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On some global solutions to 3d incompressible heat-conducting motions

Ewa Zadrzyńska and Wojciech M. Zajączkowski (Warszawa)

Abstract. We consider stability of solutions to stationary Navier–Stokes equations coupled with the heat equation in a set of solutions to the corresponding nonstationary system. The coupling is such that in the right-hand side of the Navier–Stokes equations there is a power function of temperature and in the equation for temperature there is a viscous dissipation term. We consider the non-slip boundary condition for velocity and the Dirichlet boundary condition for temperature. Moreover, the existence of a global strong-weak solution which remains close to the stationary solution for all time is proved.

1. Introduction. We consider the following initial-boundary value problem describing the motion of an incompressible heat-conducting fluid:

$$
\begin{aligned}
&v_t - \nu\Delta v + v\cdot\nabla v + \nabla p = \alpha(\theta)f && \text{in } \Omega\times\mathbb{R}_+,\\
&\operatorname{div} v = 0 && \text{in } \Omega\times\mathbb{R}_+,\\
&\theta_t - \varkappa\Delta\theta + v\cdot\nabla\theta = \nu|\mathbb{D}(v)|^2 && \text{in } \Omega\times\mathbb{R}_+,\\
&v = 0 && \text{on } S\times\mathbb{R}_+,\\
&\theta = \theta_* && \text{on } S\times\mathbb{R}_+,\\
&v|_{t=0} = v(0),\quad \theta|_{t=0} = \theta(0) && \text{in } \Omega,
\end{aligned}
\tag{1.1}
$$

where $\Omega\subset\mathbb{R}^3$ is a bounded domain, $S=\partial\Omega$, $v=(v_1(x,t),v_2(x,t),v_3(x,t))$ $\in\mathbb{R}^3$ is the velocity of the fluid, $p=p(x,t)\in\mathbb{R}$ is the pressure, $\theta=\theta(x,t)$ $\in\mathbb{R}$ is the temperature, $f=(f_1(x,t),f_2(x,t),f_3(x,t))\in\mathbb{R}^3$ is the external force field, $x=(x_1,x_2,x_3)$ is the Cartesian coordinate system, $\nu>0$ is the viscosity coefficient, $\varkappa>0$ is the constant conductivity coefficient and $\theta_*>0$

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is a constant. Moreover, by $\mathbb{D}(v) = \{v_{i,x_j} + v_{j,x_i}\}_{i,j=1,2,3} = \nabla v + \nabla v^T$ we denote the dilatation tensor.

Notice that the term $\alpha(\theta)f$ generates the motion by heating and mechanically by the external force f. The term $\nu|\mathbb{D}(v)|^2$ describes the heating of the fluid by viscous dissipation.

The purpose of the paper is to prove the existence of a strong-weak solution to problem (1.1) (see Definition 1.3) which is close to a strong-weak solution of the following stationary problem (see Definition 3.1):

$$
\begin{aligned}
&-\nu\Delta w + w\cdot\nabla w + \nabla q = \alpha(\vartheta)g && \text{in } \Omega,\\
&\operatorname{div} w = 0 && \text{in } \Omega,\\
&-\varkappa\Delta\vartheta + w\cdot\nabla\vartheta = \nu|\mathbb{D}(w)|^2 && \text{in } \Omega,\\
&w = 0 && \text{on } S,\\
&\vartheta = \theta^* && \text{on } S.
\end{aligned}
\tag{1.2}
$$

Therefore, first we prove existence of solutions to problem (1.2).

THEOREM 1.1. *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with C^2 boundary S. Let $g \in L_\infty(\Omega)$, $0 < \sigma < 1/8$, $\alpha \in C^1([\theta_*, \infty))$ and $|\alpha(\vartheta)| \le a_1 + a_2\vartheta^\sigma$, $|\alpha'(\vartheta)| \le a_3$ for $\vartheta \ge \theta_*$, where the constants a_i, $i = 1, 2, 3$, are such that $a_1 \ge 0$, $a_2 > 0$, $a_3 > 0$. Then there exists a strong-weak solution $(w, q, \vartheta) \in H^2(\Omega) \times H^1(\Omega) \times H^1(\Omega)$ to problem* (1.2) *with $\int_\Omega q(x)\,dx = 0$. Moreover,*

$$\|w\|_{H^2(\Omega)} + \|q\|_{H^1(\Omega)} + \|\vartheta - \theta_*\|_{H^1(\Omega)} \le c(\|g\|_{L_\infty(\Omega)})\|g\|_{L_\infty(\Omega)},$$

where $c = c(\|g\|_{L_\infty(\Omega)})$ is an increasing continuous function.

Theorem 1.1 is proved in Section 3 by using the Leray–Schauder fixed point theorem.

THEOREM 1.2. *Let the assumptions of Theorem* 1.1 *hold. Moreover, assume that*

$$\|g\|_{L_\infty(\Omega)} \le \delta_1,$$

where $\delta_1 > 0$ is sufficiently small. Then the solution (w, q, ϑ) of problem (1.2) *is unique.*

Now, let (w, q, ϑ) be the solution to problem (1.2) given by Theorems 1.1–1.2. We will examine its stability and then prove the global existence of a strong-weak solution to problem (1.1) which is close to (w, q, ϑ) for all time.

DEFINITION 1.3. Let $f \in L_{2,\mathrm{loc}}(\mathbb{R}_+, L_2(\Omega))$, $v(0) \in H_0^1(\Omega)$ with $\operatorname{div} v(0) = 0$, $\theta(0) \in L_2(\Omega)$, and let $T > 0$ be given. We say that the triple (v, p, θ) is a *strong-weak solution* to problem (1.1) if the function (v, θ) is a weak solution to (1.1) in $\Omega \times (kT, (k+1)T)$ with initial conditions $v|_{t=kT} = v(kT)$, $\theta|_{t=kT} = \theta(kT)$ for all $k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ such that $(v, \theta) \in L_\infty(kT, (k+1)T, L_2(\Omega)) \cap L_2(kT, (k+1)T; H^1(\Omega))$, $(v_t, \theta_t) \in (L_2(kT, (k+1)T; H^{-1}(\Omega)))^2$ for all $k \in \mathbb{N}_0$

and additionally $v \in L_\infty(kT,(k+1)T;H^1(\Omega)) \cap L_2(kT,(k+1)T;H^2(\Omega))$, $\nabla p \in L_2(kT,(k+1)T;L_2(\Omega))$ for all $k \in \mathbb{N}_0$.

