta$ be a probability measure supported on a subset $X$ of $\mathbb{R}^n$. Suppose there are positive constants $c$, $s$ and $l_0$ such that*

$$\delta(S) \leq c l^s \tag{30}$$

*for any hypercube $S \subset \mathbb{R}^n$ with side length $l \leq l_0$. Then* $\dim(X) \geq s$.

The goal in the next section is to prove that there exist constants $c$ and $l_0$ satisfying (30) with $\delta = \mu$, $X = K(R)$ and $s = n - \lambda(R)$, where $\lambda(R) \to 0$ as $R \to \infty$. Then from the mass distribution principle it will follow that $\dim(K(R)) = n$.

**6. A lower bound for** $\dim(K(R))$**.** Recall that

$$j_{\min} = \min_{1 \le i \le n} j_i.$$

Let $k_0$ be a positive integer such that

$$R^{-kj_i} < R^{-(k+1)j_{\min}} \quad \forall j_i \neq j_{\min} \text{ and } k \ge k_0. \tag{31}$$

Consider an arbitrary hypercube $S$ of side length $l \le l_0$ where $l_0$ satisfies

$$l_0 < R^{-(k_0+1)j_{\min}} \tag{32}$$

together with a second inequality to be determined later. We can choose $k > k_0$ so that

$$R^{-(k+1)j_{\min}} < l < R^{-kj_{\min}}. \tag{33}$$

From (31) it follows that

$$l > R^{-kj_i} \qquad \forall j_i \neq j_{\min}. \tag{34}$$

Then it is easy to see that $S$ intersects at most $2^n l^{n-1} \prod_{j_i \neq j_{\min}} R^{kj_i}$ hyper-rectangles $H_k \in \mathcal{F}_k$, so

$$\mu(S) \le 2^n l^{n-1} \prod_{j_i \neq j_{\min}} R^{kj_i} \mu(H_k) = 2^n l^{n-1} R^{k-kj_{\min}} \frac{1}{\#\mathcal{F}_k}.$$

Since $R^{(k+1)j_{\min}} > l^{-1}$ (see (33)), we have

$$\mu(S) \le 2^n l^n R^{j_{\min}} R^k \frac{1}{\#\mathcal{F}_k}.$$

Recall that we mentioned in Section 4 that later we would choose the parameter $R$ large enough to satisfy various conditions. We choose $R$ with

$$R^{-1} \sum_{i=1}^{n} R^{j_i} - 2^{n+2} 3^n n (1 + \log_2(R)) R^{-j_{\min}} - 4nR^{-j_{\min}} - R^{-1} \le 2^{-1}.$$

Then by (29) we have

$$\mu(S) \le 2^n l^n R^{j_{\min}} 2^k.$$

We further choose

$$k \ge \log(R) \quad \text{and} \quad \lambda(R) = \frac{1 + \log(2)}{j_{\min} \log(R)},$$

so

$$\mu(S) \le 2^n l^n R^{kj_{\min}\lambda(R)}.$$

Since $R^{kj_{\min}} < l^{-1}$ (see (33)), it follows that

$$\mu(S) \le 2^n l^{n-\lambda(R)}.$$

Finally, by applying the mass distribution principle we obtain

$$\dim(K(R)) \ge n - \lambda(R) \to n \quad \text{as } R \to \infty.$$

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