ACTA ARITHMETICA
178.1 (2017)

Non-real poles on the axis of absolute convergence of the zeta functions associated to Pascal's triangle modulo a prime

by

TOMOHIRO IKKAI (Nagoya)

1. Introduction. Let $\mathbb{N} = \{1, 2, \ldots\}$ and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. The set

$$\mathcal{T}_{\mathbb{N}_0} = \{(m, n) \in \mathbb{N}_0 \times \mathbb{N}_0 \,;\, m \geq n\}$$

can be regarded as Pascal's triangle in the sense that each point $(m, n) \in \mathcal{T}_{\mathbb{N}_0}$ corresponds to the binomial coefficient $\binom{m}{n} = m!/(n!(m-n)!)$. In this paper, we prefer to use $\mathcal{T}_{\mathbb{N}_0}$ instead of the familiar form of Pascal's triangle.

For a prime number p, we consider the set

$$\mathrm{Pas}(p) = \left\{(m, n) \in \mathcal{T}_{\mathbb{N}_0} \,;\, \binom{m}{n} \not\equiv 0 \pmod{p}\right\};$$

the distribution of its points has been studied for a long time. It is classically known that $\mathrm{Pas}(p)$ has a certain "self-similarity" as seen in Figures 1 and 2, where each filled-in point belongs to $\mathrm{Pas}(p)$. Kummer [7] gave a criterion for a power of p to divide a binomial coefficient in terms of the expansion of non-negative integers in base p. We can determine which point $(m, n) \in \mathcal{T}_{\mathbb{N}_0}$ belongs to $\mathrm{Pas}(p)$ by Kummer's criterion, or by Lucas' formula for binomial coefficients appearing in Section 4, which makes the argument simpler. In fact, such an arithmetic property causes $\mathrm{Pas}(p)$ to have "self-similarity."

Essouabri [1] defines a zeta function associated to $\mathrm{Pas}(p)$ by

$$Z_p(P, Q; s) = \sum_{\substack{(m,n)\in \mathrm{Pas}(p) \\ P(m,n)\neq 0}} \frac{Q(m, n)}{P(m, n)^{s/\deg P}}, \tag{1.1}$$

where $s = \sigma + it \in \mathbb{C}$ with sufficiently large real part σ and imaginary part t,

2010 *Mathematics Subject Classification*: Primary 11M41; Secondary 11B65.

Key words and phrases: Pascal's triangle, binomial coefficients, Dirichlet series, zeta function of fractal strings.

Received 19 November 2015; revised 1 September 2016.
Published online 6 March 2017.

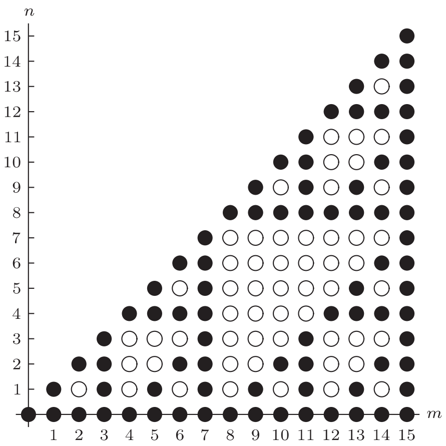

Fig. 1. $0 \le m < 16$ in Pas(2)

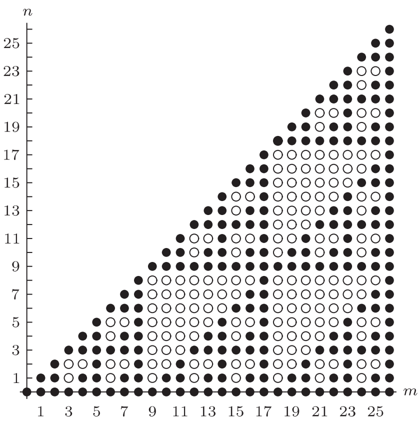

Fig. 2. $0 \le m < 27$ in Pas(3)

and P, Q are two-variable polynomials with real coefficients. Moreover, P is required to be "$\mathcal{T}$-elliptic" to ensure the convergence of (1.1); $P \in \mathbb{R}[X,Y]$ is said to be *$\mathcal{T}$-elliptic* if the highest degree part P_* of P (which means that P_* is homogeneous and $\deg(P - P_*) < \deg P$) satisfies $P_*(x,y) > 0$ for all $(x,y) \in \mathbb{R}^2$ with $x \ge y \ge 0$ and $(x,y) \ne (0,0)$.

Essouabri proved several analytic properties of $Z_p(P,Q;s)$ including meromorphic continuation, the location of possible poles and the abscissa of absolute convergence. In particular, for $P(X,Y) = X$, he proved the following intriguing fact.

THEOREM 1.1 (Essouabri, 2005 [1]). *Let $\theta_p = \log \frac{p(p+1)}{2} / \log p$. Then the function $Z_p(X,1;s)$, meromorphic on the whole complex plane, has at least two non-real poles on its axis of absolute convergence, $\{\sigma = \theta_p\}$ (the vertical line through the abscissa of absolute convergence).*

Indeed, we only have to verify that $Z_p(X,1;s)$ has one such pole since the other can be obtained by using the Schwarz reflection principle.

