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REPRESENTATIONS AND COHOMOLOGIES OF HOM-LIE–YAMAGUTI ALGEBRAS WITH APPLICATIONS

BY

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Abstract. The representation and cohomology theory of Hom-Lie–Yamaguti algebras are introduced. As an application, we study deformation and extension of Hom-Lie–Yamaguti algebras. It is proved that a 1-parameter infinitesimal deformation of a Hom-Lie–Yamaguti algebra T corresponds to a Hom-Lie–Yamaguti algebra of deformation type and a $(2,3)$-cocycle of T with coefficients in the adjoint representation. We also prove that abelian extensions of Hom-Lie–Yamaguti algebras are classified by the $(2,3)$-cohomology group.

1. Introduction. In recent years, Hom-type algebras were studied by many researchers. The first examples coming from q-deformations of Witt and Virasoro algebras are Hom-Lie algebras (see [HLS] for definition). Other types include Hom-associative algebras, Hom-Nambu–Lie algebras, Hom-Hopf algebras, etc. (see [ArMS, AtMs, AMM, CG, S, Y3, Y4] and the references therein). In [S], the general representation and cohomology theory of Hom-Lie algebras was developed.

The concept of Hom-Lie–Yamaguti algebra was introduced in [GNI]. It is a Hom-type generalization of a Lie–Yamaguti algebra of [KW, BEM], a general Lie triple system of [Y1, Y2] and a Lie triple algebra of [K]. In [MCL], the authors studied formal deformations of Hom-Lie–Yamaguti algebras, where only low dimensional deformation cohomology was defined without the help of any representation. So it is of interest to make a systematic study of Hom-Lie–Yamaguti algebras in order to give their general representation and cohomology theory. This is the aim of the present paper.

The paper is based on our recent work [Z, ZL]. In [Z], we give a new characterization of the representation and cohomology theory of Lie triple systems. In [ZL], we make a detailed study of the $(2,3)$-cohomology group associated to a representation of a Lie–Yamaguti algebra. As an applica-

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tion, we study the deformation and extension theory of Lie–Yamaguti algebras.

In this paper, we define the representation and cohomology theory of Hom-Lie–Yamaguti algebras. Then we study the deformation and extension theory of those algebras, as in [ZL]. We will see that they are classified by the $(2, 3)$-cohomology groups. The difficulty in this case is that we have a morphism $\alpha : T \to T$ and more conditions on α to be compatible with the structure of Hom-Lie–Yamaguti algebras. Fortunately, we overcome this difficulty by using an equivalent characterization of a representation and a careful analysis of the coboundary operator. The results in this paper generalize Yamaguti's representation and cohomology theory for Lie–Yamaguti algebras in [Y1, Y2].

The paper is organized as follows. In Section 2, we introduce the concept of representations of a Hom-Lie–Yamaguti algebra. Then we define the coboundary operator on the cochain complex of a Hom-Lie–Yamaguti algebra with coefficients in a representation V to produce the cohomology group. We pay special attention to the $(2, 3)$-cohomology group since it will be used latter. In Section 3, we study the infinitesimal deformation theory of Hom-Lie–Yamaguti algebras. We prove that there is a Hom-Lie–Yamaguti algebra of deformation type and a $(2, 3)$-cocycle of T with coefficients in the adjoint representation associated to a deformation. We also introduce the notion of Nijenhuis operators to describe trivial deformations. In Section 4, we study abelian extensions of Hom-Lie–Yamaguti algebras. We prove that there is a one-to-one correspondence between equivalence classes of abelian extensions of the Hom-Lie–Yamaguti algebra T by V and elements of the $(2, 3)$-cohomology group.

Throughout this paper, we work over an algebraically closed field $\mathbb{K}$ of characteristic different from 2 and 3.

2. Representations and cohomologies. In this section, we first recall some basic definitions regarding Hom-Lie–Yamaguti algebras. Then we define the representation and cohomology theory of Hom-Lie–Yamaguti algebras.

A *Hom-vector space* is a pair (V, α) in which V is a vector space and $\alpha : V \to V$ is a linear map. A morphism $(V, \alpha) \to (W, \beta)$ of Hom-vector space is a linear map $f : V \to W$ such that $\beta \circ f = f \circ \alpha$. We will often abbreviate a Hom-vector space (V, α) to V. Note that the category of Hom-vector $\mathbb{K}$-spaces, where $\mathbb{K}$ is a field, can be identified with the category of modules over the polynomial $\mathbb{K}$-algebra $\mathbb{K}[t]$

DEFINITION 2.1. A *Hom-Lie–Yamaguti algebra* (or HLYA for short) consists of a vector space T together with a linear map $\alpha : T \to T$, a bilinear map $[\cdot,\cdot] : T \times T \to T$ and a trilinear map $[\cdot,\cdot,\cdot] : T \times T \times T \to T$ such that, for all $x_i, y_i \in T$, the following conditions are satisfied:

(HLY01) $\alpha([x_1,x_2]) = [\alpha(x_1),\alpha(x_2)]$;

(HLY02) $\alpha([x_1,x_2,x_3]) = [\alpha(x_1),\alpha(x_2),\alpha(x_3)]$;

(HLY1) $[x_1,x_2] + [x_2,x_1] = 0$;

(HLY2) $[x_1,x_2,x_3] + [x_2,x_1,x_3] = 0$;

(HLY3) $[[x_1,x_2],\alpha(x_3)] + \text{c.p.} + [x_1,x_2,x_3] + \text{c.p.} = 0$;

(HLY4) $$[[x_1,x_2],\alpha(x_3),\alpha(y_1)] + [[x_2,x_3],\alpha(x_1),\alpha(y_1)] + [[x_3,x_1],\alpha(x_2),\ \alpha(y_1)] = 0;$$

(HLY5) $[\alpha(x_1),\alpha(x_2),[y_1,y_2]] = [[x_1,x_2,y_1],\alpha^2(y_2)] + [\alpha^2(y_1),[x_1,x_2,y_2]]$;

(HLY6) $$[\alpha^2(x_1),\alpha^2(x_2),[y_1,y_2,y_3]] = [[x_1,x_2,y_1],\alpha^2(y_2),\ \alpha^2(y_3)] + [\alpha^2(y_1),[x_1,x_2,y_2],\alpha^2(y_3)] + [\alpha^2(y_1),\alpha^2(y_2),[x_1,x_2,y_3]],$$

where c.p. means cyclic permutations with respect to x_1,x_2,x_3. We denote a HLYA by $(T,[\cdot,\cdot],[\cdot,\cdot,\cdot],\alpha)$ or simply by T.

A linear map α satisfying (HLY01) and (HLY02) is called an *algebraic homomorphism*. When $\alpha = \mathrm{id}$, conditions (HLY01) and (HLY02) are trivial and the other conditions (HLY1)–(HLY6) reduce to conditions (LY1)–(LY6) for a Lie–Yamaguti algebra (LYA for short), given in [ZL]. Note that conditions (HLY1) and (HLY2) are equivalent to $[x_1,x_1] = 0$ and $[x_1,x_1,x_3] = 0$ respectively.

A homomorphism between two HLYAs T and T' is a linear map $\varphi : T \to T'$ satisfying $\varphi \circ \alpha = \alpha' \circ \varphi$ and

$$\varphi([x_1,x_2]) = [\varphi(x_1),\varphi(x_2)]', \tag{2.1}$$

$$\varphi([x_1,x_2,x_3]) = [\varphi(x_1),\varphi(x_2),\varphi(x_3)]'. \tag{2.2}$$

Definition 2.2. A HLYA *of deformation type* consists of a vector space T together with a linear map $\alpha : T \to T$, a bilinear map $\nu : T \times T \to T$ and a trilinear map $\omega : T \times T \times T \to T$ satisfying all conditions in Definition 2.1 except that (HLY3) is replaced by

(HLY3′) $$\nu(\nu(x_1,x_2),\alpha(x_3)) + \text{c.p.} = 0.$$

Now we give the definition of a representation of a HLYA.

Definition 2.3. Let (T,α) be a HLYA and (V,β) be a Hom-vector space. A *representation* of (T,α) on (V,β) consists of a linear map $\rho : T \to \mathrm{End}(V)$ and bilinear maps $D,\theta : T \times T \to \mathrm{End}(V)$ such that the following conditions are satisfied:

(HR01) $\rho(\alpha(x_1))\circ\beta=\beta\circ\rho(x_1)$;

(HR02) $D(\alpha(x_1),\alpha(x_2))\circ\beta=\beta\circ D(x_1,x_2)$;

(HR03) $\theta(\alpha(x_1),\alpha(x_2))\circ\beta=\beta\circ\theta(x_1,x_2)$;

(HR31) $D(x_1,x_2)-\theta(x_2,x_1)+\theta(x_1,x_2)$
$$+\rho([x_1,x_2])\circ\beta-\rho(\alpha(x_1))\rho(x_2)+\rho(\alpha(x_2))\rho(x_1)=0;$$

(HR41) $D([x_1,x_2],\alpha(x_3))+D([x_2,x_3],\alpha(x_1))+D([x_3,x_1],\alpha(x_2))=0$;

(HR42) $\theta([x_1,x_2],\alpha(y_1))\circ\beta=\theta(\alpha(x_1),\alpha(y_1))\rho(x_2)-\theta(\alpha(x_2),\alpha(y_1))\rho(x_1)$;

(HR51) $D(\alpha(x_1),\alpha(x_2))\rho(y_2)=\rho(\alpha^2(y_2))D(x_1,x_2)+\rho([x_1,x_2,y_2])\circ\beta^2$;

(HR52) $\theta(\alpha(x_1),[y_1,y_2])\circ\beta=\rho(\alpha^2(y_1))\theta(x_1,y_2)-\rho(\alpha^2(y_2))\theta(x_1,y_1)$;

(HR61) $D(\alpha^2(x_1),\alpha^2(x_2))\theta(y_1,y_2)=\theta(\alpha^2(y_1),\alpha^2(y_2))D(x_1,x_2)$
$$+\theta([x_1,x_2,y_1],\alpha^2(y_2))\circ\beta^2+\theta(\alpha^2(y_1),[x_1,x_2,y_2])\circ\beta^2;$$

(HR62) $\theta(\alpha^2(x_1),[y_1,y_2,y_3])\circ\beta^2=\theta(\alpha^2(y_2),\alpha^2(y_3))\theta(x_1,y_1)$
$$-\theta(\alpha^2(y_1),\alpha^2(y_3))\theta(x_1,y_2)+D(\alpha^2(y_1),\alpha^2(y_2))\theta(x_1,y_3).$$

In this case, we also call V a *T-module.*

For example, given a HLYA T, there is a natural adjoint representation on itself. The corresponding representation maps ρ, D and θ are given by

$$\rho(x_1)(x_2):=[x_1,x_2],\quad D(x_1,x_2)x_3:=[x_1,x_2,x_3],\quad \theta(x_1,x_2)x_3:=[x_3,x_1,x_2].$$

The next proposition gives an equivalent characterization of a representation. The proof is omitted since it is analogous to the proof of Lemma 4.5 in Section 4.

