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Global attractors for nonclassical diffusion equations with hereditary memory and a new class of nonlinearities

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Dedicated to Prof. Nguyen Manh Hung on the occasion of his 60th birthday

Abstract. We study the existence and long-time behavior in terms of existence of global attractors of weak solutions to a class of nonclassical diffusion equations with hereditary memory and a new class of nonlinearities, which contains all nonlinearities of polynomial type, Sobolev type, and even exponential type. The main novelty of our result is that no restriction on the upper growth of the nonlinearity is imposed.

1. Introduction. The asymptotic behavior of dynamical systems arising from mechanics and physics is a fundamental issue, as it is essential, for practical applications, to be able to understand and even predict the long-time behavior of the solutions of such systems. One way to attack the problem for a dissipative dynamical system is to consider its global attractor. This is a compact invariant set, which contains much information about the long-time behavior of solutions (see e.g. the monograph of Temam [Te]).

In this paper we consider the following semilinear nonclassical diffusion equation with memory:

$$
(1.1)\qquad \begin{cases} u_t - \Delta u_t - \Delta u \\ \qquad - \int_0^\infty \kappa(s)\Delta u(t-s)\,ds + f(u) = g(x), & x\in\Omega,\ t>0,\\ u(x,t)=0, & x\in\partial\Omega,\ t>0,\\ u(x,0)=u_0(x), & x\in\Omega,\\ u(x,-s)=g_0(x,s), & x\in\Omega,\ s>0,\end{cases}
$$

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where Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$. The first equation in (1.1) arises in the classical diffusion theory when assuming that the diffusing species behaves as a linear viscous fluid, which leads to including its velocity gradient in the constitutive laws [A]. Moreover, the convolution term takes into account the influence of the past history of u on its future evolution, providing a more accurate description of the diffusive process in certain materials, such as high-viscosity liquids at low temperatures and polymers (see e.g. [J]).

In the case $\kappa \equiv 0$, we obtain the so-called nonclassical diffusion equation

$$u_t - \Delta u_t - \Delta u + f(u) = g.$$

It arises as a model to describe physical phenomena, such as non-Newtonian flows, soil mechanics and heat conduction theory (see, e.g., [A, PG, Ti, TN]). In the past years, the existence and long-time behavior of solutions to nonclassical diffusion equations has been studied extensively, for both autonomous case [LM, SWZ, SY, WLZ, XLZ, X, ZWG] and non-autonomous case [AB1, AB2, AT1, AT2, SY, To, WaZ, ZL], and even in the case with finite delay [BZ, CaM, ZS].

The speed of energy dissipation for equation (1.1) is faster than that for the usual nonclassical diffusion equation. The conduction of energy is not only affected by present external forces but also by historic external forces. In recent years, the existence and long-time behavior of solutions to nonclassical diffusion equations with memory has been addressed by a number of authors (see [CoM, CMP, WW, WYZ, WaZ]). In the most of the existing papers dealing with the memory relaxation (1.1), the function $\mu(s) := -\kappa'(s)$ is assumed to satisfy the inequality

$$\mu'(s) + \delta\mu(s) \le 0,$$

which was introduced in the seminal paper [D] and commonly adopted thereafter, and the nonlinearity is assumed to be Lipschitz continuous and satisfy a Sobolev growth condition

$$\liminf_{|u|\to\infty} \frac{f(u)}{u} > -\lambda_1,$$
$$|f'(u)| \le C(1 + |u|^\alpha) \quad \text{with } (N-2)\alpha \le 4,$$

where $\lambda_1 > 0$ is the first eigenvalue of the operator $-\Delta_D$ in Ω with the homogeneous Dirichlet boundary condition. Under the above assumption on the nonlinearities, Conti, Marchini and Pata [CMP] proved the existence of a global attractor of optimal regularity under a more general assumption on the kernel κ; in particular, their result improved the previous one in [WYZ]. By combining the techniques in [CMP, WYZ] and those in [GPM1, GPM2] for reaction-diffusion equations with memory, one can prove a similar result when the nonlinearity $f(u)$ satisfies a dissipative and growth condition of polynomial type, namely,

$$C_1|u|^p - C_0 \le f(u)u \le C_2|u|^p + C_0 \quad \text{for some } p \ge 2,$$
$$f'(u) \ge -\ell.$$

Note that for the above classes of nonlinearities, some restriction on the upper growth of the nonlinearity is imposed, and an exponential nonlinearity, for example $f(u) = e^u$, is not allowed.

In this paper we remove this restriction and prove the existence of weak solutions and then the existence of a global attractor under a general assumption on the memory kernel κ (as in [CMP]) and for a very large class of nonlinearities (see condition **(H1)** below) that in particular covers both the above classes of nonlinearities and even exponential nonlinearities. This is the main novelty of our paper.

To study problem (1.1), we assume that the initial datum $u_0 \in H_0^1(\Omega)$ is given, and the nonlinearity f and the external force g satisfy the following conditions:

(H1) $f : \mathbb{R} \to \mathbb{R}$ is a continuously differentiable function satisfying

$$f'(u) \ge -\ell, \tag{1.2}$$

$$f(u)u \ge -\beta u^2 - C_0 \quad \text{for all } u \in \mathbb{R}, \tag{1.3}$$

where ℓ, β, C_0 are positive constants, $0 < \beta < \lambda_1$ with $\lambda_1 > 0$ the first eigenvalue of the operator $-\Delta_D$.

It follows from (1.2) that $0 \le \int_0^u (f'(s)s + \ell s)\,ds$, and therefore by integrating by parts, we obtain

$$F(u) \le f(u)u + \ell u^2/2 \quad \text{for all } u \in \mathbb{R}, \tag{1.4}$$

where $F(u) = \int_0^u f(s)\,ds$ is a primitive of f.

(H2) The external force g is in $H^{-1}(\Omega)$.

(H3) The convolution (or memory) kernel κ is a nonnegative summable function having the explicit form

$$\kappa(s) = \int_s^\infty \mu(r)\,dr, \tag{1.5}$$

where $\mu \in L^1(\mathbb{R}^+)$ is a decreasing (hence nonnegative) piecewise absolutely continuous function. In particular, μ is allowed to exhibit (infinitely many) jumps. Moreover, we require that

$$\kappa(s) \le \theta\mu(s), \tag{1.6}$$

for some $\theta > 0$ and every $s > 0$. As shown in [GMPZ], this is equivalent to the requirement that

$$\mu(r+s) \le Me^{-\delta r}\mu(s) \tag{1.7}$$

for some $M \ge 1, \delta > 0$, every $r \ge 0$ and almost every $s > 0$.

As in [D], a new variable which reflects the history of (1.1) is introduced, namely

$$\eta^t(x,s) = \eta(x,t,s) = \int_0^s u(x,t-r)\,dr, \qquad s \ge 0;$$

then we can check that

$$\partial_t \eta^t(x,s) = u(x,t) - \partial_s \eta^t(x,s), \qquad s \ge 0.$$

Since $\mu(s) = -\kappa'(s)$, problem (1.1) can be transformed into

$$(1.8)\qquad \begin{cases} u_t - \Delta u_t - \Delta u \\ \quad - \int_0^\infty \mu(s)\Delta\eta^t(x,s)\,ds + f(u) = g(x), & x \in \Omega,\ t > 0, \\ \eta_t^t(x,s) = -\eta_s^t(x,s) + u(x,t), & x \in \Omega,\ t > 0,\ s \ge 0, \\ u(x,t) = 0, & x \in \partial\Omega,\ t > 0, \\ \eta^t(x,s) = 0, & x \in \partial\Omega,\ s \ge 0,\ t > 0, \\ u(x,0) = u_0(x), & x \in \Omega, \\ \eta^0(x,s) = \eta_0(x,s) := \int_0^s g_0(x,r)\,dr, & x \in \Omega,\ s \ge 0. \end{cases}$$

Now, denote

$$z(t) = (u(t), \eta^t) \quad \text{and} \quad z_0 = (u_0, \eta_0).$$

Unless otherwise specified, it is understood that we consider spaces of functions which are defined on the domain Ω. Let $\langle \cdot,\cdot \rangle$ and $\|\cdot\|$ denote the $L^2(\Omega)$-inner product and $L^2(\Omega)$-norm, respectively.