A key point in Essouabri's proof of Theorem 1.1 is to estimate how fast the points in Pas(p) multiply. More precisely, he uses certain estimates for the arithmetic function $N_p^*(u) = \#\{(m,n) \in \mathrm{Pas}(p)\,;\, m < u\}$ defined for $u \in \mathbb{N}$. Actually,

$$Z_p(X,1;s) = \int_{1^-}^{\infty} u^{-s}\, dN_p^*(\lceil u \rceil)$$

for $s \in \mathbb{C}$ at which $Z_p(X,1;s)$ converges absolutely, and the Wiener–Ikehara theorem (see [6, Chapter 5, Corollary 1] for example) would give the existence of $\lim_{u\to\infty} N_p^*(u)/u^{\theta_p}$ if $Z_p(X,1;s)$ had no non-real poles on the line in question. But this contradicts the known results on $\limsup_{u\to\infty} N_p^*(u)/u^{\theta_p}$

and $\liminf_{u\to\infty} N_p^*(u)/u^{\theta_p}$, due to Harborth [4], Stolarsky [11], Stein [10] and Wilson [12]. (Details will appear in Section 2.)

The purpose of this article is to show results parallel to Theorem 1.1 for $P(X,Y) = X + Y$ and for $P(X,Y) = X + 2Y$, $p = 2$:

THEOREM 1.2.

(a) *For any prime p, the meromorphic function $Z_p(X+Y, 1; s)$ has a non-real pole* (*hence at least two non-real poles*) *on its axis of absolute convergence $\{\sigma = \theta_p\}$.*
(b) *The meromorphic function $Z_2(X+2Y, 1; s)$ has a non-real pole* (*hence at least two non-real poles*) *on its axis of absolute convergence $\{\sigma = \theta_2\}$.*

It is significant to consider $Z_p(P, 1; s)$ in the context of fractal geometry. It can be realized as the geometric zeta function of a certain fractal string with a scale transformation of s. Such a zeta function contains some geometric information of the fractal string in its poles, which are called the complex dimensions. In particular, of importance is the relationship between the existence of non-real poles on the axis of absolute convergence of the geometric zeta function and the Minkowski measurability of the corresponding fractal string, which has some connection with the zeros of the Riemann zeta function in the critical strip via spectrum theory. (Minkowski measurability is one of the geometric properties of fractal strings. For details, see [8] for example.)

We will prove Theorem 1.2 by imitating Essouabri's argument. We investigate the behavior of two functions connected with the desired poles of $Z_p(P, 1; s)$:

$$N_p(P; u) = \#\{(m,n) \in \mathrm{Pas}(p)\,;\, P(m,n) < u\},$$
$$\psi_p(P; u) = N_p(P; u)/u^{\theta_p/d},$$

where $P \in \mathbb{Z}[X,Y]$ is a $\mathcal{T}$-elliptic polynomial of degree $d \geq 1$ and $u \in \mathbb{N}$. Of course the former is an analog of $N_p^*(u)$ and the latter corresponds to $N_p^*(u)/u^{\theta_p}$ appearing above.

The following theorem shows why we consider the function $\psi_p(P; u)$.

THEOREM 1.3. *Let $P \in \mathbb{R}[X,Y]$ be a $\mathcal{T}$-elliptic polynomial of degree $d \geq 1$. Moreover, assume that P is $\mathcal{T}$-positive in the sense that $P(x,y) \geq 0$ for every $(x,y) \in \mathbb{R}^2$ with $x \geq y \geq 0$. Then $N_p(P; u) \asymp u^{\theta_p/d}$ as $u \to \infty$.*

We give some necessary estimates for $\psi_p(P; u)$ to deduce Theorem 1.2.

THEOREM 1.4.

(a) *For every prime p, the function $\psi_p(X+Y; u)$ fails to converge as $u \to \infty$.*
(b) *The function $\psi_2(X+2Y; u)$ fails to converge as $u \to \infty$.*

We will prove that Theorem 1.4 implies Theorem 1.2 at the end of Section 2, and Theorems 1.3 and 1.4 will be established in Sections 3 and 4, respectively.

In fact, one can verify statements similar to Theorem 1.4 for some other linear cases, like $\psi_2(X+3Y;u)$ and $\psi_2(2X+Y;u)$, with the same method (see Section 4). We omit the details. On the other hand, there are no established methods that work for a polynomial of degree two or more, and the author believes that a new method will be needed to handle higher degree cases.

To prove Theorem 1.4(b), we actually observe some arithmetic properties of $N_p(X+pY;u)$. There appears an algebraic difficulty when $p \geq 3$, so that we obtain the result only for $p = 2$. It is mentioned again in Section 4 why we have to restrict p. Theorem 1.4 is derived from some elementary calculations and the Gel'fond–Schneider theorem, which gives an affirmative answer to Hilbert's seventh problem: "Does α^β become transcendental when α and β are algebraic over $\mathbb{Q}$ with $\alpha \neq 0, 1$ and $\beta \notin \mathbb{Q}$?" Our calculations are free of analytic estimates, unlike the former studies [10] and [12].