In order to formulate the stability and global existence results we introduce the functions

$$u = v - w, \qquad \eta = p - q, \qquad \chi = \theta - \vartheta, \qquad h = f - g$$

and reduce problem (1.1) to the problem

$$\begin{aligned}
& u_t - \nu \Delta u + u\cdot\nabla u + \nabla\eta = -w\cdot\nabla u - u\cdot\nabla w \\
& \qquad + [\alpha(\theta)-\alpha(\vartheta)]f + \alpha(\vartheta)h && \text{in } \Omega\times\mathbb{R}_+,\\
& \operatorname{div} u = 0 && \text{in } \Omega\times\mathbb{R}_+,\\
& \chi_t - \varkappa\Delta\chi + u\cdot\nabla\chi = -w\cdot\nabla\chi - u\cdot\nabla\vartheta + \nu|\mathbb{D}(u)|^2 \\
& \qquad + 2\nu\mathbb{D}(u):\mathbb{D}(w) && \text{in } \Omega\times\mathbb{R}_+,\\
& u = 0 && \text{on } S\times\mathbb{R}_+,\\
& \chi = 0 && \text{on } S\times\mathbb{R}_+,\\
& u|_{t=0} = v(0) - w(0) \equiv u(0) && \text{in } \Omega,\\
& \chi|_{t=0} = \theta(0) - \vartheta(0) \equiv \chi(0) && \text{in } \Omega.
\end{aligned} \tag{1.3}$$

Now, we can formulate a theorem about the stability of a strong-weak stationary solution under nonstationary perturbations.

THEOREM 1.4. *Let the assumptions of Theorem* 1.2 *hold. Let* $u(0) \in H_0^1(\Omega)$, $\operatorname{div} u(0) = 0$, $\chi(0) \in L_2(\Omega)$, $\theta(0) \ge \theta_*$, *and let* $T > 0$ *be given. Assume that* $f \in C(\mathbb{R}_+, L_\infty(\Omega))$ *and*

$$\sup_{k\in\mathbb{N}_0} \|f\|_{C([kT,(k+1)T];L_\infty(\Omega))} \le \delta_1. \tag{1.4}$$

Moreover, suppose that

$$\|\nabla u(0)\|^2_{L_2(\Omega)} + \|\chi(0)\|^2_{L_2(\Omega)} \le \gamma \tag{1.5}$$

and

$$\|h(t)\|^2_{L_4} \le \delta_2\gamma \quad \text{for all } t \in \mathbb{R}_+, \tag{1.6}$$

where δ_2, $\gamma > 0$ *are some constants. Let* (w,ϑ,q) *be the strong-weak solution of problem* (1.2) *which exists in view of Theorem* 1.1. *Let* (u,η,χ) *be a strong-weak solution to problem* (1.3). *If* γ, δ_1 *and* δ_2 *are sufficiently small then*

$$\|\nabla u(t)\|^2_{L_2(\Omega)} + \|\chi(t)\|^2_{L_2(\Omega)} \le \gamma \quad \text{for all } t \in \mathbb{R}_+ \tag{1.7}$$

and

$$\begin{aligned}
& \|u\|^2_{H^{2,1}(\Omega\times(kT,(k+1)T))} + \|\chi\|^2_{L_2(kT,(k+1)T;H^1(\Omega))} \\
& \quad + \|\chi_t\|^2_{L_2(kT,(k+1)T;H^{-1}(\Omega))} + \|\nabla\eta\|^2_{L_2(kT,(k+1)T;L_2(\Omega))} \le c(T)\gamma,
\end{aligned} \tag{1.8}$$

where $c = c(T)$ *does not depend on* k.

The proof of Theorem 1.4 is based on a nonlinear differential inequality derived in the proof of Lemma 4.2 and a contradiction argument.

Let $H^{2,1}(\Omega\times(kT,(k+1)T))$ denote the space of functions with the finite norm

$$\|u\|_{H^{2,1}(\Omega\times(kT,(k+1)T))} = \Big(\|u_t\|^2_{L_2(\Omega\times(kT,(k+1)T))} + \sum_{0\le|\alpha|\le 2}\|D^\alpha u\|^2_{L_2(\Omega\times(kT,(k+1)T))}\Big)^{1/2}.$$

Using estimates (1.7)–(1.8) derived for the Faedo–Galerkin approximations we obtain the following theorem:

THEOREM 1.5. *Let the assumptions of Theorem* 1.2 *be satisfied. Let* $v(0)\in H^1_0(\Omega)$, $\operatorname{div} v(0)=0$, $\theta(0)\in L_2(\Omega)$, $\theta(0)\ge\theta_*$. *Assume* (w,q,ϑ) *is the strong-weak solution of problem* (1.2) *which exists in view of Theorem* 1.1. *Moreover, assume that* $f\in C(\mathbb{R}_+;L_\infty(\Omega))$ *and conditions* (1.4)–(1.6) *are satisfied. Let* $T>0$ *be given. If* γ, δ_1 *and* δ_2 *are sufficiently small then there exists a strong-weak global solution* $(v,p,\theta)\in H^{2,1}(\Omega\times(kT,(k+1)T))\times L_2(kT,(k+1)T;H^1(\Omega))\times C([kT,(k+1)T];L_2(\Omega))\cap L_2(kT,(k+1)T;H^1(\Omega))$, $k\in\mathbb{N}_0$, *to problem* (1.1) *with* $\int_\Omega p\,dx=0$.

The problem under consideration arises from a complete thermodynamical model of nonstationary flows of incompressible Newtonian fluids in a bounded domain. We restrict our considerations to temperature independent material coefficients ν and $\varkappa$. The presence of the dissipative heating term $\nu|\mathbb{D}(v)|^2$ in the heat equation $(1.1)_3$ follows from the general theory of incompressible heat-conducting Navier–Stokes motions.