In view of (1.7), let $L^2_\mu(\mathbb{R}^+, L^2(\Omega))$ be the Hilbert space of functions $\varphi\colon \mathbb{R}^+ \to L^2(\Omega)$ endowed with the inner product

$$\langle \varphi_1, \varphi_2 \rangle_\mu = \int_0^\infty \mu(s)\langle \varphi_1(s), \varphi_2(s)\rangle\,ds,$$

and let $\|\varphi\|_\mu$ denote the corresponding norm. In a similar manner, we introduce the inner products $\langle\cdot,\cdot\rangle_{1,\mu}$ and $\langle\cdot,\cdot\rangle_{2,\mu}$ on $L^2_\mu(\mathbb{R}^+, H^1_0(\Omega))$ and $L^2_\mu(\mathbb{R}^+, H^2(\Omega)\cap H^1_0(\Omega))$ by

$$\langle\cdot,\cdot\rangle_{1,\mu} = \langle \nabla\cdot, \nabla\cdot\rangle_\mu, \qquad \langle\cdot,\cdot\rangle_{2,\mu} = \langle \Delta\cdot, \Delta\cdot\rangle_\mu,$$

and the corresponding norms are denoted by $\|\cdot\|_{1,\mu}$ and $\|\cdot\|_{2,\mu}$.

We now introduce the following Hilbert spaces:

$$\begin{aligned} \mathcal{H} &= L^2(\Omega) \times L^2_\mu(\mathbb{R}^+, H^1_0(\Omega)), \\ \mathcal{H}_1 &= H^1_0(\Omega) \times L^2_\mu(\mathbb{R}^+, H^1_0(\Omega)), \\ \mathcal{H}_2 &= (H^2(\Omega)\cap H^1_0(\Omega)) \times L^2_\mu(\mathbb{R}^+, H^2(\Omega)\cap H^1_0(\Omega)), \end{aligned}$$

which are endowed with the respective inner products

$$\langle w_1, w_2\rangle_{\mathcal{H}} = \langle \psi_1, \psi_2\rangle + \langle \varphi_1, \varphi_2\rangle_{1,\mu},$$
$$\langle w_1, w_2\rangle_{\mathcal{H}_1} = \langle \nabla\psi_1, \nabla\psi_2\rangle + \langle \varphi_1, \varphi_2\rangle_{1,\mu},$$
$$\langle w_1, w_2\rangle_{\mathcal{H}_2} = \langle \Delta\psi_1, \Delta\psi_2\rangle + \langle \varphi_1, \varphi_2\rangle_{2,\mu},$$

where $w_i = (\psi_i, \varphi_i) \in \mathcal{H}, \mathcal{H}_1, \mathcal{H}_2$ for $i = 1, 2$.

The norms induced on $\mathcal{H}_i$ for $i = 1, 2$ are

$$\|(\psi,\varphi)\|^2_{\mathcal{H}_1} = \|\psi\|^2_{H^1_0(\Omega)} + \int_0^\infty \mu(s)\|\nabla\varphi(s)\|^2\,ds,$$

$$\|(\psi,\varphi)\|^2_{\mathcal{H}_2} = \|\psi\|^2_{H^1_0(\Omega)} + \|\psi\|^2_{H^2(\Omega)} + \int_0^\infty \mu(s)\|\Delta\varphi(s)\|^2\,ds.$$

The paper is organized as follows. In Section 2, we prove the existence and uniqueness of weak solutions by using the Faedo–Galerkin method. In Section 3, we show the existence of a global attractor for the continuous semigroup generated by the weak solutions. The main novelty of the present paper is that the nonlinearity can grow at infinity arbitrarily fast and the assumption on the memory is very general; in particular, the results obtained here extend the corresponding ones in [CMP, WYZ]. Moreover, even in the case $\kappa \equiv 0$ (i.e. without the memory term), our results are still new for the usual nonclassical diffusion equations.

2. Existence and uniqueness of weak solutions

Definition 2.1. A function $z = (u, \eta^t)$ is called a *weak solution* of problem (1.8) on the interval $(0,T)$ with initial datum $z(0) = z_0 \in \mathcal{H}_1$ if

$$u \in C([0,T]; H^1_0(\Omega)), \quad f(u) \in L^1(Q_T),$$
$$u_t \in L^2(0,T; H^1_0(\Omega)), \quad \eta^t \in C([0,T]; L^2_\mu(\mathbb{R}^+, H^1_0(\Omega))),$$
$$\eta^t_t + \eta^t_s \in L^\infty(0,T; L^2_\mu(\mathbb{R}^+, L^2(\Omega))) \cap L^2(0,T; L^2_\mu(\mathbb{R}^+, H^1_0(\Omega))),$$

and

$$\langle u_t, \varphi\rangle + \langle \nabla u_t, \nabla\varphi\rangle + \langle \nabla u, \nabla\varphi\rangle + \langle \eta^t, \varphi\rangle_{1,\mu} + \langle f(u), \varphi\rangle_{L^1,L^\infty} = \langle g, \varphi\rangle_{H^{-1},H^1_0},$$

$$\langle \eta^t_t + \eta^t_s, \xi\rangle_{1,\mu} = \langle u, \xi\rangle_{1,\mu},$$

for all test functions $\varphi \in W = H^1_0(\Omega) \cap L^\infty(\Omega)$ and $\xi \in L^2_\mu(\mathbb{R}^+, H^1_0(\Omega))$, and for a.e. $t \in [0,T]$.

We are now ready to state the existence and uniqueness result for problem (1.8).

Theorem 2.2. *Assume that hypotheses* **(H1)–(H3)** *hold. Then for any* $z_0 = (u_0, \eta_0) \in \mathcal{H}_1$ *and* $T > 0$, *problem* (1.8) *has a unique weak solution* $z = (u, \eta^t)$ *on the interval* $(0, T)$ *satisfying*

$$z \in C([0, T]; \mathcal{H}_1).$$

Moreover, the weak solutions depend continuously on the initial data.

Proof. We use the Faedo–Galerkin method. There exists a smooth orthonormal basis $\{\omega_j\}_{j=1}^{\infty}$ of $L^2(\Omega)$ which is also orthogonal in $H_0^1(\Omega)$. For instance, one can take a complete set of normalized eigenfunctions for $-\Delta$ in $H_0^1(\Omega)$ such that $-\Delta\omega_j = \nu_j\omega_j$, where ν_j is the eigenvalue corresponding to ω_j. Next we want to select an orthonormal basis $\{\zeta_j\}_{j=1}^{\infty}$ of $L^2_\mu(\mathbb{R}^+, H_0^1(\Omega))$ with all ξ_j in $\mathcal{D}(\mathbb{R}^+, H_0^1(\Omega))$. Here and below, $\mathcal{D}(I, X)$ is the space of infinitely differentiable X-valued functions with compact support in $I \subset \mathbb{R}$, whose dual space is the distribution space on I with values in X^* (dual of X), denoted by $\mathcal{D}'(I, X^*)$. For this purpose we choose vectors of the form $l_k\omega_j$ $(k, j = 1, \dots, \infty)$, where $\{l_j\}_{j=1}^{\infty}$ is an orthonormal basis in $L^2_\mu(\mathbb{R}^+)$ which is also in $\mathcal{D}(\mathbb{R}^+)$.