In our proof of Theorem 1.4, we obtain bounds for $\limsup_{u\to\infty} \psi_p(P;u)$ and $\liminf_{u\to\infty} \psi_p(P;u)$. Those bounds are cruder than those in [4], [11], [10] and [12], but sufficient for our purposes. It seems difficult to improve the bounds since no effective formula expressing $N_p(P;u)$ is known apart from the case $P(X,Y) = X$.

2. Preliminaries. First, we give some notation.

- Let $\|\cdot\|$ denote the Euclidean norm on $\mathbb{R}^2$, i.e. $\|(x,y)\| = \sqrt{x^2+y^2}$ for $(x,y) \in \mathbb{R}^2$.
- Let $f(x)$ and $g(x)$ be functions defined on a subset of $\mathbb{R}$ which is unbounded from above, e.g. $(0,\infty)$ or $\mathbb{N}$. We write $f(x) \ll g(x)$ $(x \to \infty)$ if there exists a positive constant C such that $|f(x)| \leq Cg(x)$ for any sufficiently large x, and $f(x) \asymp g(x)\,(x \to \infty)$ if both $f(x) \ll g(x)\,(x\to\infty)$ and $g(x) \ll f(x)\,(x \to \infty)$.
- Let $\mathcal{T} = \{(x,y) \in \mathbb{R}^2\,;\, x \geq y \geq 0\}$, $\mathcal{T}_{\mathbb{N}_0} = \mathcal{T} \cap (\mathbb{N}_0 \times \mathbb{N}_0)$ and $\mathcal{T}(R) = \{(x,y) \in \mathcal{T}\,;\, \|(x,y)\| \geq R\}$ for $R > 0$.
- For a prime number p, let $I_p = \{0, 1, \ldots, p-1\}$, $\mathrm{Pas}(p) = \{(m,n) \in \mathcal{T}_{\mathbb{N}_0}\,;\, \binom{m}{n} \not\equiv 0 \pmod{p}\}$, and $\theta_p = \log\frac{p(p+1)}{2}/\log p$, or equivalently $p^{\theta_p} = p(p+1)/2$.
- We shall allow that the expansion of $a \in \mathbb{N}_0$ in base p, $a = \sum_{j=0}^{h} a_j p^j$ with $a_j \in I_p$, may have its top digits $a_h, a_{h-1}, \ldots, a_{h-k}$ zero for some $0 \leq k \leq h$.

We will introduce three functions which play leading roles in this article. We define them using some properties of polynomials.

Definition 2.1. Let $P \in \mathbb{R}[X,Y]$ be a polynomial of degree d and let $P_d \in \mathbb{R}[X,Y]$ be the d-degree part of P, that is, the homogeneous polynomial uniquely determined by $\deg(P - P_d) < \deg P$ with the convention $\deg(0) = -\infty$.

(a) P is said to be $\mathcal{T}$-*elliptic* if $P_d(x,y) > 0$ for all $(x,y) \in \mathcal{T} \setminus \{(0,0)\}$.
(b) P is said to be $\mathcal{T}$-*positive* if $P(x,y) \geq 0$ for all $(x,y) \in \mathcal{T}$.

Remark. Being $\mathcal{T}$-elliptic does not necessarily imply being $\mathcal{T}$-positive and vice versa. Indeed, for example, the polynomial Y is not $\mathcal{T}$-elliptic but it is $\mathcal{T}$-positive, and $X-1$ satisfies the contrary. However, in fact, a $\mathcal{T}$-elliptic polynomial P can be shifted by some $c > 0$ so that $P + c$ also becomes $\mathcal{T}$-positive. This actually follows from Lemma 3.1 below.

Now we define three functions of importance in this article.

Definition 2.2. Assume that $P \in \mathbb{Z}[X,Y]$ is a $\mathcal{T}$-elliptic and $\mathcal{T}$-positive polynomial of degree $d \geq 1$. For $q \in \mathbb{N}_0$ and $u \in \mathbb{N}$ define

$$\begin{aligned}
\phi_p(P;q) &= \#\{(m,n) \in \mathrm{Pas}(p)\,;\, P(m,n) = q\},\\
N_p(P;u) &= \#\{(m,n) \in \mathrm{Pas}(p)\,;\, P(m,n) < u\} = \sum_{0 \leq q < u} \phi_p(P;q),\\
\psi_p(P;u) &= N_p(P;u)/u^{\theta_p/d}.
\end{aligned}$$

We note that $\psi_p(P;u)$ is a bounded function by Theorem 1.3.

Remark. If $P \in \mathbb{R}[X,Y]$ is $\mathcal{T}$-elliptic, the region $\{P(x,y) < u\} \cap \mathcal{T}$ is bounded. Hence $\phi_p(P;q)$ and $N_p(P;u)$ are well-defined. However, $\mathcal{T}$-ellipticity is not necessary for well-definedness of $\phi_p(P;q)$ and $N_p(P;u)$: for example, consider the polynomial $XY + X$, not $\mathcal{T}$-elliptic.

The number of points in the first u columns of $\mathrm{Pas}(p)$ is expressed by $N_p^*(u) = N_p(X;u)$ in the above notation. The asymptotic behavior of $N_p^*(u)$ was often considered in former studies.