The solvability of the uncoupled three-dimensional system $(1.1)_{1,2,3}$ with $\alpha(\theta)=1$ was examined by P.-L. Lions [7]. He proved the existence of a unique renormalized solution to equation $(1.1)_3$ with the initial condition on the temperature and with the Neumann boundary condition $\frac{\partial\theta}{\partial n}=0$ on $S\times(0,T)$, $T>0$, under the assumption that v is any weak solution to the Navier–Stokes equations with the initial condition and the Dirichlet boundary condition.

The two-dimensional system $(1.1)_{1,2,3}$, where the left-hand side of equation $(1.1)_3$ contains also the term $-e_2\cdot v$, $e_2=(0,1)$, was studied by Y. Kagei [4, 5]. The equations considered by Y. Kagei are supplemented with the Dirichlet and periodic boundary conditions. Moreover, it is assumed that $\alpha(\theta)f=e_2F(\theta)$, where either $F(\theta)=\theta$ or $\sup_{\theta\in\mathbb{R}}|F^{(k)}(\theta)|<\infty$ for $k\in\{0,1,2\}$. Kagei discussed the existence, uniqueness and large time behaviour questions in dependence on assumptions about data.

Hishida [3] examined an initial-boundary value problem for the three-dimensional Boussinesq system. Under some smallness assumptions on data he proved the unique existence of a global strong solution which uniformly tends to the stationary solution as $t \to \infty$ with an exponential rate.

The large time behaviour of the three-dimensional Boussinesq system was also studied in [2] and [8]. Brandolese and Schonbek [2] studied the decay and growth of both weak and strong solutions to the three-dimensional system, while Liu and Li [8] considered the stability of global regular solutions to the three-dimensional Cauchy problem for the Boussinesq equations.

The stability of a solution of the stationary Navier–Stokes system was examined by Kanbayashi, Kozono and Okabe [6] who considered the three-dimensional Navier–Stokes equations with zero external force and nonhomogeneous Dirichlet boundary condition. They proved the existence of a weak solution to the nonstationary problem which tends to a solution of the stationary problem as $t \to \infty$.

In [11] we considered problem (1.1) in a cylinder with the slip boundary conditions instead of the Dirichlet condition $(1.1)_4$. We proved the stability of a two-dimensional solution assuming that the external force does not decay as $t \to \infty$ and that

$$\sup_{\theta\in[\theta_*,\infty)} |\alpha^{(k)}(\theta)| < \infty \quad \text{for } k = 0, 1.$$

We also proved the existence and uniqueness of a strong-weak solution of the three-dimensional problem in a cylinder which is close to the solution of the two-dimensional problem.

In comparison with [11], we generalize in this paper the properties of the coupling function α (see Theorems 1.1, 1.2, 1.4, 1.5), which requires more complicated calculations.

Other stability results for some special solutions to the Navier–Stokes equations were discussed in [10].

Notice that analogous results to Theorems 1.4–1.5 also hold for the initial-boundary value problem with the Dirichlet boundary condition for the Navier–Stokes equations.

The present paper is divided into four sections. Section 2 contains notation and auxiliary results. In Section 3 the stationary problem is considered and Theorems 1.1–1.2 are proved. Finally, Section 4 is devoted to stability of a solution to the stationary problem and to the existence of a solution to problem (1.1). In particular, Theorems 1.4 and 1.5 are proved.

2. Notation and auxiliary results. Let $\Omega \subset \mathbb{R}^3$ be an open set. We denote by $L_p(\Omega)$, $p \in [1,\infty]$, the Lebesgue space of p-integrable functions, and by $W^m_p(\Omega)$, $m \in \mathbb{N}_0$, $p \in [1,\infty]$, the Sobolev space. In the special case of $p = 2$ we use the notation $H^m(\Omega) = W^m_2(\Omega)$. The norms in $L_p(\Omega)$ and

$W_p^m(\Omega)$ are denoted by $\|u\|_{L_p}$ and $\|u\|_{W_p^m}$, respectively. Next, we write

$$V = \{u \in H_0^1(\Omega) : \operatorname{div} u = 0 \text{ in } \Omega\}.$$

Let $I \subset \mathbb{R}$ be an open interval. Then $H^{2,1}(\Omega \times I)$ denotes the space of functions with the finite norm

$$\|u\|_{H^{2,1}(\Omega\times I)} = \Big(\|u_t\|^2_{L_2(\Omega\times I)} + \sum_{0\le|\alpha|\le 2} \|D_x^\alpha u\|^2_{L_2(\Omega\times I)}\Big)^{1/2},$$

where $D_x^\alpha = \partial_{x_1}^{\alpha_1}\partial_{x_2}^{\alpha_2}\partial_{x_3}^{\alpha_3}$, $|\alpha| = \alpha_1+\alpha_2+\alpha_3$, $\alpha_i \in \mathbb{N}_0$, $i = 1,2,3$.

Let X be a Banach space. Then $L_p(I;X)$ denotes the space of all measurable functions $u : I \to X$ with the finite norm

$$\|u\|_{L_p(I;X)} = \Big(\int_I \|u(t)\|_X^p\,dt\Big)^{1/p} \quad \text{if } 1 < p < \infty,$$

and

$$\|u\|_{L_\infty(I;X)} = \operatorname*{ess\,sup}_{t\in I} \|u(t)\|_X.$$

We denote by $C(\bar I;X)$ the space of all continuous functions $u : \bar I \to X$ with the finite norm

$$\|u\|_{C(\bar I;X)} = \sup_{t\in I} \|u(t)\|_X.$$

We need the following lemma:

LEMMA 2.1 (see [8]). *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain and $T > 0$. Let $\theta(0) \in L_2(\Omega)$ and assume that $\theta(0) \ge \theta_*$ for some $\theta_* > 0$. Moreover, assume that functions v, θ such that $\nabla v \in L_{20/7}(\Omega \times (kT,(k+1)T))$ and $\theta \in L_{10/3}(\Omega\times(kT,(k+1)T))$ for all $k \in \mathbb{N}_0$ satisfy* $(1.1)_{3,5,6}$. *Then*

$$\theta \ge \theta_* \quad \text{a.e. in } \Omega\times(kT,(k+1)T) \text{ for all } k \in \mathbb{N}_0.$$