Proposition 2.4. *Let (T,α) be a HLYA and (V,β) be a Hom-vector space. Assume we have a map ρ from T to* End(V) *and maps D,θ from $T\times T$ to* End(V)*. Then (ρ,D,θ) is a representation of T on V if and only if $T\oplus V$ is a HLYA under the following maps:*

$$(\alpha+\beta)(x_1+u_1)\triangleq\alpha(x_1)+\beta(u_1), \tag{2.3}$$

$$[x_1+u_1,x_2+u_2]\triangleq[x_1,x_2]+\rho(x_1)(u_2)-\rho(x_2)(u_1), \tag{2.4}$$

$$[x_1+u_1,x_2+u_2,x_3+u_3]\triangleq[x_1,x_2,x_3]+D(x_1,x_2)(u_3) -\theta(x_1,x_3)(u_2)+\theta(x_2,x_3)(u_1). \tag{2.5}$$

In this case, the HLYA $T\oplus V$ is called a semidirect product *of T and V, denoted by $T\ltimes V$.*

Motivated by Yamaguti's cohomology for Lie–Yamaguti algebras, we define cohomology for HLYAs as follows.

Let V be a representation of HLYA T. Let us define the cohomology groups of T with coefficients in V. Let $f:T\times\cdots\times T\to V$ be an n-linear map such that

$$f(\alpha(x_1),\dots,\alpha(x_n)) = \beta(f(x_1,\dots,x_n)), \tag{2.6}$$

$$f(x_1,\dots,x_{2i-1},x_{2i},\dots,x_n) = 0 \quad \text{if } x_{2i-1}=x_{2i}. \tag{2.7}$$

The vector space spanned by such linear maps is called the space of *n-cochains* of T, denoted by $C^n(T,V)$ for $n \geq 1$.

DEFINITION 2.5. For any $(f,g) \in C^{2n}(T,V) \times C^{2n+1}(T,V)$ *the coboundary operator* $\delta : (f,g) \mapsto (\delta_{\mathrm{I}} f, \delta_{\mathrm{II}} g)$ is a map from $C^{2n}(T,V) \times C^{2n+1}(T,V)$ into $C^{2n+2}(T,V) \times C^{2n+3}(T,V)$ defined as follows:

$$\begin{aligned}
&(\delta_{\mathrm{I}} f)(x_1,\dots,x_{2n+2})\\
&\quad = \rho(\alpha^{2n}(x_{2n+1}))g(x_1,\dots,x_{2n},x_{2n+2}) - \rho(\alpha^{2n}(x_{2n+2}))g(x_1,\dots,x_{2n+1})\\
&\qquad - g(\alpha(x_1),\dots,\alpha(x_{2n}),[x_{2n+1},x_{2n+2}])\\
&\qquad + \sum_{k=1}^{n}(-1)^{n+k+1}D(\alpha^{2n-1}(x_{2k-1}),\alpha^{2n-1}(x_{2k}))\\
&\qquad\qquad\qquad \cdot f(x_1,\dots,\hat{x}_{2k-1},\hat{x}_{2k},\dots,x_{2n+2})\\
&\qquad + \sum_{k=1}^{n}\sum_{j=2k+1}^{2n+2}(-1)^{n+k}\\
&\qquad\qquad \cdot f(\alpha^2(x_1),\dots,\hat{x}_{2k-1},\hat{x}_{2k},\dots,[x_{2k-1},x_{2k},x_j],\dots,\alpha^2(x_{2n+2})),
\end{aligned}$$

$$\begin{aligned}
&(\delta_{\mathrm{II}} g)(x_1,x_2,\dots,x_{2n+3})\\
&\quad = \theta(\alpha^{2n}(x_{2n+2}),\alpha^{2n}(x_{2n+3}))g(x_1,\dots,x_{2n+1})\\
&\qquad - \theta(\alpha^{2n}(x_{2n+1}),\alpha^{2n}(x_{2n+3}))g(x_1,\dots,x_{2n},x_{2n+2})\\
&\qquad + \sum_{k=1}^{n+1}(-1)^{n+k+1}D(\alpha^{2n}(x_{2k-1}),\alpha^{2n}(x_{2k}))g(x_1,\dots,\hat{x}_{2k-1},\hat{x}_{2k},\dots,x_{2n+3})\\
&\qquad + \sum_{k=1}^{n+1}\sum_{j=2k+1}^{2n+3}(-1)^{n+k}\\
&\qquad\qquad \cdot g(\alpha^2(x_1),\dots,\hat{x}_{2k-1},\hat{x}_{2k},\dots,[x_{2k-1},x_{2k},x_j],\dots,\alpha^2(x_{2n+3})).
\end{aligned}$$

When $\alpha = \mathrm{id}$, one recovers Yamaguti's cohomology for LYA defined in [Y2].

LEMMA 2.6. *With the above notation, for any $(f,g) \in C^{2n}(T,V) \times C^{2n+1}(T,V)$, we have*

$$\delta_{\mathrm{I}} f(\alpha(x_1)\dots,\alpha(x_{2n+2})) = \beta(\delta_{\mathrm{I}} f(x_1,\dots,x_{2n+2})), \tag{2.8}$$

$$\delta_{\mathrm{II}} g(\alpha(x_1)\dots,\alpha(x_{2n+3})) = \beta(\delta_{\mathrm{II}} g(x_1,\dots,x_{2n+3})). \tag{2.9}$$

Thus we obtain a well-defined map

$$\delta = (\delta_{\mathrm{I}},\delta_{\mathrm{II}}) : C^{2n}(T,V) \times C^{2n+1}(T,V) \to C^{2n+2}(T,V) \times C^{2n+3}(T,V).$$

Proof. We only prove (2.8), since (2.9) can be verified similarly. By Definition 2.5, we have

$$
\begin{aligned}
(\delta_{\mathrm{I}} f)(\alpha(x_1),\dots,\alpha(x_{2n+2})) &= \rho(\alpha^{2n+1}(x_{2n+1}))g(\alpha(x_1),\dots,\alpha(x_{2n+2})) \\
&\quad - \rho(\alpha^{2n+1}(x_{2n+2}))g(\alpha(x_1),\dots,\alpha(x_{2n+1})) \\
&\quad - g(\alpha^2(x_1),\dots,\alpha^2(x_{2n}),\alpha([x_{2n+1},x_{2n+2}])) \\
&\quad + \sum_{k=1}^{n}(-1)^{n+k+1}D(\alpha^{2n}(x_{2k-1}),\alpha^{2n}(x_{2k})) \\
&\qquad\qquad \cdot f(\alpha(x_1),\dots,\hat{x}_{2k-1},\hat{x}_{2k},\dots,\alpha(x_{2n+2})) \\
&\quad + \sum_{k=1}^{n}\sum_{j=2k+1}^{2n+2}(-1)^{n+k} \\
&\qquad \cdot f(\alpha^3(x_1),\dots,\hat{x}_{2k-1},\hat{x}_{2k},\dots,\alpha([x_{2k-1},x_{2k},x_j]),\dots,\alpha^3(x_{2n+2})) \\
&= \rho(\alpha^{2n+1}(x_{2n+1}))\circ\beta(g(x_1,\dots,x_{2n+2})) \\
&\quad - \rho(\alpha^{2n+1}(x_{2n+2}))\circ\beta(g(x_1,\dots,x_{2n+1})) \\
&\quad - \beta(g(\alpha(x_1),\dots,\alpha(x_{2n}),[x_{2n+1},x_{2n+2}])) \\
&\quad + \sum_{k=1}^{n}(-1)^{n+k+1}D(\alpha^{2n}(x_{2k-1}),\alpha^{2n}(x_{2k})) \\
&\qquad\qquad \circ\beta(f(x_1,\dots,\hat{x}_{2k-1},\hat{x}_{2k},\dots,x_{2n+2})) \\
&\quad + \sum_{k=1}^{n}\sum_{j=2k+1}^{2n+2}(-1)^{n+k} \\
&\qquad \cdot \beta\big(f(\alpha^2(x_1),\dots,\hat{x}_{2k-1},\hat{x}_{2k},\dots,[x_{2k-1},x_{2k},x_j],\dots,\alpha^2(x_{2n+2}))\big) \\
&= \beta(\rho(\alpha^{2n}(x_{2n+1}))g(x_1,\dots,x_{2n},x_{2n+2}))) \\
&\quad - \beta(\rho(\alpha^{2n}(x_{2n+2}))g(x_1,\dots,x_{2n+1})) \\
&\quad - \beta(g(\alpha(x_1),\dots,\alpha(x_{2n}),[x_{2n+1},x_{2n+2}])) \\
&\quad + \sum_{k=1}^{n}(-1)^{n+k+1} \\
&\qquad \cdot \beta\big(D(\alpha^{2n-1}(x_{2k-1}),\alpha^{2n-1}(x_{2k}))(x_1,\dots,\hat{x}_{2k-1},\hat{x}_{2k},\dots,x_{2n+2})\big) \\
&\quad + \sum_{k=1}^{n}\sum_{j=2k+1}^{2n+2}(-1)^{n+k} \\
&\qquad \cdot \beta\big(f(\alpha^2(x_1),\dots,\hat{x}_{2k-1},\hat{x}_{2k},\dots,[x_{2k-1},x_{2k},x_j],\dots,\alpha^2(x_{2n+2}))\big) \\
&= \beta((\delta_{\mathrm{I}} f)(x_1,x_2,\dots,x_{2n+2})),
\end{aligned}
$$

where in the second equality we use condition (2.6) and in the third equality we use (HR01)–(HR03). ∎

Proposition 2.7. *The coboundary operator defined above satisfies $\delta\circ\delta = 0$, that is, $\delta_{\mathrm{I}}\circ\delta_{\mathrm{I}} = 0$ and $\delta_{\mathrm{II}}\circ\delta_{\mathrm{II}} = 0$.*

Proposition 2.7 can be proved by tedious computations. We will check a special case in Proposition 2.11.

Let $Z^{2n}(T,V)\times Z^{2n+1}(T,V)$ be the subspace of $C^{2n}(T,V)\times C^{2n+1}(T,V)$ spanned by (f,g) such that $\delta(f,g)=0$, which is called the *space of cocycles*, and let $B^{2n}(T,V)\times B^{2n+1}(T,V)=\delta(C^{2n-2}(T,V)\times C^{2n-1}(T,V))$, the *space of coboundaries.*

DEFINITION 2.8. For $n\geq 2$, the $(2n,2n+1)$-*cohomology group* of a HLYA T with coefficients in V is defined to be the quotient space

$$\begin{aligned}&H^{2n}(T,V)\times H^{2n+1}(T,V)\\&\qquad\triangleq (Z^{2n}(T,V)\times Z^{2n+1}(T,V))/(B^{2n}(T,V)\times B^{2n+1}(T,V)).\end{aligned}$$

In conclusion, we obtain a cochain complex whose cohomology group is called the cohomology group of a HLYA T with coefficients in V.

Note that in Definition 2.8 we assume $n\geq 2$. For $n=1$, we define the $(2,3)$-cohomology group of a HLYA T with coefficients in V as follows.