Clearly $N_p^*(u) = O(u^2)\,(u \to \infty)$; the first non-trivial estimate $N_p^*(u) = o(u^2)\,(u \to \infty)$ was given by Fine [3]. At present it is known that $N_p^*(u)$ is of the order of u^{θ_p}. For $p = 2$, this was proved earlier by Stolarsky [11], who showed that $u^{\theta_2}/3 < N_2^*(u) < 3u^{\theta_2}$.

Evaluations of

$$\alpha_p = \limsup_{u \to \infty} \psi_p(X;u) \quad \text{and} \quad \beta_p = \liminf_{u \to \infty} \psi_p(X;u)$$

for $p = 2$ also appeared in [11]: $1 \leq \alpha_2 \leq 1.052$ and $0.72 \leq \beta_2 \leq 3^{\theta_2}/7$ (< 0.815). Sharper evaluations of α_2 and β_2 were given by Harborth [4]: $\alpha_2 = 1$ (optimal) and $0.812556 \leq \beta_2 < 0.812557$ (the exact value computed to the sixth decimal place). The upper limit α_p for general p's was explicitly computed by Stein [10] to be $\alpha_p = 1$. On the other hand, it is too difficult at present to compute β_p exactly. A general evaluation

$$(1 - 2^{1/(1-\theta_p)})^{\theta_p - 1} \le \beta_p < \frac{3-\theta_p}{2(2-\theta_p)^{2-\theta_p}},$$

given by Wilson [12], is useful because it ensures that $\beta_p < 1$ for $p \ge 3$.

We conclude this section with a generalization of Theorem 1.1. The proof is based on Essouabri's method. We emphasize that the above results yield $\alpha_p \ne \beta_p$ for any prime p.

THEOREM 2.3. *Let $P \in \mathbb{Z}[X,Y]$ be a $\mathcal{T}$-elliptic and $\mathcal{T}$-positive polynomial of degree $d \ge 1$, and assume that $\psi_p(P;u)$ fails to converge as $u \to \infty$. Then $Z_p(P,1;s)$ has a non-real pole on its axis of absolute convergence $\{\sigma = \theta_p\}$.*

Proof. First, we remark that Landau's theorem (see [5, Theorem 10] for example) implies that $s = \theta_p$ is a singularity of $Z_p(P,1;s)$. We recall that Essouabri [1] mentioned that all singularities of $Z_p(P,1;s)$ including $s = \theta_p$ must be simple poles.

Now assume that $Z_p(P,1;s)$ has no non-real poles on its axis of absolute convergence.

Let $f(u) = N_p(P; \lceil u^{d/\theta_p} \rceil)$ for $u > 0$. Then

$$Z_p(P,1;\theta_p s) = \int_{1^-}^{\infty} u^{-s}\, df(u).$$

Apply the Wiener–Ikehara theorem (see [6, Chapter 5, Corollary 1] for example) to deduce that $f(u)/u$ converges as $u \to \infty$. This implies that $N_p(P;\lceil u \rceil)/u^{\theta_p/d}$ converges as $u \to \infty$, contrary to the assumption that $\psi_p(P;u)$ fails to converge. ■

This theorem combined with Theorem 1.4 gives Theorem 1.2, so it remains to prove Theorems 1.3 and 1.4.

3. Proof of Theorem 1.3.

We need the following lemma from Essouabri's article [1].

LEMMA 3.1 ([1, Lemma 2]). *Let $P \in \mathbb{R}[X,Y]$ be a $\mathcal{T}$-elliptic polynomial of degree $d \ge 1$. Then there exist positive real numbers c_1, c_2 and R such that*

$$c_1 \|(x,y)\|^d \le P(x,y) \le c_2 \|(x,y)\|^d \quad \text{for all } (x,y) \in \mathcal{T}(R).$$

Now we begin the proof of Theorem 1.3. First of all, the desired estimate can be easily obtained when $P(X,Y) = X$ from the above evaluations of α_p and β_p in [4], [11], [10] and [12]: for example,

$$\frac{\beta_p}{2} u^{\theta_p} \le N_p(X;u) \le 2\alpha_p u^{\theta_p} \tag{3.1}$$

for any sufficiently large $u \in \mathbb{N}$.

Let us consider the general case. By Lemma 3.1, we can find positive numbers c_1, c_2 and R such that $c_1\|(x,y)\|^d \le P(x,y) \le c_2\|(x,y)\|^d$ for all $(x,y) \in \mathcal{T}(R)$. Since any $(x,y) \in \mathcal{T}$ satisfies $x \le \|(x,y)\| \le 2x$, we have

$$\tfrac{1}{2}(P(x,y)/c_2)^{1/d} \le x \le (P(x,y)/c_1)^{1/d} \tag{3.2}$$

whenever $(x,y) \in \mathcal{T}(R)$. By (3.2), all points (x,y) on the curve segment $\{P(x,y) = u\} \cap \mathcal{T}(R)$ satisfy