3. Existence of solutions to the stationary problem. In this section we prove existence and uniqueness theorems for problem (1.2). First, we introduce the function $\bar\vartheta = \vartheta - \theta_*$ and rewrite (1.2) in the form

$$\begin{aligned} &-\nu\Delta w + w\cdot\nabla w + \nabla q = \alpha(\bar\vartheta+\theta_*)g && \text{in } \Omega,\\ &\operatorname{div} w = 0 && \text{in } \Omega,\\ &-\varkappa\Delta\bar\vartheta + w\cdot\nabla\bar\vartheta = \nu|\mathbb{D}(w)|^2 && \text{in } \Omega,\\ &w = 0 && \text{on } S,\\ &\bar\vartheta = 0 && \text{on } S. \end{aligned} \tag{3.1}$$

Let $0 < \sigma < 1/8$. Assume that

$$\alpha \in C([\theta_*,\infty)), \tag{3.2}$$

$$|\alpha(\vartheta)| \le a_1 + a_2\vartheta^\sigma \quad \text{for } \vartheta \ge \theta_*, \tag{3.3}$$

where $a_1 \ge 0$, $a_2 > 0$ are constants.

Definition 3.1. By a *weak solution* to problem (3.1) we mean a triple $(w,q,\bar\vartheta)\in V\times L_2(\Omega)\times H_0^1(\Omega)$ with $\int_\Omega q(x)\,dx=0$ such that the following integral identities hold:

$$\nu\int_\Omega \nabla w\cdot\nabla\phi_1\,dx+\int_\Omega w\cdot\nabla w\cdot\phi_1\,dx-\int_\Omega q\operatorname{div}\phi_1\,dx = \int_\Omega \alpha(\bar\vartheta+\theta_*)g\phi_1\,dx \quad \forall\phi_1\in H_0^1(\Omega), \tag{3.4}$$

and

$$\varkappa\int_\Omega \nabla\bar\vartheta\cdot\nabla\phi_2\,dx+\int_\Omega w\cdot\nabla\bar\vartheta\phi_2\,dx = \nu\int_\Omega |\mathbb{D}(w)|^2\phi_2\,dx \quad \forall\phi_2\in H_0^1(\Omega). \tag{3.5}$$

Definition 3.2. We say that a triple $(w,q,\bar\vartheta)$ is a *strong-weak solution to* (3.1) if it is a weak solution of (3.1) and $(w,q)\in(H^2(\Omega)\cap V)\times H^1(\Omega)$.

Our aim is to show the existence of a solution to problem (3.1). Therefore we start with an a priori estimate.

Lemma 3.3. *Let $0<\sigma<1/8$ and assume that conditions* (3.2)–(3.3) *are satisfied. Let $g\in L_\infty(\Omega)$, $p=4/(3\sigma)$, $p_1=1/\sigma$, $r=1/(8\sigma)$, $s=1/(4\sigma)$, $s_1=1/(2\sigma)$, $p'=p/(p-1)$, $p_1'=p_1/(p_1-1)$, $r'=r/(r-1)$, $s'=s/(s-1)$, $s_1'=s_1/(s_1-1)$. Assume that $(w,q,\bar\vartheta)$ is a strong-weak solution to problem* (3.1). *Then*

$$\|w\|_{H^2}+\|q\|_{H^1}+\|\bar\vartheta\|_{H^1}\le cF(1+F), \tag{3.6}$$

where

$$F=(\|g\|_{L_2}^2+\|g\|_{L_\infty}^{2r'}+\|g\|_{L_{3/2}}+\|g\|_{L_{(3/2)p'}}^{s'})^2+\|g\|_{L_2}+\|g\|_{L_{2p_1'}}^{s_1'},$$

and $c>0$ is a constant.

Proof. Inserting $\phi_1=w\in V\cap H^2(\Omega)$ to (3.4) we get

$$\nu\|\nabla w\|_{L_2}^2=\int_\Omega \alpha(\bar\vartheta+\theta_*)gw\,dx.$$

Using the Poincaré inequality we get

$$\|w\|_{H^1}^2\le c(\|g\|_{L_\infty}\|\bar\vartheta\|_{L_{\sigma q'}}^\sigma\|w\|_{L_q}+\|g\|_{L_2}\|w\|_{L_2}),$$

where $q'=2/\sigma>16$, $q=q'/(q'-1)$. Hence

$$\|w\|_{H^1}^2\le c(\|g\|_{L_\infty}^2\|\bar\vartheta\|_{L_2}^{2\sigma}+\|g\|_{L_2}^2)$$

and continuing we get

$$\begin{aligned}\|w\|_{H^1}^2 &\le c(\varepsilon)(\|g\|_{L_\infty}^{2r'} + \|g\|_{L_2}^2) + \varepsilon\|\bar\vartheta\|_{L_2}^{2\sigma r}\\ &= c(\varepsilon)(\|g\|_{L_\infty}^{2r'} + \|g\|_{L_2}^2) + \varepsilon\|\bar\vartheta\|_{L_2}^{1/4},\end{aligned}\tag{3.7}$$

where $\varepsilon > 0$.

Now, taking $\phi_2 = \bar\vartheta$ in (3.5) we obtain

$$\varkappa\int_\Omega |\nabla\bar\vartheta|^2\,dx = \nu\int_\Omega |\mathbb{D}(w)|^2\bar\vartheta\,dx,$$

so

$$\varkappa\|\nabla\bar\vartheta\|_{L_2}^2 \le c\|w\|_{H^2}^2\|\bar\vartheta\|_{L_2},$$

and by the Poincaré inequality we have

$$\|\bar\vartheta\|_{H^1} \le c\|w\|_{H^2}^2.\tag{3.8}$$

To find an estimate of $\|w\|_{H^2}$ rewrite $(3.1)_{1,2,4}$ in the form

$$\begin{aligned}&-\nu\Delta w + \nabla q = \alpha(\bar\vartheta + \theta_*)g - w\cdot\nabla w &&\text{in } \Omega,\\ &\operatorname{div} w = 0 &&\text{in } \Omega,\\ &w = 0 &&\text{on } S.\end{aligned}$$

