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A note on rational points near planar curves

by

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1. Introduction. Let f be a real-valued function defined on a compact interval $I = [\rho, \xi] \subseteq \mathbb{R}$. For positive real numbers $\delta \leq 1/2$ and $Q \geq 1$, define

$$\tilde{N}_f(Q,\delta) = \#\left\{ \begin{matrix} (a,b,q) \in \mathbb{Z}^3 : 1 \leq q \leq Q,\ a/q \in I,\ \gcd(a,b,q) = 1, \\ |f(a/q) - b/q| < \delta/Q \end{matrix} \right\}.$$

Roughly speaking, this counts the number of rational points with denominator at most Q that lie within δQ^{-1} of the curve $\mathcal{C}_f = \{(x, f(x)) : x \in I\}$. Huang [3, Theorem 2] estimated this quantity. As discussed in [3], such estimates are readily applied to the Lebesgue theory of metric diophantine approximation.

THEOREM 1.1 (Huang). *Let $0 < c_1 \leq c_2$. Assume that $f : I \to \mathbb{R}$ is a C^2 function satisfying*

$$c_1 \leq |f''(x)| \leq c_2 \quad (x \in I),$$

with Lipschitz second derivative. Assume further that

$$1/2 \geq \delta > Q^{\varepsilon - 1} \tag{1.1}$$

for some $\varepsilon \in (0,1)$. Then

$$\frac{2\sqrt{3}}{9\zeta(3)} + O(Q^{-\varepsilon/2}) \leq \frac{\tilde{N}_f(Q,\delta)}{|I|\delta Q^2} \leq \frac{1}{\zeta(3)} + O(Q^{-\varepsilon/2}). \tag{1.2}$$

The implied constant depends on I, c_1, c_2, ε and the Lipschitz constant; it is independent of f, δ and Q.

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Theorem 1.1 sharpened the upper bounds obtained by Huxley [4] and Vaughan–Velani [5], as well as the lower bounds obtained by Beresnevich–Dickinson–Velani [1] and Beresnevich–Zorin [2].

The purpose of this note is to squeeze together the constants in (1.2), so as to confirm Huang's conjectured asymptotic formula

$$\tilde{N}_f(Q,\delta) \sim \frac{2}{3\zeta(3)}|I|\delta Q^2 \quad (Q \to \infty), \tag{1.3}$$

within the range (1.1). The asymptotic formula (1.3) follows straightforwardly from our theorem, which we state below and establish in the next section.

Theorem 1.2. *Assume the hypotheses of Theorem* 1.1. *Let* $\eta > 0$ *and*

$$0 < \tau < \varepsilon/2.$$

Then

$$\frac{2}{3\zeta(3)} - \eta + O(Q^{-\tau}) \le \frac{\tilde{N}_f(Q,\delta)}{|I|\delta Q^2} \le \frac{2}{3\zeta(3)} + \eta + O(Q^{-\tau}).$$

The implied constant depends on $I, c_1, c_2, \varepsilon, \eta$ *and the Lipschitz constant.*

We use Landau and Vinogradov notation: for functions f and positive-valued functions g, we write $f \ll g$ or $f = O(g)$ if there exists a constant C such that $|f(x)| \le Cg(x)$ for all x. If S is a set, we denote the cardinality of S by $\#S$.

2. The count. In this section, we prove Theorem 1.2. For positive real numbers $\delta \le 1/2$ and $Q \ge 1$, define the auxiliary counting function

$$\hat{N}_f(Q,\delta) = \#\left\{ \begin{matrix} (a,b,q) \in \mathbb{Z}^3 : 1 \le q \le Q,\ a/q \in I,\ \gcd(a,b,q) = 1, \\ |f(a/q) - b/q| < \delta/q \end{matrix} \right\}.$$

With the same assumptions as in Theorem 1.1, Huang [3, Corollary 1] showed that

$$\hat{N}_f(Q,\delta) = (\zeta(3)^{-1} + O(Q^{-\varepsilon/2})) \cdot |I|\delta Q^2. \tag{2.1}$$

Let $t \in \mathbb{N}$, $1/2 < \alpha < 1$ and

$$\alpha_i = \alpha^i \quad (0 \le i \le t).$$

We will have $t \ll_\eta 1$, so the hypothesis (1.1) is satisfied with 2τ in place of ε and $(\alpha_i Q, \alpha_j \delta)$ in place of (Q,δ), whenever Q is large and $0 \le i,j \le t$. In particular (2.1) holds with these adjustments, so

$$\hat{N}_f(\alpha_i Q, \alpha_j \delta) = \left(\frac{\alpha_i^2 \alpha_j}{\zeta(3)} + O(Q^{-\tau}) \right) \cdot |I|\delta Q^2 \quad (0 \le i,j \le t). \tag{2.2}$$

