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Conjugacy classes of diffeomorphisms of the interval in $\mathcal{C}^1$-regularity

by

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Abstract. We consider the conjugacy classes of diffeomorphisms of the interval, endowed with the $\mathcal{C}^1$-topology. Given two diffeomorphisms f, g of $[0;1]$ without hyperbolic fixed points, we give a complete answer to the following two questions:

- under what conditions does there exist a sequence of smooth conjugates $h_n f h_n^{-1}$ of f tending to g in the $\mathcal{C}^1$-topology?
- under what conditions does there exist a continuous path of $\mathcal{C}^1$-diffeomorphisms h_t such that $h_t f h_t^{-1}$ tends to g in the $\mathcal{C}^1$-topology?

We also present some consequences of these results to the study of $\mathcal{C}^1$-centralizers for $\mathcal{C}^1$-contractions of $[0;\infty)$; for instance, we exhibit a $\mathcal{C}^1$-contraction whose centralizer is uncountable and abelian, but is not a flow.

1. Introduction

1.1. Conjugacy classes of diffeomorphisms of the interval. We consider diffeomorphisms of $[0;1]$ whose fixed points are precisely 0 and 1. In 1970, N. Kopell [K] showed that, for an open and $\mathcal{C}^2$-dense set of such diffeomorphisms f, the only $\mathcal{C}^1$-diffeomorphisms commuting with f are the iterates of f, i.e. the elements of the group $\{f^i : i \in \mathbb{Z}\}$: the diffeomorphism f has trivial centralizer. One of the keys to this theorem is the fact that the $\mathcal{C}^1$-centralizers of f on $[0;1)$ and on $(0;1]$ are both one parameter-groups, which will respectively be denoted by f_t^- and f_t^+. The centralizer of f on $[0;1]$ corresponds to those values of t such that $f_t^- = f_t^+$. Comparison between these two flows is described by the Mather invariant of f: this invariant

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depends continuously on f in the $\mathcal{C}^2$-topology, and vanishes when $f_t^- = f_t^+$ for all t. More details about the centralizer and the Mather invariant can be found in Section 2. For the moment, let us explain how this result is the starting point of the study presented in this paper.

The Mather invariant is a $\mathcal{C}^1$-conjugacy invariant among $\mathcal{C}^2$-diffeomorphisms. However, it does not vary continuously with respect to the $\mathcal{C}^1$-topology: in [BCVW], it is shown that every f can be $\mathcal{C}^1$-perturbed into a diffeomorphism for which the Mather invariant vanishes, and thus the perturbed diffeomorphism is embedded in a flow. The argument mentioned in [BCVW] seemed to indicate that an arbitrarily small $\mathcal{C}^1$-perturbation of f may lead to a Mather invariant having every a priori fixed value. On the other hand, the Mather invariant represents, together with the values of the derivatives at the extremities—when these are not equal to 1—a complete $\mathcal{C}^1$-conjugacy invariant among $\mathcal{C}^2$-diffeomorphisms of $[0;1]$ without fixed point in the interior.

This reasoning led the authors of [BCVW] to the following conjecture:

Conjecture 1. For all $\alpha > 1$ and $0 < \beta < 1$, consider the set $D_{\alpha,\beta}$ of diffeomorphisms $f\colon [0;1] \to [0;1]$ whose fixed points are exactly 0 and 1, and such that $Df(0) = \alpha$ and $Df(1) = \beta$. Then $\mathcal{C}^1$-conjugacy classes are all $\mathcal{C}^1$-dense in $D_{\alpha,\beta}$. In other words, given two diffeomorphisms $f, g \in D_{\alpha,\beta}$, one can find a diffeomorphism arbitrarily $\mathcal{C}^1$-close to g and conjugate to f by a diffeomorphism of $[0;1]$.

Our Theorem 1.2 confirms this conjecture: conjugacy classes are dense in $D_{\alpha,\beta}$. Actually, we show a slightly stronger result, implying that each diffeomorphism $g \in D_{\alpha,\beta}$ is *accessible* from f by a path in its differentiable conjugacy class. We will see that this stronger notion is crucial for the study of centralizers.

More precisely, the key notion for the study of centralizers will be the following:

Definition 1.1. Given $f, g \in \mathcal{D}\mathit{iff}^1([0;1])$, one says that f is *isotopic to g by $\mathcal{C}^1$-conjugacy* (or simply *isotopic by conjugacy to g*) if there exists a $\mathcal{C}^1$-continuous path $(h_t)_{t\in[0,1)}$, $h_t \in \mathcal{D}\mathit{iff}^1([0;1])$, such that $h_0 = \mathrm{id}$, and $h_t f h_t^{-1} \to g$ as $t \to 1$. Under these conditions, the path $(h_t f h_t^{-1})_{t\in[0;1)}$ will be called an *isotopy by conjugacy from f to g*.

And this is the result:

Theorem 1.2. *Let f and g be two diffeomorphisms of $[0;1]$ without fixed points, except 0 and 1, and $f, g \geq \mathrm{id}$ on $(0;1)$. Then there exists an isotopy by conjugacy from f to g if and only if f and g have the same derivatives at 0 and 1: $Df(0) = Dg(0)$ and $Df(1) = Dg(1)$.*

Continuity of the Mather invariant in the $\mathcal{C}^2$-topology implies that Theorem 1.2 is not true in the $\mathcal{C}^r$-topology, for any $r \geq 2$.

Remark. This notion of isotopy by conjugacy is *a priori* more restrictive than the fact that g is *accessible from f by a path in its conjugacy class* (i.e. g is the limit of a continuous path $(h_t f h_t^{-1})_{t\in[0;1)}$ such that $h_t \in \mathcal{D}\mathit{iff}^1([0;1])$ for all t and $h_0 = \mathrm{id}$, without requiring any continuity of h_t with respect to t). Actually, concerning diffeomorphisms of $D_{\alpha,\beta}$, the two notions are equivalent.

Indeed, Theorem 1.2 is a consequence of the more general Theorem 4.8: each continuous path of diffeomorphisms linking f to g that neither changes derivatives on the boundary nor introduces fixed points in the interior can be $\mathcal{C}^1$-approximated by an isotopy by conjugacy from f to g.

Now that we have introduced the notion of isotopy by conjugacy, we come to the natural question:

Question 1. Given two diffeomorphisms f, g of $[0;1]$, under what conditions is there an isotopy by $\mathcal{C}^1$-conjugacy from f to g?

As already seen, the case of diffeomorphisms without fixed points in $(0;1)$ was settled by Theorem 1.2, which also establishes Conjecture 1.

Let us consider now diffeomorphisms of $[0;1]$ without hyperbolic fixed points in $(0;1)$, but with no restriction on the set of nonhyperbolic fixed points. We will see that, in this context, the *existence of an isotopy by conjugacy from f to g* is a more restrictive condition than that *g is an accumulation point of the conjugacy class of f*.

Theorem 1.3 below provides a necessary and sufficient condition for each of those properties. Its statement features the notion of *signature* of a diffeomorphism f of $[0;1]$ (see Definition 7.1): the signature of f is a pair $((C = \{C_i\}_{i\in I}, \prec), \sigma)$, where:

- $(C, \prec)$ is a countable ordered set—roughly speaking, the set of maximal intervals on which the sign of f does not change, ordered by their position in $[0;1]$;
- σ is a map from C into $\{+,-\}$ which associates to each interval the sign of $f - \mathrm{id}$ on it.

The most general answer to Question 1 provided in this paper can be stated as follows:

Theorem 1.3. *Let f and g be two nondecreasing $\mathcal{C}^1$-diffeomorphisms of $[0;1]$ without hyperbolic fixed points except possibly 0 and 1, and such that $Df(0) = Dg(0)$ and $Df(1) = Dg(1)$. Let $((C, \prec), \sigma)$ and $((C', \prec'), \sigma')$ denote the respective signatures of f and g. Then:*

1. *There exists an isotopy by conjugacy from f to g if and only if there exists an injective and order preserving map $\Phi : C' \to C$ such that $\sigma'(c') = \sigma(\Phi(c'))$ for all $c' \in C'$.*

2. *There exists a sequence of conjugates of f converging to g if and only if, for every finite subset γ' of C', there exists an injective and order preserving map $\Phi : \gamma' \to C$ such that $\sigma'(c') = \sigma(\Phi(c'))$ for all $c' \in \gamma'$.*

We now give an example to show the distinction between the existence of an isotopy by conjugacy from f to g and the fact that g is an accumulation point of the conjugacy class of f.

EXAMPLE 1.4. Let $f, g \in \mathcal{D}\mathit{iff}^1([0;1])$ without hyperbolic fixed points be such that $\mathrm{Fix}(f)$ is a sequence tending to 0, and the sign of $f - \mathrm{id}$ changes at each fixed point, while $\mathrm{Fix}(g)$ consists of a sequence tending to 0 and a sequence tending to 1, and the sign of $g - \mathrm{id}$ changes at each fixed point. Namely, they can been chosen so that:

- the sign of $f - \mathrm{id}$ is:
  - < 0 on $\left(\frac{1}{2n}; \frac{1}{2n-1}\right)$,
  - > 0 on $\left(\frac{1}{2n+1}; \frac{1}{2n}\right)$ for all $n > 0$;
- the sign of $g - \mathrm{id}$ is
  - < 0 on $\left(\frac{1}{2n}; \frac{1}{2n-1}\right)$ and on $\left(1 - \frac{1}{2n-1}; 1 - \frac{1}{2n}\right)$,
  - > 0 on $\left(\frac{1}{2n+1}; \frac{1}{2n}\right)$, $\left(1 - \frac{1}{2n}; 1 - \frac{1}{2n+1}\right)$ and $\left(\frac{1}{3}; \frac{2}{3}\right)$ for all $n > 1$.

Then there exists a sequence $(h_n)_{n\in\mathbb{N}}$ of diffeomorphisms of $[0;1]$ such that $h_n f h_n^{-1}$ converges to g, but there does not exist any isotopy by conjugacy from f to g. Indeed, one can easily check that the signature of f is $-\mathbb{N}$ with $\sigma_f(-n) = (-1)^{n+1}$, whereas the signature of g is $\mathbb{Z}$ with $\sigma_g(n) = (-1)^n$.

1.2. $\mathcal{C}^r$-centralizer of contractions of $[0;\infty)$. The *centralizer* $\mathcal{C}^r(f)$, for $r \geq 0$, of a $\mathcal{C}^r$-diffeomorphism f of a manifold M is the set $\mathcal{C}^r(f) = \{g \in \mathcal{D}\mathit{iff}^r(M) : gf = fg\}$. This set is a group and contains the group $\langle f \rangle = \{f^i\}_{i\in\mathbb{Z}}$ generated by f. One says that f has *trivial centralizer* when $\mathcal{C}^r(f)$ is as small as possible, that is, $\mathcal{C}^r(f) = \langle f \rangle$.

We are interested here in diffeomorphisms of the half-line $[0;\infty)$, and more precisely in *contractions*, that is, diffeomorphisms having 0 as the only fixed point and which are strictly smaller than the identity on $(0;\infty)$. In this context, size and structure of centralizers of diffeomorphisms of $[0;\infty)$ depend essentially on the regularity of the diffeomorphisms in question.

For example, each $\mathcal{C}^0$-contraction f of $[0;\infty)$ is conjugate to the translation $x \mapsto x - 1$ of $\mathbb{R}$ (after restriction to $(0;\infty)$), so that the centralizer of f is conjugate to the group of lifts on $\mathbb{R}$ of diffeomorphisms of the circle. Consequently, the $\mathcal{C}^0$-centralizer of a contraction has large size and does not depend, up to a conjugacy, on the diffeomorphism cosidered.

Beyond the $\mathcal{C}^0$ context, however, rather opposite situations may occur: for example, if f is a homothety of $[0;\infty)$, then the only homeomorphisms commuting with f and which are differentiable at zero are homotheties. In particular, $\mathcal{C}^1(f)$ is the group of homotheties. In the case of a $\mathcal{C}^2$-contraction

f of $[0;\infty)$, the situation is quite similar: the $\mathcal{C}^1$-centralizer is then isomorphic to $\mathbb{R}$ (it is a flow). This follows from Kopell's lemma [K] and from Szekeres' theorem [Sz], or from Sternberg's theorem [St]. Consequently:

- *The $\mathcal{C}^r$-centralizer of a contraction of $[0;\infty)$, for $r \geq 2$, is isomorphic to a subgroup of $\mathbb{R}$.*

That is what was stated in Kopell's lemma: if f is a $\mathcal{C}^2$-contraction of $[0;\infty)$ and g is a $\mathcal{C}^1$-diffeomorphism commuting with f, then, if g has another fixed point besides 0, then g must be the identity. From that, one can deduce immediately that the $\mathcal{C}^1$-centralizer of f is an ordered and archimedean group, and thus from Hölder's theorem is a subgroup of $\mathbb{R}$.

The results above lead us to the following natural question:

QUESTION 2. Which subgroups of $\mathbb{R}$ appear as $\mathcal{C}^r$-centralizers of contractions of $[0;\infty)$?

Concerning this question, recall that F. Sergeraert [Se] gave explicit examples of $\mathcal{C}^\infty$-contractions of $[0;\infty)$ whose $\mathcal{C}^r$-centralizers, for $r \geq 2$, are not whole $\mathbb{R}$. Also, relying on his construction, H. Eynard [Ey] provided examples where the centralizer is a proper subgroup of $\mathbb{R}$ containing a Cantor set.

Between these two extreme situations, we find the case of the $\mathcal{C}^1$-centralizer of $\mathcal{C}^1$-contractions of $[0;\infty)$, in which we are interested in this paper. This context is rather different from the $\mathcal{C}^2$-context, in particular because, Kopell's lemma being invalid in the $\mathcal{C}^1$-context, the centralizer presents a greater flexibility than in the $\mathcal{C}^2$-context, and also because in compensation the nonvalidity of Szekeres' theorem does not ensure anymore the existence of a centralizer as big as $\mathbb{R}$.

1.3. Embedding of a group in a centralizer and isotopy by conjugacy to the identity. In this context of $\mathcal{C}^1$-centralizers of $\mathcal{C}^1$-contractions, various situations can occur; indeed:

- Togawa [T] showed that, for every map in a certain G_δ-dense subset of the set of $\mathcal{C}^1$-contractions, the centralizer is trivial.
- It is a consequence of our results that there exist contractions of $[0;\infty)$ whose centralizers contain nontrivial diffeomorphisms having fixed points different from 0 (going in this way against the conclusions of Kopell's lemma). Here we note that the existence of such examples was previously known (see in particular [FF]).

Existence of $\mathcal{C}^1$-counterexamples to Kopell's lemma leads to the following question: given a compact subinterval of $(0;\infty)$, which groups of diffeomorphisms with support in this subinterval can be embedded in the centralizer of a contraction? First we have to formulate this question in more precise words.

Definition 1.5. Given a compact interval $J \subset (0;\infty)$ and a group G of diffeomorphisms of J, the group G will be called *embeddable in the centralizer of a contraction* if there exists a $\mathcal{C}^1$-contraction f of $[0;\infty)$, with 0 as the only fixed point, and a subgroup G_0 of $\mathcal{C}^1(f)$ such that $\{g|_J : g \in G_0\} = G$.

Our first result answers the above question by characterizing groups which are embeddable in the $\mathcal{C}^1$-centralizer of a contraction. Before stating this result, we extend the notion of isotopy by conjugacy from a diffeomorphism to another diffeomorphism (Definition 1.1) to a notion of isotopy by conjugacy from a group to a diffeomorphism:

Definition 1.6. A subgroup $G \subset \mathcal{D}\mathit{iff}^1([0;1])$ is said to be *isotopic by conjugacy to a $\mathcal{C}^1$-diffeomorphism g* of $[0;1]$ if there exists a continuous path $(h_t)_{t\in[0;1)}$ of $\mathcal{C}^1$-diffeomorphisms of $[0;1]$ such that, for all $f \in G$, $(h_t f h_t^{-1})_{t\in[0;1)}$ is an isotopy by conjugacy from f to g.

Theorem 1.7. *A group of diffeomorphisms of a compact subinterval of* $(0;\infty)$ *is embeddable in the centralizer of a contraction if and only if there exists an isotopy by $\mathcal{C}^1$-conjugacy from this group to* id.

Thus we want now to answer the following question:

Question 3. Which groups of diffeomorphisms of $[0;1]$ are isotopic by conjugacy to the identity?

An obvious obstruction is the existence of a hyperbolic fixed point (i.e. with derivative different from 1) for at least one element of the group. The question is whether there exist other obstructions. In this paper, we answer this question in the case where G is generated by only one diffeomorphism, namely we shall prove:

Theorem 1.8. *A $\mathcal{C}^1$-diffeomorphism f of* $[0;1]$ *is isotopic to the identity by $\mathcal{C}^1$-conjugacy if and only if all its fixed points are tangent to the identity.*

We think that Theorem 1.8 is not true if the isotopy is required to be a $\mathcal{C}^2$-continuous path of $\mathcal{C}^2$-conjugates. As an immediate corollary of Theorem 1.8, we obtain:

Corollary 1.9. *If J is a nontrivial subinterval of* $(0;\infty)$ *and if g_J is a diffeomorphism of J whithout hyperbolic fixed points, then g_J is embeddable in the $\mathcal{C}^1$-centralizer of a contraction: there exists a contraction f of* $[0;\infty)$ *and a diffeomorphism $g \in \mathcal{C}^1(f)$ whose restriction to J is g_J.*

The following result exhibits $\mathcal{C}^1$-contractions with big centralizers:

Corollary 1.10. *Let $\mathcal{I} = \{I_i\}_{i\in\mathbb{N}}$ be a sequence of subintervals of* $[0;1]$ *whose interiors are pairwise disjoint and, for all $i \in \mathbb{N}$, let g_i be a diffeomorphism of* $[0;1]$ *with support in I_i, whose derivatives at fixed points are* 1. *Assume that, for all $i \in \mathbb{N}$, g_i tends to the identity in the $\mathcal{C}^1$-topology, i.e.* $\lim_{i\to\infty} \|g_i - \mathrm{id}\|_1 = 0$. *Then:*

- *for every bounded sequence $\bar{n} = \{n_i\}_{i\in\mathbb{N}}$ of integers, the map $g_{\bar{n}}$ which coincides with $g_i^{n_i}$ on I_i for all $i \in \mathbb{N}$ and with the identity outside the union of the I_i's is a diffeomorphism of* $[0;1]$;
- *the set of all $g_{\bar{n}}$ where $\bar{n}$ is a bounded sequence of integers is an abelian and uncountable group, denoted by $G_{\mathcal{I}}$;*
- *the group $G_{\mathcal{I}}$ is isotopic by conjugacy to the identity.*

Proof. One checks easily that G_I is a group of diffeomorphisms. From Theorem 1.8, if $\bar{n}$ is the constant sequence equal to 1, then $g_{\bar{n}}$ is isotopic by conjugacy to the identity, via the isotopy $(h_t g_{\bar{n}} h_t^{-1})_{t\in[0;1)}$. It follows that, for all i, $(h_t g_i h_t^{-1})_{t\in[0;1)}$ is an isotopy by conjugacy from g_i to the identity. Thus, for every bounded sequence $\bar{n}$ of integers, $(h_t g_{\bar{n}} h_t^{-1})_{t\in[0;1)}$ is an isotopy by conjugacy from $g_{\bar{n}}$ to the identity. ∎

After a first informal version of this paper was available, Andrès Navas showed us a clever and simple argument proving Theorem 1.8, which is based on the cohomological equation. His result can also be applied in the context of diffeomorphisms of the circle with irrational rotation number. Here is what he shows:

Theorem 1.11 (Navas). *Each diffeomorphism of* $[0;1]$ *without hyperbolic fixed points is isotopic by conjugacy to the identity. Each diffeomorphism of the circle with irrational rotation number α is isotopic by conjugacy to the rotation R_α.*

2. Background

2.1. The $\mathcal{C}^r$, $r \geq 2$, setting: Kopell's lemma and theorem, Szekeres vector field and Mather invariant. In [K], Kopell considered the centralizer of $\mathcal{C}^2$-diffeomorphisms of the interval. She proved the existence of a $\mathcal{C}^2$-dense open subset $\mathcal{O}$ of diffeomorphisms whose centralizer is trivial: if $f \in \mathcal{O}$ and g commutes with f, then $g = f^k$ for some $k \in \mathbb{Z}$. The technical point in [K], known as Kopell's lemma, concerns the centralizer of $\mathcal{C}^2$-contractions of $[0,\infty)$.

Lemma 2.1 (Kopell's lemma). *Let f be a $\mathcal{C}^2$-contraction of* $[0;\infty)$. *Let $g \in \mathcal{D}\mathit{iff}^1([0;\infty))$ commute with f. Assume that g has a fixed point in* $(0;\infty)$. *Then g is the identity map.*

The proof of Kopell's lemma uses essentially the control of the distortion of the iterates f^n on a fundamental domain $[x; f(x)]$ to prove that the derivative of any diffeomorphism g commuting with f and having a fixed point at x is bounded on $[x; f(x)]$, independently of the diffeomorphism g. As g^n, for $n \in \mathbb{Z}$, also commutes with f, this means that the derivatives of g^n, $n \in \mathbb{Z}$, are bounded on $[x; f(x)]$ independently of $n \in \mathbb{Z}$. One easily deduces that the only possibility is that g is the identity.

Kopell's lemma implies that the $\mathcal{C}^1$-centralizer $\mathcal{C}^1(f)$ of a $\mathcal{C}^2$-contraction f is naturally ordered: if $g_1, g_2 \in \mathcal{C}^1(f)$ satisfy $g_1(x_0) < g_2(x_0)$ for a point $x_0 \in [0; \infty)$, then $g_1(x) < g_2(x)$ for all $x \in [0; \infty)$. This property defines an order, which is a total archimedean order, and then Hölder's theorem implies that $\mathcal{C}^1(f)$ is isomorphic to a subgroup of $\mathbb{R}$.