$$\tfrac{1}{2}(u/c_2)^{1/d} \le x \le (u/c_1)^{1/d}. \tag{3.3}$$

If $u \in \mathbb{N}$ is sufficiently large, the same curve segment must be contained in $\mathcal{T}(R)$, and (3.3) holds for any $(x,y) \in \mathcal{T}$ with $P(x,y) = u$. This implies that

$$N_p\big(X; \lfloor \tfrac{1}{2}(u/c_2)^{1/d} \rfloor\big) \le N_p(P;u) \le N_p\big(X; \lceil (u/c_1)^{1/d} \rceil\big),$$

and we can conclude by (3.1) that, for example,

$$\frac{\beta_p}{2 \cdot 4^{\theta_p} c_2^{\theta_p/d}} u^{\theta_p/d} \le N_p(P;u) \le \frac{2^{\theta_p+1}\alpha_p}{c_1^{\theta_p/d}} u^{\theta_p/d}$$

as desired. (Note that $\lfloor u \rfloor \ge u/2$ and $\lceil u \rceil \le 2u$ when $u \ge 2$.)

4. Proof of Theorem 1.4. The theorem states that none of $\psi_p(X+Y;u)$ and $\psi_2(X+2Y;u)$ converges as $u \to \infty$. To prove this, we are going to find two distinct accumulation points of $\psi_p(P;u)$ for each of the polynomials P in question.

The following proposition plays a key role in our proof.

PROPOSITION 4.1. *Suppose that* $(m,n) \in \mathcal{T}_{\mathbb{N}_0}$ *with*

$$m = \sum_{j=0}^{h} m_j p^j \quad \text{and} \quad n = \sum_{j=0}^{h} n_j p^j.$$

Then $(m,n) \in \mathrm{Pas}(p)$ *if and only if* $m_j \ge n_j$ *for* $j = 0,1,\ldots,h$, *with the convention that* $a < b$ *implies* $\binom{a}{b} = 0$.

Proof. This is an immediate corollary of Lucas' formula (introduced in his textbook [9, Section 228])

$$\binom{m}{n} \equiv \prod_{j=0}^{h} \binom{m_j}{n_j} \pmod{p}.$$

(Note that m_j's and n_j's belong to I_p.) ∎

To prove Theorem 1.4, for simplicity, we abbreviate $\phi_p(X+Y;q)$, $N_p(X+Y;u)$ and $\psi_p(X+Y;u)$ to $\phi_p(q)$, $N_p(u)$ and $\psi_p(u)$, respectively. In addition, we extend the domain of ϕ_p to $\mathbb{Z}$ by letting $\phi_p(q) = 0$ for $q < 0$. We first prove the following lemma.

LEMMA 4.2. *Let $q \in \mathbb{N}_0$ and $r \in I_p$. Then*

$$\phi_p(pq+r) = \left\lfloor \frac{r}{2} + 1 \right\rfloor \phi_p(q) + \left\lfloor \frac{p-r}{2} \right\rfloor \phi_p(q-1). \tag{4.1}$$

Proof. First, we remark that

$$\phi_p(q) = \#\{n \in \mathbb{N}_0\,;\, (q-n,n) \in \mathrm{Pas}(p)\}.$$

In order to use Proposition 4.1, we express q, n and $q-n$ in base p:

$$q = \sum_{j=0}^{h} q_j p^j, \qquad n = \sum_{j=0}^{h} n_j p^j, \qquad q-n = \sum_{j=0}^{h} m_j p^j.$$

(We may assume that $q - n \geq 0$.) Consider

$$v_a(q) = \#\{n \in \mathbb{N}_0\,;\, n_0 = a, (q-n,n) \in \mathrm{Pas}(p)\}$$

for $a \in I_p$. Then it is clear that $\phi_p(q) = \sum_{a=0}^{p-1} v_a(q)$.

To get the desired identity we should calculate $v_a(q)$. We note that

$$m_0 = \begin{cases} q_0 - n_0 & (q_0 \geq n_0), \\ q_0 - n_0 + p & (q_0 < n_0), \end{cases}$$

so that, by Proposition 4.1, $v_a(q) = 0$ if $q_0/2 < a \leq q_0$ or $(q_0+p)/2 < a \leq p-1$. Otherwise, by Proposition 4.1, we have

$$\begin{aligned} v_a(q) &= \#\{n \in \mathbb{N}_0\,;\, n_0 = a,\, m_j \geq n_j\ (j = 1,\ldots,h)\} \\ &= \#\left\{ \frac{n-a}{p} \in \mathbb{N}_0\,;\, \left(\frac{q-n-m_0}{p}, \frac{n-a}{p} \right) \in \mathrm{Pas}(p) \right\} \\ &= \begin{cases} \#\{n' \in \mathbb{N}_0\,;\, \left(\frac{q-q_0}{p} - n', n'\right) \in \mathrm{Pas}(p)\} & (0 \leq a \leq q_0/2), \\ \#\{n' \in \mathbb{N}_0\,;\, \left(\frac{q-q_0}{p} - 1 - n', n'\right) \in \mathrm{Pas}(p)\} & (q_0 < a \leq (q_0+p)/2) \end{cases} \\ &= \begin{cases} \phi_p\left(\frac{q-q_0}{p}\right) & (0 \leq a \leq q_0/2), \\ \phi_p\left(\frac{q-q_0}{p} - 1\right) & (q_0 < a \leq (q_0+p)/2), \end{cases} \end{aligned}$$