Then (see [9])

$$\|w\|_{W_\delta^2} + \|\nabla q\|_{L_\delta} \le c(\|\alpha(\bar\vartheta + \theta_*)g\|_{L_\delta} + \|w\cdot\nabla w\|_{L_\delta}) \quad \text{for } \delta > 1.\tag{3.9}$$

Taking $\delta = 3/2$ and using (3.7) we estimate

$$\|w\cdot\nabla w\|_{L_{3/2}} \le c\|w\|_{H^1}^2 \le c(\|g\|_{L_\infty}^{2r'} + \|g\|_{L_2}^2) + \varepsilon\|\bar\vartheta\|_{L_2}^{1/4}.\tag{3.10}$$

Next,

$$\begin{aligned}\|\alpha(\bar\vartheta + \theta_*)g\|_{L_{3/2}} &\le c(\|g\|_{L_{3/2}} + \|\bar\vartheta\|_{L_{(3/2)\sigma p}}^\sigma\|g\|_{L_{(3/2)p'}})\\ &\le c(\varepsilon)(\|g\|_{L_{3/2}} + \|g\|_{L_{(3/2)p'}}^{s'}) + \varepsilon\|\bar\vartheta\|_{L_2}^{\sigma s}.\end{aligned}\tag{3.11}$$

Inserting (3.10)–(3.11) into (3.9) we get

$$\begin{aligned}\|w\|_{W_{3/2}^2} + \|\nabla q\|_{L_{3/2}} &\le \varepsilon\|\bar\vartheta\|_{L_2}^{1/4}\\ &\quad + c(\|g\|_{L_\infty}^{2r'} + \|g\|_{L_2}^2 + \|g\|_{L_{3/2}} + \|g\|_{L_{(3/2)p'}}^{s'}).\end{aligned}\tag{3.12}$$

Now, using (3.9) with $\delta = 2$ implies

$$\begin{aligned}\|w\|_{H^2} + \|\nabla q\|_{L_2} &\le c(\|g\|_{L_2} + \|\bar\vartheta\|_{L_2}^\sigma\|g\|_{2p_1'} + \|w\cdot\nabla w\|_{L_2})\\ &\le c(\varepsilon)(\|g\|_{L_2} + \|g\|_{L_{2p_1'}}^{s_1'}) + \varepsilon\|\bar\vartheta\|_{L_2}^{\sigma s_1} + c\|w\cdot\nabla w\|_{L_2}.\end{aligned}\tag{3.13}$$

Since

$$\|w\|_{L_r} \le c\|w\|_{W^2_{3/2}} \qquad \forall r < \infty,$$
$$\|\nabla w\|_{L_{r'}} \le c\|w\|_{W^2_{3/2}} \qquad \forall r' \le 3,$$

applying (3.12) we obtain

$$\|w\cdot\nabla w\|_{L_2} \le c\|w\|^2_{W^2_{3/2}} \le c\varepsilon\|\bar\vartheta\|^{1/2}_{L_2} + c(\|g\|^{2r'}_{L_\infty} + \|g\|^2_{L_2} + \|g\|_{L_{3/2}} + \|g\|^{s'}_{L_{(3/2)p'}})^2. \tag{3.14}$$

Inserting (3.14) into (3.13) and taking into account that $\sigma s_1 = 1/2$ we get

$$\|w\|_{H^2} + \|\nabla q\|_{L_2} \le c[(\|g\|^2_{L_2} + \|g\|_{L_{3/2}} + \|g\|^{2r'}_{L_\infty} + \|g\|^{s'}_{L_{(3/2)p'}})^2 + \|g\|_{L_2} + \|g\|^{s_1'}_{L_{2p_1'}}] + c\varepsilon\|\bar\vartheta\|^{1/2}_{L_2}. \tag{3.15}$$

Now, (3.15) and (3.8) yield (3.6) if we assume that ε is sufficiently small. ∎

Let $w \in H^2(\Omega)\cap V$ be given and consider the problem

$$\begin{aligned} -\varkappa\Delta\bar\vartheta + w\cdot\nabla\bar\vartheta &= \nu|\mathbb{D}(w)|^2 && \text{in } \Omega,\\ \bar\vartheta &= 0 && \text{on } S. \end{aligned} \tag{3.16}$$

Lemma 3.4. *For a given $w \in H^2(\Omega)\cap V$ there exists a unique solution $\bar\vartheta \in H^1_0(\Omega)$ to problem* (3.16) *satisfying inequality* (3.8).

Proof. Since $w \in H^2\cap V$, the existence and uniqueness of a solution $\bar\vartheta$ follows from the Lax–Milgram theorem. ∎

Now, fix $w \in H^2(\Omega)\cap V$ and define a mapping $A : H^2(\Omega)\cap V \to H^2(\Omega)\cap V$ by setting $Aw = w^*$ where $w^* \in H^2(\Omega)\cap V$ satisfies

$$\nu\int_\Omega \nabla w^*\cdot\nabla\phi\,dx + \int_\Omega w\cdot\nabla w\cdot\phi\,dx = \int_\Omega \alpha(\bar\vartheta+\theta_*)g\phi\,dx \qquad \forall\phi\in V, \tag{3.17}$$

where $\bar\vartheta \in H^1_0(\Omega)$ is the weak solution to problem (3.16). Obviously, the operator A is well-defined.

Lemma 3.5. *Let $0 < \sigma < 1/8$ and assume that conditions* (3.2)–(3.3) *are satisfied. Moreover, assume that $\alpha \in C^1([\theta_*,\infty))$ and $|\alpha'(\vartheta)| \le a_3$ for $\vartheta \ge \theta_*$, where $a_3 > 0$ is a constant. Let $w_i \in H^2(\Omega)\cap V$ for $i = 1,2$ and let $w_i^* = Aw_i$ for $i = 1,2$. Then*

$$\|w_1^* - w_2^*\|_{H^2} \le c[\|\bar\vartheta_1\|_{H^1}\|g\|_{L_4} + (\|w_1\|_{W^1_4} + \|w_2\|_{W^1_4})(\|g\|_{L_4}+1)]\|w_1 - w_2\|_{W^1_4}, \tag{3.18}$$

where $c > 0$ is a constant.