(i) *Existence.* Given an integer n, denote by P_n and Q_n the projections on the subspaces

$$\operatorname{span}(\omega_1, \dots, \omega_n) \in H_0^1(\Omega) \quad \text{and} \quad \operatorname{span}(\zeta_1, \dots, \zeta_n) \in L^2_\mu(\mathbb{R}^+, H_0^1(\Omega)),$$

respectively. We look for a function $z_n = (u_n, \eta_n^t)$ of the form

$$u_n(t) = \sum_{j=1}^{n} a_j(t)\omega_j \quad \text{and} \quad \eta_n^t(s) = \sum_{j=1}^{n} b_j(t)\zeta_j(s)$$

satisfying

$$\begin{aligned} &\langle(\partial_t u_n - \Delta\partial_t u_n, \partial_t\eta_n^t), (\omega_k, \zeta_j)\rangle_{\mathcal{H}} \\ &\quad = \Big\langle\Big(\Delta u_n + \int_0^\infty \mu(s)\Delta\eta_n^t(s)\,ds + g - f(u_n), u_n - \partial_s\eta_n^t\Big), (\omega_k, \zeta_j)\Big\rangle_{\mathcal{H}} \\ &(u_n, \eta_n^t)|_{t=0} = (P_n u_0, Q_n\eta_0), \end{aligned} \tag{2.1}$$

for a.e. $t \le T$ and every $k, j = 0, \dots, n$, where ω_0 and ζ_0 are the zero vectors in the respective spaces. Taking (ω_k, ζ_0) and (ω_0, ζ_k) in (2.1), and applying the divergence theorem to the term

$$\Big\langle \int_0^\infty \Delta\eta_n^t(s)\,ds, \omega_k\Big\rangle,$$

we get a system of ODEs in the variables $a_k(t)$ and $b_k(t)$ of the form

$$
\begin{aligned}
&\frac{d}{dt}((1+\nu_k)a_k) = -\nu_k a_k - \sum_{j=1}^{n} b_j \langle \zeta_j, \omega_k\rangle_{1,\mu} + \langle g, \omega_k\rangle - \langle f(u_n), \omega_k\rangle,\\
&\frac{d}{dt} b_k = \sum_{j=1}^{n} a_j \langle \omega_j, \zeta_k\rangle_{1,\mu} - \sum_{j=1}^{n} b_j \langle \zeta_j', \zeta_k\rangle_{1,\mu},
\end{aligned} \tag{2.2}
$$

subject to the initial conditions

$$
\begin{aligned}
a_k(0) &= \langle u_0, \omega_k\rangle_{H_0^1(\Omega)},\\
b_k(0) &= \langle \eta_0, \zeta_k\rangle_{1,\mu}.
\end{aligned} \tag{2.3}
$$

According to standard existence theory for ODEs, there exists a continuous solution of (2.2)–(2.3) on some interval $(0, T_n)$. The *a priori* estimates below imply that in fact $T_n = +\infty$.

Multiplying the first equation of (2.2) by a_k and the second by b_k, then summing over k and adding the results, we get

$$
\frac{1}{2}\frac{d}{dt}\|z_n\|_{\mathcal{H}_1}^2 = -\|\nabla u_n\|^2 - \langle \partial_s \eta_n^t, \eta_n^t\rangle_{1,\mu} + \langle g, u_n\rangle_{H^{-1},H_0^1} - \langle f(u_n), u_n\rangle. \tag{2.4}
$$

Using (1.3) and the Cauchy inequality, we have

$$
\begin{aligned}
\langle g, u_n\rangle_{H^{-1},H_0^1} &- \langle f(u_n), u_n\rangle\\
&\le \varepsilon\|\nabla u_n\|^2 + \frac{1}{4\varepsilon}\|g\|_{H^{-1}(\Omega)}^2 + \beta\|u_n\|^2 + C_0|\Omega|,
\end{aligned} \tag{2.5}
$$

where $\varepsilon > 0$ will be chosen later. From (2.4) and (2.5) we obtain

$$
\begin{aligned}
\frac{d}{dt}\|z_n\|_{\mathcal{H}_1}^2 &+ 2\langle \partial_s \eta_n^t, \eta_n^t\rangle_{1,\mu} + 2(1-\beta/\lambda_1-\varepsilon)\|\nabla u_n\|^2\\
&\le \frac{1}{2\varepsilon}\|g\|_{H^{-1}(\Omega)}^2 + 2C_0|\Omega|.
\end{aligned} \tag{2.6}
$$

Integrating by parts and using **(H3)**, we get

$$
2\langle \partial_s \eta_n^t, \eta_n^t\rangle_{1,\mu} = -\int_0^\infty \mu'(s)\|\nabla \eta_n^t(s)\|^2\, ds \ge 0. \tag{2.7}
$$

Thus, the term $2\langle \partial_s \eta_n^t, \eta_n^t\rangle_{1,\mu}$ in (2.6) can be neglected and we obtain

$$
\frac{d}{dt}\|z_n\|_{\mathcal{H}_1}^2 + 2(1-\beta/\lambda_1-\varepsilon)\|\nabla u_n\|^2 \le C(\|g\|_{H^{-1}(\Omega)}^2 + 1).
$$

Choosing $\varepsilon > 0$ small enough so that $1-\beta/\lambda_1-\varepsilon > 0$ and integrating on $(0, t)$, $t \in (0, T)$, leads to

$$
\begin{aligned}
\|z_n(t)\|_{\mathcal{H}_1}^2 &+ 2(1-\beta/\lambda_1-\varepsilon)\int_0^t \|\nabla u_n(r)\|^2\, dr\\
&\le \|z_0\|_{\mathcal{H}_1}^2 + CT(\|g\|_{H^{-1}(\Omega)}^2 + 1).
\end{aligned}
$$

Hence, in particular, we see that

$$\begin{aligned} &\{u_n\} \text{ is bounded in } L^\infty(0,T;H_0^1(\Omega)),\\ &\{\eta_n^t\} \text{ is bounded in } L^\infty(0,T;L^2_\mu(\mathbb{R}^+,H_0^1(\Omega))). \end{aligned} \tag{2.8}$$

Therefore, by the Banach–Alaoglu theorem, there exists a function $z = (u,\eta^t)$ such that

$$\begin{aligned} u_n &\rightharpoonup u \quad \text{weakly-star in } L^\infty(0,T;H_0^1(\Omega)),\\ \eta_n^t &\rightharpoonup \eta^t \quad \text{weakly-star in } L^\infty(0,T;L^2_\mu(\mathbb{R}^+,H_0^1(\Omega))), \end{aligned} \tag{2.9}$$

and

$$\begin{aligned} \Delta u_n &\rightharpoonup u \quad \text{weakly in } L^2(0,T;H^{-1}(\Omega)),\\ \Delta\eta_n^t &\rightharpoonup \eta^t \quad \text{weakly in } L^2(0,T;L^2_\mu(\mathbb{R}^+,H^{-1}(\Omega))), \end{aligned} \tag{2.10}$$

up to a subsequence.