Let $C^2(T,V)$ be the space of maps $\nu: T\times T\to V$ such that $\nu(x_1,x_2)=-\nu(x_2,x_1)$ and

$$\nu(\alpha(x_1),\alpha(x_2))=\beta\circ\nu(x_1,x_2). \tag{CC01}$$

Let $C^3(T,V)$ be the space of maps $\omega: T\times T\times T\to V$ such that $\omega(x_1,x_2,x_3)=-\omega(x_2,x_1,x_3)$ and

$$\omega(\alpha(x_1),\alpha(x_2),\alpha(x_3))=\beta\circ\omega(x_1,x_2,x_3). \tag{CC02}$$

DEFINITION 2.9. Let (T,α) be a HLYA and (V,β) a T-module. Then $(\nu,\omega)\in C^2(T,V)\times C^3(T,V)$ is called a $(2,3)$-*cocycle* if for all $x_1,x_2,y_1,y_2,y_3\in T$, we have

$$\begin{aligned}&\omega(x_1,x_2,x_3)+\text{c.p.}-\rho(\alpha(x_1))\nu(x_2,x_3)\\&\qquad\qquad-\text{c.p.}+\nu([x_1,x_2],\alpha(x_3))+\text{c.p.}=0;\end{aligned} \tag{CC1}$$

$$\theta(\alpha(x_1),\alpha(y_1))\nu(x_2,x_3)+\text{c.p.}+\omega([x_1,x_2],\alpha(x_3),\alpha(y_1))+\text{c.p.}=0; \tag{CC2}$$

$$\begin{aligned}&\omega(\alpha(x_1),\alpha(x_2),[y_1,y_2])+D(\alpha(x_1),\alpha(x_2))\omega(y_1,y_2)\\&\qquad=\nu([x_1,x_2,y_1],\alpha^2(y_2))+\nu(\alpha^2(y_1),[x_1,x_2,y_2])\\&\qquad\quad+\rho(\alpha^2(y_1))\omega(x_1,x_2,y_2)-\rho(\alpha^2(y_2))\omega(x_1,x_2,y_1);\end{aligned} \tag{CC3}$$

$$\begin{aligned}&\omega(\alpha^2(x_1),\alpha^2(x_2),[y_1,y_2,y_3])+D(\alpha^2(x_1),\alpha^2(x_2))\omega(y_1,y_2,y_3)\\&\quad=\omega([x_1,x_2,y_1],\alpha^2(y_2),\alpha^2(y_3))+\omega(\alpha^2(y_1),[x_1,x_2,y_2],\alpha^2(y_3))\\&\qquad+\omega(\alpha^2(y_1),\alpha^2(y_2),[x_1,x_2,y_3])+\theta(\alpha^2(y_2),\alpha^2(y_3))\omega(x_1,x_2,y_1)\\&\qquad-\theta(\alpha^2(y_1),\alpha^2(y_3))\omega(x_1,x_2,y_2)+D(\alpha^2(y_1),\alpha^2(y_2))\omega(x_1,x_2,y_3).\end{aligned} \tag{CC4}$$

The space of $(2,3)$-cocycles is denoted by $Z^2(T,V)\times Z^3(T,V)$.

We remark that conditions (CC3) and (CC4) are equivalent to $\delta_{\mathrm{I}}(\nu) = 0$ and $\delta_{\mathrm{II}}(\omega) = 0$ respectively. Why we add conditions (CC1) and (CC2) can be seen from the following context.

Let f be a linear map of T into a representation space V. Then f is called a *derivation* of T into V if

$$f([x_1, x_2]) = \rho(x_1)f(x_2) - \rho(x_2)f(x_1), \tag{2.10}$$

$$f([x_1, x_2, x_3]) = \theta(x_2, x_3)f(x_1) - \theta(x_1, x_3)f(x_2) + D(x_1, x_2)f(x_3). \tag{2.11}$$

Definition 2.10. Let (T, α) be a HLYA and (V, β) a T-module. Then $(\nu, \omega) \in C^2(T, V) \times C^3(T, V)$ is called a $(2,3)$-*coboundary* if there exists a map $f : T \to V$ such that

$$f \circ \alpha = \beta \circ f; \tag{BB01}$$

$$\nu(x_1, x_2) = \rho(x_1)f(x_2) - \rho(x_2)f(x_1) - f([x_1, x_2]); \tag{BB1}$$

$$\omega(x_1, x_2, x_3) = \theta(x_2, x_3)f(x_1) - \theta(x_1, x_3)f(x_2) + D(x_1, x_2)f(x_3) - f([x_1, x_2, x_3]). \tag{BB2}$$

The space of $(2,3)$-coboundaries is denoted by $B^2(T, V) \times B^3(T, V)$.

Proposition 2.11. *The space of* $(2,3)$*-coboundaries is contained in the space of* $(2,3)$*-cocycles.*

Proof. We verify that if (ν, ω) satisfies (BB01), (BB1) and (BB2), then it must satisfy (CC01), (CC02) and (CC1)–(CC4).

By definition, for (CC01), we have

$$\begin{aligned}
&\nu(\alpha(x_1), \alpha(x_2)) - \beta \circ \nu(x_1, x_2) \\
&\quad = \rho(\alpha(x_1))f(\alpha(x_2)) - \rho(\alpha(x_2))f(\alpha(x_1)) - f([\alpha(x_1), \alpha(x_2)]) \\
&\qquad - \beta \circ \{\rho(x_1)f(x_2) - \rho(x_2)f(x_1) - f([x_1, x_2])\} \\
&\quad = \underline{\rho(\alpha(x_1)) \circ \beta} \circ f(x_2) - \underline{\underline{\rho(\alpha(x_2)) \circ \beta}} \circ f(x_1) - f \circ \alpha([x_1, x_2]) \\
&\qquad - \underline{\beta \circ \rho(x_1)} \circ f(x_2) + \underline{\underline{\beta \circ \rho(x_2)}} \circ f(x_1) + \beta \circ f([x_1, x_2]) = 0,
\end{aligned}$$

where in the last equality we have used (HR01) and (BB01).

For (CC02), we have

$$\begin{aligned}
&\omega(\alpha(x_1), \alpha(x_2), \alpha(x_3)) - \beta \circ \omega(x_1, x_2, x_3) \\
&\quad = \theta(\alpha(x_2), \alpha(x_3))f(\alpha(x_1)) - \theta(\alpha(x_1), \alpha(x_3))f(\alpha(x_2)) \\
&\qquad + D(\alpha(x_1), \alpha(x_2))f(\alpha(x_3)) - f([\alpha(x_1), \alpha(x_2), \alpha(x_3)]) \\
&\qquad \cdot \beta \circ \{\theta(x_2, x_3)f(x_1) - \theta(x_1, x_3)f(x_2) + D(x_1, x_2)f(x_3) - f([x_1, x_2, x_3])\}
\end{aligned}$$

$$= \underline{\theta(\alpha(x_2),\alpha(x_3))\circ\beta}\circ f(x_1) - \underline{\underline{\theta(\alpha(x_1),\alpha(x_3))\circ\beta}}\circ f(x_2)$$
$$+ \underline{\underline{\underline{D(\alpha(x_1),\alpha(x_2))\circ\beta}}}\circ f(x_3) - f\circ\alpha([x_1,x_2,x_3])$$
$$\cdot\, \underline{\beta\circ\theta(x_2,x_3)}\circ f(x_1) - \underline{\underline{\beta\circ\theta(x_1,x_3)}}\circ f(x_2) + \underline{\underline{\underline{\beta\circ D(x_1,x_2)}}}\circ f(x_3)$$
$$- \beta\circ f([x_1,x_2,x_3]) = 0.$$

where in the last equality we have used (HR02), (HR03) and (BB01).

For (CC1), we have

$$\begin{aligned}
&\omega(x_1,x_2,x_3) + \text{c.p.} - \rho(\alpha(x_1))\nu(x_2,x_3) - \text{c.p.} + \nu([x_1,x_2],\alpha(x_3)) + \text{c.p.}\\
&= \big(\theta(x_2,x_3)f(x_1) - \theta(x_1,x_3)f(x_2) + D(x_1,x_2)f(x_3) - f([x_1,x_2,x_3])\big) + \text{c.p.}\\
&\quad - \rho(\alpha(x_1))\big(\rho(x_2)f(x_3) - \rho(x_3)f(x_2) - f([x_2,x_3])\big) + \text{c.p.}\\
&\quad + \big(\rho([x_1,x_2])f(\alpha(x_3)) - \rho(\alpha(x_3))f([x_1,x_2]) - f([[x_1,x_2],\alpha(x_3)])\big) + \text{c.p.}\\
&= \big(D(x_1,x_2) - \theta(x_2,x_1) + \theta(x_1,x_2) + \rho[x_1,x_2]\circ\beta\\
&\quad - \rho(\alpha(x_1))\rho(x_2) + \rho(\alpha(x_2))\rho(x_1)\big)f(x_3) + \text{c.p.}\\
&\quad - f\big([x_1,x_2,x_3] + \text{c.p.} + [[x_1,x_2],\alpha(x_3)] + \text{c.p.}\big) = 0.
\end{aligned}$$

The last equality follows from (HR31) and (HLY3).

By direct computations, for (CC2), we get

$$\begin{aligned}
&\theta(\alpha(x_1),\alpha(y_1))\nu(x_2,x_3) + \text{c.p.} + \omega([x_1,x_2],\alpha(x_3),\alpha(y_1)) + \text{c.p.}\\
&= \theta(\alpha(x_1),\alpha(y_1))\big(\rho(x_2)f(x_3) - \rho(x_3)f(x_2) - f([x_2,x_3])\big) + \text{c.p.}\\
&\quad + \big(\theta(\alpha(x_3),\alpha(y_1))f([x_1,x_2]) - \theta([x_1,x_2],\alpha(y_1))f(\alpha(x_3))\\
&\quad + D([x_1,x_2],\alpha(x_3))f(\alpha(y_1)) - f([[x_1,x_2],\alpha(x_3),\alpha(y_1)])\big) + \text{c.p.}\\
&= \big(D([x_1,x_2],\alpha(x_3)) + D([x_2,x_3],\alpha(x_1)) + D([x_3,x_1],\alpha(x_2))\big)f(\alpha(y_1))\\
&\quad - \big(\theta([x_1,x_2],y_1)\circ\beta - \theta(\alpha(x_1),\alpha(y_1))\rho(x_2) + \theta(\alpha(x_2),\alpha(y_1))\rho(x_1)\big)f(x_3)\\
&\quad - \text{c.p.} - f([[x_1,x_2],\alpha(x_3),\alpha(y_1)] + \text{c.p.}) = 0.
\end{aligned}$$

The last equality uses (HR41), (HR42) and (HLY4).

The other cases can checked as follows: (CC3) follows from conditions (HR51), (HR52) and (HLY5); (CC4) is a consequence of (HR61), (HR62) and (HLY6). Therefore the space of $(2,3)$-coboundaries is contained in the space of $(2,3)$-cocycles. ∎

Definition 2.12. The $(2,3)$-*cohomology group* of a HLYA T with coefficients in V is defined as the quotient space

$$H^2(T,V)\times H^3(T,V) \triangleq Z^2(T,V)\times Z^3(T,V)/B^2(T,V)\times B^3(T,V).$$

3. Infinitesimal deformations. Let T be a HLYA and $\nu : T \times T \to T$ and $\omega : T \times T \times T \to T$ be bilinear and trilinear maps. Consider a λ-parametrized family of bilinear maps and trilinear maps

$$[x_1, x_2]_\lambda \triangleq [x_1, x_2] + \lambda\nu(x_1, x_2),$$
$$[x_1, x_2, x_3]_\lambda \triangleq [x_1, x_2, x_3] + \lambda\omega(x_1, x_2, x_3).$$

If $[\cdot,\cdot]_\lambda$ and $[\cdot,\cdot,\cdot]_\lambda$ endow T with a HLYA structure which is denoted by T_λ, then we say that (ν, ω) generates a λ-*parameter infinitesimal deformation* of the HLYA T.

THEOREM 3.1. *With the above notation, (ν, ω) generates a λ-parameter infinitesimal deformation of a HLYA T if and only if:*

(i) *(ν, ω) defines a HLYA of deformation type on T;*
(ii) *(ν, ω) is a $(2, 3)$-cocycle of T with coefficients in the adjoint representation.*

Proof. Assume (ν, ω) generates a λ-parameter infinitesimal deformation of the HLYA T. Then the maps $[x_1, x_2]_\lambda$ and $[x_1, x_2, x_3]_\lambda$ defined above satisfy (HLY1)–(HLY6). From these conditions, we will deduce that (ν, ω) is a $(2, 3)$-cocycle and (ν, ω) defines a HLYA of deformation type on T.