In the opposite direction, Szekeres' theorem states that any $\mathcal{C}^2$-contraction of $[0; \infty)$ is the time-one map of a vector field X which is $\mathcal{C}^2$ on $(0; \infty)$ and $\mathcal{C}^1$ at 0. By combining Kopell's lemma and Szekeres' theorem, one find that $\mathcal{C}^1(f)$ is precisely the flow of X (and thus is isomorphic to $\mathbb{R}$). As f is assumed to be $\mathcal{C}^r$, $r \geq 2$, it is natural to consider its $\mathcal{C}^r$-centralizer. It is a subgroup of $\mathcal{C}^1(f)$, and [Se] provides an example of a $\mathcal{C}^\infty$-contraction f which has no $\mathcal{C}^2$ square root: the time-$\frac{1}{2}$ map of the Szekeres flow is not $\mathcal{C}^2$ (see also [Ey] for recent generalizations). This cannot happen if the fixed point 0 is hyperbolic (or even if it is not $\mathcal{C}^\infty$-flat). In that case, f is smoothly conjugate to a smooth normal form and hence is embedded in a smooth flow.

Let us come back now to diffeomorphisms f of $[0; 1]$ and assume for instance that $\mathrm{Fix}(f) = \{0, 1\}$ and $f(x) > x$ for $x \in (0; 1)$. In this case, the restriction $(f_-)^{-1}$ of f^{-1} to $[0; 1)$ and f_+ of f to $(0; 1]$ are conjugate to contractions of $[0; \infty)$ and therefore are the time-one maps of the Szekeres flows X_- and X_+ respectively.

The $\mathcal{C}^1$-centralizer of f is the intersection

$$\mathcal{C}^1(f) = \mathcal{C}^1(f_+) \cap \mathcal{C}^1(f_-).$$

In [K], N. Kopell shows that, for a $\mathcal{C}^2$-open and dense subset of such diffeomorphisms, $\mathcal{C}^1(f_+) \cap \mathcal{C}^1(f_-) = \{f^n : n \in \mathbb{Z}\}$. Let us quickly explain why this is natural: Start with a smooth diffeomorphism f_0 of $[0; 1]$, $f_0 > \mathrm{id}$ on $(0; 1)$, which is the time-one map of a vector field X. Choose a point $a \in (0, 1)$ and a (smooth) diffeomorphism φ supported on $[a; f_0(a)]$. Consider then the diffeomorphism $f = \varphi f_0$. One easily checks that the Szekeres vector fields X_- and X_+ coincide respectively with $\varphi_*(X)$ and X on $[a; f_0(a)] = [a; f(a)]$. If $\varphi(x) > x$ on $(a; f_0(a))$, one can see that, for every $t \in (0; 1)$, $(X_-)_t(a) > (X_+)_t(a)$. Thus $\mathcal{C}^1(f_+) \cap \mathcal{C}^1(f_-) = \{f^n : n \in \mathbb{Z}\}$.

This argument is closely related to the Mather invariant: the Szekeres flows X_- and X_+ both provide a time parametrization $\psi_-, \psi_+ \colon \mathbb{R} \to (0; 1)$

which conjugate f to the translation $T_1 : t \mapsto t+1$. These parametrizations are well defined up to composition with a translation. The Mather invariant is the change of these coordinates: it is a diffeomorphism of $\mathbb{R}$ commuting with the translation T_1, and well defined up to compositon on the right and on the left with translations.

In the example above, the Mather invariant $\mathcal{M}_f$ will be precisely the diffeomorphism induced on $\mathbb{R}$ commuting with T_1 and whose expression in the fundamental domain $\psi_-^{-1}([a; f_0(a)])$ is

$$\mathcal{M}_f|_{\psi_-^{-1}([a;f_0(a)])} = \psi_+^{-1}\varphi\psi_-.$$

Let us conclude this section by noticing that Kopell's lemma, Szekeres' theorem, and the Mather invariant are typical of $\mathcal{C}^2$-diffeomorphisms and do not hold in the $\mathcal{C}^1$-setting: for instance, Tsuboi [Ts] builds counterexamples to the $\mathcal{C}^{1+\alpha}$ Kopell's lemma. Indeed, as can be noticed from our Theorems 1.7 and 1.8, Kopell's lemma is not true anymore in these settings. These two theorems together give $\mathcal{C}^1$-counterexamples to Kopell's lemma: they show that every diffeomorphism supported on a compact interval included in $(0;\infty)$ can be embedded in the $\mathcal{C}^1$-centralizer of a $\mathcal{C}^1$-contraction.

In the same spirit, Kopell's lemma implies that the $\mathcal{C}^1$-centralizer of a $\mathcal{C}^2$-diffeomorphism f of $[0;1]$ is not abelian if and only if f coincides with the identity on some nonempty open interval. This is no more true if f is $\mathcal{C}^1$: [BoFa] shows that there exist diffeomorphisms of $[0;1]$ without fixed point in $(0;1)$ whose centralizer may contain the free group $\mathbb{F}_2$.

Therefore, it is somewhat surprising that the Mather invariant is actually invariant under $\mathcal{C}^1$-conjugacies (among $\mathcal{C}^2$-diffeomorphisms). The reason is that the $\mathcal{C}^1$-centralizer is preserved under $\mathcal{C}^1$-conjugacies, hence the Szekeres flows are conjugate. Nevertheless, [BCVW] shows that one can obtain a trivial Mather invariant by $\mathcal{C}^1$-small pertubations.

In this paper, we will define and use (in Section 4.3) a notion of Mather invariant in the $\mathcal{C}^1$-setting. Let us give an idea of this tool. We consider diffeomorphisms f, g of $[0;1]$ without fixed points in $(0;1)$ such that g coincides with f in a neighbourhood of 0 and of 1. As a consequence, there is a unique diffeomorphism h_- of $[0;1)$ which conjugates f to g and which is the identity map in a neighbourhood of 0. In the same way, there is a unique diffeomorphism h_+ of $(0;1]$ which conjugates f to g and which is the identity map in a neighbourhood of 1. The Mather invariant of g with respect to f is the diffeomorphism $\mathcal{M}_f(g) = h_+^{-1}h_- : (0;1) \to (0;1)$, which commutes with f. This Mather invariant vanishes if and only if f and g are conjugate by a diffeomorphism h which is the identity in a neighbourhood of 0 and 1. Our main result consists in some sense in cancelling this relative Mather invariant by small perturbations of g.

2.2. Historical motivation in foliation theory. Kopell's lemma has been an important tool for the study of codimension 1 $\mathcal{C}^2$-foliations, in particular on 3-manifolds. In particular, due to the Novikov theorem and Thurston's construction of foliations, compact leaves diffeomorphic to a torus $\mathbb{T}^2$ play a key role in this study. For such leaves, the holonomy group is generated by two commuting diffeomorphisms of a transverse section which is a segment.

Let us mention a straightforward consequence of Kopell's theorem: on $\mathbb{T}^3$, there are foliations whose leaves are cylinders and finitely many compact tori, which are $\mathcal{C}^2$-structurally stable: every $\mathcal{C}^2$-foliation $\mathcal{C}^2$-close to them is conjugate to them by a homeomorphism. Such a foliation $\mathcal{F}$ may be constructed as the suspension of two commuting diffeomorphisms f, g of the circle: f is a Morse–Smale diffeomorphism with exactly two hyperbolic fixed points, and g is the identity. One chooses f so that its restriction to each segment between two successive fixed points belongs to the $\mathcal{C}^2$-open set given by Kopell's result for which the centralizer is trivial. Then any foliation $\mathcal{C}^2$-close to $\mathcal{F}$ is conjugate to the suspension of two commuting diffeomorphisms f_1 and g_1, which are $\mathcal{C}^2$-close to f and g respectively. As the centralizer of f_1 is trivial and g_1 is close to the identity, one deduces that $g_1 = \mathrm{id}$ and one concludes using the fact that f is structurally stable. The fact that the Mather invariant can be made trivial by small $\mathcal{C}^1$-perturbations of f shows that none of these foliations are $\mathcal{C}^1$-structurally stable.

Let us give a more recent application of the study of centralizers in foliation theory. The topology of the space of foliations remains mostly a mistery: one does not know whether this space is arc-connected, locally connected, etc. Recently, H. Eynard announced the connectedness of the space of $\mathcal{C}^\infty$ codimension 1 foliations on compact 3-manifolds whose tangent bundle belongs to a given homotopy class. More precisely, in her PhD thesis, she reduced the connectedness problem to the problem of connectedness of the space of $\mathcal{C}^\infty$-actions of $\mathbb{Z}^2$ on $[0,1]$; this last step has been done in [BE].

3. Groups which are embeddable in a centralizer: proof of Theorem 1.7. Let us begin with the following useful lemma:

LEMMA 3.1. *Let G be a group of diffeomorphisms of a segment J. There exists an isotopy from G to* id *if and only if there exists a sequence $(\varphi_n)_{n\in\mathbb{N}}$ of diffeomorphisms of this segment, converging to the identity in the $\mathcal{C}^1$-topology, such that $\varphi_0 = \mathrm{id}$ and such that, for all $g \in G$, the sequence $\varphi_n \ldots \varphi_1\varphi_0 g\varphi_0^{-1}\varphi_1^{-1} \ldots \varphi_n^{-1}$ converges to the identity in the $\mathcal{C}^1$-topology.*

Proof. Let us assume that there exists an isotopy $(h_t g h_t^{-1})_{t\in[0;1)}$ from each element g of G to the identity.

Claim. *There exists a sequence* $(\varepsilon_n)_{n\in\mathbb{N}}$ *of strictly positive numbers converging to* 0 *and an increasing sequence* $(t_n)_{n\in\mathbb{N}}$ *in* $[0;1)$ *converging to* 1 *such that* $t_0=0$ *and* $\|h_{t_{n+1}}h_{t_n}^{-1}-\mathrm{id}\|_1<\varepsilon_n$ *for all* $n\in\mathbb{N}$.

Proof of the claim. Given an increasing sequence $(t'_n)_{n\geq 1}$ in $[0;1)$ converging to 1, by compactness of $\{h_t : t\in[t'_n;t'_{n+1}]\}$, for all $n\geq 1$ there exist an integer K_n and a sequence $(t_k^n)_{k\in[\![1;K_n]\!]}$ of real numbers in $[0;1)$ such that $\|h_{t_{k+1}^n}h_{t_k^n}^{-1}-\mathrm{id}\|_1<1/n$ for all $k\in[\![1;K_n-1]\!]$. By concatenating the sequences $(t_k^n)_{k\in[\![1;K_n]\!]}$ on one hand, and on the other hand the constant sequences equal to $1/n$ for $n\geq 1$, we obtain the desired sequences $(t_n)_{n\in\mathbb{N}}$ and $(\varepsilon_n)_{n\in\mathbb{N}}$. ∎

Let $(\varepsilon_n)_{n\in\mathbb{N}}$ and $(t_n)_{n\in\mathbb{N}}$ be as in the claim above, and define the following $\mathcal{C}^1$-diffeomorphisms of $[0;1]$: $\varphi_0=h_{t_0}$ and $\varphi_n=h_{t_n}h_{t_{n-1}}^{-1}$ for $n\geq 1$. Then the sequence $(\varphi_n)_{n\in\mathbb{N}}$ converges to the identity in the $\mathcal{C}^1$-topology, as also does the sequence $(\varphi_n\ldots\varphi_1\varphi_0 g\varphi_0^{-1}\varphi_1^{-1}\ldots\varphi_n^{-1})_{n\in\mathbb{N}}=(h_{t_n}gh_{t_n}^{-1})_{n\in\mathbb{N}}$, since $h_tgh_t^{-1}$ is assumed to converge to id when $t\to 1$.

Conversely, assume that there exists a sequence $(\varphi_n)_{n\in\mathbb{N}}$ of diffeomorphisms of J converging to id as $n\to\infty$ such that $\varphi_0=\mathrm{id}$ and, for all $g\in G$, $\varphi_n\ldots\varphi_1\varphi_0g\varphi_0^{-1}\varphi_1^{-1}\ldots\varphi_n^{-1}$ converges to the identity in the $\mathcal{C}^1$-topology.

For all $n\in\mathbb{N}$ and $t\in[0;1]$, define $\varphi_{n,t}=\mathrm{id}+t(\varphi_n-\mathrm{id})$.

For all $n\in\mathbb{N}$, denote by H_n the diffeomorphism of $[0;1]$ defined by $H_n=\varphi_n\circ\cdots\circ\varphi_0$, and, for $T\in\mathbb{R}$, define a diffeomorphism h_T of $[0;1]$ by $h_T=\varphi_{n+1,t}\circ H_n$, where $t\in[0;1)$ is the fractional part of T, and $n\in\mathbb{N}$ its integer part.

Since $\varphi_{n,t}H_{n-1}\to\varphi_nH_{n-1}=H_n=\mathrm{id}\circ H_n=\varphi_{n,0}H_n$ as $t\to 1$, and because of the continuity of the path $(\varphi_{n,t})_{t\in[0;1]}$ for all $n\in\mathbb{N}$, the path $(h_T)_{T\in\mathbb{R}}$ is continuous. By hypothesis, if $g\in G$, then $H_{n-1}gH_{n-1}^{-1}$ converges to id as $n\to\infty$. One also knows that $\|\varphi_{n,t}-\mathrm{id}\|_1\leq\|\varphi_n-\mathrm{id}\|_1$, where $\|\varphi_n-\mathrm{id}\|_1\to 0$ as $n\to\infty$. Consequently, $\|\varphi_{n,t}-\mathrm{id}\|_1$ tends to 0 uniformly with respect to $t\in[0;1)$. Thus $D(\varphi_{n,t}H_{n-1}gH_{n-1}^{-1}\varphi_{n,t}^{-1})$ tends to 0 uniformly on J. As $\varphi_{n,t}H_{n-1}gH_{n-1}^{-1}\varphi_{n,t}^{-1}$ has fixed points at the extremities of J, it follows that this diffeomorphism also converges to id in the $\mathcal{C}^1$-topology as $n\to\infty$. Furthermore, if $g\in G$, then $h_Tgh_T^{-1}=\varphi_{n,t}H_{n-1}gH_{n-1}^{-1}\varphi_{n,t}^{-1}$, where n still denotes the integer part of T and t its fractional part. As a consequence, $h_Tgh_T^{-1}$ converges to id as $T\to\infty$. ∎

Sufficient condition of Theorem 1.7. We show that a group which is isotopic to the identity by conjugacy is embeddable in a centralizer.

Let us consider a closed subinterval J of $[0;\infty)$, a group G of diffeomorphisms of this interval, and a $\mathcal{C}^1$-continuous path $(h_t)_{t\in[0;1)}$ of diffeomorphisms of J such that $h_0=\mathrm{id}$ and $h_tgh_t^{-1}\to\mathrm{id}$ as $t\to\infty$ for all $g\in G$. From Lemma 3.1, there exists a sequence $(\varphi_n)_{n\in\mathbb{N}}$ of diffeomorphisms of J

converging to id as $n \to \infty$, and satisfying
$$\varphi_n \dots \varphi_0 g \varphi_0^{-1} \dots \varphi_n^{-1} \xrightarrow[n\to\infty]{} \mathrm{id} \quad \text{for all } g \in G.$$

Let now f_0 be a homothety with ratio $\alpha < 1$ such that the iterated images of J under f_0 are pairwise disjoint. One defines a contraction f of $(0;\infty)$ by
$$\begin{gathered} f|_J = f_0 \circ \varphi_0, \\ f|_{f_0(J)} = f_0 \circ (f_0\varphi_1 f_0^{-1});\ \dots;\ f|_{f_0^n(J)} = f_0 \circ (f_0^n \varphi_n f_0^{-n}) \quad \text{for all } n \geq 1, \\ f = f_0 \quad \text{elsewhere.} \end{gathered}$$
Since $\varphi_n \xrightarrow{\mathcal{C}^1} \mathrm{id}$ as $n \to \infty$, we know that $D(f_0^n \varphi_n f_0^{-n})(f_0^n(x))$ converges to 1 uniformly with respect to $x \in J$. Then $Df|_{f_0^n(J)}$ tends to α as $n \to \infty$, and thus $Df(x) \to \alpha$ as $x \to 0$. Consequently, f can be extended to a $\mathcal{C}^1$-diffeomorphism of $[0;\infty)$.

Then we extend each $g_J \in G$ to a diffeomorphism g of $(0;\infty)$ in such a way that it commutes with f, i.e. $g = f^n g_J f^{-n}$ on $f_0^n(J)$, where $n \in \mathbb{Z}$, and $g = \mathrm{id}$ elsewhere. For all $n \in \mathbb{N}$, we then have $g|_{f_0^n(J)} = f_0^n \varphi_n \dots \varphi_0 g_J \varphi_0^{-1} \dots \varphi_n^{-1} f_0^{-n}$, where $\varphi_n \dots \varphi_0 g_J \varphi_0^{-1} \dots \varphi_n^{-1}$ tends to the identity in the $\mathcal{C}^1$-topology. So, as f_0 is a homothety, the derivative of $g|_{f_0^n(J)}$ also tends to 1 as $n \to \infty$, and thus $Dg(x) \to 1$ as $x \to 0$. It follows that g extends in a differentiable way at 0.

This ends the proof of the sufficient condition of Theorem 1.7.

Necessary condition of Theorem 1.7. We now show that given a group G of diffeomorphisms of some $J = [a;b] \subset [0;\infty)$, there exists a contraction f of $[0;\infty)$ in whose centralizer G is embeddable.

First note that, by replacing if necessary f by f^k with k some large enough integer, one can assume that J is included in a fundamental domain $[f(x_0);x_0]$ of f.

By assumption, each element g_J of G has an extension commuting with f. Denoting by g such an extension, for all $n \in \mathbb{N}$ we have $g|_{f^n(J)} = f^n g_J f^{-n}$. If $n \in \mathbb{N}$, we denote by A_n the increasing affine map from $f^n(J)$ into J and by g_n the $\mathcal{C}^1$-diffeomorphism of J defined by $g_n = A_n g|_{f^n(J)} A_n^{-1}$.

Claim. *$\|g_n - \mathrm{id}\|_1 \to 0$ as $n \to \infty$.*

Indeed, the orbits under g of fixed points of g_J which are the extremities a and b of J accumulate at 0 and consist of fixed points of g. By continuity of the derivative of g at 0, the derivative of g tends to $Dg(0)$ at 0; moreover $Dg(0) = \lim_{n\to\infty} g(f^n(a))/f^n(a) = 1$. Thus, if n is large enough, $g|_{f^n(J)}$ has derivative near 1 between two fixed points, and as a consequence is $\mathcal{C}^1$-close to id. By conjugating this diffeomorphism by the affine map A_n, one does not modify its derivative, which enables us to conclude that g_n is also $\mathcal{C}^1$-close to the identity if n is large enough. ∎

From the expression of $g|_{f^n(J)}$ given above, one has

$$g_n = A_n f^n g_J f^{-n} A_n^{-1}.$$

Defining $\varphi_0 = \mathrm{id}|_J$; $\varphi_1 = A_1 f|_J$; $\varphi_n = (A_n f^n|_J)(A_{n-1} f^{n-1}|_{f^{n-1}(J)})^{-1}$ for $n \geq 2$, one has $g_n = \varphi_n \ldots \varphi_1 \varphi_0 g_0 \varphi_0^{-1} \varphi_1^{-1} \ldots \varphi_n^{-1}$. From the previous claim and Lemma 3.1, if we prove the following result, we will have proved that there exists an isotopy by conjugacy from g_J to id and thus completed the proof of Theorem 1.7:

Claim. *$\varphi_n \to \mathrm{id}$ as $n \to \infty$ in the $\mathcal{C}^1$-topology.*

Indeed, by definition, $\varphi_n = A_n f|_{f^{n-1}(J)} A_{n-1}^{-1}$. Furthermore, as Df is continuous at 0 and $Df(0) \neq 0$, one has

$$\sup_{x,y \in f^{n-1}(J)} \frac{Df(x)}{Df(y)} \xrightarrow[n\to\infty]{} 1.$$

Consequently, $D\varphi_n(x)/D\varphi_n(y)$ tends to 1 uniformly with respect to $x, y \in J$ as $n \to \infty$. Since $D\varphi_n$ is equal to 1 at one point of J at least, it follows that $D\varphi$ tends to 1 uniformly on J. The claim follows. ∎

4. Isotopies by conjugacy

4.1. Statement of the result. In this section, we consider the set

$$D_{\alpha,\beta} = \{f \in \mathcal{D}\mathit{iff}^1([0;1]) : \mathrm{Fix}(f) = \{0;1\},\ f \geq \mathrm{id},\ Df(0) = \alpha,\ Df(1) = \beta\},$$

well-defined for all $\alpha \geq 1$ and $0 < \beta \leq 1$, and we show that the conjugacy classes of such diffeomorphisms are dense in this set in the $\mathcal{C}^1$-topology.

Let now f and g be two diffeomorphisms in $D_{\alpha,\beta}$. We will show that in each $\mathcal{C}^1$-neighbourhood of g, there is a conjugate of f. This will give

Theorem 4.1. *For each diffeomorphism $f \in D_{\alpha,\beta}$, the differentiable conjugacy class of f is dense in $D_{\alpha,\beta}$.*

Let $f, g \in D_{\alpha,\beta}$, and $\mathcal{U}$ be a $\mathcal{C}^1$-neighbourhood of g in $D_{\alpha,\beta}$. In order to find a conjugate of f in $\mathcal{U}$, we will perturb g by sufficiently small diffeomorphisms so that we do not go out of $\mathcal{U}$, till we find a conjugate of f.