where $n' = (n-a)/p$. Hence

$$\begin{aligned} \phi_p(q) &= \sum_{a=0}^{\lfloor q_0/2 \rfloor} \phi_p\left(\frac{q-q_0}{p} \right) + \sum_{a=q_0+1}^{\lfloor (q_0+p)/2 \rfloor} \phi_p\left(\frac{q-q_0}{p} - 1 \right) \\ &= \left\lfloor \frac{q_0}{2} + 1 \right\rfloor \phi_p\left(\frac{q-q_0}{p} \right) + \left\lfloor \frac{p-q_0}{2} \right\rfloor \phi_p\left(\frac{q-q_0}{p} - 1 \right). \end{aligned}$$

Replacing q with $pq+r$, we obtain the conclusion. ∎

LEMMA 4.3. *Let $q \in \mathbb{N}$ and $k \in \mathbb{N}_0$. Then*

$$\phi_p(p^k q - 1) = \left\lfloor \frac{p+1}{2} \right\rfloor^k \phi_p(q-1). \tag{4.2}$$

Proof. We use induction on k and (4.1), noting that $p^k q-1 = p(p^{k-1}q-1) + (p-1)$. ∎

We now evaluate N_p by using (4.2).

LEMMA 4.4. *Let $u \in \mathbb{N}$ and $k \in \mathbb{N}_0$. Then*

$$N_p(p^k u) = \begin{cases} 2^{k\theta_2} N_2(u) - \frac{3^k-1}{2}\phi_2(u-1) & (p=2), \\ p^{k\theta_p} N_p(u) - \frac{p^k-1}{2}\cdot\left(\frac{p+1}{2}\right)^k \phi_p(u-1) & (p\ge 3). \end{cases} \tag{4.3}$$

Proof. By (4.1), we can calculate as follows:

$$\begin{aligned} N_p(pu) &= \sum_{q=0}^{u-1}\sum_{r=0}^{p-1}\phi_p(pq+r) \\ &= \sum_{q=0}^{u-1}\phi_p(q)\sum_{r=0}^{p-1}\left\lfloor \frac{r}{2}+1\right\rfloor + \sum_{q=0}^{u-1}\phi_p(q-1)\sum_{r=0}^{p-1}\left\lfloor\frac{p-r}{2}\right\rfloor \\ &= N_p(u)\sum_{r=0}^{p-1}\left(\left\lfloor\frac{r}{2}+1\right\rfloor + \left\lfloor\frac{p-r}{2}\right\rfloor\right) - \phi_p(u-1)\sum_{r=0}^{p-1}\left\lfloor\frac{p-r}{2}\right\rfloor. \end{aligned}$$

We can check that $\sum_{r=0}^{p-1}(\lfloor\frac{r}{2}+1\rfloor + \lfloor\frac{p-r}{2}\rfloor) = p^{\theta_p}$ and $\sum_{r=0}^{p-1}\lfloor\frac{p-r}{2}\rfloor = C_p$, where

$$C_p = \begin{cases} 1 & (p=2), \\ (p-1)(p+1)/4 & (p\ge 3), \end{cases}$$

so that

$$N_p(pu) = p^{\theta_p}N_p(u) - C_p\phi_p(u-1). \tag{4.4}$$

Hence

$$N_p(p^k u) = p^{k\theta_p}N_p(u) - C_p\sum_{l=0}^{k-1}p^{l\theta_p}\phi_p(p^{k-l-1}u-1)$$

by induction on k. Then we use (4.2) to finish the proof. ∎

The equality below gives some accumulation points of $\psi_p(u)$.

LEMMA 4.5. *Let $u\in\mathbb{N}$. Then*

$$\lim_{k\to\infty}\psi_p(p^k u) = \psi_p(u) - \frac{\phi_p(u-1)}{2u^{\theta_p}}. \tag{4.5}$$

Proof. Dividing both sides of (4.3) by $(p^k u)^{\theta_p}$, we obtain

$$\psi_p(p^k u) = \begin{cases} \psi_2(u) - \dfrac{1-3^{-k}}{2u^{\theta_2}}\phi_2(u-1) & (p=2), \\ \psi_p(u) - \dfrac{1-p^{-k}}{2u^{\theta_p}}\phi_p(u-1) & (p\ge 3), \end{cases}$$

which yields the conclusion. ∎

Now we complete the proof of Theorem 1.4. Our aim is to find two distinct accumulation points of $\psi_p(u)$. Substituting $u = 1$ into (4.5), we obtain $\lim_{k\to\infty} \psi_p(p^k) = 1/2$. Next we substitute $u = 2$ to get $\lim_{k\to\infty} \psi_p(2p^k) = 3/2^{\theta_p+1}$. Since $\theta_p > \theta_2$ for $p \ge 3$, we obtain $3/2^{\theta_p+1} < 3/2^{\theta_2+1} = 1/2$, which gives the second accumulation point $3/2^{\theta_p+1}$ of $\psi_p(u)$.