From (HLY01), we have

$$\begin{aligned}
&\alpha([x_1, x_2]_\lambda) - [\alpha(x_1), \alpha(x_2)]_\lambda \\
&\quad = \alpha[x_1, x_2] - [\alpha(x_1), \alpha(x_2)] + \lambda\{\alpha \circ \nu(x_1, x_2) - \nu(\alpha(x_1), \alpha(x_2))\} = 0.
\end{aligned}$$

thus we get

$$\alpha \circ \nu(x_1, x_2) = \nu(\alpha(x_1), \alpha(x_2)). \tag{3.1}$$

From (HLY02), we have

$$\begin{aligned}
\alpha([x_1, x_2, x_3]_\lambda) &- [\alpha(x_1), \alpha(x_2), \alpha(x_3)]_\lambda \\
&= \alpha[x_1, x_2, x_3] - [\alpha(x_1), \alpha(x_2), \alpha(x_3)] \\
&\quad + \lambda\{\alpha \circ \nu(x_1, x_2, x_3) - \nu(\alpha(x_1), \alpha(x_2), \alpha(x_3))\} = 0,
\end{aligned}$$

thus we obtain

$$\alpha \circ \nu(x_1, x_2, x_3) = \nu(\alpha(x_1), \alpha(x_2), \alpha(x_3)). \tag{3.2}$$

From (HLY3), we have

$$\begin{aligned}
[x_1, x_2, x_3]_\lambda &+ \text{c.p.} + [[x_1, x_2]_\lambda, \alpha(x_3)]_\lambda + \text{c.p.} \\
&= [x_1, x_2, x_3] + \text{c.p.} + [[x_1, x_2], \alpha(x_3)] + \text{c.p.} \\
&\quad + \lambda\{\omega(x_1, x_2, x_3) + \text{c.p.} + \nu([x_1, x_2], \alpha(x_3)) + \text{c.p.} \\
&\quad + [\nu(x_1, x_2), \alpha(x_3)] + \text{c.p.}\} \\
&\quad + \lambda^2\{\nu(\nu(x_1, x_2), \alpha(x_3)) + \text{c.p.}\} = 0,
\end{aligned}$$

thus we get

$$\omega(x_1,x_2,x_3)+\text{c.p.}+\nu([x_1,x_2],\alpha(x_3))+\text{c.p.} + [\nu(x_1,x_2),\alpha(x_3)]+\text{c.p.}=0, \tag{3.3}$$

$$\nu(\nu(x_1,x_2),\alpha(x_3))+\text{c.p.}=0. \tag{3.4}$$

From (HLY4), we have

$$\begin{aligned}
&[[x_1,x_2]_\lambda,\alpha(x_3),\alpha(y_1)]_\lambda+\text{c.p.}\\
&\quad=[[x_1,x_2],\alpha(x_3),\alpha(y_1)]+\text{c.p.}\\
&\qquad+\lambda\{\omega([x_1,x_2],\alpha(x_3),\alpha(y_1))+\text{c.p.}+[\nu(x_1,x_2),\alpha(x_3),\alpha(y_1)]+\text{c.p.}\}\\
&\qquad+\lambda^2\{\omega(\nu(x_1,x_2),\alpha(x_3),\alpha(y_1))+\text{c.p.}\}=0,
\end{aligned}$$

thus we get

$$\omega([x_1,x_2],\alpha(x_3),\alpha(y_1))+\text{c.p.}+[\nu(x_1,x_2),\alpha(x_3),\alpha(y_1)]+\text{c.p.}=0, \tag{3.5}$$

$$\omega(\nu(x_1,x_2),\alpha(x_3),\alpha(y_1))+\text{c.p.}=0. \tag{3.6}$$

From (HLY5), we have

$$[\alpha(x_1),\alpha(x_2),[y_1,y_2]_\lambda]_\lambda=[[x_1,x_2,y_1]_\lambda,\alpha^2(y_2)]_\lambda+[\alpha^2(y_1),[x_1,x_2,y_2]_\lambda]_\lambda;$$

the left hand side is equal to

$$\begin{aligned}
&[\alpha(x_1),\alpha(x_2),[y_1,y_2]+\lambda\nu(y_1,y_2)]_\lambda\\
&\quad=[\alpha(x_1),\alpha(x_2),[y_1,y_2]]\\
&\qquad+\lambda\{\omega(\alpha(x_1),\alpha(x_2),[y_1,y_2])+[\alpha(x_1),\alpha(x_2),\nu(y_1,y_2)]\}\\
&\qquad+\lambda^2\omega(\alpha(x_1),\alpha(x_2),\nu(y_1,y_2)),
\end{aligned}$$

and the right hand side is equal to

$$\begin{aligned}
&[[x_1,x_2,y_1]_\lambda,\alpha^2(y_2)]_\lambda+[\alpha^2(y_1),[x_1,x_2,y_2]_\lambda]_\lambda\\
&\quad=[[x_1,x_2,y_1],\alpha^2(y_2)]+[\alpha^2(y_1),[x_1,x_2,y_2]]\\
&\qquad+\lambda\{[\omega(x_1,x_2,y_1),\alpha^2(y_2)]+\nu([x_1,x_2,y_1],\alpha^2(y_2))\\
&\qquad+[\alpha^2(y_1),\omega(x_1,x_2,y_2]+\nu(y_1,[x_1,x_2,y_2])\}\\
&\qquad+\lambda^2\{\nu(\omega(x_1,x_2,y_1),\alpha^2(y_2))+\nu(\alpha^2(y_1),\omega(x_1,x_2,y_2))\}.
\end{aligned}$$

Hence we obtain

$$\begin{aligned}
&\omega(\alpha(x_1),\alpha(x_2),[y_1,y_2])+[\alpha(x_1),\alpha(x_2),\nu(y_1,y_2)]\\
&\quad=[\omega(\alpha(x_1),\alpha(x_2),y_1),\alpha^2(y_2)]+\nu([x_1,x_2,y_1],\alpha^2(y_2))\\
&\qquad+[\alpha^2(y_1),\omega(x_1,x_2,y_2)]+\nu(\alpha^2(y_1),[x_1,x_2,y_2])
\end{aligned} \tag{3.7}$$

and

$$\omega(\alpha(x_1),\alpha(x_2),\nu(y_1,y_2)) = \nu(\omega(x_1,x_2,y_1),\alpha^2(y_2)) + \nu(\alpha^2(y_1),\omega(x_1,x_2,y_2)). \tag{3.8}$$

From (HLY6), we have

$$\begin{aligned}[\alpha^2(x_1),\alpha^2(x_2),[y_1,y_2,y_3]_\lambda]_\lambda &= [[x_1,x_2,y_1]_\lambda,\alpha^2(y_2),\alpha^2(y_3)]_\lambda + [\alpha^2(y_1),[x_1,x_2,y_2]_\lambda,\alpha^2(y_3)]_\lambda \\ &\quad + [\alpha^2(y_1),\alpha^2(y_2),[x_1,x_2,y_3]_\lambda]_\lambda;\end{aligned}$$

the left hand side is equal to

$$\begin{aligned}&[\alpha^2(x_1),\alpha^2(x_2),[y_1,y_2,y_3]+\lambda\omega(y_1,y_2,y_3)]_\lambda \\ &= [\alpha^2(x_1),\alpha^2(x_2),[y_1,y_2,y_3]] + \lambda\omega(\alpha^2(x_1),\alpha^2(x_2),[y_1,y_2,y_3]) \\ &\quad + [\alpha^2(x_1),\alpha^2(x_2),\lambda\omega(y_1,y_2,y_3)] + \lambda\omega(\alpha^2(x_1),\alpha^2(x_2),\lambda\omega(y_1,y_2,y_3)) \\ &= [\alpha^2(x_1),\alpha^2(x_2),[y_1,y_2,y_3]] \\ &\quad + \lambda\{\omega(\alpha^2(x_1),\alpha^2(x_2),[y_1,y_2,y_3]) + [\alpha^2(x_1),\alpha^2(x_2),\omega(y_1,y_2,y_3)]\} \\ &\quad + \lambda^2\omega(\alpha^2(x_1),\alpha^2(x_2),\omega(y_1,y_2,y_3)),\end{aligned}$$

and the right hand side is equal to

$$\begin{aligned}&[[x_1,x_2,y_1]+\lambda\omega(x_1,x_2,y_1),\alpha^2(y_2),\alpha^2(y_3)]_\lambda \\ &\quad + [\alpha^2(y_1),[x_1,x_2,y_2]+\lambda\omega(x_1,x_2,y_2),\alpha^2(y_3)]_\lambda \\ &\quad + [\alpha^2(y_1),\alpha^2(y_2),[x_1,x_2,y_3]+\lambda\omega(x_1,x_2,y_3)]_\lambda \\ &= [[x_1,x_2,y_1],\alpha^2(y_2),\alpha^2(y_3)] + [y_1,[x_1,x_2,y_2],y_3] \\ &\quad + [\alpha^2(y_1),\alpha^2(y_2),[x_1,x_2,y_3]] \\ &\quad + \lambda\{\omega([x_1,x_2,y_1],\alpha^2(y_2),\alpha^2(y_3)) + [\omega(x_1,x_2,y_1),\alpha^2(y_2),\alpha^2(y_3)] \\ &\qquad + \omega(\alpha^2(y_1),[x_1,x_2,y_2],\alpha^2(y_3)) + [\alpha^2(y_1),\omega(x_1,x_2,y_2),\alpha^2(y_3)] \\ &\qquad + \omega(\alpha^2(y_1),\alpha^2(y_2),[x_1,x_2,y_3]) + [\alpha^2(y_1),\alpha^2(y_2),\omega(x_1,x_2,y_3)]\} \\ &\quad + \lambda^2\{\omega(\omega(x_1,x_2,y_1),\alpha^2(y_2),\alpha^2(y_3)) + \omega(y_1,\omega(x_1,x_2,y_2),y_3) \\ &\quad + \omega(\alpha^2(y_1),\alpha^2(y_2),\omega(x_1,x_2,y_3))\}.\end{aligned}$$

Thus we get

$$\begin{aligned}&\omega(\alpha^2(x_1),\alpha^2(x_2),[y_1,y_2,y_3]) + [\alpha^2(x_1),\alpha^2(x_2),\omega(y_1,y_2,y_3)] \\ &= \omega([x_1,x_2,y_1],\alpha^2(y_2),\alpha^2(y_3)) + \omega(\alpha^2(y_1),[x_1,x_2,y_2],\alpha^2(y_3)) \\ &\quad + \omega(\alpha^2(y_1),\alpha^2(y_2),[x_1,x_2,y_3]) + [\omega(x_1,x_2,y_1),\alpha^2(y_2),\alpha^2(y_3)] \\ &\quad + [\alpha^2(y_1),\omega(x_1,x_2,y_2),\alpha^2(y_3)] + [\alpha^2(y_1),\alpha^2(y_2),\omega(x_1,x_2,y_3)]\end{aligned} \tag{3.9}$$

and

$$\begin{aligned}(3.10)\quad &\omega(\alpha^2(x_1),\alpha^2(x_2),\omega(y_1,y_2,y_3))\\ &= \omega(\omega(x_1,x_2,y_1),\alpha^2(y_2),\alpha^2(y_3)) + \omega(\alpha^2(y_1),\omega(x_1,x_2,y_2),\alpha^2(y_3))\\ &\quad + \omega(\alpha^2(y_1),\alpha^2(y_2),\omega(x_1,x_2,y_3)).\end{aligned}$$