As announced in the introduction, we actually show the stronger result stated in Theorem 1.2, which we will deduce from the following theorem:

Theorem 4.2. *Let $f \in D_{\alpha,\beta}$ without fixed points in $(0;1)$ and let $(\varepsilon_t)_t \subset (0;1)$ be a continuous path. Let $(f_t)_{t\in[0;1]}$ be a $\mathcal{C}^1$-continuous path in $D_{\alpha,\beta}$ such that:*

- *$f_0 = f$;*
- *f_t has no fixed point in $(0;1)$ for all $t < 1$.*

Then there exists a $\mathcal{C}^1$*-continuous path* $(h_t)_{t\in[0;1)}$ *of diffeomorphisms of* $[0;1]$*, each coinciding with* id *on a neighbourhood of* 0 *and of* 1*, such that* $h_0 = \mathrm{id}$ *and* $\|h_t f h_t^{-1} - f_t\|_1 < \varepsilon_t$ *for all* $t \in [0;1)$*.*

Remarks. Since f_t has no fixed point in $(0;1)$ for all $t < 1$, the diffeomorphism f_1 can have in $(0;1)$ only fixed points with derivative equal to 1.

By choosing (ε_t) converging to 0, under the hypotheses of this theorem, one obtains a continuous path $(h_t)_{t\in[0;1)}$ such that $\lim_{t\to 1} h_t f h_t^{-1} = \lim_{t\to 1} f_t$.

Proof of Theorem 1.2 from Theorem 4.2. Given $f, g \in D_{\alpha,\beta}$, one can always easily exhibit a $\mathcal{C}^1$-continuous path $(f_t)_{t\in[0;1)} \subset D_{\alpha,\beta}$ such that $f_0 = f$ and $f_t \to g$ as $t \to 1$, for example, $f_t = (1-t)f + tg$. If $(\varepsilon_t)_{t\in[0;1)}$is a continuous path of strictly positive numbers converging to 0, Theorem 4.2 provides a continuous path $(h_t)_{t\in[0;1)}$ of diffeomorphisms of $[0;1]$ such that $h_0 f h_0^{-1} = f$ and $h_t f h_t^{-1} \to g$ as $t \to 1$. Then $(h_t f h_t^{-1})_{t\in[0;1)}$ is an isotopy by conjugacy from f to g. ■

4.2. Two "gluing lemmas". From now on, Φ will denote a fixed $\mathcal{C}^\infty$-diffeomorphism of $[0;1]$ which is decreasing, equal to 1 on $[0;1/2]$ and equal to 0 on a neighbourhood of 1. Its derivative, continuous on $[0;1]$, is thus bounded. We define $M_\Phi = \max_{x\in[0;1]} |D\Phi| > 1$.

Gluing at the extremities. Given two $\mathcal{C}^1$-diffeomorphisms of $0;1]$, f and g, having the same derivatives at 0 and at 1, and an $\varepsilon > 0$, Corollary 4.5 below enables us to obtain a new $\mathcal{C}^1$-diffeomorphism which coincides with f on a neighourhood of 0 and on a neighourhood of 1, and which is ε-close to g in the $\mathcal{C}^1$-topology on the whole interval $[0;1]$.

Definition 4.3. Let $g \in \mathcal{D}\mathit{iff}^1([0;1])$. For every $\varepsilon > 0$ and $a, b \in (0;1)$, we denote by $\mathcal{U}^g_{\varepsilon,a,b}$ the set of $f \in \mathcal{D}\mathit{iff}^1([0;1])$ such that:

- $\|f|_{[0;a]} - g|_{[0;a]}\|_1 < \varepsilon$;
- $\|f|_{[b;1]} - g|_{[b;1]}\|_1 < \varepsilon$.

We set $\mathcal{U}^g_\varepsilon = \bigcup_{a,b\in(0;1)} \mathcal{U}^g_{\varepsilon,a,b}$.

Lemma 4.4. *Let* $\varepsilon > 0$*. Then there exists* $\tilde\varepsilon > 0$ *such that for all* $g \in \mathcal{D}\mathit{iff}^1([0;1])$ *and* $a, b \in (0;1)$*, if* $f \in \mathcal{U}^g_{\tilde\varepsilon,a,b}$*, then there exists* $f_0 \in \mathcal{D}\mathit{iff}^1([0;1])$ *such that:*

(i) $\|f_0 - g\|_1 < \varepsilon$;
(ii) $f_0|_{[0;a/2]\cup[(1+b)/2;1]} = f|_{[0;a/2]\cup[(1+b)/2;1]}$;
(iii) $f_0|_{[a;b]} = g|_{[a;b]}$.

Proof. Let $\varepsilon > 0$, and $0 < \tilde\varepsilon < \varepsilon/(2M_\Phi)$. Let $g \in \mathcal{D}\mathit{iff}^1([0;1])$, $a, b \in (0;1)$ and $f \in \mathcal{U}^g_{\tilde\varepsilon,a,b}$. If $a \geq b$, it suffices to let f_0 be f. Thus, we assume $a < b$.

We now define a map Φ_0 of $[0;1]$, which is obtained from Φ as follows:

- if $x \in [0;a]$, then $\Phi_0(x) = \Phi(x/a)$, so Φ_0 is equal to 1 on $[0;a/2]$ and to 0 on a neighbourhood of a in $[0;a]$;
- $\Phi_0(x) = 0$ if $x \in [a;b]$;
- $\Phi_0(x) = \Phi\big(\frac{-x+1}{1-b}\big)$ if $x \in [b;1]$, so Φ_0 is 0 on a neighbourhood of b in $[b;1]$, and 1 on $[(1+b)/2;1]$.

Set now $f_0 = \Phi_0 f + (1-\Phi_0)g$, and notice that the hypothesis that "$\|f|_{[0;a]} - g|_{[0;a]}\|_1 < \tilde{\varepsilon}$" implies, by integrating the inequality $|Df - Dg| < \tilde{\varepsilon}$ on $[0;x]$, that $|f(x) - g(x)| < \tilde{\varepsilon}x$ for all $x \in [0;a]$. By the same method, one obtains $|f(x) - g(x)| < \tilde{\varepsilon}(1-x)$ for each $x \in [b;1]$. As (i) and (iii) follow immediately from the construction of f_0, one has now only to show that $|f_0 - g| < \varepsilon$ and $|Df_0 - Dg| < \varepsilon$, which is a simple calculation. ∎

From this lemma, we deduce the following corollary:

COROLLARY 4.5. *Let $\varepsilon > 0$ and $f, g \in D_{\alpha,\beta}$. Then there exist $a, b \in (0;1)$ and $f_0 \in D_{\alpha,\beta}$ such that:*

(i) $\|f_0 - g\|_1 < \varepsilon$;
(ii) $f_0|_{[0;a/2]\cup[(1+b)/2;1]} = f|_{[0;a/2]\cup[(1+b)/2;1]}$;
(iii) $f_0|_{[a;b]} = g|_{[a;b]}$.

Proof. Indeed, considering $\tilde{\varepsilon} > 0$ as in Lemma 4.4, since f and g have the same derivatives at 0 and at 1, there exist $a, b \in (0;1)$ such that $f \in \mathcal{U}^g_{\tilde{\varepsilon},a,b}$. Now apply Lemma 4.4. ∎

This result will be useful in proving the density of conjugacy classes of diffeomorphisms from $D_{\alpha,\beta}$ in that set. However, in order to obtain an isotopy by conjugacy from a diffeomorphism of $D_{\alpha,\beta}$ to another, a parameter version will be needed:

LEMMA 4.6. *Let $f \in D_{\alpha,\beta}$ and $(f_t)_{t\in[0;1)}$ a $\mathcal{C}^1$-continuous path of diffeomorphisms in $D_{\alpha,\beta}$. Then, for every continuous path $(\varepsilon_t)_{t\in[0;1)}$ in $(0;\infty)$, there exist continuous paths $(a_t)_{t\in[0;1)}$ and $(b_t)_{t\in[0;1)}$ in $(0;1)$ and a new $\mathcal{C}^1$-continuous path $(\tilde{f}_t)_{t\in[0;1)}$ in $D_{\alpha,\beta}$ such that:*

- $\tilde{f}_t|_{[0;a_t/2]\cup[(b_t+1)/2;1]} = f|_{[0;a_t/2]\cup[(b_t+1)/2;1]}$ *for all $t < 1$;*
- $\|\tilde{f}_t - f_t\|_1 < \varepsilon_t$ *for all $t < 1$.*

Moreover, if $f_0 = f$, then we can also require that $\tilde{f}_0 = f$.

Let us first give the following result:

LEMMA 4.7. *Let $(\varepsilon_t)_{t\in[0;1)}$ be a continuous path in $(0;\infty)$, $f \in D_{\alpha,\beta}$ and $(f_t)_{t\in[0;1)}$ a $\mathcal{C}^1$-continuous path in $D_{\alpha,\beta}$. Then there exist continuous paths $(a_t)_{t\in[0;1)}$ and $(b_t)_{t\in[0;1)}$ in $(0;1)$ such that $\|f_t|_{[0;a_t]\cup[b_t;1]} - f|_{[0;a_t]\cup[b_t;1]}\|_1 < \varepsilon_t$ for all $t < 1$.*

Proof. Given an increasing sequence $(t_n)_{n\in\mathbb{N}}$ in $[0;1)$ converging to 1, one uses the continuity of $(f_t)_{t\in[0;1)}$ on each compact set $[t_n;t_{n+1}]$ to obtain a partition of this interval into finitely many subintervals such that, for all t in one of these subintervals I, only one $0 < a_I < 1$ satisfies $\|f_t|_{[0;a_I]} - f|_{[0;a_I]}\|_1 < \min_{t\in[t_n;t_{n+1}]}\varepsilon_t$. All what is needed now is finding a continuous map φ such that, for all t in I, $a_t = \varphi(t)$ is smaller than a_I, which can easily be done.

We conclude by the same token as regards the path (b_t). ∎

Proof of Lemma 4.6. Lemma 4.7 gives us two continuous paths (a_t) and (b_t) such that for all t,

$$\|f_t|_{[0;a_t]\cup[b_t;1]} - f|_{[0;a_t]\cup[b_t;1]}\|_1 < \inf\left(\frac{\varepsilon_t}{2}, \frac{\varepsilon_t}{4M_\Phi}\right).$$

Considering the construction done in Lemma 4.4, one obtains the desired path $(\tilde{f}_t)$; its continuity follows from the continuity of (a_t) and (b_t). ∎

To prove Theorem 4.2, it is now sufficient to establish the following theorem:

THEOREM 4.8. *Let $f \in D_{\alpha,\beta}$ without fixed points in $(0;1)$ and $(\varepsilon_t)_{t\in[0;1)}$ be a continuous path in $(0;\infty)$. Let $(f_t)_{t\in[0;1)}$ be a $\mathcal{C}^1$-continuous path in $D_{\alpha,\beta}$ such that:*

- *$f_0 = f$;*
- *f_t has no fixed point in $(0;1)$ for all $t < 1$;*
- *for all $t < 1$, there exist $a_t > 0$ and $b_t < 1$, continuously depending on t, such that f_t coincides with f on $[0,a_t]\cup[b_t,1]$.*

Then there exists a continuous path $(h_t)_{t\in[0;1)}$ of diffeomorphisms of $[0;1]$, each coinciding with id *on a neighbourhood of* 0 *and of* 1, *such that $h_0 = \mathrm{id}$ and $\|h_t f h_t^{-1} - f_t\|_1 < \varepsilon_t$ for all $t \in [0;1)$.*

Proof of Theorem 4.2 from Theorem 4.8. For $(f_t)_{t\in[0;1)}$, thanks to Lemma 4.6, one can exhibit continuous paths $(a_t)_{t\in[0;1)}$ and $(b_t)_{t\in[0;1)}$ in $(0;1)$ and a continuous path $(\tilde{f}_t)_{t\in[0;1)}$ of diffeomorphisms such that:

- $\tilde{f}_0 = f_0$;
- $\tilde{f}_t|_{[0;a_t/2]\cup[(b_t+1)/2;1]} = f|_{[0;a_t/2]\cup[(b_t+1)/2;1]}$ for all $t < 1$;
- $\|\tilde{f}_t - f_t\|_1 < \varepsilon_t/2$ for all $t < 1$.

One then applies Theorem 4.8 to obtain a $\mathcal{C}^1$-continuous path $(h_t)_{t\in[0;1)}$ of diffeomorphisms of $[0;1]$, with $h_0 = \mathrm{id}$, such that h_t coincides with id on a neighbourhood of 0 and of 1, and $\|h_t f h_t^{-1} - \tilde{f}_t\|_1 < \varepsilon_t/2$ for all $t \in [0;1)$. Then $\|h_t f h_t^{-1} - f_t\|_1 < \varepsilon_t$ for all $t \in [0;1)$. ∎

Let us also point out a corollary of Lemma 4.4 which will be useful in Section 5.2. If f and g are $\mathcal{C}^1$-diffeomorphisms of $[0;1]$ coinciding at one point, this corollary produces a new $\mathcal{C}^1$-diffeomorphism coinciding with f in a neighbourhood of this point, and arbitrarily close to g.

Corollary 4.9. *Let $\varepsilon > 0$. Then there exists $\tilde{\varepsilon} > 0$ such that for all $g \in \mathcal{D}\mathit{iff}^1([0;1])$, $x_0 \in (0;1)$, and $\eta \in (0;\min(x_0/2,(1-x_0)/2))$, if f is a diffeomorphism of $[0;1]$ such that:*

- $f(x_0) = g(x_0)$;
- $\|f|_{[x_0-2\eta;x_0+2\eta]} - g|_{[x_0-2\eta;x_0+2\eta]}\|_1 < \tilde{\varepsilon}$,

then there exists $g_0 \in \mathcal{D}\mathit{iff}^1([0;1])$ such that:

- $g_0|_{[x_0-\eta;x_0+\eta]} = f|_{[x_0-\eta;x_0+\eta]}$;
- $g_0|_{[0;x_0-2\eta]\cup[x_0+2\eta;1]} = g|_{[0;x_0-2\eta]\cup[x_0+2\eta;1]}$;
- $\|g_0 - g\|_1 < \varepsilon$.

Here is the parameter version of this corollary:

Corollary 4.10. *Let $(\varepsilon_t)_{t\in[0;1)}$ be a continuous path in $(0;\infty)$. Then there exists a continuous path $(\tilde{\varepsilon}_t)_{t\in[0,1)}$ in $(0;\infty)$ such that for all $\mathcal{C}^1$-continuous paths $(f_t)_{t\in[0;1)}$ in $\mathcal{D}\mathit{iff}^1([0;1])$, all continuous paths $(x_t)_{t\in[0,1)}$ in $(0;1)$, and all $\eta_n \in (0;\min(x_t/2,(1-x_t)/2))$, if f is a diffeomorphism of $[0;1]$ such that:*

- $f(x_t) = f_t(x_t)$ *for all* $t < 1$;
- $\|f|_{[x_t-2\eta_t;x_t+2\eta_t]} - f_t|_{[x_t-2\eta_t;x_t+2\eta_t]}\|_1 < \tilde{\varepsilon}_t$,

then there exists a $\mathcal{C}^1$-continuous path $(g_t)_{t\in[0,1)}$ in $\mathcal{D}\mathit{iff}^1([0;1])$ such that:

- $g_t|_{[x_t-\eta_t;x_t+\eta_t]} = f|_{[x_t-\eta_t;x_t+\eta_t]}$;
- $g_t|_{[0;x_t-2\eta_t]\cup[x_t+2\eta_t;1]} = f_t|_{[0;x_t-2\eta_t]\cup[x_t+2\eta_t;1]}$;
- $\|g_t - f_t\|_1 < \varepsilon_t$.

The proofs of Corollaries 4.9 and 4.10 are absolutely similar to the one of Lemma 4.7, thus we omit them.

"Partial gluing" near an extremity

Definition 4.11. If $g \in \mathcal{D}\mathit{iff}^1([0;1])$, $\varepsilon > 0$ and $a \in (0;1)$, we willl denote by $\mathcal{U}^g_{\varepsilon,a}$ the set of maps $f \in \mathcal{H}\mathit{omeo}([0;1])$ such that:

- $f|_{[a;1]} \in \mathcal{D}\mathit{iff}^1([a;1],[f(a);1])$;
- $\|f|_{[a;1]} - g|_{[a;1]}\|_1 < \varepsilon$.

We will see in the proof of the following lemma how, from a $\mathcal{C}^1$-diffeomorphism g of $[0;1]$ and a homeomorphism $f \in \mathcal{U}^g_{\varepsilon,a}$, one can produce a map coinciding with f in a neighbourhood of the extremities and which is arbitrarily close to g.

Lemma 4.12. *Let $\varepsilon > 0$. Then there exists $\tilde{\varepsilon} > 0$ such that for all $g \in \mathcal{D}\mathit{iff}^1([0;1])$, $a \in (0;1)$, $1 > b > (a+1)/2$, and $f \in \mathcal{U}^g_{\tilde{\varepsilon},a}$, there exists $g_0 \in \mathcal{D}\mathit{iff}^1([0;1])$ satisfying:*

- $g_0|_{[(a+1)/2;b]} = f|_{[(a+1)/2;b]}$;
- $g_0|_{[0;a]\cup[(b+1)/2;1]} = g|_{[0;a]\cup[(b+1)/2;1]}$;
- $\|g_0 - g\|_1 < \varepsilon$.

Proof. Let $\varepsilon > 0$ and $\tilde{\varepsilon} < \min(\varepsilon/2, \varepsilon/(4M_\Phi))$. Let $g \in \mathcal{D}\mathit{iff}^1([0;1])$, $a \in (0;1)$, $f \in \mathcal{U}^g_{\tilde{\varepsilon},a}$, and let Φ_0 be a $\mathcal{C}^\infty$-map of $[0;1]$, constructed from Φ (see Section 4.2) in the following way:

- on $[0;a]$ and on $[(b+1)/2;1]$, Φ_0 is constant, equal to 1;
- if $x \in [a;(a+1)/2]$, then $\Phi_0(x) = \Phi\big(\frac{x}{(1-a)/2}\big)$;
- on $[(a+1)/2;b]$, Φ_0 is constant, equal to 0;
- if $x \in [b;(b+1)/2]$, then $\Phi_0(x) = \Phi\big(\frac{1-x}{(1-b)/2}\big)$.

Then one defines $g_0 = \Phi_0\, g + (1-\Phi_0) f$, and one checks that $\|g - g_0\|_1$ is bounded by ε. The other conclusions can be checked by simple calculation, using the fact that the integration of the inequality $|Df - Dg| < \tilde{\varepsilon}$ gives $|f(x) - g(x)| < \tilde{\varepsilon}(1-x)$ for all $x \in [a;1]$. ■

One can notice that, in Lemmas 4.12 and 4.4, $\tilde{\varepsilon}$ depends only on ε. That justifies the following definition:

Definition 4.13. Given $\varepsilon > 0$, we denote by $\mathrm{Marg}(\varepsilon)$ the set of strictly positive real numbers satisfying the conditions satisfied by $\tilde{\varepsilon}$ in Lemmas 4.4 and 4.12 and in Corollary 4.9. In other words,

$$\mathrm{Marg}(\varepsilon) = \bigg(0; \min\bigg(\frac{\varepsilon}{2}, \frac{\varepsilon}{4M_\Phi}\bigg)\bigg).$$

Just as for the first "gluing" lemma, we state a parameter version of Lemma 4.12:

Lemma 4.14. *Let $(\varepsilon_t)_{t\in[0;1)}$ be a continuous path in $(0;\infty)$, $f \in D_{\alpha,\beta}$ and $(f_t)_{t\in[0;1)}$ a continuous path in $D_{\alpha,\beta}$. Lemma 4.6 ensures the existence of a continuous path $(b_t)_{t\in[0;1)}$ in $(0;1)$ such that $\|f_t|_{[b_t;1]} - f|_{[b_t;1]}\|_1 < \mathrm{Marg}(\varepsilon_t)$. Then, for all paths $(c_t)_{t\in[0;1)}$ and $(d_t)_{t\in[0;1)}$ such that $b_t < c_t < 1$ and $(c_t+1)/2 < d_t < 1$, there exists a $\mathcal{C}^1$-continuous path $(\tilde{f}_t)_{t\in[0;1)}$ in $D_{\alpha,\beta}$ satisfying:*

- $\tilde{f}_t|_{[0;c_t]\cup[(d_t+1)/2;1]} = f_t|_{[0;c_t]\cup[(d_t+1)/2;1]}$;
- $\tilde{f}_t|_{[(c_t+1)/2;d_t]} = f|_{[(c_t+1)/2;d_t]}$;
- $\|\tilde{f}_t - f_t\|_1 < \varepsilon_t$.

Once again, the proof is analogous to the one of Lemma 4.7, and so we omit it.

4.3. Mather invariant. If $f \in D_{\alpha,\beta}$ and $a, b \in (0;1)$, we set

$$D_{f,a,b} = \{g \in D_{\alpha,\beta} : f|_{[0;a]} = g|_{[0;a]} \text{ and } f|_{[b;1]} = g|_{[b;1]}\}, \qquad D_f = \bigcup_{a,b\in(0;1)} D_{f,a,b}.$$

If $f \in D_{\alpha,\beta}$, $a, b \in (0;1)$, $g \in D_{f,a,b}$ and $x_0 < a$ is such that $f(x_0) < a$, then $f|_{[0;f(x_0)]} = g|_{[0;f(x_0)]}$, and then h defined on $[0; f(x_0)]$ by $h = \mathrm{id}$ conjugates f to g on $[0; x_0]$. The Remark below states that this diffeomorphism extends uniquely onto $[0;1)$ in such a way that it conjugates f to g.

Remark. For all $f \in D_{\alpha,\beta}$ and $g \in D_f$, there exists a unique diffeomorphism h of $[0;1)$ which conjugates f to g and coincides with id on a neighbourhood of 0.