Now, we estimate $\partial_t z_n$. From (2.4) and (2.7), we get

$$\frac{d}{dt}\|z_n\|^2_{\mathcal{H}} + \|\nabla u_n\|^2 + 2\int_\Omega f(u_n)u_n\,dx \le \|g\|^2_{H^{-1}(\Omega)}. \tag{2.11}$$

Integrating (2.11) from 0 to T, we obtain

$$\|z_n(T)\|^2_{\mathcal{H}} + \int_0^T \|\nabla u_n(t)\|^2\,dt + 2\int_{Q_T} f(u_n)u_n\,dx\,dt \le \|z_0\|^2_{\mathcal{H}_1} + T\|g\|^2_{H^{-1}(\Omega)}.$$

In particular,

$$\int_{Q_T} f(u_n)u_n\,dx\,dt \le C. \tag{2.12}$$

In the first equation in (2.1), replacing (ω_k,ζ_j) by $(\partial_t u_n,\partial_t\eta_n^t) = \partial_t z_n$, and then using the Cauchy inequality and the fact that $\eta^t_{nt} = u_n - \eta^t_{ns}$, we get

$$\begin{aligned} &\frac{d}{dt}\Big(\|\nabla u_n\|^2 + 2\int_\Omega F(u_n)\,dx\Big) + 2\|\partial_t z_n\|^2_{\mathcal{H}_1} + 2\|\partial_t\nabla u_n\|^2\\ &\quad\le C(\varepsilon)\int_0^\infty \mu(s)\|\nabla\eta_n^t\|^2\,ds + \varepsilon\|\mu\|_{L^1(\mathbb{R}^+)}\|\partial_t\nabla u_n\|^2 + 2\|\partial_t\eta_n^t\|^2_{1,\mu}\\ &\qquad + \varepsilon\|\partial_t\nabla u_n\|^2 + C(\varepsilon)\|g\|^2_{H^{-1}(\Omega)}. \end{aligned}$$

Hence, by choosing ε small enough, we arrive at

$$\|\partial_t\nabla u_n\|^2 + \frac{d}{dt}\Big(\|\nabla u_n\|^2 + 2\int_\Omega F(u_n)\,dx\Big) \le C(\|\eta_n^t\|^2_{1,\mu} + \|g\|^2_{H^{-1}(\Omega)}).$$

Integrating from 0 to t and using (1.4), (2.12) and (2.8), we deduce that

$$\{\partial_t u_n\} \text{ is bounded in } L^2(0,T;H_0^1(\Omega)),$$

so, up to a subsequence,

$$\begin{aligned} \partial_t u_n &\rightharpoonup u_t \qquad \text{weakly in } L^2(0,T;H_0^1(\Omega)), \\ \Delta\partial_t u_n &\rightharpoonup \Delta u_t \qquad \text{weakly in } L^2(0,T;H^{-1}(\Omega)). \end{aligned} \tag{2.13}$$

We now prove that $\{f(u_n)\}$ is bounded in $L^1(Q_T)$. Set $h(s) = f(s) - f(0) + \gamma s$, where $\gamma > \ell$. Note that $h(s)s = (f(s) - f(0))s + \gamma s^2 = f'(c)s^2 + \gamma s^2 \geq (\gamma - \ell)s^2 \geq 0$ for all $s \in \mathbb{R}$, we have

$$\begin{aligned} \int_{Q_T} |h(u_n)|\,dx\,dt &\leq \int_{Q_T\cap\{|u_n|>1\}} |h(u_n)u_n|\,dx\,dt + \int_{Q_T\cap\{|u_n|\leq 1\}} |h(u_n)|\,dx\,dt \\ &\leq \int_{Q_T} h(u_n)u_n\,dx\,dt + \sup_{|s|\leq 1}|h(s)|\,|Q_T| \\ &\leq \int_{Q_T} f(u_n)u_n\,dx\,dt + |f(0)|\,\|u_n\|_{L^1(Q_T)} + \gamma\|u_n\|^2_{L^2(Q_T)} \\ &\quad + \sup_{|s|\leq 1}|h(s)|\,|Q_T| \\ &\leq C, \end{aligned}$$

where we have used (2.8), (2.12) and the boundedness of $\{u_n\}$ in the space $L^\infty(0,T;H_0^1(\Omega))$. Hence $\{h(u_n)\}$, and therefore $\{f(u_n)\}$, is bounded in $L^1(Q_T)$.

Using the Aubin–Lions lemma [Lio], we can suppose that $u_n \to u$ strongly in $L^2(0,T;L^2(\Omega))$. Hence $u_n \to u$ a.e. in Q_T, up to a subsequence. Moreover, using the definition of $h(s)$ and (2.11), (2.8), we obtain

$$\int_{Q_T} h(u_n)u_n\,dx\,dt \leq C.$$

Therefore, by [G, Lemma 6.1], we find that $h(u) \in L^1(Q_T)$ and for all test functions $\varphi \in C_0^\infty([0,T];H_0^1(\Omega)\cap L^\infty(\Omega))$,

$$\int_{Q_T} h(u_n)\varphi\,dx\,dt \to \int_{Q_T} h(u)\varphi\,dx\,dt.$$

Hence, $f(u) \in L^1(Q_T)$ and

$$\int_{Q_T} f(u_n)\varphi\,dx\,dt \to \int_{Q_T} f(u)\varphi\,dx\,dt \quad \text{for all } \varphi \in C_0^\infty([0,T];H_0^1(\Omega)\cap L^\infty(\Omega)). \tag{2.14}$$

We are now ready to show that the limit $z = (u,\eta^t)$ is a weak solution of (1.8). Choose an arbitrary test function

$$\phi = (\varphi,\xi) \in \mathcal{D}([0,T],H_0^1(\Omega)\cap L^\infty(\Omega)) \times \mathcal{D}([0,T],\mathcal{D}(\mathbb{R}^+,H_0^1(\Omega)))$$

of the form

$$\varphi(t)=\sum_{j=1}^{m} a_j(t)\omega_j \quad\text{and}\quad \xi(t)=\sum_{j=1}^{m} b_j(t)\zeta_j,$$

where m is a fixed integer, and $\{a_j\}_{j=1}^m$ and $\{b_j\}_{j=1}^m$ are given functions in $\mathcal{D}((0,T))$. Then (2.1) holds with $(v(t),\xi(t))$ in place of (ω_k,ζ_j). Integrating the resulting equation over $(0,T)$ and passing to the limits, in view of (2.9), (2.10), (2.13) and (2.14), we get

$$\begin{aligned}\int_0^T [\langle u_t,\varphi\rangle + \langle\nabla u_t,\nabla\varphi\rangle + \langle \eta_t^t,\xi\rangle_{1,\mu}]\,dt \\ = -\int_0^T [\langle\nabla u,\nabla\varphi\rangle + \langle\eta^t,\varphi\rangle_{1,\mu}]\,dt - \int_0^T\Big[\int_\Omega f(u)\varphi\,dx - \langle g,\varphi\rangle_{H^{-1},H_0^1}\Big]\,dt \\ + \int_0^T [-\langle\partial_s\eta^t,\xi\rangle_{1,\mu} + \langle u,\xi\rangle_{1,\mu}]\,dt.\end{aligned}$$

Using a density argument, we conclude that $z=(u,\eta^t)$ satisfies the equation in the weak sense. Moreover, by standard arguments, we can check that z satisfies the initial condition $z(0)=z_0$. This implies that $z(\cdot)$ is a weak solution of problem (1.8).

(ii) *Uniqueness and continuous dependence on the initial data.* We assume that $z_1=(u_1,\eta_1^t)$ and $z_2=(u_2,\eta_2^t)$ are solutions of (1.8) with initial data z_{10} and z_{20}, respectively. Denote $w=z_1-z_2=(u_3,\eta_3^t)$. Then

$$\partial_t u_3 - \Delta\partial_t u_3 - \Delta u_3 - \int_0^\infty \mu(s)\Delta\eta_3^t(x,s)\,ds + (\hat f(u_1(t) - \hat f(u_2)) - \ell u_3 = 0 \tag{2.15}$$

for all $t>0$,where $\hat f(s)=f(s)+\ell s$. Here because $u_3(t)$ does not belong to $W=H_0^1(\Omega)\cap L^\infty(\Omega)$, we cannot choose $u_3(t)$ as a test function. Consequently, the proof will be more involved than that in [CMP, WYZ, WaZ].