Proof. Let $\bar\vartheta_i \in H^1_0(\Omega)$, $i = 1,2$, be the solutions to problem (3.16) corresponding to w_i. We will estimate $\|\bar\vartheta_1 - \bar\vartheta_2\|_{L_2}$. Subtracting (3.5) for

$i=2$ from (3.5) for $i=1$ and inserting $\phi_3=\bar\vartheta_1-\bar\vartheta_2$ we get

$$\varkappa\|\nabla\bar\vartheta_1-\nabla\bar\vartheta_2\|_{L_2}^2+\int_\Omega (w_1-w_2)\cdot\nabla\bar\vartheta_1(\bar\vartheta_1-\bar\vartheta_2)\,dx = \nu\int_\Omega (|\mathbb{D}(w_1)|^2-|\mathbb{D}(w_2)|^2)(\bar\vartheta_1-\bar\vartheta_2)\,dx.$$

Hence

$$\begin{aligned}\|\nabla\bar\vartheta_1-\nabla\bar\vartheta_2\|_{L_2}^2 \le c(&\|w_1-w_2\|_{L_4}\|\nabla\bar\vartheta_1\|_{L_2}\|\bar\vartheta_1-\bar\vartheta_2\|_{L_4}\\ &+\|\nabla w_1-\nabla w_2\|_{L_4}\|\nabla w_1\|_{L_4}\|\bar\vartheta_1-\bar\vartheta_2\|_{L_2}\\ &+\|\nabla w_2\|_{L_4}\|\nabla w_1-\nabla w_2\|_{L_4}\|\bar\vartheta_1-\bar\vartheta_2\|_{L_2}).\end{aligned}$$

Using the Poincaré inequality and the imbedding $H^1(\Omega)\to L_4(\Omega)$ we get

$$\|\bar\vartheta_1-\bar\vartheta_2\|_{H^1}\le c(\|\bar\vartheta_1\|_{H^1}+\|w_1\|_{W_4^1}+\|w_2\|_{W_4^1})\|w_1-w_2\|_{W_4^1}. \tag{3.19}$$

Since $w_i^*\in H^2(\Omega)\cap V$, $i=1,2$, satisfies (3.17), there exist $q_i\in H^1(\Omega)$ such that

$$\begin{aligned}&-\nu\Delta(w_1^*-w_2^*)+\nabla(q_1-q_2)=\alpha'(\beta\bar\vartheta_1+(1-\beta)\bar\vartheta_2)\cdot(\bar\vartheta_1-\bar\vartheta_2)g\\ &\qquad\qquad+w_2\cdot\nabla w_2-w_1\cdot\nabla w_1 && \text{in } \Omega,\\ &\operatorname{div}(w_1^*-w_2^*)=0 && \text{in } \Omega,\\ &w_1^*-w_2^*=0 && \text{on } S,\end{aligned}$$

where $\beta\in(0,1)$. Therefore,

$$\begin{aligned}&\|w_1^*-w_2^*\|_{H^2}+\|\nabla(q_1-q_2)\|_{L_2}\\ &\quad\le c\big(\|(\bar\vartheta_1-\bar\vartheta_2)g\|_{L_2}+\|(w_1-w_2)\cdot\nabla w_1\|_{L_2}+\|w_2\cdot\nabla(w_1-w_2)\|_{L_2}\big)\\ &\quad\le c[\|\bar\vartheta_1-\bar\vartheta_2\|_{H^1}\|g\|_{L_4}+\|w_1-w_2\|_{W_4^1}(\|w_1\|_{W_4^1}+\|w_2\|_{W_4^1})].\end{aligned}$$

Thus, from (3.19) the estimate (3.18) follows. ∎

Proof of Theorem 1.1. We apply the Leray–Schauder fixed point theorem. By Lemma 3.5 the operator $A:H^2(\Omega)\cap V\to H^2(\Omega)\cap V$ is continuous. Moreover, it is compact since the imbedding $H^2(\Omega)\cap V\to W_4^1(\Omega)$ is compact. Now, let $w\in H^2(\Omega)\cap V$ be a solution of the equation

$$w=\lambda Aw,$$

where $\lambda\in[0,1]$. Using Lemma 3.3 we get estimate (3.6), the right-hand side of which is independent of λ. This ends the proof of Theorem 1.1. ∎

Proof of Theorem 1.2. Let $(w_i,q_i,\bar\vartheta_i)$, $i=1,2$, be two solutions to problem (3.1). Repeating the proof of Lemma 3.5 we obtain the estimates

$$\|\bar\vartheta_1-\bar\vartheta_2\|_{H^1}\le c(\|\bar\vartheta_1\|_{H^1}+\|w_1\|_{W_4^1}+\|w_2\|_{W_4^1})\|w_1-w_2\|_{W_4^1}$$

and

$$\begin{aligned}&\|w_1-w_2\|_{H^2}+\|\nabla(q_1-q_2)\|_{L_2}\\&\quad\le c[\|\bar\vartheta_1\|_{H^1}\|g\|_{L_4}+(\|w_1\|_{W_4^1}+\|w_2\|_{W_4^1})(\|g\|_{L_4}+1)]\|w_1-w_2\|_{H^2}.\end{aligned}$$

Adding the above inequalities gives

$$\begin{aligned}\|\bar\vartheta_1-\bar\vartheta_2\|_{H^1}&+\|w_1-w_2\|_{H^2}+\|\nabla(q_1-q_2)\|_{L_2}\\&\le c[(\|\bar\vartheta_1\|_{H^1}+\|w_1\|_{H^2}+\|w_2\|_{H^2})(1+\|g\|_{L_4})]\|w_1-w_2\|_{H^2}.\end{aligned}\tag{3.20}$$

By Lemma 3.3 we have the estimate

$$\|w_1\|_{H^2}+\|w_2\|_{H^2}+\|\bar\vartheta_1\|_{H^1}\le cF(1+F).$$

Therefore, the assumption $\|g\|_{L_\infty}\le\delta_1$ for sufficiently small δ_1 implies that $cF(1+F)(1+\|g\|_{L_4})<1$. In view of (3.20) this gives the uniqueness of a solution to problem (1.2). ■

4. Stability of a solution to the stationary problem (1.2) and existence of solutions to problem (1.1). Let (w,q,ϑ) be the solution to problem (1.2) which exists in view of Theorems 1.1 and 1.2. In this section we will prove its stability under a nonstationary perturbation.