Proof. If h conjugates f to g, the relation $hfh^{-1} = g$ gives

$$h_n|_{[f^n(x_0);f^{n+1}(x_0)]} = g^n h_0 f^{-n} = g^n f^{-n} \quad \text{for all integers } n.$$

One checks that this formula defines a $\mathcal{C}^1$-diffeomorphism of $[0;1)$. ∎

Definition 4.15. The homeomorphism of $[0;1]$ coinciding with the diffeomorphism h mentioned in the Remark above on $[0;1)$ and with value 1 at 1 will be called the *unitary conjugacy from g to f*, and denoted by h_g. It is the only conjugacy from g to f which coincides with id on a neighbourhood of 0.

Lemma 4.16. *Let f be a $\mathcal{C}^1$-diffeomorphism of $[0;1]$, $(a_t)_{t\in[0;1)}$ and $(b_t)_{t\in[0;1)}$ be continuous paths in $[0;1]$, and $(f_t)_{t\in[0;1)}$ be a $\mathcal{C}^1$-continuous path of diffeomorphisms of $[0;1]$ such that $f_t \in D_{f,a_t,b_t}$ for all $t \in [0;1)$. Then, for every compact interval I in $(0;1)$, the path $(h_{f_t}|_I)_{t\in[0;1)}$ is continuous in the $\mathcal{C}^1$-topology.*

Proof. Let $I = [v_0; v_1]$. Since $v_1 < 1$, and by continuity of $(a_t)_t$, for every $t \in [0;1)$ there exists a neighbourhood $\mathcal{V}_t$ of t and an integer n_t such that, for all $t' \in \mathcal{V}_t$, $h_{f_{t'}}$ is defined by $h_{f_{t'}} = f_{t'}^{n_t} f^{-n_t}$ and so depends continuously on $f_{t'}$ in the $\mathcal{C}^1$-topology. The continuity of $h_{f_t}|_I$ follows. ∎

If the extension of h_g to $[0;1]$ has $\mathcal{C}^1$-regularity, then g is $\mathcal{C}^1$-conjugate to f by h_g, and the conclusion of Theorem 4.1 is satisfied. However, generally, h_g is not differentiable at 1. In order to show that the conjugacy class of f is arbitrarily close to g, one modifies g on $\mathcal{U} \cap D_f$ in such a way that h_g coincides with id on a neighbourhood of 1.

Proposition 4.17. *If $f \in D_{\alpha,\beta}$, $g \in D_f$, and $\mathcal{U}$ is a $\mathcal{C}^1$-neighbourhood of g, then there exists $g' \in D_f \cap \mathcal{U}$ such that $h_{g'} = h'$ coincides with* id *in a neighbourhood of* 1.

A parameter version of this statement is the following:

Proposition 4.18. *Let $f \in D_{\alpha,\beta}$; $(a_t)_{t\in[0;1)}$ and $(b_t)_{t\in[0;1)}$ continuous paths in $(0;1)$; $(\varepsilon_t)_{t\in[0;1)}$ a continuous path in $(0;\infty)$. Let lastly $(f_t)_{t\in[0;1)}$ be* *a $\mathcal{C}^1$-continuous path in $D_{\alpha,\beta}$ such that $f_t \in D_{f,a_t,b_t}$ for all $t \in [0;1)$. Then there exist continuous paths $(\tilde{b}_t)_{t\in[0;1)}$ and $(\tilde{c}_t)_{t\in[0;1)}$ in $(0;1)$, as well as a $\mathcal{C}^1$-continuous path $(\tilde{f}_t)_{t\in[0;1)}$ in $D_{f,a_t,\tilde{b}_t}$ such that, for all $t \in [0;1)$, we have:*

- $h_{\tilde{f}_t} = \mathrm{id}$ *on* $[\tilde{c}_t; 1]$;
- $\|\tilde{f}_t - f_t\|_1 < \varepsilon_t$.

In particular, this yields:

Corollary 4.19. *Under the hypotheses of this proposition, the map*

$$[0;1) \to \mathcal{D}\mathit{iff}^1([0;1]), \qquad t \mapsto h_{\tilde{f}_t},$$

is continuous in the $\mathcal{C}^1$-topology.

Indeed, we saw in Lemma 4.16 that, for every compact interval I in $(0;1)$, the diffeomorphism $h_{\tilde{f}_t}|_I$ depends continuously on t in the $\mathcal{C}^1$-topology. Thus it is sufficient to choose I in such a way that $h_{\tilde{f}_t}$ coincides with id on the complement of I so as to ensure that $\|h_{\tilde{f}_{t'}}|_I - h_{\tilde{f}_t}|_I\|_1$, as well as $\|h_{\tilde{f}_{t'}}|_{\complement I} - h_{\tilde{f}_t}|_{\complement I}\|_1$ are arbitrarily small provided that t and t' are sufficiently close to each other.

From these two propositions, one can prove Theorems 4.1 and 4.8 rather easily. Proposition 4.18 will also be useful in Section 6 in the proof of Lemma 6.1.

Proof of Theorem 4.1 from Proposition 4.17. Given two diffeomorphisms f, g of $D_{\alpha,\beta}$, we prove that there exists a conjugate of f arbitrairily $\mathcal{C}^1$-close to g. Let $\varepsilon > 0$. From Corollary 4.5, there exist $1 > a, b > 0$ and $g_0 \in D_{f,a,b}$ such that $\|g_0 - g\|_1 < \varepsilon/2$. From Proposition 4.17, there exists $\tilde{g}_0 \in D_f$ such that $\|\tilde{g}_0 - g_0\|_1 < \varepsilon/2$ and $h_{\tilde{g}_0} = \mathrm{id}$ in a neighbourhood of 1. Then $h_{\tilde{g}_0}$ is a $\mathcal{C}^1$-diffeomorphism of $[0;1]$ coinciding with the identity on a neighbourhood of 0 and of 1, conjugating f to $\tilde{g}_0$, and $\|\tilde{g}_0 - g\|_1 < \varepsilon$. ∎

Proof of Theorem 4.8 from Proposition 4.18. If $(f_t)_{t\in[0;1)}$ is a continuous path in D_f, then $(\tilde{f}_t)_{t\in[0;1)} = (h_{\tilde{f}_t} f h_{\tilde{f}_t}^{-1})_{t\in[0;1)}$ given by Proposition 4.18 is a $\mathcal{C}^1$-continuous path of conjugates of f such that the conjugacies $h_{\tilde{f}_t}$ all coincide with the identity near 0 and 1 and $\|h_{\tilde{f}_t} f h_{\tilde{f}_t}^{-1} - f_t\|_1 < \varepsilon_t$ for all $t < 1$. Moreover, Corollary 4.19 ensures the $\mathcal{C}^1$-continuity of the path $(h_{\tilde{f}_t})_{t\in[0;1)}$. ∎

From now on, the aim is to reword the problem in order to obtain a formulation similar to the $\mathcal{C}^2$-problem. For that, we will introduce an equivalent of the $\mathcal{C}^2$-Mather invariant. This will enable us to measure how much h_g differs from id near 1.

Definition 4.20. Let $f \in D_{\alpha,\beta}$ and $g \in D_f$. We define the *Mather invariant of g with respect to f*, denoted by $\mathcal{M}_f(g)$, to be the unique homeomorphism of $[0;1]$ commuting with f and coinciding with h_g on a neighbourhood of 1.

REMARK. The Mather invariant is well-defined since h_g commutes with f on a neighbourhood of 1. By choosing a fundamental domain I included in this neighbourhood and by pushing $h_g|_I$ by the dynamic of f, one obtains the unique homeomorphism $\mathcal{M}_f(g)|_{(0;1)}$ of $(0;1]$ which commutes with f and coincides with h_g on a neighbourhood of 1. By finally setting $\mathcal{M}_f(g)(0) = 0$, one obtains a diffeomorphism of $[0;1]$.

LEMMA 4.21. *If $f \in D_{\alpha,\beta}$ and $g, g' \in D_f$ are such that $g' \geq g$ and $g' > g$ on a closed interval containing a fundamental domain of g', then $\mathcal{M}_f(g') > \mathcal{M}_f(g)$.*

Proof. Let $a, b \in (0;1)$ be such that $g, g' \in D_{f,a,b}$. By definition of $\mathcal{M}_f$, it is sufficient to show that $h_{g'} > h_g$ on a fundamental domain $I_0 = [x_0; f(x_0)]$ of f included in $[b;1]$. In other words, it is sufficient to show that $h_{g'}h_g^{-1} > \mathrm{id}$ on $h_g(I_0)$, or that $g'^n g^{-n} > \mathrm{id}$ on $h_g(I_0)$ if n is so large that $g^{-n}(x) < a$. Let $x \in h_g(I_0)$. Since $g' \geq g$, we know that $g'^n g^{-n}(x) \geq x_0$, and so the suborbit $\{g'^i(g^{-n}(x))\}_{0 \leq i < n}$ lies in the fundamental domain of g' on which $g' > g$. Consequently, $g'^n(g^{-n}(x)) > g^n(g^{-n}(x)) > x$. ∎

LEMMA 4.22. *Let $f \in D_{\alpha,\beta}$ and let $(a_t)_{t\in[0;1)}, (b_t)_{t\in[0;1)}$ be continuous paths in $(0;1)$. Let $(f_t)_{t\in[0;1)}$ be a path of $\mathcal{C}^1$-diffeomorphisms with $f_t \in D_{f,a_t,b_t}$ for all $t < 1$. Then, for every compact subinterval I of $(0;1)$, the path $(\mathcal{M}_f(f_t)|_I)_{t\in[0;1)}$ is continuous in the $\mathcal{C}^1$-topology.*

Proof. Let $t \in [0;1)$ and $\mathcal{V}$ be a neighbourhood of t. From Lemma 4.16, we know that $h_{f_{t'}}$ depends continuously on $t' \in \mathcal{V}$ on a fundamental domain I_0 of f included in $[\sup_{t'\in\mathcal{V}} b_{t'}; 1)$. As I is compact, there exists an integer n such that at each point $x \in I$, $\mathcal{M}_f(f_{t'})$ is obtained by conjugating $h_{f_{t'}}|_{I_0}$ by f less than n times. Thus it suffices that $h_{f_{t'}}|_{I_0}$ is sufficiently close to $h_{f_t}|_{I_0}$, in other words that t' is sufficiently close to t for $\mathcal{M}_f(f_{t'})$ to be close to $\mathcal{M}_f(f_t)$ on the whole I. ∎

5. Proofs of Theorems 4.1 and 4.8. This section is dedicated to finishing the proof of the main result of this paper, that is, Theorem 4.8, preceded by its discrete and more comprehensible version, Theorem 4.1. We will first use the material presented in the previous section to reword Propositions 4.17 and 4.18. Let us recall in this connection that proving these two propositions will complete the proofs of Theorems 4.1 and 4.8.

Propositions 5.1 and 5.2 below are rewordings of Propositions 4.17 and 4.18 respectively.

PROPOSITION 5.1. *If $f \in D_{\alpha,\beta}$, $g \in D_f$, and $\mathcal{U}$ is a $\mathcal{C}^1$-neighbourhood of g, then there exists $g' \in D_{\alpha,\beta} \cap \mathcal{U}$ such that $\mathcal{M}_f(g') = \mathrm{id}$.*

Proposition 5.2. *Let* $f \in D_{\alpha,\beta}$ *and let* $(a_t)_{t\in[0;1)}$ *and* $(b_t)_{t\in[0;1)}$ *be continuous paths in* $(0;1)$*, and* $(\varepsilon_t)_{t\in[0;1)}$ *a continuous path in* $(0;\infty)$*. Let lastly* $(f_t)_{t\in[0;1)}$ *be a* $\mathcal{C}^1$*-continuous path in* $D_{\alpha,\beta}$ *such that* $f_t \in D_{f,a_t,b_t}$ *for all* $t \in [0;1)$*. Then there exist continuous paths* $(\tilde{b}_t)_{t\in[0;1)}$ *and* $(\tilde{c}_t)_{t\in[0;1)}$ *in* $(0;1)$*, as well as a* $\mathcal{C}^1$*-continuous path* $(\tilde{f}_t)_{t\in[0;1)}$ *in* $D_{f,a_t,\tilde{b}_t}$ *such that, for all* $t \in [0;1)$*, we have:*

- $\mathcal{M}_f(\tilde{f}_t) = \mathrm{id}$*;*
- $\|\tilde{f}_t - f_t\|_1 < \varepsilon_t$*.*

Thus we will try, in the following subsections, to cancel the Mather invariant of a diffeomorphism (i.e. to make it coincide with the identity) while remaining in an arbitrarily small neighbourhood of it.

5.1. To make the unitary conjugate have a fixed point. In order to prove Proposition 5.1, given $f \in D_{\alpha,\beta}$ and $g \in D_f$, we will first make $\mathcal{M}_f(g)$ have fixed points by modifying g by small perturbations. More precisely, Proposition 5.3 will enable us to prescribe the point which will become a fixed point of $\mathcal{M}_f(g)$ after having perturbed g:

Proposition 5.3. *Let* $f \in D_{\alpha,\beta}$*;* $a, b \in (0;1)$ *with* $a < b$*;* $g \in D_{f,a,b}$*;* $\mathcal{U}$ *be a* $\mathcal{C}^1$*-neighbourhood of* g *in* $D_{\alpha,\beta}$*; and* $p \in (0;1)$*. Then there exist* $\tilde{b} > b$ *and* $\tilde{g} \in \mathcal{U} \cap D_{f,a,\tilde{b}}$ *such that* p *is a fixed point of* $\mathcal{M}_f(\tilde{g})$*.*

The parameter version follows:

Proposition 5.4. *Let* f *be a diffeomorphism of* $[0;1]$ *without fixed points in* $(0;1)$*. Let* $(f_t)_{t\in[0;1)}$ *be a* $\mathcal{C}^1$*-continuous path of diffeomorphisms of* $[0;1]$ *such that:*

- $f_0 = f$*;*
- f_t *has no fixed point in* $(0;1)$ *for all* $t < 1$*;*
- *for all* $t < 1$*, there exist* $a_t > 0$ *and* $b_t < 1$*, depending continuously on* t*, such that* f_t *coincides with* f *on* $[0, a_t] \cup [b_t, 1]$ *(i.e.* $f_t \in D_{f,a_t,b_t}$*).*

Then, for every path $(\varepsilon_t)_{t\in[0;1)}$ *in* $(0;\infty)$ *and every continuous path* $(p_t)_{\in[0;1)}$ *in* $(0;1)$*, there exists a* $\mathcal{C}^1$*-continuous path* $(\tilde{f}_t)_{t\in[0;1)}$ *of diffeomorphisms of* $[0;1]$ *such that:*

- $\tilde{f}_0 = f$*;*
- *for all* $t < 1$*, there exists* $\tilde{b}_t < 1$*, depending continuously on* t*, such that* $\tilde{f}_t$ *coincides with* f *on* $[0;a_t] \cup [\tilde{b}_t;1]$*;*
- *for all* $t \in [0;1)$*,* $\|\tilde{f}_t - f_t\|_1 < \varepsilon_t$*;*
- *for all* $t \in [0;1)$*,* p_t *is a fixed point of* $\mathcal{M}_f(\tilde{f}_t)$*.*

The idea of the proof is to perturb each f_t in a continuous way with respect to t. So, one will obtain a new path which will also be continuous with respect to t, and such that the unitary conjugacies will have a fixed point for all t. For that, we will consider two diffeomorphisms f_+ and f_- going respectively "arbitrarily faster" and "arbitrarily slower" than f—the meaning of these expressions will be made rigorous in Lemma 5.5 below—and we will make the f_t's coincide with these diffeomorphisms for a sufficiently long time for them to catch up or lose their lead with respect to f.

First, we will show that such diffeomorphisms f_+ and f_- exist:

LEMMA 5.5. *Let f be a $\mathcal{C}^1$-diffeomorphism of* $[0;1]$ *such that* $f > \mathrm{id}$. *There exist two $\mathcal{C}^1$-diffeomorphisms f_+ and f_- of* $[0;1]$ *such that:*

- $f_+ > f$, $\mathrm{id} < f_- < f$;
- $Df_-(1) = Df_+(1) = Df(1)$;
- *for all $x \in (0;1)$ and all $n_0 \in \mathbb{N}$, there exists $k \in \mathbb{N}$ such that*

$$(f_+)^k(x) \geq f^{n_0+k}(x) \quad \text{and} \quad (f_-)^{n_0+k}(x) \leq f^k(x); \tag{1}$$

  consequently, for all $k' \geq k$, the inequalities $(f_+)^{k'}(x) \geq f^{n_0+k'}(x)$ and $(f_-)^{n_0+k'}(x) \leq f^{k'}(x)$ are still satisfied;
- *the lead of f_+ with respect to f and the delay of f_- with respect to f are decreasing, i.e., given n_0, the smallest integer k satisfying the last condition is increasing with respect to x.*

Notation. If $f \in D_{\alpha,\beta}$, $x \in (0;1)$ and $n_0 \in \mathbb{N}$, we will denote by $k(n_0, x)$ the smallest integer k satisfying condition (1) above.

The proof of Lemma 5.5 is quite long and deferred to the Annex.

Let us now recall the definition of translation number of a diffeomorphism with respect to another commuting with it:

LEMMA-DEFINITION 5.6. If f, h are two increasing $\mathcal{C}^1$-diffeomorphisms of $[0;1]$ such that f has no fixed point on $(0;1)$, $f > \mathrm{id}$ and f and h commute on $[0;1]$, then we can consider, given $x \in [0;1]$ and $n \in \mathbb{N}$, the integer $m(n)$ defined by

$$f^{m(n)}(x) \leq h^n(x) < f^{m(n)+1}(x).$$

Then the limit $\lim_{n\to\infty} m(n)/n$ exists and is independent of x. We call it the *translation number of h with respect to f*, and denote it by $\tau_f(h)$. Furthermore, if $(h_t)_{t\in[0;1)}$ is a path of $\mathcal{C}^1$-diffeomorphisms which varies $\mathcal{C}^1$-continuously on compact sets in $(0;1)$, then the translation number $\tau_f(h_t)$ depends continuously on t.

One can consult [BF] for a more general proof in the case of local homeomorphisms, or [N] for a similar proof in the context of rotation numbers.

Let us also introduce the following definition:

Definition 5.7. Let f, g be increasing $\mathcal{C}^1$-diffeomorphisms of $[0;1]$, with no fixed point in $(0;1)$, such that g coincides with f on a neighbourhood $\mathcal{V}_0$ of 0 and on a neighbourhood $\mathcal{V}_1$ of 1. Then the *delay of g with respect to f* is the integer $r_f(g) = [|\tau_f(\mathcal{M}_f(g))|]$, that is, the integer part of the absolute value of the translation number of the Mather invariant of g with respect to f.

Lemma 5.8. *Let f be an increasing $\mathcal{C}^1$-diffeomorphism of $[0;1]$ with no other fixed point than 0 and 1; $(a_t)_{t\in[0;1)}$ and $(b_t)_{t\in[0;1)}$ be continuous paths in $(0;1)$; and $(g_t)_{t\in[0;1)}$ be a $\mathcal{C}^1$-continuous path of increasing diffeomorphisms $g_t \in D_{f,a_t,b_t}$. Then $\tau_f(\mathcal{M}_f(g_t))$ depends continuously on t, and the delay of g_t with respect to f is upper semicontinuous with respect to t.*

Proof. From Lemma 4.22, we know that the Mather invariant of g_t with respect to f varies continuously on compact sets in $[0;1]$. Thus, from Lemma-Definition 5.6, the translation number $\tau_f(\mathcal{M}_f(g))$ depends continuously on t. One concludes thanks to the continuity of the absolute value and the upper semicontinuity of the integer part. ∎

Lemma 5.9. *Let $f \in D_{\alpha,\beta}$, $(r_t)_{t\in[0;1)}$ be an upper semicontinuous collection of integers and $(x_t)_{t\in[0;1)}$ be a continuous path in $(0;1)$. Then the collection $(\ell_t)_{t\in[0;1)} = (f_+^{k(r_t,x_t)}(x_t))_{t\in[0;1)}$ is locally upper bounded; in other words: for all $t \in (0;1)$, there exists $\varepsilon > 0$ such that $\sup_{s\in[t-\varepsilon;t+\varepsilon]} \ell_s < 1$. The same holds for the collection $(f_-^{k(r_t,x_t)}(x_t))_{t\in[0;1)}$.*

Proof. Let $t \in [0;1)$ and $\varepsilon > 0$. If $s \in [t-\varepsilon; t+\varepsilon]$ and ε is small enough, then by semicontinuity, r_s varies from $r_t - 1$ to $r_t + 1$. Thus $k(r_s, x_s)$ is bounded by $\max_{s\in[t-\varepsilon;t+\varepsilon]} k(r_t+1, x_s)$. Since f_+ has been defined in Lemma 5.5 in such a way that $k(n_0, x)$ increases with respect to x, one can conclude that $k(r_s, x_s)$ is bounded by $k(r_t + 1, x_{t+\varepsilon})$. Finally, the increasing of f_+ ensures that $\ell_s \leq f_+^{k(r_t+1,x_{t+\varepsilon})}(x_{t+\varepsilon})$. ∎

Lemma 5.10. *Let $(\ell_t)_{t\in[0;1)}$ be a locally upper bounded collection of real numbers in $[0;1)$, that is, for all $t \in (0;1)$, there exists $\varepsilon_t > 0$ such that $\sup_{s\in[t-\varepsilon_t;t+\varepsilon_t]} \ell_s < 1$. Then there exists a continuous path $(d_t)_{t\in[0;1)}$ in $(0;1)$ such that $1 > d_t > \ell_t$ for all $t \in [0;1)$.*

Proof. Each compact set C in $[0;1)$ is covered by $\bigcup_{t\in C}[t-\varepsilon_t; t+\varepsilon_t]$. From this union, one can extract a finite subcover. Since $\ell_t|_{[t-\varepsilon_t;t+\varepsilon_t]}$ is bounded by M_t, $(\ell_t)_{t\in C}$ is bounded by the maximum of these constants. So ℓ_t is bounded on each compact set in $[0;1)$. Let $(t_n)_{n\in\mathbb{N}}$ be an increasing sequence converging to 1. On each interval $[t_n; t_{n+1}]$, ℓ_t is bounded by M_n, so there exists a continuous path $(d_t)_{t\in[0;1)}$ such that, on $[t_n; t_{n+1}]$, one has $d_t > M_n$, and the lemma is proved. ∎

Proof of Proposition 5.4. Since f_t and $f_\pm$ have the same derivative at 1, from Lemma 4.7, one can find a continuous path $(c_t)_{t\in[0;1)}$ in $(0;1)$ such that, for all $t<1$,

$$c_t > b_t, \qquad f^{-1}\left(\frac{c_t+1}{2}\right) > b_t, \qquad \|f_t|_{[c_t;1]} - f_\pm|_{[c_t;1]}\|_1 < \tilde{\varepsilon}_t = \frac{\varepsilon_t}{4M_\Phi}.$$

From Lemmas 5.8–5.10, one can in this way produce a continuous path $(d_t)_{t\in[0;1)}$ in $(0;1)$ such that, for all $t\in[0;1)$,

$$d_t > f_+^{k(r_t+1, f_+(\frac{c_t+1}{2}))+1}\left(\frac{c_t+1}{2}\right).$$

For all $t<1$, from Lemma 4.14, there exist diffeomorphisms $f_{t,+}$ and $f_{t,-}$ such that:

- $f_{t,\pm}$ coincide with f_t on $[0;c_t]\cup[(d_t+1)/2;1]$;
- $f_{t,\pm}$ coincide with $f_\pm$ on $[(c_t+1)/2;d_t]$;
- $\|f_{t,\pm}-f_t\|_1 < \varepsilon_t$.