We use some ideas in [GK]. Let

$$B_k(s)=\begin{cases} k & \text{if } s>k,\\ s & \text{if } |s|\le k,\\ -k & \text{if } s<-k.\end{cases}$$

Consider the corresponding Nemytskiĭ mapping $\hat B_k : W\to W$ defined by

$$\hat B_k(u_3)(x) = B_k(u_3(x)) \quad \text{for all } x\in\Omega.$$

By [KZPS, Theorem 4.7] (see also [GK, Lemma 2.3]), we know that $\|\hat B_k(u_3)-u_3\|_W\to 0$ as $k\to\infty$. Now multiplying (2.15) by $\hat B_k(u_3)$, then integrating over Ω we get

$$\frac{d}{dt}\Big(\int_\Omega u_3\hat{B}_k(u_3)\,dx+\int_\Omega \nabla u_3\nabla\hat{B}_k(u_3)\,dx-\frac{1}{2}(\|\hat{B}_k(u_3)\|^2+\|\nabla\hat{B}_k(u_3)\|^2)\Big)$$
$$+\int_\Omega \nabla u_3\nabla\hat{B}_k(u_3)\,dx+\int_0^\infty \mu(s)\int_\Omega \nabla\eta_3^t\nabla\hat{B}_k(u_3)\,dx\,ds$$
$$+\int_\Omega(\hat{f}(u_1)-\hat{f}(u_2))\hat{B}_k(u_3)\,dx-\ell\int_\Omega u_3\hat{B}_k(u_3)\,dx=0.$$

Thus,

$$\frac{d}{dt}\Big(\int_\Omega u_3\hat{B}_k(u_3)\,dx+\int_\Omega \nabla u_3\nabla\hat{B}_k(u_3)\,dx-\frac{1}{2}(\|\hat{B}_k(u_3)\|^2+\|\nabla\hat{B}_k(u_3)\|^2)\Big)$$
$$+\int_{\{x\in\Omega:\,|u(x,t)|\le k\}}|\nabla u_3|^2\,dx+\int_0^\infty \mu(s)\int_{\{x\in\Omega:\,|u(x,t)|\le k\}}\nabla\eta_3^t\nabla u_3\,dx\,ds$$
$$+\int_\Omega \hat{f}'(\xi)u_3\hat{B}_k(u_3)\,dx=\ell\int_\Omega u_3\hat{B}_k(u_3)\,dx. \tag{2.16}$$

Note that $\hat{f}'(s)\ge 0$ and $sB_k(s)\ge 0$ for all $s\in\mathbb{R}$, we have

$$\int_\Omega \hat{f}'(\xi)u_3\hat{B}_k(u_3)\,dx\ge 0.$$

Moreover,

$$\int_{\{x\in\Omega:\,|u(x,t)|\le k\}}|\nabla u_3|^2\,dx\ge 0,$$

and

$$\int_0^\infty \mu(s)\int_{\{x\in\Omega:\,|u(x,t)|\le k\}}\nabla\eta_3^t\nabla u_3\,dx\,ds$$
$$=\int_0^\infty \mu(s)\int_{\{x\in\Omega:\,|u(x,t)|\le k\}}\nabla\eta_3^t\partial_t\nabla\eta_3^t\,dx\,ds$$
$$+\int_0^\infty \mu(s)\int_{\{x\in\Omega:\,|u(x,t)|\le k\}}\nabla\eta_3^t\partial_s\nabla\eta_3^t\,dx\,ds$$
$$=\int_0^\infty \mu(s)\int_{\{x\in\Omega:\,|u(x,t)|\le k\}}\nabla\eta_3^t\partial_t\nabla\eta_3^t\,dx\,ds$$
$$-\frac{1}{2}\int_0^\infty \mu'(s)\int_{\{x\in\Omega:\,|u(x,t)|\le k\}}|\nabla\eta_3^t|^2\,dx\,ds$$
$$\ge\frac{1}{2}\int_0^\infty \mu(s)\int_{\{x\in\Omega:\,|u(x,t)|\le k\}}\frac{d}{dt}|\nabla\eta_3^t|^2\,dx\,ds.$$

From the above inequalities and (2.16) we deduce that

$$\frac{d}{dt}\Big(\int_\Omega u_3\hat{B}_k(u_3)\,dx + \int_\Omega \nabla u_3\nabla\hat{B}_k(u_3)\,dx - \frac{1}{2}(\|\hat{B}_k(u_3)\|^2 + \|\nabla\hat{B}_k(u_3)\|^2)\Big) + \frac{1}{2}\int_0^\infty \mu(s) \int_{\{x\in\Omega:\,|u(x,t)|\le k\}} \frac{d}{dt}|\nabla\eta_3^t|^2\,dx\,ds \le \ell\int_\Omega u_3\hat{B}_k(u_3)\,dx.$$

Integrating from 0 to t, where $t\in(0,T)$, then letting $k\to\infty$, we obtain

$$\begin{aligned}\|u_3(t)\|^2 &+ \|\nabla u_3(t)\|^2 + \|\eta_3^t\|_{1,\mu}^2\\ &\le \|u_3(0)\|^2 + \|\nabla u_3(0)\|^2 + \|\eta_3^0\|_{1,\mu}^2 + 2\ell\int_0^t \|u_3(s)\|^2\,ds\\ &\le \|u_3(0)\|^2 + \|\nabla u_3(0)\|^2 + \|\eta_3^0\|_{1,\mu}^2 + 2\ell\int_0^t (\|u_3\|^2 + \|\nabla u_3\|^2 + \|\eta_3^s\|_{1,\mu}^2)\,ds.\end{aligned}$$

By the Gronwall inequality in integral form, we get

$$\|w(t)\|_{\mathcal{H}_1}^2 \le \|w(0)\|_{\mathcal{H}_1}^2 e^{2\ell t} \le \|w(0)\|_{\mathcal{H}_1}^2 e^{2\ell T} \quad \text{for all } t\in[0,T].$$

Hence we get the continuous dependence of the solutions on the initial data, and in particular the uniqueness when $w(0)=0$. ■

3. Existence of a global attractor. Theorem 2.2 allows us to define a continuous semigroup $S(t):\mathcal{H}_1\to\mathcal{H}_1$ associated to problem (1.8) by the formula

$$S(t)z_0 := z(t),$$

where $z(\cdot)$ is the unique global weak solution of (1.8) with the initial datum $z_0\in\mathcal{H}_1$. The aim of this section is to prove the following theorem.

THEOREM 3.1. *Assume that* **(H1)–(H3)** *hold. Then the semigroup* $\{S(t)\}_{t\ge0}$ *possesses a compact global attractor in* $\mathcal{H}_1$.

To prove this theorem, by the classical abstract results on existence of global attractors (see e.g. [Te, Theorem 1.1]), we need to show that the semigroup $S(t)$ has a bounded absorbing set B_0 in $\mathcal{H}_1$ and $S(t)$ is asymptotically compact in $\mathcal{H}_1$, that is, for any $t>0$, it can be decomposed in the form

$$S(t) = S_1(t) + S_2(t),$$

where for any bounded subset B in $\mathcal{H}_1$, we have

(i) $S_1(t)$ is a continuous mapping from $\mathcal{H}_1$ into itself and

$$r_B(t) = \sup_{y\in B} \|S_1(t)y\|_{\mathcal{H}_1} \to 0 \quad \text{as } t \to \infty;$$

(ii) the operators $S_2(t)$ are uniformly compact for t large, i.e., $\bigcup_{t\ge t_0} S_2(t)B$ is relatively compact in $\mathcal{H}_1$ for some $t_0 > 0$.

To prove (ii) we only need to show that for some $T > 0$, $S_2(T)B_0$ is relatively compact in $\mathcal{H}_1$, with B_0 a bounded absorbing set of $S(t)$.