Therefore by (3.1), (3.2), (3.4), (3.6), (3.8) and (3.10), (ν,ω) defines a HLYA of deformation type on T. Furthermore, by (3.1), (3.2), (3.3), (3.5), (3.7) and (3.9), we conclude that (ν,ω) is a $(2,3)$-cocycle of T with coefficients in the adjoint representation. ■

A deformation is said to be *trivial* if there exists a linear map $N : T \to T$ such that for $\varphi_\lambda = \mathrm{id} + \lambda N : T_\lambda \to T$ we have

$$(3.11)\qquad \varphi_\lambda[x_1,x_2]_\lambda = [\varphi_\lambda x_1,\varphi_\lambda x_2],$$

$$(3.12)\qquad \varphi_\lambda[x_1,x_2,x_3]_\lambda = [\varphi_\lambda x_1,\varphi_\lambda x_2,\varphi_\lambda x_3].$$

It follows from (3.11) and (3.12) that N must satisfy

$$(3.13)\qquad N[Nx_1,x_2] + N[x_1,Nx_2] - N^2[x_1,x_2] = [Nx_1,Nx_2],$$

and

$$\begin{aligned}(3.14)\quad & N[Nx_1,x_2,x_3] + N[x_1,Nx_2,x_3] + N[x_1,x_2,Nx_3] - N^2[x_1,x_2,x_3]\\ &= [Nx_1,Nx_2,x_3] + [Nx_1,x_2,Nx_3] + [x_1,Nx_2,Nx_3].\end{aligned}$$

DEFINITION 3.2. A linear operator $N : T \to T$ is called a *Nijenhuis operator* of a HLYA T if (3.13) and (3.14) hold.

An important property of a Nijenhuis operator is that it gives a trivial deformation.

THEOREM 3.3. *Let N be a Nijenhuis operator for T. Then a deformation of T can be obtained by setting*

$$(3.15)\qquad \nu(x_1,x_2) = [Nx_1,x_2] + [x_1,Nx_2] - N[x_1,x_2],$$

$$\begin{aligned}(3.16)\qquad \omega(x_1,x_2,x_3) &= [Nx_1,x_2,x_3] + [x_1,Nx_2,x_3] + [x_1,x_2,Nx_3]\\ &\quad - N[x_1,x_2,x_3].\end{aligned}$$

Furthermore, this deformation is trivial.

4. Abelian extensions. In this section, we study abelian extensions of HLYAs. It is showed that abelian extensions are classified by the $(2,3)$-cohomology group. We will build a bijection map from the set $\mathrm{Ext}(T,V)$ of equivalence classes of abelian extensions to $H^2(T,V)\times H^3(T,V)$.

DEFINITION 4.1. Let $(T,[\cdot,\cdot],[\cdot,\cdot,\cdot],\alpha)$, $(V,[\cdot,\cdot]_V,[\cdot,\cdot,\cdot]_V,\beta)$ and $(\hat{T},[\cdot,\cdot]_{\hat{T}}, [\cdot,\cdot,\cdot]_{\hat{T}},\hat{\alpha})$ be HLYAs, and let $i : V \to \hat{T}$ and $p : \hat{T} \to T$ be homomorphisms.

If the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & V & \xrightarrow{i} & \hat{T} & \xrightarrow{p} & T & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \hat{\alpha}} & & \downarrow{\scriptstyle \alpha} & & \\ 0 & \longrightarrow & V & \xrightarrow{i} & \hat{T} & \xrightarrow{p} & T & \longrightarrow & 0 \end{array} \tag{4.1}$$

commutes and its horizontal lines (denoted by $E_{\hat{T}}$) are short exact sequences, then we call $\hat{T}$ an *extension* of T by V. It is called an *abelian extension* if V is an abelian ideal of $\hat{T}$, i.e. $[u,v]_{\hat{T}} = 0$ and $[u,v,\cdot]_{\hat{T}} = [u,\cdot,v]_{\hat{T}} = [\cdot,u,v]_{\hat{T}} = 0$, for all $u, v \in V$.

From the left square in the diagram we deduce that if we choose an element $u \in V$, then

$$\hat{\alpha} \circ i(u) = i \circ \alpha_v(u). \tag{4.2}$$

Since i is injective, we can identify V with its image in $\hat{T}$, and thus

$$\hat{\alpha}(u) = \hat{\alpha}|_V(u) = \beta(u). \tag{4.3}$$

A *section* $\sigma : T \to \hat{T}$ of $p : \hat{T} \to T$ is a linear map such that

$$p \circ \sigma = \mathrm{id}_T \quad \text{and} \quad \hat{\alpha} \circ \sigma = \sigma \circ \alpha. \tag{4.4}$$

Definition 4.2. Two extensions of HLYAs $E_{\hat{T}} : 0 \to V \xrightarrow{i} \hat{T} \xrightarrow{p} T \to 0$ and $E_{\tilde{T}} : 0 \to V \xrightarrow{j} \tilde{T} \xrightarrow{q} T \to 0$ are called *equivalent* if there exists a HLYA homomorphism $F : \hat{T} \to \tilde{T}$ such that the following diagram commutes:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & V & \xrightarrow{i} & \hat{T} & \xrightarrow{p} & T & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle F} & & \downarrow{\scriptstyle \mathrm{id}} & & \\ 0 & \longrightarrow & V & \xrightarrow{j} & \tilde{T} & \xrightarrow{q} & T & \longrightarrow & 0 \end{array} \tag{4.5}$$

The set of equivalence classes of extensions of T by V is denoted by $\mathrm{Ext}(T,V)$.

Let $\hat{T}$ be an abelian extension of T by V. Define maps ρ from T to $\mathrm{End}(V)$ and D, θ from $T \times T$ to $\mathrm{End}(V)$ by

$$\rho(x_1)(u) \triangleq [\sigma(x_1), u]_{\hat{T}}, \tag{4.6}$$

$$D(x_1,x_2)(u) \triangleq [\sigma(x_1), \sigma(x_2), u]_{\hat{T}}, \tag{4.7}$$

$$\theta(x_1,x_2)(u) \triangleq [u, \sigma(x_1), \sigma(x_2)]_{\hat{T}}. \tag{4.8}$$

Lemma 4.3. *With the above notation, (ρ, D, θ) is a representation of T on V and does not depend on the choice of the section σ. Moreover, equivalent abelian extensions give the same representation.*

Proof. First, the fact that ρ, D, θ are independent of the choice of σ is easy to check. For details, see [ZL].

Second, we show that (ρ, D, θ) is a representation of T on V.

By the equality

$$\begin{aligned}\rho(\alpha(x_1))\circ\beta(u) &= [\sigma(\alpha(x_1)),\beta(u)]_{\hat{T}} = [\hat{\alpha}\circ\sigma(x_1),\hat{\alpha}(u)]_{\hat{T}}\\ &= \hat{\alpha}([\sigma(x_1),u]_{\hat{T}}) = \beta([\sigma(x_1),u]_{\hat{T}}) = \beta\circ\rho(x_1)(u),\end{aligned}$$

we obtain (HR01):

$$\rho(\alpha(x_1))\circ\beta = \beta\circ\rho(x_1). \tag{4.9}$$

By the equality

$$\begin{aligned}D(\alpha(x_1),\alpha(x_2))\circ\beta(u) &= [\sigma(\alpha(x_1)),\sigma(\alpha(x_2)),\beta(u)]_{\hat{T}}\\ &= [\hat{\alpha}\circ\sigma(x_1),\hat{\alpha}\circ\sigma(x_2),\hat{\alpha}(u)]_{\hat{T}} = \hat{\alpha}([\sigma(x_1),\sigma(x_2),u]_{\hat{T}})\\ &= \beta([\sigma(x_1),\sigma(x_2),u]_{\hat{T}}) = \beta\circ D(x_1,x_2)(u),\end{aligned}$$

we obtain (HR02):

$$D(\alpha(x_1),\alpha(x_2))\circ\beta(u) = \beta\circ D(x_1,x_2)(u). \tag{4.10}$$

By the equality

$$\begin{aligned}&[\sigma(x_1),\sigma(x_2),u]_{\hat{T}} + [\sigma(x_2),u,\sigma(x_1)]_{\hat{T}} + [u,\sigma(x_1),\sigma(x_2)]_{\hat{T}}\\ &\quad + [[\sigma(x_1),\sigma(x_2)]_{\hat{T}},\beta(u)]_{\hat{T}} + [[\sigma(x_2),u]_{\hat{T}},\hat{\alpha}\circ\sigma(x_1)]_{\hat{T}}\\ &\quad + [[u,\sigma(x_1)],\hat{\alpha}\circ\sigma(x_2)]_{\hat{T}} = 0,\end{aligned}$$

we obtain (HR31):

$$\begin{aligned}D(x_1,x_2) - \theta(x_2,x_1) + \theta(x_1,x_2) + \rho([x_1,x_2])\circ\beta&\\ - \rho(\alpha(x_1))\rho(x_2) + \rho(\alpha(x_2))\rho(x_1) &= 0.\end{aligned} \tag{4.11}$$

By the equality

$$\begin{aligned}&[[\sigma(x_1),\sigma(x_2)]_{\hat{T}},\hat{\alpha}\circ\sigma(x_3),\beta(u)]_{\hat{T}} + [[\sigma(x_2),\sigma(x_3)]_{\hat{T}},\hat{\alpha}\circ\sigma(x_1),\beta(u)]_{\hat{T}}\\ &\quad + [[\sigma(x_3),\sigma(x_1)]_{\hat{T}},\hat{\alpha}\circ\sigma(x_2),\beta(u)]_{\hat{T}} = 0,\end{aligned}$$

we have (HR41):

$$D([x_1,x_2],\alpha(x_3)) + D([x_2,x_3],\alpha(x_1)) + D([x_3,x_1],\alpha(x_2)) = 0. \tag{4.12}$$

By the equality

$$\begin{aligned}&[[\sigma(x_1),\sigma(x_2)]_{\hat{T}},\beta(u),\hat{\alpha}\circ\sigma(y_1)]_{\hat{T}} + [[\sigma(x_2),u]_{\hat{T}},\hat{\alpha}\circ\sigma(x_1),\hat{\alpha}\circ\sigma(y_1)]_{\hat{T}}\\ &\quad + [[u,\sigma(x_1)]_{\hat{T}},\hat{\alpha}\circ\sigma(x_2),\hat{\alpha}\circ\sigma(y_1)]_{\hat{T}} = 0,\end{aligned}$$

we have (HR42):

$$\theta([x_1,x_2],\alpha(y_1))\circ\beta = \theta(\alpha(x_1),\alpha(y_1))\rho(x_2) - \theta(\alpha(x_2),\alpha(y_1))\rho(x_1). \tag{4.13}$$

By the equality

$$[\hat{\alpha}\circ\sigma(x_1),\hat{\alpha}\circ\sigma(x_2),[u,\sigma(y_2)]_{\hat{T}}]_{\hat{T}} = [[\sigma(x_1),\sigma(x_2),u]_{\hat{T}},\hat{\alpha}^2\circ\sigma(y_2)]_{\hat{T}} + [\beta^2(u),[\sigma(x_1),\sigma(x_2),\sigma(y_2)]_{\hat{T}}]_{\hat{T}},$$

we have (HR51):

$$D(\alpha(x_1),\alpha(x_2))\rho(y_2) = \rho(\alpha^2(y_2))D(x_1,x_2) + \rho([x_1,x_2,y_2])\circ\beta^2. \tag{4.14}$$