Let us denote $k_t = k(r_t+1, f_+((c_t+1)/2))$.

CLAIM 1. *For all $t<1$, one has the following inequalities on $(d_t;1)$:*

$$\mathcal{M}_f(f_{t,+}) > \mathrm{id}, \qquad \mathcal{M}_f(f_{t,-}) < \mathrm{id}.$$

Proof of Claim 1. Let $t<1$, $x>d_t$, and n a sufficiently great integer so that $f^{-n}(x) < a_t$. Let n_1 be the smallest integer such that $f^{-n_1}(x) < (c_t+1)/2$. By construction of d_t, one can decompose n_1 as $n_1 = r_f(f_t)+1+k_t+1+n'$, where $n'\in\mathbb{N}^*$. Then

$$\mathcal{M}_f(f_{t,+})(x) = f_{t,+}^{n_1}\mathcal{M}_f(f_t)f^{-n_1}(x). \tag{2}$$

Moreover,

$$\mathcal{M}_f(f_t)(f^{-n_1}(x)) \ge f^{[\tau_f(\mathcal{M}_f(f_t))]}(f^{-n_1}(x))$$

by definition of $\tau_f(\mathcal{M}_f(f_t))$. Thus

$$f_{t,+}^{n_1}(\mathcal{M}_f(f_t)f^{-n_1}(x)) \ge f_{t,+}^{n_1}f^{[\tau_f(\mathcal{M}_f(f_t))]}(f^{-n_1}(x)). \tag{3}$$

Since $f_{t,+}\ge f$ on $[b_t;1]$ and $f^{-n_1}(x) > b_t$, one also has

$$f_{t,+}^{r_f(f_t)+1}f^{[\tau_f(\mathcal{M}_f(f_t))]}(f^{-n_1}(x)) \ge f^{-n_1}(x).$$

So

$$f_{t,+}^{n_1}f^{[\tau_f(\mathcal{M}_f(f_t))]}(f^{-n_1}(x)) \ge f_{t,+}^{k_t+1+n'}(f^{-n_1}(x)).$$

Hence $f_{t,+}^{k_t+1+n'}f^{-n_1}(x) = f_{t,+}^{n'}f_{t,+}^{k_t}(f_{t,+}f^{-n_1}(x))$; furthermore $f_{t,+}f^{-n_1}(x)\in\left[\frac{c_t+1}{2}; f_+\left(\frac{c_t+1}{2}\right)\right)$, thus $f_{t,+}$ coincides with f_+ on the interval $[f_{t,+}f^{-n_1}(x); f_+^{k_t-1}(f_{t,+}f^{-n_1}(x)))$, which enables us to write

$$f_{t,+}^{k_t+1+n'}f^{-n_1}(x) = f_{t,+}^{n'}f_+^{k_t}(f_{t,+}f^{-n_1}(x)).$$

Since $f_{t,+}f^{-n_1}(x) \in \left[\frac{c_t+1}{2}; f_+\left(\frac{c_t+1}{2}\right)\right)$, and since k is increasing with respect to its second variable, $f_{t,+}^{k_t+1+n'} f^{-n_1}(x) > f_{t,+}^{n'} f^{r_f(f_t)+1+k_t}(f_{t,+}f^{-n_1}(x))$. By noticing that $f_{t,+} = f_+$ is greater than f on $\left[\frac{c_t+1}{2}; f_+\left(\frac{c_t+1}{2}\right)\right)$, one obtains

$$f_{t,+}^{k_t+1+n'} f^{-n_1}(x) > f_{t,+}^{n'} f^{r_f(f_t)+1+k_t} f^{-n_1+1}(x) \geq f_{t,+}^{n'} f^{-n'}(x) \geq x.$$

From these calculations it follows that $\mathcal{M}_f(f_{t,+}) >$ id on $(d_t; 1)$. One can show similarly that $\mathcal{M}_f(f_{t,-}) <$ id on $(d_t; 1)$. ∎

Thus, $\mathcal{M}_f(f_+)$ (resp. $\mathcal{M}_f(f_-)$) is strictly greater (resp. smaller) than the identity on at least one fundamental domain of f, and is defined elsewhere by its commuting relation with f. It is then strictly greater (resp. smaller) than the identity on the whole $(0; 1)$; in particular $\mathcal{M}_f(f_+)(p_t) > p_t$ (resp. $\mathcal{M}_f(f_-)(p_t) < p_t$).

One now considers, for all $t < 1$, the path $(f_{t,s})_{s\in[0;1]}$ of diffeomorphisms defined by $f_{t,s} = sf_{t,+} + (1-s)f_{t,-}$.

Claim 2. *For all $t < 1$, there exists a unique $s_t \in [0; 1]$ such that $\mathcal{M}_f(f_{t,s_t})(p_t) = p_t$.*

It should be noted immediately that the property $\|f_{t,\pm} - f_t\|_1 < \varepsilon_t$, which has been stated above, implies that, for all $s \in [0; 1]$,

$$\|f_{t,s} - f_t\|_1 < \varepsilon_t.$$

Moreover, setting $\tilde{b}_t = (d_t + 1)/2$, we have

$$f_{t,s} \in D_{f,a_t,\tilde{b}_t}.$$

Proof of Claim 2. By construction of $f_{t,+}$, this diffeomorphism coincides with f_+ on at least one fundamental domain, and is also strictly greater than $f_{t,-}$ on this domain, since $f_{t,-}$ coincides with f_- there. Consequently, if $s, s' \in [0; 1]$ with $s < s'$, then $f_{t,s'} > f_{t,s}$ on at least one fundamental domain of $f_{t,+}$, and as a consequence also on at least one fundamental domain of $f_{t,s'}$. One can thus use Lemma 4.21 to state that, if $s < s'$, then $\mathcal{M}_f(f_{t,s}) < \mathcal{M}_f(f_{t,s'})$. On the other hand, given $t \in [0; 1)$, one knows that $\mathcal{M}_f(f_{t,s})$ depends continuously on s on each compact subset of $(0; 1)$, and $\mathcal{M}_f(f_{t,+})(p_t) > p_t$ and $\mathcal{M}_f(f_{t,-})(p_t) < p_t$. The result follows. ∎

Claim 3. *The real number s_t depends continuously on t.*

Proof of Claim 3. Given $t \in [0; 1)$ and a sequence $(t_n)_{n\in\mathbb{N}}$ converging to t, there exists an integer N such that $f^{-N}(p_{t_n}) = f_{t_n,s_{t_n}}^{-N}(p_{t_n})$ for n large enough. Moreover, by continuity of the path $(p_t)_t$ and of the diffeomorphism f^{-N}, one knows that $f^{-N}(p_{t_n})$ converges to $f^{-N}(p_t)$, which is equal to $f_{t,s_t}^{-N}(p_t)$. So $f_{t_n,s_{t_n}}^{-N}(p_{t_n})$ converges to $f_{t,s_t}^{-N}(p_t)$, and by uniqueness of the parameter s satisfying this property, one can deduce that $s_{t_n} \to s_t$ as $n \to \infty$, which proves the continuity of $(s_t)_{t\in[0;1)}$. ∎

Claim 3 implies the continuity of the path $(\tilde{f}_t)_{t\in[0;1)} = (f_{t,s_t})_{t\in[0;1)}$, so the proof of Proposition 5.4 is now complete. ∎

5.2. Creating a degenerate fixed point of h_g

LEMMA 5.11. *Let $f \in D_{\alpha,\beta}$; $a, b \in (0;1)$ with $a < b$; $g \in D_{f,a,b}$ such that $\mathcal{M}_f(g)$ has a fixed point p in $(b;1)$; and $\mathcal{U}$ a $\mathcal{C}^1$-neighbourhood of g in $D_{\alpha,\beta}$. Then there exist $\tilde{b} > b$ and $\tilde{g} \in \mathcal{U} \cap D_{f,a,\tilde{b}}$ such that p is a fixed point of $\mathcal{M}_f(\tilde{g})$, with derivative equal to 1.*

This time again, we state a parameter version of this lemma:

LEMMA 5.12. *Let $f \in D_{\alpha,\beta}$, $(f_t)_{t\in[0;1)}$ be a $\mathcal{C}^1$-continuous path in $D_{\alpha,\beta}$, $(p_t)_{t\in[0;1)}$ a continuous path in $(0;1)$ and $(\varepsilon_t)_{t\in[0;1)}$ a continuous path in $(0;\infty)$, such that:*

- *$f_0 = f$;*
- *there exist continuous paths $(a_t)_{t\in[0;1)}$ and $(b_t)_{t\in[0;1)}$ in $(0;1)$ such that $f_t \in D_{f,a_t,b_t}$ for all $t < 1$;*
- *for all $t < 1$, $p_t > b_t$ and p_t is a fixed point of $\mathcal{M}_f(f_t)$.*

Then there exist a $\mathcal{C}^1$-continuous path $(\tilde{f}_t)_{t\in[0;1)}$ in $D_{\alpha,\beta}$ and a continuous path $(\tilde{b}_t)_{t\in[0;1)}$ in $(0;1)$ such that, for all t:

- *$\tilde{f}_0 = f$;*
- *$\tilde{f}_t \in D_{f,a_t,\tilde{b}_t}$;*
- *p_t is a fixed point of $\mathcal{M}_f(\tilde{f}_t)$ with derivative equal to 1;*
- *$\|\tilde{f}_t - f_t\|_1 < \varepsilon_t$.*

Here again, since the proof of Lemma 5.12 is quite simple, one gives directly the proof of the parameter version.

Proof of Lemma 5.12. The idea is to perturb locally, along the orbit of p_t, the diffeomorphism f_t by composition on the right with an affine map which will, fundamental domain by fundamental domain, make the derivative of $h_f(f_t)$ at these fixed points become closer and closer to 1. For that, one chooses a continuous path $(\varepsilon'_t)_{t\in[0;1)}$ in $(0;\infty)$ such that, for all $t < 1$, for all $\mathcal{C}^1$-diffeomorphisms φ_t of $[0;1]$ with $\|\varphi_t - \mathrm{id}\|_1 < \varepsilon'_t$, one has $\|f_t \circ \varphi - f_t\|_1 < \tilde{\varepsilon_t}$. For all $t \in [0;1)$, one can consider the smallest integer k_t such that $Dh_f(f_t)(p_t)(1-\varepsilon'_t)^{k_t} < 1$ if $Dh_f(f_t)(p_t) > 1$ (resp. $Dh_f(f_t)(p_t)(1+\varepsilon'_t)^{k_t} > 1$ if $Dh_f(f_t)(p_t) < 1$). One considers the composition of f_t with the affine map $H^i_{1-\varepsilon'_t}(x) = (1-\varepsilon'_t)(x - f^i(p_t)) + f^i(p_t)$, resp. $H^i_{1+\varepsilon'_t}(x) = (1+\varepsilon'_t)(x - f^i(p_t)) + f^i(p_t)$ on a neighbourhood of the fixed points $f^i(p_t)$ where $i = 0, \ldots, k_t - 2$, and then with the affine map

$$\frac{1}{(1-\varepsilon'_t)^{k_t-1} Dh - f(f_t)(p_t)}(x - f^{k_t-2}(p_t)) + f^{k_t-2}(p_t)$$

resp.

$$\frac{1}{(1+\varepsilon_t')^{k_t-1}Dh-f(f_t)(p_t)}(x-f^{k_t-2}(p_t))+f^{k_t-2}(p_t),$$

on a neighbourhood of the fixed point $f^{k_t-2}(p_t)$. One can notice that, as p_t has been chosen to be greater than b_t, the diffeomorphism $h_f(\tilde{f}_t)$ at $f^i(p_t)$ is obtained by conjugating $H^{i-1}_{1\pm\varepsilon_t'}\circ h_f(\tilde{f}_t)$ by f at $f^{i-1}(p_t)$. This ensures the preservation of the improvement given to the derivative of $h_f(\tilde{f}_t)$ along the orbit of p_t, and thus enables us to conclude that, after having worked as explained above, the derivative of $h_f(\tilde{f}_t)$ at the fixed point $f^k(p_t)$ is equal to 1.

The $\mathcal{C}^1$-diffeomorphism $\tilde{f}_t$ is then given by re-gluing these local perturbations to the initial diffeomorphism f_t following Corollary 4.10. The continuity of this new path $(\tilde{f}_t)_{t\in[0;1)}$ follows from the continuity of the path $(p_t)_{t\in[0;1)}$ of fixed points, of the path $(\varepsilon_t)_{t\in[0;1)}$, and the continuity of $h_f(f_t)$ with respect to t on each compact subset of $(0;1)$. ∎

5.3. To squash h_g in successive fundamental domains: end of the proof of Theorems 4.1 and 4.8. In this section, we finish the proof of Proposition 5.1 by establishing the following proposition:

Proposition 5.13. *Let $(\alpha_n)_{n\in\mathbb{N}}$ and $(\beta_n)_{n\in\mathbb{N}}$ be sequences in $[0;1]$, $(f_n)_{n\in\mathbb{N}}$ be a sequence in D_{α_n,β_n} converging to* id *in the $\mathcal{C}^1$-topology and h_0 be an increasing $\mathcal{C}^1$-diffeomorphism of $[0;1]$ with $Dh_0(0)=1=Dh_0(1)$. Let $\varepsilon>0$. Then there exists a sequence $(\tilde{f}_n)_{n\in\mathbb{N}}$, where $\tilde{f}_n\in D_{\alpha_n,\beta_n}\cap\mathcal{B}_{f_n}(\varepsilon)$ for all n, such that the sequence $(h_n)_{n\in\mathbb{N}}$ of $\mathcal{C}^1$-diffeomorphisms of $[0;1]$, defined by h_0 and $h_n=\tilde{f}_{n-1}h_{n-1}f_{n-1}^{-1}$ for $n\in\mathbb{N}^*$ is stationary, equal to* id *for all n large enough.*

Proposition 5.2 will follow from its parameter version:

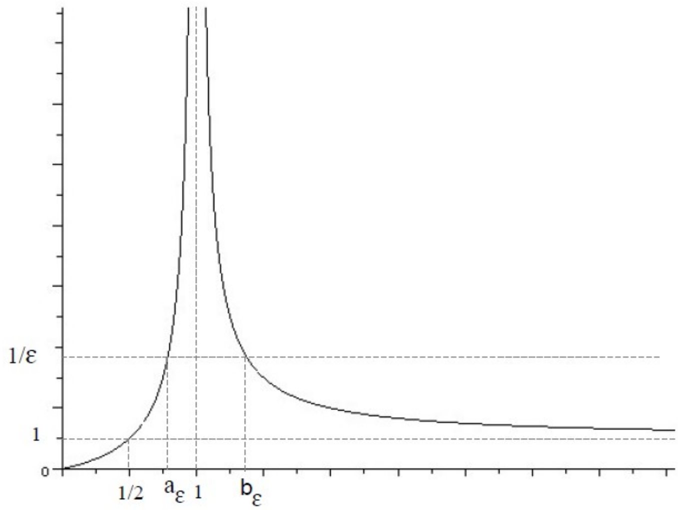
Proposition 5.14. *Let $(f_{t,n})_{(t,n)\in[0;1)\times\mathbb{N}}$ be a collection of diffeomorphisms in D_{α_n,β_n} such that:*

- *for all n, $(f_{t,n})_{t\in[0;1)}$ is a $\mathcal{C}^1$-continuous path,*
- *for all $t\in[0;1)$, $(f_{t,n})_{n\in\mathbb{N}}$ converges to the identity in the $\mathcal{C}^1$-topology as $n\to\infty$.*

Let also $(h_{t,0})_{t\in[0;1)}$ be a continuous path of increasing $\mathcal{C}^1$-diffeomorphisms of $[0;1]$ such that $Dh_{t,0}(0)=1=Dh_{t,0}(1)$ for all $t\in[0;1)$, and lastly let $(\varepsilon_t)_{t\in[0;1)}$ be a continuous path in $(0;\infty)$. Then there exists a collection $(\tilde{f}_{t,n})_{(t,n)\in[0;1)\times\mathbb{N}}$ such that $\tilde{f}_{t,n}\in D_{\alpha_n,\beta_n}\cap\mathcal{B}_{f_{t,n}}(\varepsilon_t)$ for all $(t,n)\in[0;1)\times\mathbb{N}$ and the collection $(h_{t,n})_{n\in\mathbb{N}}$ of $\mathcal{C}^1$-diffeomorphisms of $[0;1]$ defined by $h_{t,0}$ and $h_{t,n}=\tilde{f}_{t,n-1}h_{t,n-1}f_{t,n-1}^{-1}$ for $n\in\mathbb{N}^*$ is stationary for all t, equal to id *for all n greater than N large enough.*

Let us make a technical remark before beginning the proof of this proposition.

REMARK. The graph of the map $F : (0;\infty] \to (0;\infty]$, $x \mapsto \left|\frac{x}{1-x}\right|$, is given by the figure below. In particular, F is increasing on $(0;1)$ and decreasing on $(1;\infty)$; its limits at 1^- and at 1^+ are ∞; it has value 0 at 0 and tends to 1 at ∞; its value at $1/2$ is 1 and at $1/3$ is $1/2$.



NOTATIONS. If $\varepsilon > 0$, we will denote by $0 < a_\varepsilon < 1 < b_\varepsilon$ two real numbers such that $F(x) > 1/\varepsilon$ if $a_\varepsilon < x < b_\varepsilon$, and $F(x) \le 1/\varepsilon$ if $x \le a_\varepsilon$ or $x \ge b_\varepsilon$.

In order to expound clearly the reasoning, and as the adaptation to the parameter version does not present any extra difficulty, we show here only how to deduce Proposition 5.1 from Proposition 5.13.

Proof of Proposition 5.1 from Proposition 5.13. Let $g \in D_{f,a,b}$ and $p \in (0;1)$. From Proposition 5.3, there exist $\tilde{b} > b$ and $\tilde{g} \in \mathcal{U} \cap D_{f,a,\tilde{b}}$ such that p is a fixed point of $\mathcal{M}_f(\tilde{g})$. From Lemma 5.11, one can suppose that $D\mathcal{M}_f(\tilde{g})(p) = 1$. Let m be a sufficiently large integer such that $f^m(p) > \tilde{b}$. One considers the sequence $(f_n)_{n\in\mathbb{N}}$, where f_n is the normalization of the diffeomorphism $f|_{[f^{m+n}(p);f^{m+n+1}(p)]}$ from $[f^{m+n}(p); f^{m+n+1}(p)]$ on its image. One also defines a $\mathcal{C}^1$-diffeomorphism h_0 to be the normalization on $[0;1]$ of $\mathcal{M}_f(g)|_{[f^m(p);f^{m+1}(p)]}$. Then this sequence converges uniformly to the identity in the $\mathcal{C}^1$-topology, and one can apply Proposition 5.13. By perturbing g on the successive fundamental domains of f in such a way that $\tilde{f}_n$ is the normalization on $[0;1]$ of $g|_{[f^{m+n}(p);f^{m+n+1}(p)]}$, one will obtain a perturbation $\tilde{g}$ of g on $[0;1]$ such that $\mathcal{M}_f(\tilde{g}) = \mathrm{id}$. ∎

Proof of Proposition 5.14. We will write $\tilde{f}_{t,n}$ as perturbations $f_{t,n} \circ \psi_{t,n}$. By noticing that $h_{t,n+1}$ is then given by $f_{t,n} \circ \psi_{t,n} h_{t,n} f_{t,n}^{-1}$, we can interpret the transition from $h_{t,n}$ to $h_{t,n+1}$, where t is given, as the transformation of $h_{t,n}$ into $\psi_{t,n} \circ h_{t,n}$, followed by conjugacy by $f_{t,n}$. Since our aim is to obtain $h_{t,n}$ equal to the identity, it appears that the best perturbation $\psi_{t,n}$ would be $h_{t,n}^{-1}$. However, the permitted perturbations have size bounded by $\tilde{\varepsilon}_t$, which is small enough for $\tilde{f}_{t,n} = f_{t,n} \circ \psi_{t,n} \in \mathcal{B}_{f_{t,n}}(\varepsilon_t)$ to hold. This real number does not depend on $f_{t,n}$, but only on ε_t, and one can choose it in such a way that the path $(\tilde{\varepsilon}_t)_{t\in[0;1)}$ is continuous.