For $p = 2$ we need a more sophisticated argument. Substituting $u = 3$ into (4.5), we obtain $\lim_{k\to\infty} \psi_2(3 \cdot 2^k) = 3^{1-\theta_2}$. If $3^{1-\theta_2} = 1/2$, we would find that $\theta_2(1 - \theta_2) = -1$, contrary to the Gel'fond–Schneider theorem (see [2, Theorems 3.1–3.2] for example), which yields the transcendence of θ_2; hence $3^{1-\theta_2}$ is the second accumulation point of $\psi_2(u)$. This completes the proof of Theorem 1.4(a).

Remark. Numerical calculations show that $\lim_{k\to\infty} \psi_2(3\cdot 2^k) = 3^{1-\theta_2} = 0.525898\ldots$ and $\lim_{k\to\infty} \psi_2(17 \cdot 2^k) = 0.487836\ldots$. Thus we expect that $\liminf_{u\to\infty} \psi_2(u) < 1/2 < \limsup_{u\to\infty} \psi_2(u)$.

To prove Theorem 1.4(b), we use the same method although calculations slightly differ from those above. Thus we often omit the details of the proofs of the following lemmas. We only need to calculate $\phi_2(X + 2Y; q)$, $N_2(X + 2Y; u)$ and $\psi_2(X + 2Y; u)$, but some of the lemmas could be easily generalized to $\phi_p(X + pY; q)$.

Here we change the meaning of our symbols: from now on, we abbreviate $\phi_p(X + pY; q)$, $N_p(X + pY; u)$ and $\psi_p(X + pY; u)$ to $\phi_p(q)$, $N_p(u)$ and $\psi_p(u)$, respectively. In addition, we again define $\phi_p(q) = 0$ for every negative integer q.

First, we prove a formula corresponding to (4.1).

Lemma 4.6. *Let $q \in \mathbb{N}_0$ and $r \in I_p$. Then*

$$\phi_p(pq + r) = \sum_{a=0}^{r} \phi_p(q - a). \tag{4.6}$$

Proof. The proof will proceed similarly to that of Lemma 4.2. First,

$$\begin{aligned}\phi_p(q) &= \#\{n \in \mathbb{N}_0 \,;\, (q - pn, n) \in \mathrm{Pas}(p)\} \\ &= \sum_{a=0}^{p-1} (\#\{n \in \mathbb{N}_0 \,;\, n_0 = a,\ (q - pn, n) \in \mathrm{Pas}(p)\}),\end{aligned}$$

where

$$q = \sum_{i=0}^{h} q_i p^i, \qquad n = \sum_{i=0}^{h} n_i p^i, \qquad q - pn = \sum_{i=0}^{h} m_i p^i$$

in base p. We should calculate

$$v_a(q) = \#\{n \in \mathbb{N}_0 \,;\, n_0 = a,\ (q - pn, n) \in \mathrm{Pas}(p)\}$$

for every $a \in I_p$. Noting that $m_0 = q_0$, we find that

$$v_a(q) = \#\left\{\frac{n-a}{p} \in \mathbb{N}_0\,;\ \left(\frac{q-pn-m_0}{p}, \frac{n-a}{p}\right) \in \mathrm{Pas}(p)\right\}$$
$$= \#\left\{n' \in \mathbb{N}_0\,;\ \left(\frac{q-q_0}{p} - a - pn', n'\right) \in \mathrm{Pas}(p)\right\} = \phi_p\left(\frac{q-q_0}{p} - a\right)$$

if $q_0 \ge a$, where $n' = (n-a)/p$, and $v_a(q) = 0$ otherwise. We conclude the proof by replacing q with $pq + r$. ∎

Lemma 4.7. *Let $q \in \mathbb{N}$ and $k \in \mathbb{N}_0$. Set $\gamma_+ = (1+\sqrt{5})/2$ and $\gamma_- = (1-\sqrt{5})/2 = -\gamma_+^{-1}$. Then*

$$\phi_2(2^k q - 1) = \frac{(\gamma_+^{k+1} - \gamma_-^{k+1})\phi_2(q-1) + (\gamma_+^k - \gamma_-^k)\phi_2(q-2)}{\gamma_+ - \gamma_-}. \tag{4.7}$$

Proof. From (4.6) we get the "Fibonacci-like" recurrence formula

$$\phi_2(2^k q - 1) = \phi_2(2^{k-1} q - 1) + \phi_2(2^{k-2} q - 1),$$

and then we obtain the conclusion by easy calculations. ∎

What we have to do next is to sum the $\phi_p(q)$'s.