First, we formulate the following lemma:

LEMMA 4.1. *Let the assumptions of Theorem* 1.2 *hold and let* $T>0$ *be given. Let* $f\in C([kT,(k+1)T];L_\infty(\Omega))$ *for all* $k\in\mathbb{N}_0$. *Moreover, assume* (w,q,ϑ) *is the solution to problem* (1.2) *which exists in view of Theorem* 1.1. *Let* (u,η,χ) *be a sufficiently regular solution to problem* (1.3). *Then*

$$\begin{aligned}&\frac{d}{dt}(\|\nabla u\|_{L_2}^2+\|\chi\|_{L_2}^2)+c_1(\|u\|_{H^2}^2+\|\chi\|_{H^1}^2)\\&\le c_2[\|\nabla u\|_{L_2}^6+\|w\|_{W_{3^+}^1}^2(\|\nabla u\|_{L_2}^2+\|\chi\|_{L_2}^2)+\|\nabla\bar\vartheta\|_{L_2}^2\|u\|_{H^1}^2+\|\nabla u\|_{L_2}^4\|\chi\|_{L_2}^2\\&\qquad+\|f\|_{L_\infty}^2\|\chi\|_{L_2}^2+\|h\|_{L_4}^2\|\bar\vartheta\|_{L_2}+\|h\|_{L_2}^2]\quad\text{for } t>0,\end{aligned}\tag{4.1}$$

where $3^+>3$ *but close to* 3, *and* $c_1,c_2>0$ *are constants.*

Proof. Multiplying equation $(1.3)_1$ by $-\Delta u$, integrating by parts and using the boundary conditions we get

$$\begin{aligned}\frac12\frac{d}{dt}\|\nabla u\|_{L_2}^2+\nu\|\Delta u\|_{L_2}^2&=-\int_\Omega u\cdot\nabla u\cdot\Delta u\,dx\\&\quad-\int_\Omega w\cdot\nabla u\cdot\Delta u\,dx-\int_\Omega u\cdot\nabla w\cdot\Delta u\,dx\\&\quad+\int_\Omega\alpha'(\beta\theta+(1-\beta)\vartheta)\chi f\cdot\Delta u\,dx+\int_\Omega\alpha(\vartheta)h\cdot\Delta u\,dx\end{aligned}\tag{4.2}$$

for all $\beta\in(0,1)$.

We will estimate the terms on the right-hand side of (4.2). First, we examine

$$
\begin{aligned}
-\int_\Omega & u\cdot\nabla u\cdot\Delta u\,dx\\
&= -\sum_{k=1}^3\int_\Omega\frac{\partial}{\partial x_k}\Big(u\cdot\nabla u\cdot\frac{\partial}{\partial x_k}u\Big)\,dx+\sum_{k=1}^3\int_\Omega\frac{\partial}{\partial x_k}(u\cdot\nabla u)\cdot\frac{\partial u}{\partial x_k}\,dx\\
&= \sum_{k,l,m=1}^3\int_\Omega\frac{\partial u_l}{\partial x_k}\,\frac{\partial u_m}{\partial x_l}\,\frac{\partial u_m}{\partial x_k}\,dx+\sum_{k,l=1}^3\int_\Omega u\cdot\nabla\frac{\partial u_l}{\partial x_k}\,\frac{\partial u_l}{\partial x_k}\,dx\\
&= \sum_{k,l,m=1}^3\int_\Omega\frac{\partial u_l}{\partial x_k}\,\frac{\partial u_m}{\partial x_l}\,\frac{\partial u_m}{\partial x_k}\,dx\le c\|\nabla u\|_{L_3}^3,
\end{aligned}
$$

where we have used the boundary condition $(1.3)_4$ and equation $(1.3)_2$. Applying the interpolation inequality (see [1])

$$\|\nabla u\|_{L_3}\le c\|\nabla u\|_{H^1}^{1/2}\nabla u\|_{L_2}^{1/2}$$

we obtain

$$-\int_\Omega u\cdot\nabla u\cdot\Delta u\,dx\le c\|\nabla u\|_{H^1}^{3/2}\|\nabla u\|_{L_2}^{3/2}\le\varepsilon\|\nabla u\|_{H^1}^2+c(\varepsilon)\|\nabla u\|_{L_2}^6.$$

Next, we have

$$-\int_\Omega w\cdot\nabla u\cdot\Delta u\,dx\le\varepsilon\|\Delta u\|_{L_2}^2+c(\varepsilon)\|w\|_{W_{3+}^1}^2\|\nabla u\|_{L_2}^2$$

and

$$-\int_\Omega u\cdot\nabla w\cdot\Delta u\,dx\le\varepsilon\|\Delta u\|_{L_2}^2+c(\varepsilon)\|\nabla w\|_{L_3}^2\|u\|_{L_6}^2.$$

Finally,

$$\int_\Omega\alpha'(\beta\theta+(1-\beta)\vartheta)\chi f\cdot\Delta u\,dx\le\varepsilon\|\Delta u\|_{L_2}^2+c(\varepsilon)\|f\|_{L_\infty}^2\|\chi\|_{L_2}^2$$

and

$$
\begin{aligned}
\int_\Omega\alpha(\vartheta)h\cdot\Delta u\,dx&\le\varepsilon\|\Delta u\|_{L_2}^2+c(\varepsilon)\Big[\|h\|_{L_4}^2\Big(\int_\Omega\bar\vartheta^{4\sigma}\,dx\Big)^{1/2}+\|h\|_{L_2}^2\Big]\\
&\le\varepsilon\|\Delta u\|_{L_2}^2+c(\varepsilon)(\|h\|_{L_4}^2\|\bar\vartheta\|_{L_2}+\|h\|_{L_2}^2),
\end{aligned}
$$

where we have used the fact that $4\sigma<2$.