Given $n \in \mathbb{N}$ and $t \in [0;1)$, one defines $\psi_{t,n}(x) = x + K_{t,n}\Psi_{t,n}(x)$, where $h_{t,n}^{-1}(x) = x + \Psi_{t,n}(x)$ and

$$K_{t,n} = \inf\left(1; \frac{\tilde{\varepsilon}_t}{\max_{x\in[0;1]} |D\Psi_{t,n}(x)|}\right) = \inf\left(1; \tilde{\varepsilon}_t \cdot \min_{x\in[0;1]} \frac{1}{|D\Psi_{t,n}(x)|}\right).$$

One can then check that $\|\psi_{t,n} - \mathrm{id}\|_1 < \tilde{\varepsilon}_t$, and

$$K_{t,n} = \inf\left(1; \tilde{\varepsilon}_t \cdot \min_{[0;1]}\left|\frac{Dh_{t,n}}{1 - Dh_{t,n}}\right|\right).$$

As already announced, one also defines, for $(t,n) \in [0;1) \times \mathbb{N}$, the diffeomorphism $\tilde{f}_{t,n}$ to be $\tilde{f}_{t,n} = f_{t,n} \circ \psi_{t,n}$, and lastly $h_{t,n+1} = \tilde{f}_{t,n} h_{t,n} f_{t,n}^{-1}$.

Let us introduce the following operator P, defined on $\{f_{t,n}\}_{(t,n)\in[0;1)\times\mathbb{N}} \times \mathcal{D}\mathit{iff}^{\,1}_{+}([0;1])$ and with values in the same set:

$$P : (f_{t,n}, h) \mapsto (f_{t,n+1}, \tilde{f}_{t,n} h f_{t,n}^{-1}).$$

In particular, if $(t,n) \in [0;1) \times \mathbb{N}$, one has $P(f_{t,n}, h_{t,n}) = (f_{t,n+1}, h_{t,n+1})$.

Remark. The operator P is continuous with respect to t.

Proof. This follows from the continuity of the paths $(f_{t,n})_{t\in[0;1)}$ and $(h_{t,n})_{t\in[0;1)}$ for n fixed, the latter proved by induction. Indeed, $(h_{t,0})_{t\in[0;1)}$ is assumed to be continuous, and if $(h_{t,n})_{t\in[0;1)}$ is continuous, then so are $\Psi_{t,n}$ and $K_{t,n}$ (from the continuity of $(\tilde{\varepsilon}_t)_{t\in[0;1)}$), and consequently $\psi_{t,n}$ as well. This proves the continuity of $(\tilde{f}_{t,n})_{t\in[0;1)}$, and hence of $(h_{t,n})_t$. ■

Notice now that if there exists $(t,n) \in [0;1) \times \mathbb{N}$ such that $\min_{[0;1]}\left|\frac{Dh_{t,n}}{1-Dh_{t,n}}\right| \geq 1/\tilde{\varepsilon}_t$, then $K_{t,n} = 1$ and $h_{t,n+1} = \mathrm{id}$. Moreover, one can easily check that if n is such that $h_{t,n} = \mathrm{id}$, then $h_{t,k} = \mathrm{id}$ for all $k \geq n$.

So, the proof will be complete once we prove the following:

Claim 1. *For all $t \in [0;1)$, there exists $M_t \in \mathbb{N}$ such that*

$$\min_{[0;1]}\left|\frac{Dh_{t,M_t}}{1 - Dh_{t,M_t}}\right| \geq \frac{1}{\tilde{\varepsilon}_t}.$$

The rest of the proof is devoted to proving this result. For that, we will first use the convergence of $(f_{t,n})_{n\in\mathbb{N}}$ to the identity to clarify the calculations. Let $(t,n) \in [0;1) \times \mathbb{N}$. Let $\eta_t > 0$ be such that $(1+\tilde{\varepsilon}_t)(1-\eta_t) > 1+\tilde{\varepsilon}_t/2$ and $(1-\tilde{\varepsilon}_t)(1+\eta_t) < 1-\tilde{\varepsilon}_t/2$, and also such that

$$\eta_t < 1 - \frac{2}{3(1+\tilde{\varepsilon}_t/2)}.$$

There exists $N_t > 0$ such that, for all $n \geq N_t$ and all $x, y \in [0;1]$, one has

$$1 - \eta_t < \frac{Df_{t,n}(x)}{Df_{t,n}(y)} < 1 + \eta_t.$$

Define now $\eta_{N_t} = \eta_t$. The uniform convergence of $f_{t,n}$ to id as $n \to \infty$ implies also the existence of a sequence $(\eta_{t,n})_{n \geq N_t}$ converging to 0 and such that, for all $n \geq N_t$ and $x, y \in [0;1]$, one has

$$1 - \eta_{t,n} < \frac{Df_{t,n}(x)}{Df_{t,n}(y)} < 1 + \eta_{t,n}.$$

One can also require that $\eta_{t,n} \leq \eta_t$ for all $t < 1$ and all $n \geq N_t$.

We have the following result:

Claim 2. *For all $t \in [0;1)$, there exists $n \geq N_t$ such that*

$$\min_{[0;1]} \left| \frac{Dh_{t,n}}{1 - Dh_{t,n}} \right| > 1.$$

Proof of Claim 2. Let $x, y \in [0;1]$ and $n \geq N_t$. Assume that the minimum is ≤ 1. We can calculate

$$D(\psi_{t,n} \circ h_{t,n})(x) = Dh_{t,n}(x) + K_{t,n}(1 - Dh_{t,n}(x)). \tag{4}$$

Notice that:

(i) If $Dh_{t,n}(x) = 1$, then $1 - Dh_{t,n}(x) = 0$, so the derivative at this point does not change when $h_{t,n}$ gets composed with $\psi_{t,n}$.
(ii) If $Dh_{t,n}(x) < 1$ (resp. $Dh_{t,n}(x) > 1$), then $1 - Dh_{t,n}(x) > 0$ (resp. $1 - Dh_{t,n}(x) < 0$), thus the derivative becomes greater (resp. smaller) after being composed with $\psi_{t,n}$.
(iii) If $Dh_{t,n}(x) < 1$, then $D(\psi_{t,n} \circ h_{t,n})(x)$ is also < 1, and conversely.
(iv) If $Dh_{t,n}(x) < Dh_{t,n}(y)$, then $D(\psi_{t,n} \circ h_{t,n})(x) < D(\psi_{t,n} \circ h_{t,n})(y)$.

The graph of F indicates that

$$\min_{[0;1]} \left| \frac{Dh_{t,n}}{1 - Dh_{t,n}} \right| = \frac{\min Dh_{t,n}}{|1 - \min Dh_{t,n}|}.$$

On the other hand, from (iv) it follows that

$$\operatorname{argmin} D(\psi_{t,n} \circ h_{t,n}) = \operatorname{argmin} Dh_{t,n}.$$

One can now calculate that $\min D(\psi_{t,n} \circ h_{t,n}) = (1 + \tilde{\varepsilon}_t) \min Dh_{t,n}$, i.e. the minima of $D(\psi_{t,n} \circ h_{t,n})$ and $Dh_{t,n}$ are reached at the same point. It follows immediately that, for all $x \in [0;1]$,

$$D(\psi_{t,n} \circ h_{t,n})(x) \geq (1+\tilde{\varepsilon}_t) \min_{[0;1]} Dh_{t,n}. \tag{5}$$

So to know the derivative of h_{n+1}, one still has to conjugate $\psi_{t,n} \circ h_{t,n}$ by $f_{t,n}$, which leads to

$$Dh_{t,n+1}(f_{t,n}(x)) = \frac{Df_{t,n}(\psi_{t,n} h_{t,n}(x))}{Df_{t,n}(x)} \cdot D(\psi_{t,n} h_{t,n})(x).$$

From this expression and from (5), and according to our choice for η_t, we deduce that for all $x \in [0;1]$,

$$Dh_{t,n+1}(x) > (1+\tilde{\varepsilon}_t)(1-\eta_{t,n}) \min_{[0;1]} Dh_{t,n} > (1+\tilde{\varepsilon}_t/2) \min_{[0;1]} Dh_{t,n}. \tag{6}$$

Let now k be an integer such that $(1+\tilde{\varepsilon}_t/2)^k \min_{[0;1]} Dh_{t,n} > 1$. Two cases can then occur: either

- there exists an integer $n < k' \leq k$ such that $\min\left|\frac{Dh_{t,k'}}{1-Dh_{t,k'}}\right| > 1$; in this case our statement is proved; or
- for each integer k' between n and k, $\min_{[0;1]}\left|\frac{Dh_{t,k'}}{1-Dh_{t,k'}}\right| \leq 1$; in this case, from the previous calculation, one obtains

$$Dh_{n+k} > (1+\tilde{\varepsilon}_t/2)^k \min Dh_{t,n} > 1,$$

and from the graph of the map F we can conclude that Claim 2 is proved. ∎

Claim 3. *Let $t \in [0;1)$. If $n \geq N_t$ is such that $1/\tilde{\varepsilon}_t > \min\left|\frac{Dh_{t,n}}{1-Dh_{t,n}}\right| > 1$, then $\min\left|\frac{Dh_{t,k}}{1-Dh_{t,k}}\right| > 1$ for all $k \geq n$.*

Proof of Claim 3. We argue by induction. If $n \geq N_t$ and $\min\left|\frac{Dh_{t,n}}{1-Dh_{t,n}}\right| > 1$, two cases can occur:

If this minimum is reached at a point $x \in [0;1]$ such that $Dh_{t,n}(x) < 1$, then, since $1/\tilde{\varepsilon}_t > \min\left|\frac{Dh_{t,n}}{1-Dh_{t,n}}\right|$, similar calculations to those above lead to inequality (6). In particular, $\min Dh_{t,n+1} > \min Dh_{t,n}$, and we deduce, with the help of the graph of F, that $\min\left|\frac{Dh_{t,n+1}}{1-Dh_{t,n+1}}\right| > \min\left|\frac{Dh_{t,n}}{1-Dh_{t,n}}\right| > 1$.

Otherwise, this minimum is reached at a point x such that $Dh_{t,n}(x) > 1$. Thus, from the graph of F, it is reached at $\max Dh_{t,n}$, that is,

$$\min_{[0;1]}\left|\frac{Dh_{t,n+1}}{1-Dh_{t,n+1}}\right| = \frac{\max Dh_{t,n}}{\max Dh_{t,n} - 1}.$$

Furthermore, as above we can use (iv) to conclude that $\operatorname{argmax}(D\psi_{t,n} \circ h_{t,n}) = \operatorname{argmax} Dh_{t,n}$, which gives $\max D(\psi_{t,n} \circ h_{t,n}) = (1+\tilde{\varepsilon}_t) \max Dh_{t,n}$. Consequently, for all $x \in [0;1]$,

$$Dh_{t,n+1}(x) < (1-\tilde{\varepsilon}_t)(1+\eta_{t,n}) \max_{[0;1]} Dh_{t,n} < \max_{[0;1]} Dh_{t,n}. \tag{7}$$

On the other hand, the hypothesis $\min\left|\frac{Dh_n}{1-Dh_n}\right| > 1$ implies that $Dh_{t,n}(x) > 1/2$ for each t. Thus, for all $x \in [0;1]$,

$$D(\psi_{t,n} \circ h_{t,n})(x) = Dh_{t,n}(1-K_{t,n}) + K_{t,n} > \frac{1}{2}(1-K_{t,n}) + K_{t,n},$$

and then, using $\min\left|\frac{Dh_n}{1-Dh_n}\right| > 1$ again, one obtains $D(\psi_{t,n} \circ h_{t,n})(x) > \frac{1}{2}(1+\tilde{\varepsilon}_t)$. After conjugating $\psi_{t,n} \circ h_{t,n}$ by $f_{t,n}$, one has

$$Dh_{t,n+1}(x) > (1-\eta_{t,n})D(\psi_{t,n} \circ h_{t,n})(x) > \frac{1}{2}\left(1+\frac{\tilde{\varepsilon}_t}{2}\right) > \frac{1}{2}.$$

Each $Dh_{t,n+1}(x)$ is then bounded by $1/2$ on one side and by $\max Dh_{t,n} > 1$ on the other side, and thus $\min\left|\frac{Dh_{t,n+1}}{1-Dh_{t,n+1}}\right| > 1$. ∎

Claim 4. *For all $t \in [0;1)$, one has $Dh_{t,n} \to 1$ as $t \to \infty$.*

Proof of Claim 4. From the reasoning above, one can assume that

$$\min\left|\frac{Dh_n}{1-Dh_n}\right| > 1 \quad \text{for all } n > n_0.$$

Let $t \in [0;1)$ and $n \geq n_0$. From (4), we can then deduce the following:

- If $Dh_{t,n}(x) < 1$, then $D(\psi_{t,n}h_{t,n})(x) > Dh_{t,n}(x) + \tilde{\varepsilon}_t(1-Dh_{t,n})$, and so $Dh_{t,n+1}(f(x)) > (1-\eta_{t,n})(Dh_{t,n}(x) + \tilde{\varepsilon}_t(1-Dh_{t,n}))$;
- If $Dh_{t,n}(x) > 1$, then $D(\psi_{t,n}h_{t,n})(x) < Dh_{t,n}(x) + \tilde{\varepsilon}_t(1-Dh_{t,n})$, and so $Dh_{t,n+1}(f(x)) < (1+\eta_{t,n})(Dh_{t,n}(x) + \tilde{\varepsilon}_t(1-Dh_{t,n}))$.

Denoting by $d_{t,n}$ the maximal distance from $Dh_{t,n}$ to 1 and using (iv), one deduces

$$\begin{aligned} d_{n+1} &< \max_{[0;1]}\big(\eta_{t,n} + (1-\eta_{t,n})d_n(1-\tilde{\varepsilon}_t); \eta_{t,n} + (1+\eta_{t,n})d_{t,n}(1-\tilde{\varepsilon}_t)\big) \\ &= \eta_{t,n} + (1+\eta_{t,n})d_{t,n}(1-\tilde{\varepsilon}_t) < d_{t,n}(1-\tilde{\varepsilon}/2) + \eta_{t,n}. \end{aligned}$$

Notice that $\frac{\eta_{t,n}}{\tilde{\varepsilon}_t/2}$ is an attracting fixed point of the affine map $x \mapsto x(1-\tilde{\varepsilon}_t/2) + \eta_{t,n}$, and $\frac{\eta_{t,n}}{\tilde{\varepsilon}_t/2}$ converges to 0 as $n \to \infty$, when $t \in [0;1)$ is given. Consequently, the same holds for the maximal distance $d_{t,n}$ from $Dh_{t,n}$ to 1, and thus Claim 4 is proved.

One can easily check that $D\psi_{t,n}(0) = 1 = D\psi_{t,n}(1)$ for all $(t,n) \in [0;1)\times\mathbb{N}$, which follows from $D\tilde{f}_{t,n}(0) = Df_{t,n}(0)$ and $D\tilde{f}_{t,n}(1) = Df_{t,n}(1)$. ∎

Now, since $F(x) \to \infty$ as $x \to 1$, Claim 4 implies that $\min_{[0;1]}\left|\frac{Dh_{t,n}}{1-Dh_{t,n}}\right| \to \infty$ as $n \to \infty$, and thus Claim 1 is proved. ∎

This ends the proof of Theorem 4.8.

6. Isotopy by conjugacy to the identity. In this section, we prove Theorem 1.8.

It is clear that if $Df(x)\neq 1$ for some fixed point x of f, then $D(h_t f h_t^{-1})(x) = Df(x) \neq 1$ for all $t \in [0;1)$, and consequently f cannot converge to id in the $\mathcal{C}^1$-topology.

Thus, we will show now that the condition is sufficient.

For that, we will first consider the case where f has no other fixed point than 0 and 1.

Lemma 6.1. *Let f be an increasing diffeomorphism of $[0;1]$ such that $Df(0) = 1 = Df(1)$ and $\mathrm{Fix}(f) = \{0,1\}$. Let g be an increasing $\mathcal{C}^1$-diffeomorphism of $[0;1]$ without hyperbolic fixed points and with $(f-\mathrm{id})(g-\mathrm{id}) \geq 0$ on the whole $[0;1]$. Then there exists an isotopy by conjugacy $(f_t)_{t\in[0;1)}$ from f to g such that $\|f_t - g\|_1 < 2\|f-g\|_1$ for all $t \in [0;1)$.*

Proof. Let $(\tilde{f}_t)_{t\in[0;1)}$ be the continuous path of increasing $\mathcal{C}^1$-diffeomorphisms of $[0;1]$ defined by $\tilde{f}_t = (1-t)f + tg$. Then $\|\tilde{f}_t - g\|_1 \leq \|f-g\|_1$. Furthermore, if $t \in [0;1)$, then $\tilde{f}_t$ has no fixed point, $\tilde{f}_0 = f$ and $\tilde{f}_t \to g$ as $t \to 1$. One can thus apply Theorem 4.2 for a path $(\varepsilon_t)_{t\in[0;1)}$ in $(0;\infty)$ bounded by $\|f-g\|_1$, converging to 0 as $t \to 1$, and in this way obtain the existence of an isotopy by conjugacy from f to g, denoted by $(f_t)_{t\in[0;1)}$, such that $\|\tilde{f}_t - f_t\|_1 < \varepsilon_t$ for $t<1$. By the triangle inequality, one has the desired control for $\|f_t - g\|_1$. ■

Proof of Theorem 1.8. First notice that the set of connected components of $[0;1]\setminus \mathrm{Fix}(f)$ is countable, and fix a numbering $C_1, C_2, \ldots$ of the closures of these components.

For all $n \geq 1$, denoting by Φ_n the affine map from C_n into $[0;1]$, we define $f_n = \Phi_n f|_{C_n} \Phi_n^{-1}$. Then $\|f_n - \mathrm{id}\|_1 \to 0$ as $n \to \infty$. Indeed, by uniform continuity of $Df|_{C_n}$, and since $\mathrm{diam}(C_n) \to 0$, one finds that $Df|_{C_n}$ converges to $Df|_{C_n}(\partial C_n) = 1$ on C_n, and consequently $f|_{C_n}$ converges to id.

Let $t_0 = 0 < t_1 < t_2 < \cdots$ with $t_n \to 1$ as $n \to \infty$.

For all $n \geq 1$, let $((f_n)_s)_{s\in[0;1]}$ be an isotopy by conjugacy from f_n to id satisfying the conclusions of Lemma 6.1. One defines, for all $t \in [0;1]$, the diffeomorphism f_t by setting for all $n \geq 1$,

$$f_t|_{C_n} = \begin{cases} f|_{C_n} & \text{if } t < t_{n-1}, \\ \Phi_n^{-1}(f_n)_{\frac{t-t_{n-1}}{1-t_{n-1}}}\Phi_n & \text{if } t_{n-1} \leq t \leq 1, \end{cases}$$

and $f_t(1) = 1$. The path $(f_t)_{t\in[0;1)}$ is then continuous: this follows in each component C_n from the continuity of the isotopies $((f_n)_s)_{s\in[0;1]}$, and in the neighbourhood of fixed points, from the fact that these isotopies coincide with $f|_{C_n}$ at these points.

The continuity at $t = 1$ can be proved as follows: If $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $\|f|_{C_n} - \mathrm{id}\|_1 < \varepsilon/2$ for all $n \geq n_0$. Then, for all $t \geq t_{n_0-1}$, f_t coincides either with f, or with id, or with $(f_n)_s$ where $s \in [0;1)$, depending on the component C_n in which we work. In each of these cases, from Lemma 6.1 one has $\|f_t|_{C_n} - \mathrm{id}\|_1 < 2 \cdot \frac{\varepsilon}{2} = \varepsilon$. ∎

7. Generalization: proof of Theorem 1.3

7.1. Signature of a diffeomorphism of $[0;1]$**.** For instance, we saw that two $\mathcal{C}^1$-diffeomorphisms f and g of $[0;1]$ with no other fixed point than 0 and 1, with the same derivatives at 0 and at 1 and such that $(f-\mathrm{id})(g-\mathrm{id}) \geq 0$ are isotopic by conjugacy. We also saw that each $\mathcal{C}^1$-diffeomorphism of $[0;1]$ with derivative 1 at each of its fixed points is isotopic by conjugacy to the identity.

We would now like to group together these two results in a more general statement, as well as distinguish the cases in which there exists an isotopy by conjugacy from f to g from the cases in which such an isotopy does not exist, but in which it is yet possible to obtain a sequence of conjugates of f converging to g.

Definition 7.1. Let $f \in \mathcal{D}\mathit{iff}^1_+([0;1])$ without hyperbolic fixed points, except possibly 0 and/or 1. A countable and well-ordered set $(\{C_i\}_{i\in I}, \prec)$ endowed with a map $\sigma : \{C_i\}_{i\in I} \to \{+,-\}$ will be called a *signature of* f if there exists an increasing, one-to-one map $\Psi : \{C_i\}_{i\in I} \to [0;1] \setminus \mathrm{Fix}(f)$, $C_i \mapsto x_i$, such that:

- For all $i \in I$, $f(x_i) - x_i$ has the same sign as $\sigma(C_i)$.
- If $i, j \in I$ are such that $x_i < x_j$, then there exists $k \in I$ such that $x_i < x_k \leq x_j$ and $(f(x_k) - x_k)(f(x_i) - x_i) < 0$.
- For all $x \in [0;1] \setminus \mathrm{Fix}(f)$, there exists $i \in I$ such that, for all $y \in [x; x_i]$ (an unoriented interval), $(f(y) - y)(f(x_i) - x_i) \geq 0$.

Proposition 7.2. *For every orientation-preserving* $\mathcal{C}^1$*-diffeomorphism* f *of* $[0;1]$*, without hyperbolic fixed points other than possibly* 0 *and* 1*, a signature of* f *exists and is unique up to an orientation-preserving isomorphism.*

Proof. Existence. Given a $\mathcal{C}^1$-diffeomorphism f of $[0;1]$, we will first specify the meaning of the expression “maximal interval on which the sign of f does not change”, used in the Introduction.