3.1. Existence of an absorbing set

LEMMA 3.2. *Let* **(H1)**–**(H3)** *hold. Then there exists a bounded absorbing set in $\mathcal{H}_1$ for the semigroup $S(t)$.*

Proof. Multiplying the first equation of (1.8) by $u(t)$ and integrating over Ω, we obtain

$$\frac{1}{2}\frac{d}{dt}(\|u\|^2 + \|\nabla u\|^2 + \|\eta^t\|_{1,\mu}^2) + \|\nabla u\|^2 + \langle \eta^t(s), \eta_s^t(s)\rangle_{1,\mu} + \int_\Omega f(u)u\,dx = \langle g, u\rangle_{H^{-1}(\Omega),H_0^1(\Omega)}.$$

Using the hypothesis (1.3) and the Cauchy inequality, we have

$$\int_\Omega f(u)u\,dx \ge -\beta\|u\|^2 - C_0|\Omega|,$$

$$\langle g, u\rangle_{H^{-1}(\Omega),H_0^1(\Omega)} \le \varepsilon\|\nabla u\|^2 + \frac{1}{4\varepsilon}\|g\|_{H^{-1}(\Omega)}^2.$$

By integrating by parts,

$$2\langle \eta^t(s), \eta_s^t(s)\rangle_{1,\mu} = \int_0^\infty \mu(s)\frac{d}{ds}\|\nabla\eta^t(s)\|^2\,ds = -\int_0^\infty \mu'(s)\|\nabla\eta^t(s)\|^2\,ds \ge 0.$$

Therefore,

$$\frac{d}{dt}(\|u\|^2 + \|\nabla u\|^2 + \|\eta^t\|_{1,\mu}^2) + 2\left(1 - \frac{\beta}{\lambda_1} - \varepsilon\right)\|\nabla u\|^2 \le \frac{1}{2\varepsilon}\|g\|_{H^{-1}(\Omega)}^2 + 2C_0|\Omega|.$$

By (1.6) we get

$$\int_0^\infty \kappa(s)\|\eta^t(s)\|_{H_0^1(\Omega)}^2\,ds \le \theta\|\eta^t\|_{1,\mu}^2 \le \theta(\|u\|^2 + \|\nabla u\|^2 + \|\eta^t\|_{1,\mu}^2).$$

Moreover, using the second equation of (1.8) and exploiting (1.6) again yields

$$
\begin{aligned}
&\frac{d}{dt}\Big(\int_0^\infty \kappa(s)\|\eta^t(s)\|^2_{H_0^1(\Omega)}\,ds\Big)\\
&= -2\int_0^\infty \kappa(s)\int_\Omega \nabla\eta^t_s\nabla\eta^t\,dx\,ds + 2\int_0^\infty \kappa(s)\langle\eta^t(s),u\rangle_{H_0^1(\Omega)}\,ds\\
&= -\int_0^\infty \kappa(s)\frac{d}{ds}\|\nabla\eta^t(s)\|^2\,ds + 2\int_0^\infty \kappa(s)\langle\eta^t(s),u\rangle_{H_0^1(\Omega)}\,ds\\
&= -\kappa(s)\|\nabla\eta^t(s)\|^2\Big|_{s=0}^{s=\infty} + \int_0^\infty \kappa'(s)\|\nabla\eta^t(s)\|^2\,ds + 2\int_0^\infty \kappa(s)\langle\eta^t(s),u\rangle_{H_0^1(\Omega)}\,ds\\
&= -\|\eta^t\|^2_{1,\mu} + 2\int_0^\infty \kappa(s)\langle\eta^t(s),u\rangle_{H_0^1(\Omega)}\,ds\\
&\le -\|\eta^t\|^2_{1,\mu} + 2\theta\int_0^\infty \mu(s)\int_\Omega \nabla\eta^t\cdot\nabla u\,dx\,ds\\
&\le -\|\eta^t\|^2_{1,\mu} + 2\theta\Big(\int_0^\infty \mu(s)\|\nabla\eta^t\|^2\,ds\Big)^{1/2}\Big(\int_0^\infty \mu(s)\|\nabla u\|^2\,ds\Big)^{1/2}\\
&\le -\tfrac{1}{2}\|\eta^t\|^2_{1,\mu} + 2\theta^2\|\nabla u\|^2\int_0^\infty \kappa'(s)\,ds = -\tfrac{1}{2}\|\eta^t\|^2_{1,\mu} + 2\theta^2\kappa(0)\|\nabla u\|^2.
\end{aligned}
$$

Here we have used assumption **(H3)** and the Cauchy inequality.

Now, for $\gamma > 0$ to be specified, we define the functional

$$\Phi(t) = \|u\|^2 + \|\nabla u\|^2 + \|\eta^t\|^2_{1,\mu} + 4\gamma\int_0^\infty \kappa(s)\|\eta^t(s)\|^2_{H_0^1(\Omega)}\,ds.$$

It satisfies the differential inequality

$$\frac{d}{dt}\Phi + 2\Big(1-\frac{\beta}{\lambda_1}-\varepsilon-4\gamma\theta^2\kappa(0)\Big)\|\nabla u\|^2 + 2\gamma\|\eta^t\|^2_{1,\mu} \le C\|g\|^2_{H^{-1}(\Omega)} + 2C_0|\Omega|.$$

Choosing $\varepsilon,\gamma > 0$ small enough such that

$$\gamma\Big(1+\frac{1}{\lambda_1}\Big) \le 1-\frac{\beta}{\lambda_1}-\varepsilon-4\gamma\theta^2\kappa(0),$$

we have

$$\frac{d}{dt}\Phi + 2\gamma(\|u\|^2 + \|\nabla u\|^2 + \|\eta^t\|^2_{1,\mu}) \le C\|g\|^2_{H^{-1}(\Omega)} + 2C_0|\Omega|.$$

Up to further reducing γ, we also have

$$\|u\|^2 + \|\nabla u\|^2 + \|\eta^t\|^2_{1,\mu} \le \Phi \le 2(\|u\|^2 + \|\nabla u\|^2 + \|\eta^t\|^2_{1,\mu}).$$

Hence

$$\frac{d}{dt}\Phi + \gamma\Phi \le C\|g\|^2_{H^{-1}(\Omega)} + 2C_0|\Omega|.$$

By the Gronwall inequality, we get

$$\Phi(t) \le \Phi(0)e^{-\gamma t} + \int_0^t e^{-\gamma(t-s)}(C\|g\|^2_{H^{-1}(\Omega)} + 2C_0|\Omega|)\,ds.$$

Thus,

$$\begin{aligned} y(t) &\le 2e^{-\gamma t}y(0) + e^{-\gamma t}\int_0^t e^{\gamma s}(C\|g\|^2_{H^{-1}(\Omega)} + 2C_0|\Omega|)\,ds \\ &\le 2e^{-\gamma t}(\|u_0\|^2 + \|\nabla u_0\|^2 + \|\eta^0\|^2_{1,\mu}) + C_3(\|g\|^2_{H^{-1}(\Omega)} + 1), \end{aligned}$$

where

$$y(t) = \|u\|^2 + \|\nabla u\|^2 + \|\eta^t\|^2_{1,\mu}.$$

Hence there exists $\rho_0 > 0$ such that

$$\|z(t)\|^2_{\mathcal{H}_1} \le \rho_0 \tag{3.1}$$

for all $z_0 \in B$ and $t \ge t_0 = t_0(B)$, where B is an arbitrary bounded subset of $\mathcal{H}_1$. This completes the proof. ∎