By the equality

$$[\hat{\alpha}\circ\sigma(x_1),\beta(u),[\sigma(y_1),\sigma(y_2)]_{\hat{T}}]_{\hat{T}} = [[\sigma(x_1),u,\sigma(y_1)]_{\hat{T}},\hat{\alpha}^2\circ\sigma(y_2)]_{\hat{T}} + [\hat{\alpha}^2\circ\sigma(y_1),[\sigma(x_1),u,\sigma(y_2)]_{\hat{T}}]_{\hat{T}},$$

we have (HR52):

$$\theta(\alpha(x_1),[y_1,y_2])\circ\beta = \rho(\alpha^2(y_1))\theta(x_1,y_2) - \rho(\alpha^2(y_2))\theta(x_1,y_1). \tag{4.15}$$

By the equality

$$\begin{aligned}[\hat{\alpha}^2\circ\sigma(x_1),\hat{\alpha}^2\circ\sigma(x_2),[u,\sigma(y_1),\sigma(y_2)]_{\hat{T}}]_{\hat{T}} &= [[\sigma(x_1),\sigma(x_2),u]_{\hat{T}},\hat{\alpha}^2\circ\sigma(y_1),\hat{\alpha}^2\circ\sigma(y_2)]_{\hat{T}} \\ &\quad + [\beta^2(u),[\sigma(x_1),\sigma(x_2),\sigma(y_1)]_{\hat{T}},\hat{\alpha}^2\circ\sigma(y_2)]_{\hat{T}} \\ &\quad + [\beta^2(u),\hat{\alpha}^2\circ\sigma(y_1),[\sigma(x_1),\sigma(x_2),\sigma(y_2)]_{\hat{T}}]_{\hat{T}},\end{aligned}$$

we have (HR61):

$$\begin{aligned}D(\alpha^2(x_1),\alpha^2(x_2))\theta(y_1,y_2) &= \theta(\alpha^2(y_1),\alpha^2(y_2))D(x_1,x_2) \\ &\quad + \theta([x_1,x_2,y_1],\alpha^2(y_2))\circ\beta^2 + \theta(\alpha^2(y_1),[x_1,x_2,y_2])\circ\beta^2.\end{aligned} \tag{4.16}$$

By the equality

$$\begin{aligned}[\beta^2(u),\hat{\alpha}^2\circ\sigma(x_1),[\sigma(y_1),\sigma(y_2),\sigma(y_3)]_{\hat{T}}]_{\hat{T}} &= [[u,\sigma(x_1),\sigma(y_1)]_{\hat{T}},\hat{\alpha}^2\circ\sigma(y_2),\hat{\alpha}^2\circ\sigma(y_3)]_{\hat{T}} \\ &\quad + [\hat{\alpha}^2\circ\sigma(y_1),[u,\sigma(x_1),\sigma(y_2)]_{\hat{T}},\hat{\alpha}^2\circ\sigma(y_3)]_{\hat{T}} \\ &\quad + [\hat{\alpha}^2\circ\sigma(y_1),\hat{\alpha}^2\circ\sigma(y_2),[u,\sigma(x_1),\sigma(y_3)]_{\hat{T}}]_{\hat{T}},\end{aligned}$$

we have (HR62):

$$\begin{aligned}\theta(\alpha^2(x_1),[y_1,y_2,y_3])\circ\beta^2 &= \theta(\alpha^2(y_2),\alpha^2(y_3))\theta(x_1,y_1) \\ &\quad - \theta(\alpha^2(y_1),\alpha^2(y_3))\theta(x_1,y_2) + D(\alpha^2(y_1),\alpha^2(y_2))\theta(x_1,y_3).\end{aligned} \tag{4.17}$$

Therefore we see that (ρ, D, θ) is a representation of T on V.

Finally, suppose that $\mathrm{E}_{\hat{T}}$ and $\mathrm{E}_{\tilde{T}}$ are equivalent abelian extensions, and $F:\hat{T}\to\tilde{T}$ is a HLYA homomorphism satisfying $F\circ i = j$, $q\circ F = p$. Choose linear sections σ and σ' of p and q. We get $qF\sigma(x_i) = p\sigma(x_i) = x_i = q\sigma'(x_i)$, so $F\sigma(x_i) - \sigma'(x_i) \in \mathrm{Ker}(q) \cong V$. Thus, we have

$$[u,\sigma(x_1),\sigma(x_2)]_{\hat{T}} = [u,F\sigma(x_1),F\sigma(x_2)]_{\tilde{T}} = [u,\sigma'(x_1),\sigma'(x_2)]_{\tilde{T}}.$$

Therefore, equivalent abelian extensions give the same θ. Similarly, one can prove that equivalent abelian extensions give the same D and ρ. ∎

Let $\sigma : T \to \hat{T}$ be a section of an abelian extension. Define

$$\nu(x_1, x_2) \triangleq [\sigma(x_1), \sigma(x_2)]_{\hat{T}} - \sigma([x_1, x_2]), \tag{4.18}$$

$$\omega(x_1, x_2, x_3) \triangleq [\sigma(x_1), \sigma(x_2), \sigma(x_3)]_{\hat{T}} - \sigma([x_1, x_2, x_3]). \tag{4.19}$$

LEMMA 4.4. *Let $0 \to V \to \hat{T} \to T \to 0$ be an abelian extension of T by V. Then (ν, ω) defined by* (4.18) *and* (4.19) *is a* $(2,3)$*-cocycle of T with coefficients in V.*

Proof. First, we claim that the image of ν is contained in V, that is, $p \circ \nu(x_1, x_2) = 0$. In fact, since p is an algebraic homomorphism, we have

$$p \circ \nu(x_1, x_2) = [p \circ \sigma(x_1), p \circ \sigma(x_2)]_{\hat{T}} - p \circ \sigma([x_1, x_2]) = 0.$$

Next, one checks that ν and ω defined above satisfy (CC01) and (CC02). For example

$$\begin{aligned}
\nu(\alpha(x_1), \alpha(x_2)) &= [\sigma(\alpha(x_1)), \sigma(\alpha(x_2))]_{\hat{T}} - \sigma([\alpha(x_1), \alpha(x_2)]) \\
&= [\sigma \circ \alpha(x_1), \sigma \circ \alpha(x_2)]_{\hat{T}} - \sigma \circ \alpha([x_1, x_2]) \\
&= ([\hat{\alpha} \circ \sigma(x_1), \hat{\alpha} \circ \sigma(x_2)]_{\hat{T}}) - \hat{\alpha} \circ \sigma([x_1, x_2]) \\
&= \hat{\alpha}([\sigma(x_1), \sigma(x_2)]_{\hat{T}}) - \sigma([x_1, x_2])) \\
&= \beta([\sigma(x_1), \sigma(x_2)]_{\hat{T}} - \sigma([x_1, x_2])) = \beta(\nu(x_1, x_2)).
\end{aligned}$$

Finally, we verify that ν and ω satisfy (CC1)–(CC4).

By the equality

$$[\sigma x_1, \sigma x_2, \sigma x_3]_{\hat{T}} + \text{c.p.} + [[\sigma x_1, \sigma x_2]_{\hat{T}}, \hat{\alpha}(\sigma(x_3))]_{\hat{T}} + \text{c.p.} = 0,$$

we obtain

$$\begin{aligned}
\{\omega([x_1, x_2, x_3) + \sigma[x_1, x_2, x_3]_{\hat{T}}\} + \text{c.p.} + \{[\nu(x_1, x_2), \sigma(\alpha(x_3))]_{\hat{T}} \\
+ \nu([x_1, x_2], \alpha(x_3)) + \sigma([[x_1, x_2], \alpha(x_3)])\} + \text{c.p.} = 0.
\end{aligned}$$

Thus we have (CC1):

$$\begin{aligned}
\omega(x_1, x_2, x_3) + \text{c.p.} - \rho(\alpha(x_3))\nu(x_1, x_2) - \text{c.p.} \\
+ \nu([x_1, x_2], \alpha(x_3)) + \text{c.p.} = 0.
\end{aligned} \tag{4.20}$$

By the equality

$$[[\sigma x_1, \sigma x_2]_{\hat{T}}, \hat{\alpha}(\sigma x_3), \hat{\alpha}(\sigma y_1)]_{\hat{T}} + \text{c.p.} = 0,$$

we get

$$\begin{aligned}
\{[\nu(x_1, x_2), \sigma\alpha(x_3), \sigma\alpha(y_1)]_{\hat{T}} + \omega([x_1, x_2], \alpha(x_3), \alpha(y_1)) \\
+ \sigma[[x_1, x_2], \alpha(x_3), \alpha(y_1)]\} + \text{c.p.} = 0.
\end{aligned}$$

Thus we have (CC2):

$$\theta(\alpha(x_3),\alpha(y_1))\nu(x_1,x_2)+\text{c.p.}+\omega([x_1,x_2],\alpha(x_3),\alpha(y_1))+\text{c.p.}=0. \tag{4.21}$$

We have the equality

$$\begin{aligned}&[\hat{\alpha}(\sigma x_1),\hat{\alpha}(\sigma x_2),[\sigma y_1,\sigma y_2]_{\hat{T}}]_{\hat{T}}\\&\qquad=[[\sigma x_1,\sigma x_2,\sigma y_1]_{\hat{T}},\hat{\alpha}^2(\sigma y_2)]_{\hat{T}}+[\hat{\alpha}^2(\sigma y_1),[\sigma x_1,\sigma x_2,\sigma y_2]_{\hat{T}}]_{\hat{T}}.\end{aligned}$$

The left hand side is equal to

$$\begin{aligned}&[\sigma\alpha(x_1),\sigma\alpha(x_2),[\sigma y_1,\sigma y_2,]_{\hat{T}}]_{\hat{T}}\\&\qquad=[\sigma\alpha(x_1),\sigma\alpha(x_2),\nu(y_1,y_2)+\sigma([y_1,y_2]_T)]_{\hat{T}}\\&\qquad=D(\alpha(x_1),\alpha(x_2))\nu(y_1,y_2)+[\sigma\alpha(x_1),\sigma\alpha(x_2),\sigma([y_1,y_2])]_{\hat{T}}\\&\qquad=D(\alpha(x_1),\alpha(x_2))\nu(y_1,y_2)+\omega(\alpha(x_1),\alpha(x_2),[y_1,y_2])\\&\qquad\quad+\sigma([\alpha(x_1),\alpha(x_2),[y_1,y_2]]),\end{aligned}$$

and similarly the right hand side is equal to

$$\begin{aligned}&[[\sigma x_1,\sigma x_2,\sigma y_1]_{\hat{T}},\sigma\alpha^2(y_2)]_{\hat{T}}+[\sigma\alpha^2(y_1),[\sigma x_1,\sigma x_2,\sigma y_2]_{\hat{T}}]_{\hat{T}}\\&\quad=[\omega(x_1,x_2,y_1)+\sigma[x_1,x_2,y_1],\sigma y_2]_{\hat{T}}+\nu([x_1,x_2,y_1],y_2)+\sigma[[x_1,x_2,y_1],y_2]\\&\quad=-\rho(\alpha^2(y_2))\omega(x_1,x_2,y_1)+\nu([x_1,x_2,y_1],\alpha^2(y_2))+\sigma([[x_1,x_2,y_1],\alpha^2(y_2)])\\&\qquad+\rho(\alpha^2(y_1))\omega(x_1,x_2,y_2)+\nu(\alpha^2(y_1),[x_1,x_2,y_2])+\sigma([\alpha^2(y_1),[x_1,x_2,y_2]]).\end{aligned}$$