For all $x \in [0;1]$ such that $f(x) \neq x$, one considers the set $\mathcal{I}_x$ of all intervals $(a;b) \subset [0,1]$ such that

- $x \in (a,b)$,
- $(f(x) - x)(f(y) - y) \geq 0$ for all $y \in (a;b)$,

- a does not belong to the interior of $\mathrm{Fix}(f)$ with respect to the induced topology on $[a;1]$. In other words, a is neither in the interior of $\mathrm{Fix}(f)$ nor the lower extremity of a connected component of this interior.
- b does not belong to the interior of $\mathrm{Fix}(f)$ with respect to the induced topology on $[b;1]$.

One can then check that, for all $x \in [0;1] \setminus \mathrm{Fix}(f)$, the interval $I_x = \bigcup_{I \in \mathcal{I}_x} I$ belongs to $\mathcal{I}_x$: it is thus the maximal element of this set with respect to inclusion.

Consider now the set $\{I_x : x \in [0;1] \setminus \mathrm{Fix}(f)\}$. If x and x' are in the same connected component of $[0;1] \setminus \mathrm{Fix}(f)$, then $I_x = I_{x'}$. It follows in particular that this set is countable. We therefore denote from now on

$$\{I_x : x \in [0;1] \setminus \mathrm{Fix}(f)\} = \{C_i\}_{i \in I} = \{I_x\}_{x \in I'},$$

where I is a countable set, and I' is a countable subset of $[0;1]$. Now define a map Φ by

$$\Phi : \{C_i\}_{i \in I} \to \{+;-\}, \qquad I_x = C_i \mapsto \text{sign of } f(x) - x,$$

and a map Ψ by

$$\Psi : \{C_i\}_{i \in I} \to [0;1], \qquad I_x = C_i \mapsto x.$$

If $x, x' \in I'$ and $x < x'$, then $I_x \cap I_{x'} = \emptyset$. So $\{I_x\}_{x \in I'}$ is well-ordered, in the same order as the real numbers $x \in I'$, which implies that Ψ is increasing and injective.

By construction, the sign of $\Phi(I_x)$ where $x \in I'$ is the one of $f(\Psi(I_x)) - \Psi(I_x)$.

Let $x, x' \in I'$ with $x < x'$ and $(f(x) - x)(f(x') - x') > 0$. Assume that $(f(x) - x)(f(y) - y) \geq 0$ for all $y \in [x; x']$. Then $I_x \cup I_{x'}$ would belong to $\mathcal{I}_x$, which contradicts the maximality of I_x in $\mathcal{I}_x$. Therefore there exists $y \in [x; x']$ such that $(f(y) - y)(f(x) - x) < 0$. In particular y is not a fixed point, so there exists $x'' \in I'$ such that $I_y = I_{x''}$. Since Ψ is increasing and injective, from $x < y < x'$ one deduces $I_x < I_y < I_{x'}$, and then $x < x'' < x'$, and, since $x'' \in I_y$, one also has $(f(x'') - x'')(f(x) - x) < 0$.

If $x \in [0;1] \setminus \mathrm{Fix}(f)$, then there exists $\tilde{x} \in I'$ such that $x \in I_x = I_{\tilde{x}}$. For all $y \in [x; \tilde{x}]$, one has $y \in I_{\tilde{x}}$, so $(f(y) - y)(f(\tilde{x}) - \tilde{x}) \geq 0$.

Uniqueness. Assume that $((C = \{C_i\}_{i \in I}, \prec), \sigma)$ and $((C' = \{C'_i\}_{i \in I'}, \prec'), \sigma')$ are two signatures of a $\mathcal{C}^1$-diffeomorphism f of $[0;1]$. Let $i \in I$ and $x_i \in [0;1] \setminus \mathrm{Fix}(f)$ be the image of i by the map Ψ defined in Definition 7.1. Then, since $((C' = \{C'_i\}_{i \in I'}, \prec'), \sigma')$ is a signature of f, there exists a unique x'_i in the image of Ψ' such that the sign of f does not change on the whole unoriented interval $[x_i; x'_i]$. We will denote this real number by $\varphi(x_i)$. We then define $\phi(i) = \Psi'^{-1}(x'_i)$. In other words, ϕ is the map from I to I' defined by $\phi = \Psi'^{-1} \varphi \Psi$. Now we will show that ϕ is isomorphic and order-preserving.

- ϕ *is injective and order-preserving*: This follows directly from the same properties of Ψ, Ψ', and φ.
- ϕ *is onto*: Let us consider C_i' in C'. Set $x_i' = \Psi'(C_i')$, and let x_i in the image of Ψ be such that the sign of f does not change on the unoriented interval $[x_i; x_i']$. Then x_i' has the same property as $\varphi(x_i)$, and thus $x_i' = \varphi(x_i)$. Consequently, C_i' is in the image of ϕ. ∎

DEFINITION 7.3. From now on, the signature of f *in the sense A* will refer to the signature of f as constructed in the proof of the existence of a signature above.

7.2. Proof of Theorem 1.3. First, let us state some properties concerning isotopies by conjugacy:

NOTATION If f, g are two orientation-preserving $\mathcal{C}^1$-diffeomorphisms of $[0;1]$, we will write $f \rightsquigarrow g$ if there exists an isotopy by conjugacy from f to g.

PROPOSITION 7.4. *The relation $\rightsquigarrow$ is reflexive and transitive.*

Proof. One immediately checks that $\rightsquigarrow$ is reflexive.

Concerning the transitivity, assume that $(H_t)_{t\in[0;1)}$ is an isotopy by conjugacy from an orientation-preserving $\mathcal{C}^1$-diffeomorphism f_0 of $[0;1]$ to another, which we will denote by f. Assume also that $(h_t f h_t^{-1})_{t\in[0;1)}$ is an isotopy by conjugacy from f to a diffeomorphism $g \in \mathcal{D}\mathit{iff}^1_+([0;1])$, and consider a continuous path $(\varepsilon_t)_{t\in[0;1)}$ in $(0;\infty)$.

Denote by $\eta_t > 0$ the greatest constant such that if $\|\varphi - \psi\|_1 < \eta_t$, then $\|h_t\varphi h_t^{-1} - h_t\psi h_t^{-1}\|_1 < \varepsilon_t$. The continuity of the path $(h_t)_{t\in[0;1)}$ ensures that the collection $(\eta_t)_{t\in[0;1)}$ is locally bounded; more precisely, for each $t \in [0;1)$, there exists a neighbourhood $\mathcal{V}$ of t and $\tilde{\eta}_{t,\mathcal{V}} > 0$ such that $\tilde{\eta}_{t,\mathcal{V}} < \eta_{t'}$ for all $t' \in \mathcal{V}$.

On the other hand, with each given $t_0 \in [0;1)$, by convergence of H_t to f, one can associate the smallest real number $\tilde{r}_{t_0}$ of $[0;1)$ such that $\|f - H_{\tilde{r}_{t_0}}\|_1 < \eta_{t_0}$. Since the collection $(\eta_t)_{t\in[0;1)}$ is locally bounded, one has a similar property for $(\tilde{r}_t)_{t\in[0;1)}$: for each $t \in [0;1)$, there exists a neighbourhood $\mathcal{V}$ of t and $r_{t,\mathcal{V}} \in [0;1)$ such that $\tilde{r}_t < r_{t,\mathcal{V}}$ for all $t' \in \mathcal{V}$. Lemma 5.10 now yields a continuous path $(r(t))_{t\in[0;1)}$ in $[0;1)$ satisfying, for all $t \in [0;1)$, $\|H_{r(t)} - f\|_1 < \eta_t$, and hence $\|h_t H_{r(t)} h_t^{-1} - h_t f h_t^{-1}\|_1 < \varepsilon_t$. By choosing $\varepsilon_t \to 0$ as $t \to 1$, one obtains the convergence of $h_t H_{r(t)} h_t^{-1}$ to g as $t \to 1$ in the $\mathcal{C}^1$-topology, and consequently an isotopy by conjugacy from f_0 to g. ∎

PROPOSITION 7.5. *Let f, g be increasing $\mathcal{C}^1$-diffeomorphisms of* $[0;1]$ *without hyperbolic fixed points, and $\mathcal{B} = \{B_i\}_{i\in I}$ be the set of connected components of* $[0;1] \setminus \mathrm{Fix}(f)$. *Assume that* $(g|_{B_i} - \mathrm{id})(f|_{B_i} - \mathrm{id}) \geq 0$ *for all*

$i \in I$, and $g = \mathrm{id}$ outside $\bigcup_{i\in I} B_i$. Then there exists an isotopy by conjugacy from f to g.

Proof. The proof is similar to the one of Theorem 1.8, by using Lemma 6.1. ∎

Proof of Theorem 1.3. Case 1. First, we assume that there exists an isotopy from f to g, denoted by $(h_t f h_t^{-1})_{t\in[0;1)}$, where $(h_t)_{t\in[0;1)}$ is continuous. One defines Ψ (resp. Ψ') to be the increasing and injective map $\{C_i\}_{i\in I} \to [0;1]\setminus \mathrm{Fix}(f)$ (resp. $\{C_i'\}_{i\in I'} \to [0;1]\setminus \mathrm{Fix}(g)$) satisfying the conditions listed in the definition of the signature, and if $i \in I$ (resp. $i \in I'$), one defines $x_i = \Psi(C_i)$ (resp. $x_i' = \Psi'(C_i')$).

Notice that if $x_i' \in \Psi'(C')$, then, by convergence of $h_t f h_t^{-1}$ to g, there exists $T > 0$ such that $(h_t f h_t^{-1}(x_i') - x_i)(g(x_i) - x_i) > 0$ for all $t \geq T$. So, $f - \mathrm{id}$ has at $h_t^{-1}(x_i')$ the same sign as g at x_i', for all $t \geq T$. By definition of the signature of f, there exists a unique $x_{\varphi(i)} \in \Psi(C)$ such that $f - \mathrm{id}$ has constant sign on the whole unoriented interval $[h_T^{-1}(x_i'); x_{\varphi(i)}]$. The continuity of $t \mapsto h_t^{-1}(x_i')$ ensures that $f - \mathrm{id}$ has constant sign on each unoriented interval of the kind $[h_t^{-1}(x_i'); x_{\varphi(i)}]$, where $t \geq T$ (in particular, φ is well-defined: it does not depend on the choice of t, provided that it satisfies the above mentioned conditions). Then $\Phi(C_i') = \Psi^{-1}(x_{\varphi(i)})$ defines a map $\Phi : C' \to C$, and one shows that:

• *Φ preserves the signs*: The explanation above shows that $\sigma(\Phi(C_i')) = \sigma'(C_i')$ for all $i \in I'$.

• *Φ is non-decreasing*: Let $C_i', C_k' \in C'$ with $C_i' \prec C_k'$; then $x_i' < x_k'$. If $g - \mathrm{id}$ had the same sign at x_i' and at x_k', then, by definition of the signature of g, there would exist $x_l' \in \Psi'(C')$ with $x_i' < x_l' < x_k'$ such that $(g(x_i') - x_i)(g(x_l') - x_l') < 0$. Thus it is sufficient to consider the case where $(g(x_i') - x_i)(g(x_k') - x_k') < 0$. For each $t \in [0;1)$ sufficiently close to 1 we have $h_t^{-1}(x_i') < h_t^{-1}(x_k')$, and $(f(h_t^{-1}(x_i')) - h_t^{-1}(x_i'))(f(h_t^{-1}(x_k')) - h_t^{-1}(x_k')) < 0$. Assume now that $x_{\varphi(k)} \leq x_{\varphi(i)}$. Three cases can occur:

If $x_{\varphi(k)} \leq h_t^{-1}(x_i')$, then $f - \mathrm{id}$ must have constant sign on $[x_{\varphi(k)}; h_t^{-1}(x_k')]$; yet $h_t^{-1}(x_i')$ belongs to this interval. That leads to a contradiction.

If $x_{\varphi(i)} \geq h_t^{-1}(x_k')$, then $f - \mathrm{id}$ must have constant sign on $[h_t^{-1}(x_i); x_{\varphi(i)}]$, which contradicts the fact that $h_t^{-1}(x_k')$ belongs to this interval.

If $h_t^{-1}(x_i) < x_{\varphi(k)} \leq x_{\varphi(i)} < h_t^{-1}(x_k')$, then since $f - \mathrm{id}$ has to have constant sign on $[h_t^{-1}(x_i'); x_{\varphi(i)}]$ and on $[x_{\varphi(k)}; h_t^{-1}(x_k')]$, it follows that it has constant sign on $[h_t^{-1}(x_i'); h_t^{-1}(x_k')]$. This time again, this contradicts the fact that $f - \mathrm{id}$ does not have the same sign at $h_t^{-1}(x_i')$ and at $h_t^{-1}(x_k')$.

Thus $x_{\varphi(i)} < x_{\varphi(k)}$, and we get the assertion since Ψ^{-1} is increasing.

• *Φ is one-to-one*: This follows directly from monotonicity.

CASE 2. Let now $\{C'_i\}_{i\in[\![1;N]\!]}$ be a finite subset of $\{C'_i\}_{i\in I'}$. Assume that there exists a sequence of conjugates of f, denoted by $(h_n f h_n^{-1})_{n\in\mathbb{N}}$, converging to g as $n\to\infty$. Let m be a sufficiently large integer so that, for all $i\in[\![1;N]\!]$, $h_n f h_n^{-1}(x'_i)-x'_i$ has the same sign as $g(x'_i)-x'_i$. Then $f-\mathrm{id}$ has the same sign at $h_m^{-1}(x'_i)$ as $g-\mathrm{id}$ has at x'_i. By definition of the signature of f, for all $i\in[\![1;N]\!]$, there exists $x_{\varphi(i)}\in\Psi(C)$ such that $f-\mathrm{id}$ has constant sign on the unoriented interval $[x_{\varphi(i)};h_m^{-1}(x'_i)]$. One then defines $\Phi(C'_i)=\Psi^{-1}(x_{\varphi(i)})$ and one shows, similarly to Case 1, that Φ is an increasing and sign-preserving one-to-one map. Here, unlike the case where we had an isotopy by conjugacy from f to g, φ depends on the chosen integer m, so the proof would not work if we had not restricted it to a finite subset of C'.

PROPOSITION 7.6. *Let f and g be increasing $\mathcal{C}^1$-diffeomorphisms of $[0;1]$ without hyperbolic fixed points. Denote by $((C=\{C_i\}_{i\in I},\prec),\sigma)$ and $((C'=\{C'_i\}_{i\in I'},\prec'),\sigma')$ their respective signatures. Then:*

1. *There exists an isotopy by conjugacy from f to g if and only if there exists a one-to-one and order-preserving map $\Phi: C'\to C$ such that $\sigma(C'_i)=\sigma'(\Phi(C'_i))$ for all $i\in I'$.*
2. *There exists a sequence of conjugates of f converging to g if and only if, for every finite subset J' of I', there exists a one-to-one and order-preserving map $\Phi:\{C'_i\}_{i\in J'}\to C$ such that $\sigma(C'_i)=\sigma'(\Phi(C'_i))$ for all $i\in J'$.*

End of the proof of Theorem 1.3 assuming Proposition 7.6. Consider the signature in the sense A of f, $((\tilde{C}=\{\tilde{C}_i\}_{i\in I},\tilde{\prec}),\tilde{\sigma})$. In each $\tilde{C}_i$, choose a point x_i which is not a fixed point of f, and associate to it the connected component C_i of $[0;1]\setminus\mathrm{Fix}(f)$ to which it belongs. Define $\prec$ by: if $i,j\in I$, then $C_i\prec C_j$ if and only if $C'_i\prec' C'_j$, and the map σ by: if $i\in I$, then $\sigma(C_i)=\tilde{\sigma}(\tilde{C}_i)$. Then choose $((\{C_i\}_{i\in I},\prec),\sigma)$ to be the description of the signature of f, and call such a description of the signature of a diffeomorphism of $\mathcal{D}\mathit{iff}^1_+([0;1])$ the signature *in the sense B*. We will denote by $((C',\prec'),\sigma')$ the signature in the sense B of g.

Assume that there exists a one-to-one and increasing map $\Phi: C'\to C$ such that $\sigma(C'_i)=\sigma'(\Phi(C'_i))$ for all $i\in I'$. If $Df(0)=Dg(0)\neq 1$, then one can consider the smallest element of the ordered set C', denoted by $(0;a_g)$; then $\Phi((0;a_g))$ is the smallest element of C, denoted by $(0;a_f)$. Otherwise, $Df(0)=Dg(0)=1$ and we define $a_f=a_g=0$.

Similarly, if $Df(1)=Dg(1)\neq 1$, then we can consider the greatest element of C', denoted by $(b_g;1)$; then $\Phi((b_g;1))$ is the smallest element of C, denoted by $(b_f;1)$. Here again, if $Df(1)=Dg(1)=1$, then define $b_f=b_g=1$. Then $f|_{[a_f;b_f]}$ and $g|_{[a_g;b_g]}$ are increasing $\mathcal{C}^1$-diffeomorphisms without hyperbolic fixed points. Moreover, Φ is a one-to-one and increasing map from $C' \setminus ((0;a_g) \cup (b_g;1))$ to $C \setminus ((0;a_f) \cup (b_f;1))$, compatible with the signs σ' and σ on these sets. Normalizing the diffeomorphisms $f|_{[a_f;b_f]}$ and $g|_{[a_g;b_g]}$ so as to obtain diffeomorphisms defined on $[0;1]$, one can use Proposition 7.6 to obtain an isotopy by conjugacy from one to the other. Let now h be an increasing $\mathcal{C}^1$-diffeomorphism of $[0;1]$ such that $h(a_f) = a_g$, $h(b_f) = b_g$ and h is an affine map on $[a_f;b_f]$. We define $g_1 = hgh^{-1}$. Then there exists an isotopy by conjugacy from $f|_{[a_f;b_f]}$ to $g_1|_{[a_f;b_f]}$.

On $(0;a_f)$, $g_1 - \mathrm{id}$ has sign $\sigma'([0;a_g]) = \sigma([0;a_f])$, and on $(b_f;1)$, $g_1 - \mathrm{id}$ has sign $\sigma'([b_g;1]) = \sigma([b_f;1])$. Moreover $g_1 - \mathrm{id}$ has no fixed point on these intervals. From Theorem 1.2 there exists an isotopy by conjugacy from $f|_{[0;a_f]}$ to $g_1|_{[0;a_f]}$, as well as from $f|_{[b_f;1]}$ to $g_1|_{[b_f;1]}$. From these three isotopies by conjugacy, by following the same method as in the proof of Theorem 1.8, one can construct an isotopy by conjugacy from f to g_1, and the transitivity of the relation $\rightsquigarrow$ enables us to conclude.

The proof in the case of a sequence of conjugates follows exactly the same scheme. ■

Proposition 7.7. *Let f, g be increasing $\mathcal{C}^1$-diffeomorphisms of $[0;1]$ such that:*

- $(f - \mathrm{id})(g - \mathrm{id}) \geq 0$*;*
- $Df(0) = Dg(0) = Df(1) = Dg(1) = 1$*;*
- *g has no fixed point in $(0;1)$;*
- *if $x, y \notin \mathrm{Fix}(f)$, then $z \notin \mathrm{Fix}(f)$ for all $x \leq z \leq y$.*

Then there exists an isotopy by conjugacy from f to g, denoted by $(h_t f h_t^{-1})_{t\in[0;1)}$, and a constant $C > 0$ such that, for all $t \in [0;1)$,

$$\|h_t f h_t^{-1} - g\|_1 < C\|f - g\|_1.$$

Proof. We prove the result in the case where f has an interval of fixed points $[0;a]$ and has no fixed point in $(a;1)$. The other cases can be handled similarly.

Let $(a_t)_{t\in[0;1)}$ be a continuous path with $a_0 = a$ and $a_t \to 0$ as $t \to 1$, and $(h_t)_{t\in[0;1)}$ be a continuous path in $\mathcal{D}\mathit{iff}^1_+([0;1])$ such that $h_t(a) = a_t$ and $\{\|h_t\|_1 : t \in [0;1)\}$ is bounded (for example one can choose $\delta < \min((1-a)/2, a/2)$ and choose h_t coinciding on $[0;a-\delta]$ with the affine map from $[0;a]$ into $[0;a_t]$, and on $[a+\delta;1]$ with the affine map from $[a;1]$ into $[a_t;1]$, and then smooth the map on $[a-\delta;a+\delta]$).

Then $h_t f h_t^{-1}$ converges to f_1, where f_1 has no fixed points in $(0;1)$ and $(f - \mathrm{id})(f_1 - \mathrm{id}) \geq 0$ on $[0;1]$. Then, by using Theorem 1.2 to obtain an isotopy by conjugacy from f_1 to g, and then Proposition 7.4 to combine the two isotopies, we obtain an isotopy by conjugacy from f to g. ■

By a similar method to the one presented in the proof of Theorem 1.8, one can deduce the following corollary:

COROLLARY 7.8. *Let f, g be increasing $\mathcal{C}^1$-diffeomorphisms of $[0;1]$ without hyperbolic fixed points, and $\mathcal{B} = \{B_i\}_{i\in I}$ be the set of connected components of $[0;1] \setminus \mathrm{Fix}(g)$. Assume that $(g|_{B_i} - \mathrm{id})(f|_{B_i} - \mathrm{id}) \geq 0$ for all $i \in I$, and $f = \mathrm{id}$ outside $\bigcup_{i\in I} B_i$. Then there exists an isotopy by conjugacy from f to g.*

Proof of Proposition 7.6. Let $((\{C'_i\}_{i\in I'}, \prec'), \sigma')$ be the signature of g in the sense A. The set $((\{C_i\}_{i\in I}, \prec), \sigma)$ will be considered as the signature of f in the sense B.