3.2. Asymptotic compactness. Recall that we only assume that $g \in H^{-1}(\Omega)$. However, we know that for any $g \in H^{-1}(\Omega)$ and $\varepsilon > 0$, there is a $g^\varepsilon \in L^2(\Omega)$, which depends on g and ε, such that

$$\|g - g^\varepsilon\|_{H^{-1}(\Omega)} < \varepsilon. \tag{3.2}$$

3.2.1. *Decomposition of the equation.* To make the asymptotic estimates, we decompose the solution $S(t)z_0 = z(t)$ of problem (1.8) as

$$S(t)z_0 = S_1(t)z_0 + S_2(t)z_0,$$

where $S_1(t)z_0 = z_1(t)$ and $S_2(t)z_0 = z_2(t)$, that is, $z = (u, \eta^t) = z_1 + z_2$, with

$$\begin{aligned} u = v^\varepsilon + w^\varepsilon, &\quad \eta^t = \zeta^{t\varepsilon} + \xi^{t\varepsilon}, \\ z_1 = (v^\varepsilon, \zeta^{t\varepsilon}), &\quad z_2 = (w^\varepsilon, \xi^{t\varepsilon}), \end{aligned}$$

where $z_1(t)$ is the unique solution of the problem

$$\begin{cases} v_t^\varepsilon - \Delta v_t^\varepsilon - \Delta v^\varepsilon + f(u) \\ \qquad -f(w^\varepsilon) - \int_0^\infty \mu(s)\Delta\zeta^{t\varepsilon}(s)\,ds + \lambda v^\varepsilon = g - g^\varepsilon, \quad \lambda > \ell, \\ \partial_t \zeta^{t\varepsilon} = -\partial_s \zeta^{t\varepsilon} + v^\varepsilon, \\ v^\varepsilon(x,t)|_{\partial\Omega} = 0, \quad v^\varepsilon(x,t)|_{t=0} = u_0(x), \\ \zeta^{t\varepsilon}(x,s)|_{\partial\Omega} = 0, \quad \zeta^0(x,s) = \zeta_0(x,s) := \int_0^s g_0(x,r)\,dr, \end{cases} \tag{3.3}$$

and $z_2(t)$ is the unique solution of the problem

$$(3.4)\qquad \begin{cases} w_t^\varepsilon - \Delta w_t^\varepsilon - \Delta w^\varepsilon + f(w^\varepsilon) \\ \qquad - \int_0^\infty \mu(s)\Delta \xi^{t\varepsilon}(s)\,ds - \lambda(u - w^\varepsilon) = g^\varepsilon, \quad \lambda > \ell, \\ \partial_t \xi^{t\varepsilon} = -\partial_s \xi^{t\varepsilon} + w^\varepsilon, \\ w^\varepsilon(x,t)|_{\partial\Omega} = 0, \quad w^\varepsilon(x,t)|_{t=0} = 0, \\ \xi^{t\varepsilon}(x,s)|_{\partial\Omega} = 0, \quad \xi^0(x,s) = \xi_0(x,s) = 0. \end{cases}$$

By using similar arguments to the proof of Theorem 2.2, one can prove the existence and uniqueness of solutions to problems (3.3) and (3.4). Moreover, for problem (3.4), because $g^\varepsilon \in L^2(\Omega)$ and the initial data are zero (so belong to $\mathcal{H}_2 := \big(H^2(\Omega) \cap H_0^1(\Omega)\big) \times L^2_\mu(\mathbb{R}^+, H^2(\Omega) \cap H_0^1(\Omega)))$, we can show that the solution $(w^\varepsilon, \xi^{t\varepsilon})$ is in fact a strong solution. In particular, we will have $w^\varepsilon \in C([0,T]; H^2(\Omega) \cap H_0^1(\Omega))$ for any $T > 0$, and this will be used in the proof of Lemma 3.4 below.

3.2.2. *The first a priori estimate.* We begin with the decay estimate for solutions of (3.3).

LEMMA 3.3. *Let hypotheses* **(H1)**–**(H3)** *hold. Then for any $\varepsilon > 0$, the solutions of problem* (3.3) *satisfy the following estimates: there is a constant d_0, which depends on λ_1, ℓ, such that for every $t \ge 0$,*

$$\|S_1(t)z_0\|^2_{\mathcal{H}_1} \le Q(\|z_0\|_{\mathcal{H}_1})e^{-d_0 t} + \varepsilon.$$

Proof. Multiplying the first equation of (3.3) by v^ε we get

$$\frac{1}{2}\frac{d}{dt}(\|v^\varepsilon\|^2 + \|\nabla v^\varepsilon\|^2) + \lambda\|v^\varepsilon\|^2 + \|\nabla v^\varepsilon\|^2 + \int_0^\infty \mu(s)\int_\Omega \nabla\zeta^{t\varepsilon}\nabla v^\varepsilon\,dx\,ds \\ + \langle f(u) - f(w^\varepsilon), v^\varepsilon\rangle = \langle g - g^\varepsilon, v^\varepsilon\rangle_{H^{-1},H_0^1}.$$

Noting that $\zeta_t^{t\varepsilon} = -\zeta_s^{t\varepsilon} + v^\varepsilon$, we have

$$\int_0^\infty \mu(s)\int_\Omega \nabla\zeta^{t\varepsilon}\nabla v^\varepsilon\,dx\,ds \\ = \int_0^\infty \mu(s)\int_\Omega \nabla\zeta^{t\varepsilon}\nabla\zeta_t^{t\varepsilon}\,dx\,ds + \int_0^\infty \mu(s)\int_\Omega \nabla\zeta^{t\varepsilon}\nabla\zeta_s^{t\varepsilon}\,dx\,ds \\ = \frac{1}{2}\frac{d}{dt}\|\zeta^{t\varepsilon}\|^2_{1,\mu} + \int_0^\infty \mu(s)\int_\Omega \nabla\zeta^{t\varepsilon}\nabla\zeta_s^{t\varepsilon}\,dx\,ds,$$

and by the Cauchy inequality,

$$\langle g - g^\varepsilon, v^\varepsilon\rangle_{H^{-1},H_0^1} \le \tfrac{1}{2}\|\nabla v^\varepsilon\|^2 + \tfrac{1}{2}\|g - g^\varepsilon\|^2_{H^{-1}(\Omega)}.$$

Therefore, noticing that $f'(s) \geq -\ell$, we have

$$\frac{d}{dt}(\|v^\varepsilon\|^2 + \|\nabla v^\varepsilon\|^2 + \|\zeta^{t\varepsilon}\|^2_{1,\mu}) + \|\nabla v^\varepsilon\|^2 + 2(\lambda - \ell)\|v^\varepsilon\|^2 + 2\int_0^\infty \mu(s)\int_\Omega \nabla\zeta^{t\varepsilon}\nabla\zeta^{t\varepsilon}_s\,dx\,ds \leq \|g - g^\varepsilon\|^2_{H^{-1}(\Omega)}.$$

Similarly to the proof of Lemma 3.2 we obtain, for some $d_0 > 0$,

$$\|S_1(t)z_0\|^2_{\mathcal{H}_1} \leq Q(\|z_0\|_{\mathcal{H}_1})e^{-d_0t} + \frac{1}{C}\|g - g^\varepsilon\|^2_{H^{-1}(\Omega)}.$$

Taking $\varepsilon^2 \leq C\varepsilon$ in (3.2) we conclude that

$$\|S_1(t)z_0\|^2_{\mathcal{H}_1} \leq Q(\|z_0\|_{\mathcal{H}_1})e^{-d_0t} + \varepsilon. \ \blacksquare$$

3.2.3. *The second a priori estimate.* For the solution $z_2(t)$ of (3.4), we have