Thus we obtain (CC3):

$$\begin{aligned}&\omega(\alpha(x_1),\alpha(x_2),[y_1,y_2])+D(\alpha(x_1),\alpha(x_2))\omega(y_1,y_2)\\&\qquad=\nu([x_1,x_2,y_1],\alpha^2(y_2))+\nu(\alpha^2(y_1),[x_1,x_2,y_2]))\\&\qquad\quad+\rho(\alpha^2(y_1))\omega(x_1,x_2,y_2)-\rho(\alpha^2(y_2))\omega(x_1,x_2,y_1).\end{aligned} \tag{4.22}$$

We have the equality

$$\begin{aligned}&[\hat{\alpha}^2(\sigma x_1),\hat{\alpha}^2(\sigma x_2),[\sigma y_1,\sigma y_2,\sigma y_3]_{\hat{T}}]_{\hat{T}}\\&\qquad=[[\sigma x_1,\sigma x_2,\sigma y_1]_{\hat{T}},\hat{\alpha}^2(\sigma y_2),\hat{\alpha}^2(\sigma y_3)]_{\hat{T}}\\&\qquad\quad+[\hat{\alpha}^2(\sigma y_1),[\sigma x_1,\sigma x_2,\sigma y_2]_{\hat{T}},\hat{\alpha}^2(\sigma y_3)]_{\hat{T}}\\&\qquad\quad+[\hat{\alpha}^2(\sigma y_1),\hat{\alpha}^2(\sigma y_2),[\sigma x_1,\sigma x_2,\sigma y_3]_{\hat{T}}]_{\hat{T}}.\end{aligned}$$

The left hand side is equal to

$$\begin{aligned}&[\sigma\alpha^2(x_1),\sigma\alpha^2(x_2),[\sigma y_1,\sigma y_2,\sigma y_3]_{\hat{T}}]_{\hat{T}}\\&\quad=[\sigma\alpha^2(x_1),\sigma\alpha^2(x_2),\omega(y_1,y_2,y_3)+\sigma([y_1,y_2,y_3])]_{\hat{T}}\\&\quad=D(\alpha^2(x_1),\alpha^2(x_2))\omega(y_1,y_2,y_3)+[\sigma\alpha^2(x_1),\sigma\alpha^2(x_2),\sigma([y_1,y_2,y_3])]_{\hat{T}}\\&\quad=D(\alpha^2(x_1),\alpha^2(x_2))\omega(y_1,y_2,y_3)+\omega(\alpha^2(x_1),\alpha^2(x_2),[y_1,y_2,y_3])\\&\qquad+\sigma([\alpha^2(x_1),\alpha^2(x_2),[y_1,y_2,y_3]]).\end{aligned}$$

Similarly, the right hand side is equal to

$$\begin{aligned}&\theta(\alpha^2(y_2),\alpha^2(y_3))\omega(x_1,x_2,y_1)+\omega([x_1,x_2,y_1],\alpha^2(y_2),\alpha^2(y_3))\\&\quad+\sigma([[x_1,x_2,y_1],\alpha^2(y_2),\alpha^2(y_3)])-\theta(\alpha^2(y_1),\alpha^2(y_3))\omega(x_1,x_2,y_2)\\&\quad+\omega(\alpha^2(y_1),[x_1,x_2,y_2],\alpha^2(y_3))+\sigma([\alpha^2(y_1),[x_1,x_2,y_2],\alpha^2(y_3)])\\&\quad+D(\alpha^2(y_1),\alpha^2(y_2))\omega(x_1,x_2,y_3)+\omega(\alpha^2(y_1),\alpha^2(y_2),[x_1,x_2,y_3])\\&\quad+\sigma([\alpha^2(y_1),\alpha^2(y_2),[x_1,x_2,y_3]]).\end{aligned}$$

Thus we obtain (CC4):

$$\begin{aligned}&\omega(\alpha^2(x_1),\alpha^2(x_2),[y_1,y_2,y_3])+D(\alpha^2(x_1),\alpha^2(x_2))\omega(y_1,y_2,y_3)\\&\quad=\omega([x_1,x_2,y_1],\alpha^2(y_2),\alpha^2(y_3))+\omega(\alpha^2(y_1),[x_1,x_2,y_2],\alpha^2(y_3))\\&\qquad+\omega(\alpha^2(y_1),\alpha^2(y_2),[x_1,x_2,y_3])+\theta(\alpha^2(y_2),\alpha^2(y_3))\omega(x_1,x_2,y_1)\\&\qquad-\theta(\alpha^2(y_1),\alpha^2(y_3))\omega(x_1,x_2,y_2)+D(\alpha^2(y_1),\alpha^2(y_2))\omega(x_1,x_2,y_3).\end{aligned}\tag{4.23}$$

Therefore we get all the $(2,3)$-cocycle conditions of Definition 2.9. ∎

From Lemmas 4.3 and 4.4, we see that an abelian extensiona of HLYA T by V gives rise to a representation of T on V and $(2,3)$-cocycle of T with coefficients in V. Conversely, given a representation and a $(2,3)$-cocycle, we can obtain a HLYA structure on the space $T\oplus V$.

LEMMA 4.5. *Let T be a HLYA, (ρ,D,θ) a representation of T on V, and (ν,ω) a $(2,3)$-cocycle of T with coefficients in V. Then $T\oplus V$ is a HLYA under the following bilinear and trilinear maps:*

$$(\alpha+\beta)(x_1+u_1)\triangleq\alpha(x_1)+\beta(u_1),\tag{4.24}$$

$$\begin{aligned}[x_1+u_1,x_2+u_2]_\nu\triangleq{}&[x_1,x_2]+\nu(x_1,x_2)+\rho(x_1)(u_2)\\&-\rho(x_2)(u_1),\end{aligned}\tag{4.25}$$

$$\begin{aligned}[x_1+u_1,x_2+u_2,x_3+u_3]_\omega\triangleq{}&[x_1,x_2,x_3]+\omega(x_1,x_2,x_3)\\&+D(x_1,x_2)(u_3)-\theta(x_1,x_3)(u_2)\\&+\theta(x_2,x_3)(u_1),\end{aligned}\tag{4.26}$$

This HLYA is denoted by $E_{(\nu,\omega)}=T\oplus_{(\nu,\omega)}V$.

Proof. We will verify that conditions (HLY01)–(HLY02) and (HLY1)–(HLY6) hold for maps defined on $T\oplus V$ by (4.24)–(4.26).

Now condition (HLY01) becomes

$$\begin{aligned}&(\alpha+\beta)([x_1+u_1,x_2+u_2])\\&\qquad=[(\alpha+\beta)(x_1+u_1,),(\alpha+\beta)(x_2+u_2)].\end{aligned}\tag{4.27}$$

The left hand is equal to

$$(\alpha+\beta)([x_1,x_2]+\nu(x_1,x_2)+\rho(x_1)(u_2)-\rho(x_2)(u_1))$$
$$=\alpha([x_1,x_2])+\beta\circ\nu(x_1,x_2)+\beta\circ\rho(x_1)(u_2)-\beta\circ\rho(x_2)(u_1),$$

and the right hand is equal to

$$[\alpha(x_1)+\beta(u_1),\alpha(x_2)+\beta(u_2)]$$
$$=[\alpha(x_1),\alpha(x_2)]+\nu(\alpha(x_1),\alpha(x_2))+\rho(\alpha(x_1))\circ\beta(u_2)-\rho(\alpha(x_2))\circ\beta(u_1).$$

Since α is an algebraic homomorphism, and by conditions (HR01), (CC01), we obtain (4.27).

Condition (HLY02) becomes

$$(4.28)\qquad (\alpha+\beta)([x_1+u_1,x_2+u_2,x_3+u_3])$$
$$=[(\alpha+\beta)(x_1+u_1),(\alpha+\beta)(x_2+u_2),(\alpha+\beta)(x_3+u_3)].$$

The left hand is equal to

$$(\alpha+\beta)([x_1,x_2,x_3]+D(x_1,x_2)(u_3)-\theta(x_1,x_3)(u_2)+\theta(x_2,x_3)(u_1))$$
$$=\alpha([x_1,x_2,x_3])+\beta\circ\omega(x_1,x_2,x_3)+\beta\circ D(x_1,x_2)(u_3)-\beta\circ\theta(x_1,x_3)(u_2)$$
$$+\beta\circ\theta(x_2,x_3)(u_1),$$

and the right hand is equal to

$$[\alpha(x_1)+\beta(u_1),\alpha(x_2)+\beta(u_2),\alpha(x_3)+\beta(u_3)]$$
$$=[\alpha(x_1),\alpha(x_2),\alpha(x_3)]+\omega(\alpha(x_1),\alpha(x_2),\alpha(x_3))+D(\alpha(x_1),\alpha(x_2))\circ\beta(u_3)$$
$$-\theta(\alpha(x_1),\alpha(x_3))\circ\beta(u_2)+\theta(\alpha(x_2),\alpha(x_3))\circ\beta(u_1).$$

Since α is an algebraic homomorphism, and by conditions (HR02), (CC02), we obtain (4.28).

For (HLY1) and (HLY2), by definition we have

$$[x_1+u_1,x_1+u_1]=[x_1,x_1]+\rho(x_1)(u_1)-\rho(x_1)(u_1)=0,$$
$$[x_1+u_1,x_1+u_1,x_3+u_3]=[x_1,x_1,x_3]+D(x_1,x_1)(u_3)-\theta(x_1,x_3)(u_1)$$
$$+\theta(x_1,x_3)(u_1)=0.$$

For (HLY3), we have

$$[x_1+u_1,x_2+u_2,x_3+u_3]_\omega+\text{c.p.}$$
$$=\{[x_1,x_2,x_3]+\underline{\omega(x_1,x_2,x_3)}+D(x_1,x_2)(u_3)-\theta(x_1,x_3)(u_2)$$
$$+\theta(x_2,x_3)(u_1)\}+\text{c.p.}$$

and

$$
\begin{aligned}
&[[x_1+u_1,x_2+u_2]_\nu,\alpha(x_3)+\beta(u_3)]_\nu+\text{c.p.}\\
&\quad=[[x_1,x_2]+\rho(x_1)(u_2)-\rho(x_2)(u_1),\alpha(x_3)+\beta(u_3)]_\nu+\text{c.p.}\\
&\quad=\{[[x_1,x_2],\alpha(x_3)]+\underline{\nu([x_1,x_2],x_3)}+\rho([x_1,x_2])\circ\beta(u_3)\\
&\qquad\underline{-\rho(\alpha(x_3))\nu(x_1,x_2)}-\rho(\alpha(x_3))\rho(x_1)(u_2)+\rho(\alpha(x_3))\rho(x_2)(u_1)\}+\text{c.p.}
\end{aligned}
$$

Thus by (HR31) and (CC1) we obtain

$$
\begin{aligned}
\text{(4.29)}\qquad &[x_1+u_1,x_2+u_2,x_3+u_3]_\omega+\text{c.p.}\\
&\qquad+[x_1+u_1,x_2+u_2]_\nu,\alpha(x_3)+\beta(u_3)]_\nu+\text{c.p.}=0.
\end{aligned}
$$

For (HLY4), we have

$$
\begin{aligned}
&[[x_1+u_1,x_2+u_2]_\nu,\alpha(x_3)+\beta(u_3),\alpha(y_1)+\beta(v_1)]_\omega+\text{c.p.}\\
&\quad=\{[[x_1,x_2],\alpha(x_3),\alpha(y_1)]+\underline{\omega([x_1,x_2],\alpha(x_3),\alpha(y_1))}\\
&\qquad+D([x_1,x_2],\alpha(x_3))(\beta(v_1))-\theta([x_1,x_2],\alpha(y_1))(\beta(u_3))\\
&\qquad+\underline{\theta(\alpha(x_3),\alpha(y_1))(\nu(x_1,x_2)}+\rho(x_1)(u_2)-\rho(x_2)(u_1))\}+\text{c.p.}=0,
\end{aligned}
$$

where the last equality follows from (CC2), (HR41) and (HR42).