Let $\varepsilon > 0$, and denote by J_ε the finite subset of I' defined by: $i \in J_\varepsilon$ if there exists $x \in C'_i$ such that $\max(|g(x) - x|, |Dg(x) - 1|) \geq \varepsilon$, so that $\|g|_{\complement \bigcup_{i\in J_n} C'_i} - \mathrm{id}|_{\complement \bigcup_{i\in J_\varepsilon} C'_i}\|_1 < \varepsilon$.

Let $\varPhi_\varepsilon$ be a one-to-one map from $\{C'_i\}_{i\in J_\varepsilon}$ to $\{C_i\}_{i\in I}$, and define an increasing $\mathcal{C}^1$-diffeomorphism f_0 of $[0;1]$ as follows:

- on each $\varPhi_\varepsilon(C'_i) = (a_i; b_i)$, $f_0 = \mathrm{id}$ on $\big(a_i; \frac{2}{3}a_i + \frac{1}{3}b_i\big] \cup \big[\frac{1}{3}a_i + \frac{2}{3}b_i; b_i\big)$; f_0 has no fixed point on $\big(\frac{2}{3}a_i + \frac{1}{3}b_i; \frac{1}{3}a - i + \frac{2}{3}b_i\big)$; and on this latter interval, $f_0 - \mathrm{id}$ has the same sign as $f - \mathrm{id}$;
- $f_0 = \mathrm{id}$ elsewhere.

Proposition 7.5 again yields an isotopy by conjugacy from f to f_0.

Define now $f_1 \in \mathcal{D}\mathit{iff}^1_+([0;1])$ by:

- for all $i \in J_\varepsilon$, $f_1|_{C'_i}$ is conjugate to $f_0|_{\varPhi_\varepsilon(C'_i)}$ (by an increasing and affine $\mathcal{C}^1$-diffeomorphism, denoted by $\tilde{h}_i$);
- $f_1 = \mathrm{id}$ outside $\bigcup_{i\in J_\varepsilon} C'_i$.

Note that the signature of f_1 in the sense B is $(\{C'_i\}_{i\in J_\varepsilon}, \sigma')$.

CLAIM. *There exists an isotopy by conjugacy from f_0 to f_1.*

Proof of the Claim. Define $h_0 = \mathrm{id}$, and, if $i \geq 1$, define:

- $\varphi(i) < i$ to be the integer satisfying $a_{\varphi(i)} = \max\{a_j : j < i$ and $a_j < a_i\}$; if this set is empty, then $\varphi(i) = 0$;
- $\psi(i) < i$ to be the integer satisfying $a_{\psi(i)} = \min\{a_j : j < i$ and $a_j > a_i\}$; if this set is empty, then $\psi(i) = 1$;
- a conjugacy $h_i \in \mathcal{D}\mathit{iff}^1_+([0;1])$ in such a way that $h_i|_{[\frac{2}{3}a'_i + \frac{1}{3}b'_i; \frac{1}{3}a'_i + \frac{2}{3}b'_i]}$ coincides with $\tilde{h}_i$; on $[b'_{\varphi(i)}; a'_i]$, h_i coincides with the affine map from $\big[\frac{1}{3}a'_{\varphi(i)} + \frac{2}{3}b'_{\varphi(i)}; \frac{2}{3}a'_i + \frac{1}{3}b'_i\big]$ to $\big[\frac{1}{3}a_{\varphi(i)} + \frac{2}{3}b_{\varphi(i)}; \frac{2}{3}a_i + \frac{1}{3}b_i\big]$; on $[b'_i; a'_{\psi(i)}]$, h_i coincides with the affine map from $\big[\frac{1}{3}a'_i + \frac{2}{3}b'_i; \frac{2}{3}a'_{\psi(i)} + \frac{1}{3}b'_{\psi(i)}\big]$ to $\big[\frac{1}{3}a_i + \frac{2}{3}b_i; \frac{2}{3}a_i\psi(i) + \frac{1}{3}b_{\psi(i)}\big]$; $h_i = h_{i-1}$ on $\big[0; \frac{1}{3}a'_{\varphi(i)} + \frac{2}{3}b'_{\varphi(i)}\big] \cup \big[\frac{2}{3}a'_{\psi(i)} + \frac{1}{3}b'_{\psi(i)}; 1\big]$.

For all $i \in \mathbb{N}^*$ and $t \in [0;1)$, define $h_{t,i} = th_i + (1-t)h_{i-1}$. Choose a non-decreasing sequence $(t_n)_{n\in\mathbb{N}} \subset [0;1)$ with $t_0 = 0$ and $t_n \to 1$ as $n \to \infty$, and, given $t_i \le t < t_{i+1}$, define $h_t = h_{\frac{t-t_i}{t_{i+1}-t_i},i+1}$. One can check that the path of $\mathcal{C}^1$-diffeomorphisms constructed in this way is continuous. Furthermore, $h_t f_0 h_t^{-1} \to f_1$ as $t \to 1$ in the $\mathcal{C}^1$-topology.

Indeed, let $\varepsilon > 0$. From the regularity of f_0 and f_1, and since their fixed points, except 0 and 1, are not hyperbolic, there exists $n \in \mathbb{N}$ such that $\|f_0|_{C_i} - \mathrm{id}\|_1 < \varepsilon/2$ and $\|f_1|_{C_i} - \mathrm{id}\|_1 < \varepsilon/2$ for all $i \ge n$. In particular, if $t \ge 0$, since h_t is an affine map on the intervals where $f_0 \ne \mathrm{id}$, one also has $\|h_t f_0 h_t^{-1}|_{h_t(C_i)} - \mathrm{id}\|_1 < \varepsilon/2$.

On the other hand, if $t \ge t_{n-1}$, then $h_t f_0 h_t^{-1} = f_1$ on $\bigcup_{k<n} C'_k$. On the complement of this set, $h_t f_0 h_t^{-1}$ and f_1 either coincide with the identity (the former outside $\bigcup_{k\ge n} h_t(C_k)$ and the latter outside $\bigcup_{k\ge n} C'_k$), or, as we just saw, are $\varepsilon/2$-close to the identity in the $\mathcal{C}^1$-norm. Consequently, $\|h_t f_0 h_t^{-1} - f_1\|_1 < \varepsilon$ for all $t \ge t_{n-1}$, and the proof is complete. ∎

Let g_ε be the increasing $\mathcal{C}^1$-diffeomorphism coinciding with g on $\bigcup_{i\in J_\varepsilon} C'_i$ and with id elsewhere, and define a $\mathcal{C}^1$-diffeomorphism $\tilde{g}_\varepsilon$ by: $\tilde{g}_\varepsilon$ has no fixed point and has the same sign as $\sigma'(C'_i)$ on C'_i, and coincides with id outside $\bigcup_{i\in J_\varepsilon} C'_i$.

By Corollary 7.8, there exists an isotopy by conjugacy from f_1 to $\tilde{g}_\varepsilon$, and then Theorem 1.8 ensures the existence of an isotopy by conjugacy from $\tilde{g}_\varepsilon$ to g_ε.

Hence, by Proposition 7.4, there exists an isotopy by conjugacy from f to g_ε. As a consequence, there exists a conjugate of f ε-close to g_ε, thus also 2ε-close to g, and that is true for all $\varepsilon > 0$; the second item of the theorem is thus proved.

The first item can be proved by following the same reasoning; the only difference is that one does not "cancel the waves" of g which are smaller than ε before applying the described method. Consequently, one uses the one-to-one map which maps C' to C, which in this case does exist without the hypothesis of finiteness of the signature of g, and, by using the intermediate diffeomorphisms as in Case 2 (except g_ε), one obtains the existence of isotopies that one combines by using Proposition 7.4, to obtain an isotopy by conjugacy from f to g.

8. Annex

8.1. Statement and structure of the proof. Here we prove Lemma 5.5. The method of proof has been kindly suggested by C. Bonatti.

For convenience, we first transpose the problem to the following equivalent proposition, whose proof is the subject of this Annex:

PROPOSITION 8.1. *Let f be a $\mathcal{C}^1$-diffeomorphism of $[0;\infty)$ without fixed points except 0, such that f is a contraction (i.e. $f-\mathrm{id}<0$ on $(0;\infty)$). Then there exist $\mathcal{C}^1$-contractions f_+ and f_- of $[0;\infty)$ such that:*

- $Df_+(0)=Df(0)=Df_-(0)$;
- $f_-(x)<f(x)<f_+(x)$ *for all* $x>0$;
- *for each* $x>0$, *there exists* $n\in\mathbb{N}^*$ *such that* $f_-^n(x)<f^{n+1}(x)<f^n(x)<f^{n-1}(x)<f_+^n(x)$.

REMARK. If f_+ and f_- are $\mathcal{C}^1$-contractions of $[0;\infty)$ with $f_-<f<f_+$ and satisfying the conclusions of the proposition only on a neighbourhood of 0, then f_+ and f_- satisfy the conclusions on the whole half-line. Thus, it will be sufficient to construct these two contractions on a neighbourhood of 0, and then to extend them to contractions remaining respectively greater and smaller than f.

The first step consists in proving that, if f is embeddable in a $\mathcal{C}^1$-flow, then f satisfies the conclusions of Proposition 8.1.

LEMMA 8.2. *Let f be a $\mathcal{C}^1$-contraction of $[0;\infty)$ such that f is the time-one map of a $\mathcal{C}^1$-vector field X on $[0;\infty)$. Then there exist $\mathcal{C}^1$-contractions f_+ and f_- of $[0;\infty)$ satisfying the conclusions of Proposition* 8.1.

We will hence reduce the initial problem to the following result:

LEMMA 8.3. *If f is a $\mathcal{C}^1$-contraction of $[0;\infty)$, then there exist $\mathcal{C}^1$-contractions g_+,g_- of $[0;\infty)$ and $\mathcal{C}^1$-vector fields X_+,X_- of $[0;\infty)$ such that:*

- $g_-<f<g_+$ *on* $[0;\infty)$;
- g_+,g_- *are the respective time-one maps of* X_+,X_-;
- $Dg_-(0)=Df(0)=Dg_+(0)$.

Indeed, by applying Lemma 8.2 to the contractions g_+ and g_-, one can easily find the desired contractions f_+ and f_- for f.

8.2. Proof of Lemma 8.2. Let $\varphi(t,x)$ be the relevant flow. By considering 1 as origin, one can write each $x\in(0;1]$ in the form $x=\varphi(t(x),1)$. Then $x\mapsto t(x)$ is a positive, strictly decreasing map, and tends to ∞ as $x\to 0$. Moreover, it is differentiable with derivative $1/X(x)$ for all $x\in(0;1]$.

Define f_+ and f_- on $(0;f(1))$ by

$$f_+(x)=\varphi(1-1/t(x),x)\quad\text{and}\quad f_-(x)=\varphi(1+1/t(x),x)$$

One can check that $f_-<f<f_+<\mathrm{id}$ on $(0;f(1))$. Let us now show that these two formulas define $\mathcal{C}^1$-contractions of $(0;f(1))$ with the same derivative at 0 as f.

Indeed, for f_- one has

$$\begin{aligned}&\frac{d}{dx}\left(\varphi\left(1+\frac{1}{t(x)},x\right)\right)\\&\quad=X\left(\varphi\left(1+\frac{1}{t(x)},x\right)\right)\cdot\frac{1}{X(x)}-X\left(\varphi\left(1+\frac{1}{t(x)}\right)\right)\cdot\frac{1}{X(x)}\cdot\frac{1}{t^2(x)},\end{aligned}$$

the first term coming from the derivative of φ with respect to the second variable, and the second term from the derivative of φ with respect to time.

Notice that the right hand side above is strictly positive, as $t(x)>1$ on $(0;f(1))$. Hence f_- is a diffeomorphism from $(0;f(1))$ on its image.

It remains to show that f_- is $\mathcal{C}^1$ at 0 and that $Df_-(0)=Df(0)$. For that, just notice that the first term above has the same limit when $x\to 0$ as $X(f(x))/x$, and this expression tends to $Df(0)$. The second term tends to 0 by the same reasoning. The case of f_+ is analogous.

It remains to prove the third conclusion of Proposition 8.1. Write $f^n(x)=\varphi(n,x)$ for all $x\in(0;1]$ and $n\in\mathbb{N}$. Set $f_-^n(x)=x_n$. Then

$$x_n=\varphi(t_n,x),\quad\text{where}\quad t_n=n+\frac{1}{t(x_{n-1})}+\frac{1}{t(x_{n-2})}+\cdots+\frac{1}{t(x_1)}+\frac{1}{t(x)}.$$

It suffices to show that

$$\frac{1}{t(x_{n-1})}+\frac{1}{t(x_{n-2})}+\cdots+\frac{1}{t(x_1)}+\frac{1}{t(x)}>1.$$

We argue by contradiction: if this sum were bounded by some constant C, then each $t(x_n)$ would be smaller than $n+C$, hence

$$\frac{1}{t(x_{n-1})}+\frac{1}{t(x_{n-2})}+\cdots+\frac{1}{t(x_1)}+\frac{1}{t(x)}\geq\sum_{k=1}^{n}\frac{1}{k+C},$$

which can be arbitrarily large, a contradiction.

Finally, we extend the constructed diffeomorphisms f_+ and f_- restricted to a small neighbourhood of 0 to $\mathcal{C}^1$-contractions of $[0;\infty)$ in such a way that the inequality $f_-<f<f_+$ is preserved. ■

8.3. Proof of Lemma 8.3. In the case where $Df(0)=1$, the proof is as follows. We define the following $\mathcal{C}^1$-vector fields on $[0;1]$:

$$X_-(x)=\frac{1}{2}(f(x)-x)\frac{d}{dx},\qquad X_+(x)=2(f(x)-x)\frac{d}{dx}\ .$$

One can check that they have the desired properties, by using the fact that f is $\mathcal{C}^1$-close to the identity on a neighbourhood of 0.

The case where $Df(0)\neq 1$ is more difficult, and is handled in two steps.

First, we notice that if the derivative of f is monotonic, then f satisfies the conclusions of Lemma 8.3:

Lemma 8.4. *Let f be a $\mathcal{C}^1$-contraction of $[0;\infty)$ such that its derivative is a monotonic continuous map which is not equal to 1 at 0. Then there exist $\mathcal{C}^1$-contractions g_+, g_- of $[0;\infty)$ and $\mathcal{C}^1$-vector fields X_+, X_- on $[0;\infty)$ satisfying the conclusions of Lemma* 8.3.

Finally, the following lemma will enable us to complete the proof:

Lemma 8.5. *Let f be a $\mathcal{C}^1$-contraction of $[0;\infty)$ such that $Df(0) \neq 1$. Then there exist $\mathcal{C}^1$-contractions h_+, h_- of $[0;\infty)$ such that:*

- *$h_- < f < h_+$ on some interval $(0;\varepsilon]$, where $\varepsilon > 0$;*
- *$Dh_-(0) = Df(0) = Dh_+(0)$;*
- *h_- and h_+ have monotonic derivatives.*

Indeed, given h_+, h_-, one can apply Lemma 8.4 to each of them to obtain $\mathcal{C}^1$-contractions which will be the time-one maps of $\mathcal{C}^1$-vector fields on $[0;\infty)$ and be respectively greater and smaller than f.

Proof of Lemma 8.4. Assume that Df is increasing. Since f is greater than the homothety $x \mapsto Df(0) \cdot x$, one can define $g_-(x) = Df(0) \cdot x$, which is the time-one map of the linear vector field $X_-(x) = \log(Df(0)) \cdot x\frac{d}{dx}$ and satisfies the desired conditions. As regards g_+, consider the vector field $X_+(x) = \log\big(\frac{x}{f^{-1}(x)}\big) \cdot x\frac{d}{dx}$ on $(0; f(1))$, and define g_+ as its time-one map. Since Df is increasing, so is $f(x)/x$, and hence $|\log(y/f^{-1}(y))| \leq |\log(f(x)/x)|$ for all $y \in [f(x); x]$. Given $x_0 \in (0; f(1))$, for all $y \in [f(x_0); x_0]$, the norm of $X^+(y)$ is smaller than the one of $\log(f(x_0)/x_0) \cdot x\frac{d}{dx}$, whose time-one map maps x_0 to $f(x_0)$. It follows that x_0 takes a time longer than 1 to reach $f(x_0)$ along the orbit of X_+. Hence $g_+(x_0) \in [f(x_0); x_0]$, and thus $g_+(x_0) \geq f(x_0)$: the map g_+ is greater than f.

One has now to check that g_+ is $\mathcal{C}^1$ also at 0, and that its derivative at this point is the same as that of f; this follows from the expression of the derivative of X_+ by letting $x \to 0$.

In the case where Df is decreasing, we follow the same reasoning with $g_+(x) = Df(0) \cdot x$ and g_- being the time-one map of the vector field $X_-(x) = \log(x/f^{-1}(x)) \cdot x\frac{d}{dx}$. ∎

Proof of Lemma 8.5. On a neighbourhood of 0 define

$$h_+(x) = \int_0^x \inf_{y \in [0;z]} Df(y)\, dz \quad \text{and} \quad h_-(x) = \int_0^x \sup_{y \in [0;z]} Df(y)\, dz.$$

These are $\mathcal{C}^1$-maps whose derivatives are strictly smaller than 1, so they define $\mathcal{C}^1$-contractions on a neighbourhood of 0. Their derivative at 0 is $Df(0)$. It remains to extend h_+ and h_- as $\mathcal{C}^1$-diffeomorphisms of the half-line.

9. Appendix. We present here the proof, due to A. Navas, of Theorem 1.11. The proof is based on ergodic theory and reduces the problem to the approximate solving of a cohomological equation. The starting point is the following remark:

Let f be a $\mathcal{C}^1$-diffeomorphism of $[0;1]$ without hyperbolic fixed points. We want to find a $\mathcal{C}^1$-diffeomorphism φ which conjugates f to a $\mathcal{C}^1$-diffeomorphism close to the identity. This amounts to finding φ such that $\log D(\varphi f \varphi^{-1})$ is close to 0, in other words such that $\log(D\varphi) - \log(D\varphi) \circ f$ is close to $\log Df$. Thus we want to find approximate and continuous solutions to the cohomological equation

$$\rho - \rho \circ f = \log Df.$$

After adding to ρ a constant if necessary, the map $\varphi(x) = \int_0^x \exp(\rho(u))\,du$ becomes a $\mathcal{C}^1$-diffeomorphism of $[0;1]$ satisfying the desired conditions. More precisely, that argument shows:

LEMMA 9.1. *Let f be a diffeomorphism of $[0;1]$ or of the circle $S^1 = \mathbb{R}/\mathbb{Z}$. Let $(\rho_t)_{t\in[0;1)}$ be a continuous path of continuous maps such that $\rho_t - \rho_t \circ f$ converges uniformly to $\log Df$ as $t \to 1$ and such that $\int_0^1 \exp \rho_t(u)\,du = 1$. Define*

$$h_t(x) = \int_0^x \exp(\rho_t(u))\,du.$$

Then $(h_t)_{t\in[0;1)}$ is a continuous path of $\mathcal{C}^1$-diffeomorphisms of the interval (or of the circle) and is an isotopy by conjugacy from f to the identity (in the case of the interval) or to the rotation of the same rotation number as f (in the case of the circle).

Then we have to prove the existence of the path ρ_t. The following proposition ensures the existence of approximate solutions to the cohomological equation.

PROPOSITION 9.2. *Let f be a $\mathcal{C}^1$-diffeomorphism of $[0;1]$ without hyperbolic fixed points (resp. a diffeomorphism of the circle with irrational rotation number), and $\varepsilon > 0$. Then there exists a continuous map ρ of $[0;1]$ (resp. of S^1) such that $\|\rho - \rho \circ f - \log(Df)\|_\infty < \varepsilon$.*

Proof. Given an integer $k \geq 1$ and a map φ from $[0;1]$ to $\mathbb{R}$, consider the kth Birkhoff sum $S_k(\varphi) = \sum_{i=0}^{k-1} \varphi(f^i)$, and, given $n \in \mathbb{N}$, consider the map ρ_n defined by

$$\rho_n = \frac{S_1(\log(Df)) + S_2(\log(Df)) + \cdots + S_n(\log(Df))}{n}.$$

One can then calculate that

$$\rho_n - \rho_{n+1} \circ f = \log(Df) - \frac{S_n(\log(Df)) \circ f}{n}.$$

One concludes by using the following claim:

Claim. *$S_n(\log(Df))/n$ converges uniformly to* 0 *as $n \to \infty$.*

Proof of the Claim: This is a consequence of the fact that the hypotheses "no hyperbolic fixed points" and "irrational rotation number" imply that, for every probability measure μ which is invariant under f and ergodic, the Lyapunov exponent of f with respect to μ, given by $\lambda(\mu) = \int \log(Df)d\mu$, is equal to 0. See [PS] or [H] for more details. ∎

Now we are able to construct a sequence $(h_n = \int_0^x \exp(\rho(u))\,du)_{n\in\mathbb{N}}$ of conjugacies such that $h_n f h_n^{-1}$ converges to the identity in the $\mathcal{C}^1$-topology. We still have to obtain not only a sequence of conjugates, but an isotopy by conjugacy from f to the identity.

Given $n \in \mathbb{N}$ and $\lambda \in (0;1)$, if $t = \frac{\lambda}{n} + \frac{1-\lambda}{n+1}$, we denote by ρ_t the map (of the interval or of the circle) defined by

$$\rho_t = \lambda\rho_n + (1-\lambda)\rho_{n+1}.$$

One can now conclude the proof by applying the following lemma, whose proof is immediate:

Lemma 9.3. *$(\rho_t)_{t\in[0;1)}$ is a continuous path of continuous maps such that $\int_0^1 \exp\rho_t(u)\,du = 1$ for all t, and $\rho_t - \rho_t \circ f$ converges uniformly to $\log Df$ as $t \to 1$.*

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