LEMMA 3.4. *Let* **(H1)**–**(H3)** *hold. Then for any $\varepsilon > 0$, there is $M > 0$ such that for any $z_0 \in \mathcal{H}_1$, there exists $T > 0$ large enough, which depends on $\|g\|^2_{H^{-1}(\Omega)}$, ε, $\|z_0\|^2_{\mathcal{H}_1}$, such that*

$$\|S_2(t)z_0\|^2_{\mathcal{H}_2} \leq M \quad \text{for all } t \geq T. \tag{3.5}$$

Proof. Multiplying the first equation of (3.4) by $-\Delta w^\varepsilon$, then using (1.2) and the Cauchy inequality, we obtain

$$\begin{aligned}\frac{d}{dt}(\|\nabla w^\varepsilon\|^2 + \|\Delta w^\varepsilon\|^2 + \|\xi^{t\varepsilon}\|^2_{2,\mu}) + \|\Delta w^\varepsilon\|^2 + 2(\lambda - \ell)\|\nabla w^\varepsilon\|^2 \\ + 2\int_0^\infty \mu(s)\int_\Omega \Delta\zeta^{t\varepsilon}\Delta\zeta^{t\varepsilon}_s\,dx\,ds \\ \leq \|g^\varepsilon\|^2_{H^{-1}(\Omega)} + \lambda^2\|u\|^2 \leq C(\|g^\varepsilon\|^2_{H^{-1}(\Omega)} + \rho_0)\end{aligned}$$

when $t \geq t_0(B)$, where we have used (3.1). Notice that

$$\int_0^\infty \mu(s)\int_\Omega \Delta\zeta^{t\varepsilon}\Delta\zeta^{t\varepsilon}_s\,dx\,ds = -\int_0^\infty \mu'(s)\|\Delta\zeta^{t\varepsilon}\|^2\,ds \geq 0,$$

so we can omit this term in the above inequality. Hence, similarly to the proof of Lemma 3.2, we obtain a number $T > 0$ large enough such that

$$\|S_2(t)z_0\|^2_{\mathcal{H}_2} \leq M \quad \text{for all } t \geq T. \ \blacksquare$$

In addition, for any $\xi_0 \in L^2_\mu(\mathbb{R}^+, H^1_0(\Omega))$, the Cauchy problem (see e.g. [BP, PZ])

$$\begin{cases}\partial_t\xi^t = -\partial_s\xi^t + w, \quad t > 0,\\ \xi^0 = \xi_0,\end{cases}$$

has a unique solution $\xi^t \in C(\mathbb{R}^+; L^2_\mu(\mathbb{R}^+, H^1_0(\Omega)))$, and

$$\xi^t(s) = \begin{cases} \int_0^s w(t-r)\,dr, & 0 < s \le t, \\ \xi_0(s-t) - \xi_0(0) + \int_0^t w(t-r)\,dr, & s > t. \end{cases} \tag{3.6}$$

So, for (3.6), thanks to $\xi^0(x,s) = 0$, we have

$$\xi^{t\varepsilon}(s) = \begin{cases} \int_0^s w^\varepsilon(t-r)\,dr, & 0 < s \le t, \\ \int_0^t w^\varepsilon(t-r)\,dr, & s > t. \end{cases} \tag{3.7}$$

Let B_0 be the bounded absorbing set obtained in Lemma 3.2. We will prove the following result.

LEMMA 3.5. *Assume* **(H1)**–**(H3)** *hold. Set*

$$\mathcal{K}^\varepsilon_T = PS_2(T)B_0$$

for $T > 0$ large enough, where $\{S_2(t)\}_{t\ge 0}$ is the solution semigroup of (3.4), *and $P : \mathcal{H}_1 \to L^2_\mu(\mathbb{R}^+, H^1_0(\Omega))$ is the canonical projection operator. Then there is a positive constant $M_1 = M_1(\|B_0\|_{\mathcal{H}_1})$ such that*

(i) *$\mathcal{K}^\varepsilon_T$ is bounded in $L^2_\mu(\mathbb{R}^+, H^2(\Omega) \cap H^1_0(\Omega)) \cap H^1_\mu(\mathbb{R}^+, H^1_0(\Omega))$;*
(ii) $\sup_{\xi \in \mathcal{K}^\varepsilon_T} \|\xi(s)\|^2_{H^1_0(\Omega)} \le M_1$.

Moreover, $\mathcal{K}^\varepsilon_T$ is relatively compact in $L^2_\mu(\mathbb{R}^+, H^1_0(\Omega))$.

Proof. From (3.7) we have

$$\partial_s \xi^{t\varepsilon}(s) = \begin{cases} w^\varepsilon(t-s), & 0 < s \le t, \\ 0, & s > t, \end{cases}$$

which, combined with Lemma 3.4, implies (i).

Using now (3.7) once again, we easily deduce that

$$\|\xi^{T\varepsilon}(s)\|^2_{H^1_0(\Omega)} \le \begin{cases} \int_0^s \|w^\varepsilon(T-r)\|^2_{H^1_0(\Omega)}\,dr \le \int_0^T \|w^\varepsilon(T-r)\|^2_{H^1_0(\Omega)}\,dr, & 0 < s \le T, \\ \int_0^T \|w^\varepsilon(T-r)\|^2_{H^1_0(\Omega)}\,dr, & s > T. \end{cases}$$

By (3.5), we know that (ii) holds. Because $H^2(\Omega) \cap H^1_0(\Omega) \hookrightarrow H^1_0(\Omega)$ compactly, we conclude that $\mathcal{K}_T$ is relatively compact in $L^2_\mu(\mathbb{R}^+, H^1_0(\Omega))$ thanks to the following lemma.

LEMMA 3.6 ([PZ]). *Assume that $\mu \in C^1(\mathbb{R}^+) \cap L^1(\mathbb{R}^+)$ is a nonnegative function and satisfies the following condition: if there exists $s_0 \in \mathbb{R}^+$ such that $\mu(s_0) = 0$, then $\mu(s) = 0$ for all $s \geq s_0$. Moreover, let X_0, X_1, X_2 be Banach spaces, where X_0, X_2 are reflexive and satisfy*

$$X_0 \hookrightarrow X_1 \hookrightarrow X_2,$$

and the embedding $X_0 \hookrightarrow X_1$ is compact. Let $\mathcal{C} \subset L^2_\mu(\mathbb{R}^+, X_1)$ satisfy

(i) *$\mathcal{C}$ is a subset in $L^2_\mu(\mathbb{R}^+, X_0) \cap H^1_\mu(\mathbb{R}^+, X_2)$;*
(ii) *$\sup_{\eta \in \mathcal{C}} \|\eta(s)\|^2_{X_1} \leq h(s)$ for all $s \in \mathbb{R}^+$, where $h \in L^1_\mu(\mathbb{R}^+)$.*

Then $\mathcal{C}$ is relatively compact in $L^2_\mu(\mathbb{R}^+, X_1)$. ■

Note that $\mathcal{H}_1 = H^1_0(\Omega) \times L^2_\mu(\mathbb{R}^+, H^1_0(\Omega))$, and since the embedding $H^2(\Omega) \cap H^1_0(\Omega) \hookrightarrow H^1_0(\Omega)$ is compact, we obtain

LEMMA 3.7. *Let $\{S_2(t)\}_{t\geq 0}$ be the solution semigroup of* (3.4). *Then under the assumption of Lemma* 3.5, *for $T > 0$ large enough, $S_2(T)B$ is relatively compact in $\mathcal{H}_1$.*

3.3. Proof of Theorem 3.1. By Lemma 3.2, the semigroup $S(t)$ has a bounded absorbing set B_0 in $\mathcal{H}_1$. Moreover, $S(t)$ is asymptotically compact in $\mathcal{H}_1$ by Lemmas 3.3 and 3.7. Therefore, the ω-limit set $\mathcal{A} = \omega(B_0)$ is the compact global attractor for $S(t)$ in $\mathcal{H}_1$.

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