For (HLY5), we have

$$
\begin{aligned}
&[\alpha(x_1)+\beta(u_1),\alpha(x_2)+\beta(u_2),[y_1+v_1,y_2+v_2]_\nu]_\omega\\
&\qquad=[\alpha(x_1),\alpha(x_2),[y_1,y_2]]+\underline{\omega(\alpha(x_1),\alpha(x_2),[y_1,y_2])}\\
&\qquad\quad+\underline{D(\alpha(x_1),\alpha(x_2))\big(\nu(y_1,y_2)}+\rho(y_1)(v_2)-\rho(y_2)(v_1)\big)\\
&\qquad\quad-\theta(x_1,[y_1,y_2])(u_2)+\theta(x_2,[y_1,y_2])(u_1)
\end{aligned}
$$

and

$$
\begin{aligned}
&[[x_1+u_1,x_2+u_2,y_1+v_1]_\omega,\alpha^2(y_2)+\beta^2(v_2)]_\nu\\
&\qquad+[\alpha^2(y_1)+\beta^2(v_1),[x_1+u_1,x_2+u_2,y_2+v_2]_\omega]_\nu\\
&\quad=[[x_1,x_2,y_1],\alpha^2(y_2)]+\underline{\nu([x_1,x_2,y_1],\alpha^2(y_2))}+\rho([x_1,x_2,y_1])(v_2)\\
&\qquad\underline{-\rho(y_2)\big(\omega(x_1,x_2,y_1)}+D(x_1,x_2)(v_1)-\theta(x_1,y_1)(u_2)+\theta(x_2,y_1)(u_1)\big)\\
&\qquad+[\alpha^2(y_1),[x_1,x_2,y_2]]+\underline{\nu(\alpha^2(y_1),[x_1,x_2,y_2])}\\
&\qquad+\underline{\rho(y_1)\big(\omega(x_1,x_2,y_2)}+D(x_1,x_2)(v_2)-\theta(x_1,y_2)(u_2)+\theta(x_2,y_2)(u_1)\big)\\
&\qquad-\rho([x_1,x_2,y_2])(v_1).
\end{aligned}
$$

Thus by (CC3), (HR51) and (HR52) we obtain

$$
\begin{aligned}
&[\alpha(x_1)+\beta(u_1),\alpha(x_2)+\beta(u_2),[y_1+v_1,y_2+v_2]_\nu]_\omega \\
&\qquad = [[x_1+u_1,x_2+u_2,y_1+v_1]_\omega,\alpha^2(y_2)+\beta^2(v_2)]_\nu \\
&\qquad\quad + [\alpha^2(y_1)+\beta^2(v_1),[x_1+u_1,x_2+u_2,y_2+v_2]_\omega]_\nu
\end{aligned}
$$

Therefore (HLY5) is valid.

Now it remains to verify (HLY6). By definition,

$$
\begin{aligned}
&[\alpha^2(x_1)+\beta^2(u_1),\alpha^2(x_2)+\beta^2(u_2),[y_1+v_1,y_2+v_2,y_3+v_3]] \\
&\quad = [\alpha^2(x_1),\alpha^2(x_2),[y_1,y_2,y_3]] + \underline{\omega(\alpha^2(x_1),\alpha^2(x_2),[y_1,y_2,y_3])} \\
&\quad\quad - \theta(\alpha^2(x_1),[y_1,y_2,y_3])(\beta^2(u_2)) + \theta(\alpha^2(x_2),[y_1,y_2,y_3])(\beta^2(u_1)) \\
&\quad\quad + \underline{D(\alpha^2(x_1),\alpha^2(x_2))\big(\omega(y_1,y_2,y_3)} \\
&\quad\quad\quad + D(y_1,y_2)(v_3) - \theta(y_1,y_3)(v_2) + \theta(y_2,y_3)(v_1)\big),
\end{aligned}
$$

$$
\begin{aligned}
&[[x_1+u_1,x_2+u_2,y_1+v_1],\alpha^2(y_2)+\beta^2(v_2),\alpha^2(y_3)+\beta^2(v_3)] \\
&\quad = [[x_1,x_2,y_1],\alpha^2(y_2),\alpha^2(y_3)]] + \underline{\omega([x_1,x_2,y_1],\alpha^2(y_2),\alpha^2(y_3)])} \\
&\quad\quad - D([x_1,x_2,y_1],\alpha^2(y_2))(\beta^2(v_3)) + \theta([x_1,x_2,y_1],\alpha^2(y_3))(\beta^2(u_1)) \\
&\quad\quad + \underline{\theta(\alpha^2(y_2),\alpha^2(y_3))\big(\omega(x_1,x_2,y_1)} \\
&\quad\quad\quad + D(x_1,x_2)(v_1) - \theta(x_1,y_1)(u_2) + \theta(x_2,y_1)(u_1)\big),
\end{aligned}
$$

$$
\begin{aligned}
&[\alpha^2(y_1)+\beta^2(v_1),[x_1+u_1,x_2+u_2,y_2+v_2],\alpha^2(y_3)+\beta^2(v_3)] \\
&\quad = [\alpha^2(y_1),[x_1,x_2,y_2],\alpha^2(y_3)] + \omega(\alpha^2(y_1),[x_1,x_2,y_2],\alpha^2(y_3)) \\
&\quad\quad + D(\alpha^2(y_1),[x_1,x_2,y_2])(\beta^2(v_3)) + \theta([x_1,x_2,y_2],\alpha^2(y_3))(\beta^2(v_1)) \\
&\quad\quad \underline{-\theta(\alpha^2(y_1),\alpha^2(y_3))\big(\omega(x_1,x_2,y_2)} \\
&\quad\quad\quad + D(x_1,x_2)(v_2) - \theta(x_1,y_2)(u_2) + \theta(x_2,y_2)(u_1)\big),
\end{aligned}
$$

$$
\begin{aligned}
&[\alpha^2(y_1)+\beta^2(v_1),\alpha^2(y_2)+\beta^2(v_2),[x_1+u_1,x_2+u_2,\alpha^2(y_3)+\beta^2(v_3)]] \\
&\quad = [y_1+v_1,y_2+v_2,[x_1,x_2,y_3]+D(x_1,x_2)(v_3)-\theta(x_1,y_3)(u_2)+\theta(x_2,y_3)(u_1)] \\
&\quad = [\alpha^2(y_1)+\alpha^2(y_2),[x_1,x_2,y_3]] + \underline{\omega(\alpha^2(y_1)+\alpha^2(y_2),[x_1,x_2,y_3])} \\
&\quad\quad - \theta(\alpha^2(y_1),[x_1,x_2,y_3])(\beta^2(v_2)) + \theta(\alpha^2(y_2),[x_1,x_2,y_3])(\beta^2(v_1)) \\
&\quad\quad + \underline{D(\alpha^2(y_1),\alpha^2(y_2))\big(\omega(x_1,x_2,y_3)} \\
&\quad\quad\quad + D(x_1,x_2)(v_3) - \theta(x_1,y_3)(u_2) + \theta(x_2,y_3)(u_1)\big).
\end{aligned}
$$

It follows that

$$\begin{aligned}
&[\alpha^2(x_1)+\beta^2(u_1),\alpha^2(x_2)+\beta^2(u_2),[y_1+v_1,y_2+v_2,y_3+v_3]]\\
&\quad=[[x_1+u_1,x_2+u_2,y_1+v_1],\alpha^2(y_2)+\beta^2(v_2),\alpha^2(y_3)+\beta^2(v_3)]\\
&\qquad+[\alpha^2(y_1)+\beta^2(v_1),[x_1+u_1,x_2+u_2,y_2+v_2],y_3+v_3]\\
&\qquad+[\alpha^2(y_1)+\beta^2(v_1),\alpha^2(y_2)+\beta^2(v_2),[x_1+u_1,x_2+u_2,\alpha^2(y_3)+\beta^2(v_3)]]
\end{aligned}$$

by (CC4), (HR61) and (HR62). Therefore we obtain a HLYA on $T\oplus V$ under the maps (4.24)–(4.26). The proof is complete. ∎

LEMMA 4.6. *Two abelian extensions of HLYAs* $0\to V\to T\oplus_{(\nu,\omega)}V\to T\to 0$ *and* $0\to V\to T\oplus_{(\nu',\omega')}V\to T\to 0$ *are equivalent if and only if* (ν,ω) *and* (ν',ω') *are in the same cohomology class.*

Proof. Assume the two extensions are equivalent, and let $F:T\oplus_{(\nu,\omega)}V\to T\oplus_{(\nu',\omega')}V$ be the corresponding homomorphism. Then

$$F[x_1,x_2]_\nu=[F(x_1),F(x_2)]_{\nu'}, \tag{4.30}$$

$$F[x_1,x_2,x_3]_\omega=[F(x_1),F(x_2),F(x_3)]_{\omega'}. \tag{4.31}$$

Since F is an equivalence of extensions, there exists $f:T\to V$ such that

$$F(x_i+u)=x_i+f(x_i)+u,\quad \forall x_i\in T.$$

Now equality (4.30) reads

$$\begin{aligned}
&[x_1,x_2]+f([x_1,x_2])+\nu(x_1,x_2)\\
&\qquad=[x_1,x_2]+\nu'(x_1,x_2)+\rho(x_1)f(x_2)-\rho(x_2)f(x_1).
\end{aligned}$$

Thus

$$(\nu-\nu')(x_1,x_2)=\rho(x_1)f(x_2)-\rho(x_2)f(x_1)-f([x_1,x_2]). \tag{4.32}$$

Equality (4.31) is equivalent to

$$\begin{aligned}
&[x_1,x_2,x_3]+\omega(x_1,x_2,x_3)+f([x_1,x_2,x_3])\\
&\qquad=[x_1,x_2,x_3]+\omega'(x_1,x_2,x_3)\\
&\qquad\quad+D(x_1,x_2)f(x_3)-\theta(x_1,x_3)f(x_2)+\theta(x_2,x_3)f(x_1).
\end{aligned}$$

Thus

$$\begin{aligned}
(\omega-\omega')(x_1,x_2,x_3)&=D(x_1,x_2)f(x_3)-\theta(x_1,x_3)f(x_2)\\
&\quad+\theta(x_2,x_3)f(x_1)-f([x_1,x_2,x_3]).
\end{aligned} \tag{4.33}$$

Therefore (ν,ω) and (ν',ω') are in the same cohomology class. Conversely, if (ν,ω) and (ν',ω') are in the same cohomology class, then we can show that F is an equivalence. We omit the details. ∎

Finally, we obtain the main result of this section:

THEOREM 4.7. *Let T be a HLYA and V a T-module. Then there is a one-to-one correspondence between the set of equivalence classes of abelian extensions of the HLYA and the $(2,3)$-cohomology group. More precisely, there is a bijection*

$$\mathrm{Ext}(T,V) \to H^2(T,V) \times H^3(T,V).$$

Therefore, the abelian extensions of T by V are classified by the $(2,3)$-cohomology group.

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