Lemma 4.8. *Let $u \in \mathbb{N}$ and $k \in \mathbb{N}_0$. Then*

$$N_p(p^k u) = p^{k\theta_p} N_p(u) - \sum_{l=0}^{k-1} p^{l\theta_p} \sum_{b=1}^{p-1} \frac{(p-b)(p-b+1)}{2} \phi_p(p^{k-l-1}u - b). \tag{4.8}$$

In particular,

$$N_2(2^k u) = 2^{k\theta_2} N_2(u) - \sum_{l=0}^{k-1} 2^{l\theta_2} \phi_2(2^{k-l-1}u - 1). \tag{4.9}$$

Proof. First, we transform the summatory function $N_p(pu)$ using (4.6):

$$N_p(pu) = \sum_{q=0}^{u-1}\sum_{r=0}^{p-1}\sum_{a=0}^{r} \phi_p(q-a) = \sum_{a=0}^{p-1}(p-a)\sum_{q=0}^{u-1}\phi_p(q-a)$$
$$= \sum_{a=0}^{p-1}(p-a)\Big(N_p(u) - \sum_{b=1}^{a}\phi_p(u-b)\Big)$$
$$= p^{\theta_p} N_p(u) - \sum_{b=1}^{p-1} \frac{(p-b)(p-b+1)}{2}\phi_p(u-b).$$

We obtain the desired identities by induction on k. ∎

We combine (4.7) with (4.9) to obtain the following formula for $\psi_2(u)$.

Lemma 4.9. *Let $u \in \mathbb{N}$. Then*

$$\lim_{k\to\infty} \psi_2(2^k u) = \psi_2(u) - \frac{3\phi_2(u-1) + \phi_2(u-2)}{5u^{\theta_2}}. \tag{4.10}$$

Proof. By (4.7) and (4.8), we have

$$\psi_2(2^k u) = \psi_2(u) - \frac{1}{(\gamma_+ - \gamma_-)u^{\theta_2}} \times \sum_{l=0}^{k-1} \frac{(\gamma_+^{k-l} - \gamma_-^{k-l})\phi_2(u-1) + (\gamma_+^{k-l-1} - \gamma_-^{k-l-1})\phi_2(u-2)}{2^{(k-l)\theta_2}}.$$

Hence we immediately obtain the desired formula by noting that $2^{\theta_2} = 3 > |\gamma_\pm|$. ∎

We can now complete the proof of Theorem 1.4(b). Substituting $u = 1$ and $n = 9$ into (4.10), we obtain $\lim_{k\to\infty} \psi_2(2^k) = 2/5$ and $\lim_{k\to\infty} \psi_2(9\cdot 2^k) = 64/(5\cdot 9^{\theta_2})$. If they were equal, we would have $2\theta_2^2 - 5 = 0$, contradicting the Gel'fond–Schneider theorem again. Thus we have found two distinct accumulation points of $\psi_2(u)$.

Remark. Numerical calculations show that $64/(5\cdot 9^{\theta_2}) = 0.393342\ldots$, thus we expect that $\liminf_{u\to\infty} \psi_2(u) < 2/5 \le \limsup_{u\to\infty} \psi_2(u)$.

Remark. We would obtain results analogous to Theorem 1.4(b) for general p if we could find explicit forms of $\phi_p(p^k u - b)$ for $b \in I_p$ like (4.7). The sequences $\{\phi_p(p^k u - b)\}_{k=0}^{\infty}$ for $b \in I_p$ are determined by a system of p linear relations. Although, in fact, it can be solved explicitly when $p = 3$, the form of the solution is very complicated.

Acknowledgements. The author expresses his gratitude to Prof. Kohji Matsumoto, Mr. Hiroki Fujino, Mr. Shinya Kadota and Mr. Yuta Suzuki for their helpful comments. The author also thanks the referee(s) for useful comments.

References

[1] D. Essouabri, *Zeta functions associated to Pascal's triangle mod p*, Japan. J. Math. 31 (2005), 157–174.

[2] N. I. Fel'dman and Yu. V. Nesterenko, *Number Theory, Vol. IV, Transcendental Numbers*, Encyclopaedia Math. Sci. 44, Springer, 1998.

[3] N. J. Fine, *Binomial coefficients modulo a prime*, Amer. Math. Monthly 54 (1947), 589–592.

[4] H. Harborth, *Number of odd binomial coefficients*, Proc. Amer. Math. Soc. 62 (1977), 19–22.

[5] G. H. Hardy and M. Riesz, *The General Theory of Dirichlet's Series*, Dover, 2005.

[6] E. Hlawka, J. Schoißengeier and R. Taschner, *Geometric and Analytic Number Theory*, Springer, 1991.

[7] E. E. Kummer, *Über die Ergänzungssätze zu den allgemeinen Reciprocitätsgesetzen*, J. Reine Angew. Math. 44 (1852), 93–146.
[8] M. L. Lapidus and M. van Frankenhuijsen, *Fractal Geometry, Complex Dimensions and Zeta Functions*, 2nd ed., Springer, 2013.
[9] E. Lucas, *Théorie des Nombres*, Librairie Sci. et Tech., Albert Blanchard, 1961.
[10] A. H. Stein, *Binomial coefficients not divisible by a prime*, in: Number Theory, Lecture Notes in Math. 1383, Springer, 1989, 170–177.
[11] K. B. Stolarsky, *Power and exponential sums of digital sums related to binomial coefficient parity*, SIAM J. Appl. Math. 32 (1977), 717–730.
[12] B. Wilson, *Asymptotic behavior of Pascal's triangle modulo a prime*, Acta Arith. 83 (1998), 105–116.

Tomohiro Ikkai
Graduate School of Mathematics
Nagoya University
464-8602 Nagoya, Japan
E-mail: m13006z@math.nagoya-u.ac.jp