Employing (2.2), we have

$$\begin{aligned}
&\tilde{N}_f(Q,\delta) \\
&\qquad \ge \sum_{i=1}^{t} \#\left\{ \begin{array}{c} (a,b,q) \in \mathbb{Z}^3 : \alpha_i Q < q \le \alpha_{i-1} Q,\ a/q \in I,\ \gcd(a,b,q) = 1, \\ |f(a/q) - b/q| < \alpha_i \delta/q \end{array} \right\} \\
&\qquad = \sum_{i=1}^{t} \big(\hat{N}_f(\alpha_{i-1}Q, \alpha_i\delta) - \hat{N}_f(\alpha_i Q, \alpha_i \delta)\big) \\
&\qquad = \sum_{i=1}^{t} \left(\frac{\alpha_{i-1}^2 \alpha_i - \alpha_i^3}{\zeta(3)} + O(Q^{-\tau}) \right) \cdot |I|\delta Q^2.
\end{aligned}$$

Now

$$\tilde{N}_f(Q,\delta) \ge \left(\frac{X(\boldsymbol{\alpha})}{\zeta(3)} + O(tQ^{-\tau}) \right) \cdot |I|\delta Q^2, \tag{2.3}$$

where

$$X(\boldsymbol{\alpha}) = \sum_{i \le t} (\alpha_{i-1}^2 \alpha_i - \alpha_i^3).$$

We compute

$$\begin{aligned}
X(\boldsymbol{\alpha}) &= (\alpha - \alpha^3) \sum_{j=0}^{t-1} (\alpha^3)^j = \frac{(\alpha - \alpha^3)(1 - \alpha^{3t})}{1 - \alpha^3} \\
&= (1 - \alpha^{3t})(1 - (1 + \alpha + \alpha^2)^{-1}).
\end{aligned}$$

Choosing α close to 1, and then choosing $t \ll_\eta 1$ large, gives

$$X(\boldsymbol{\alpha}) \ge 2/3 - \zeta(3)\eta.$$

Substituting this into (2.3) yields the desired lower bound.

We attack the upper bound in a similar fashion, but there is an extra term to consider. By (2.2), we have

$$\begin{aligned}
&\tilde{N}_f(Q,\delta) - \tilde{N}_f(\alpha_t Q, \alpha_t \delta) \\
&\qquad \le \sum_{i=1}^{t} \#\left\{ \begin{array}{c} (a,b,q) \in \mathbb{Z}^3 : \alpha_i Q < q \le \alpha_{i-1} Q,\ a/q \in I,\ \gcd(a,b,q) = 1, \\ |f(a/q) - b/q| < \alpha_{i-1} \delta/q \end{array} \right\} \\
&\qquad = \sum_{i=1}^{t} \big(\hat{N}_f(\alpha_{i-1}Q, \alpha_{i-1}\delta) - \hat{N}_f(\alpha_i Q, \alpha_{i-1} \delta)\big) \\
&\qquad = \sum_{i=1}^{t} \left(\frac{\alpha_{i-1}^3 - \alpha_{i-1} \alpha_i^2}{\zeta(3)} + O(Q^{-\tau}) \right) \cdot |I|\delta Q^2.
\end{aligned}$$

Now

$$\tilde{N}_f(Q,\delta) - \tilde{N}_f(\alpha_t Q, \alpha_t \delta) \le \left(\frac{Y(\boldsymbol{\alpha})}{\zeta(3)} + O(tQ^{-\tau}) \right) \cdot |I|\delta Q^2,$$

where

$$Y(\boldsymbol{\alpha}) = \sum_{i \le t} (\alpha_{i-1}^3 - \alpha_{i-1}\alpha_i^2).$$

Notice that

$$Y(\boldsymbol{\alpha}) = \alpha^{-1} X(\boldsymbol{\alpha}) \le \frac{1-\alpha^2}{1-\alpha^3} = \frac{1+\alpha}{1+\alpha+\alpha^2}.$$

Choosing α close to 1 gives $Y(\boldsymbol{\alpha}) \le 2/3 + \zeta(3)\eta/2$, and so

$$\tilde{N}_f(Q,\delta) \le \tilde{N}_f(\alpha_t Q, \alpha_t \delta) + \left(\frac{2}{3\zeta(3)} + \frac{\eta}{2} + O(tQ^{-\tau}) \right) \cdot |I|\delta Q^2. \tag{2.4}$$

For the first term on the right hand side of (2.4), we bootstrap Huang's upper bound (1.2). This gives

$$\tilde{N}_f(\alpha_t Q, \alpha_t \delta) \le \left(\frac{\alpha_t^3}{\zeta(3)} + O(Q^{-\tau}) \right) \cdot |I|\delta Q^2.$$

Choosing $t \ll_\eta 1$ large, so that $\alpha_t^3 \le \zeta(3)\eta/2$, we now have

$$\tilde{N}_f(\alpha_t Q, \alpha_t \delta) \le \left(\frac{\eta}{2} + O(Q^{-\tau}) \right) \cdot |I|\delta Q^2.$$

Substituting this into (2.4) provides the sought upper bound, completing the proof of the theorem.

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