Taking into account the above estimates, assuming that ε is sufficiently small and using the equivalence of the norms $\|\Delta u\|_{L_2}$ and $\|u\|_{H^2}$ for u in

$H_0^1(\Omega)\cap H^2(\Omega)$ we get

$$
\frac{d}{dt}\|\nabla u\|_{L_2}^2 + c\|u\|_{H^2}^2 \le c(\|\nabla u\|_{L_2}^6 + \|w\|_{W_{3+}^1}^2\|u\|_{H^1}^2 \quad [2pt] + \|f\|_{L_\infty}^2\|\chi\|_{L_2}^2 + \|h\|_{L_4}^2\|\bar\vartheta\|_{L_2} + \|h\|_{L_2}^2). \tag{4.3}
$$

Now, multiplying $(1.3)_3$ by χ and integrating over Ω gives

$$
\frac{1}{2}\frac{d}{dt}\|\chi\|_{L_2}^2 + \varkappa\|\nabla\chi\|_{L_2}^2 = -\int_\Omega w\cdot\nabla\chi\chi\,dx - \int_\Omega u\cdot\nabla\vartheta\chi\,dx + \nu\int_\Omega |\mathbb{D}(u)|^2\chi\,dx + 2\nu\int_\Omega \mathbb{D}(u):\mathbb{D}(w)\chi\,dx. \tag{4.4}
$$

The first two terms on the r.h.s. of (4.4) are estimated as follows:

$$
\begin{aligned}
-\int_\Omega w\cdot\nabla\chi\chi\,dx &\le \varepsilon\|\nabla\chi\|_{L_2}^2 + c(\varepsilon)\|w\|_{L_\infty}^2\|\chi\|_{L_2}^2\\
&\le \varepsilon\|\nabla\chi\|_{L_2}^2 + c(\varepsilon)\|w\|_{W_{3+}^1}^2\|\chi\|_{L_2}^2,\\
-\int_\Omega u\cdot\nabla\vartheta\chi\,dx &\le \varepsilon\|\nabla\chi\|_{L_2}^2 + c(\varepsilon)\|\nabla\bar\vartheta\|_{L_2}^2\|u\|_{L_3}^2.
\end{aligned}
$$

Next, using the interpolation inequality

$$
\|\chi\|_{L_3} \le c\|\chi\|_{H^1}^{1/2}\|\chi\|_{L_2}^{1/2}
$$

(see [1]) we obtain

$$
\begin{aligned}
\int_\Omega |\mathbb{D}(u)|^2\chi\,dx &\le \|\nabla u\|_{H^1}\|\nabla u\|_{L_2}\|\chi\|_{L_3}\\
&\le \varepsilon\|\nabla u\|_{H^1}^2 + c(\varepsilon)\|\nabla u\|_{L_2}^2\|\chi\|_{L_3}^2\\
&\le \varepsilon\|\nabla u\|_{H^1}^2 + c(\varepsilon)\|\nabla u\|_{L_2}^2\|\chi\|_{H^1}\|\chi\|_{L_2}\\
&\le \varepsilon(\|\nabla u\|_{H^1}^2 + \|\nabla\chi\|_{L_2}^2) + c(\varepsilon)\|\nabla u\|_{L_2}^4\|\chi\|_{L_2}^2.
\end{aligned}
$$

Finally,

$$
\int_\Omega \mathbb{D}(u):\mathbb{D}(w)\chi\,dx \le \varepsilon\|\chi\|_{L_6}^2 + c(\varepsilon)\|\nabla u\|_{L_2}^2\|\nabla w\|_{L_3}^2.
$$

Using the above estimates in (4.4) we obtain

$$
\begin{aligned}
\frac{d}{dt}\|\chi\|_{L_2}^2 + c\|\nabla\chi\|_{L_2}^2 &\le \varepsilon(\|\nabla u\|_{H^1}^2 + \|\nabla\chi\|_{L_2}^2 + \|\chi\|_{L_6}^2)\\
&\quad + c(\varepsilon)\big(\|w\|_{W_{3+}^1}^2\|\chi\|_{L_2}^2 + \|\nabla\bar\vartheta\|_{L_2}^2\|u\|_{L_3}^2\\
&\quad + \|\nabla u\|_{L_2}^4\|\chi\|_{L_2}^2 + \|\nabla u\|_{L_2}^2\|\nabla w\|_{L_3}^2\big).
\end{aligned} \tag{4.5}
$$

Inequalities (4.3) and (4.5) with sufficiently small ε imply (4.1). ■

We introduce the notation:

$$X(t) = \|\nabla u(t)\|_{L_2}^2 + \|\chi(t)\|_{L_2}^2,$$
$$A(t) = \|w\|_{W_{3+}^1}^2 + \|\nabla\bar\vartheta\|_{L_2}^2 + \|f(t)\|_{L_\infty}^2,$$
$$G(t) = \|h(t)\|_{L_4}^2.$$

LEMMA 4.2. *Let the assumptions of Lemma* 4.1 *hold. Assume that*

$$\sup_{k\in\mathbb{N}_0} \|f\|_{C([kT,(k+1)]T;L_\infty(\Omega))} \le \delta_1, \quad X(0) \le \gamma,$$
$$G(t) \le \delta_2\gamma \quad \textit{for all } t\in\mathbb{R}_+.$$

If the constants γ *and* δ_i $(i = 1, 2)$ *are sufficiently small then*

$$X(t) \le \gamma \quad \textit{for all } t\in[kT,(k+1)T],\ k\in\mathbb{N}_0. \tag{4.6}$$

Proof. In view of Theorems 1.1 and 1.2 the differential inequality (4.1) can be rewritten as

$$\frac{dX}{dt} + c_1X \le c_2(X^3 + AX + G) \quad \text{for all } t\in(kT,(k+1)T],\ k\in\mathbb{N}_0,$$

where $c_1, c_2 > 0$ are constants. By the assumptions about f and g and by Theorem 1.1 it follows that

$$A(t) \le c_3\delta_1 \quad \text{for all } t\in[kT,(k+1)T],\ k\in\mathbb{N}_0,$$

where $c_3 > 0$. Therefore, if δ_1 is so small that $c_2c_3\delta_1 \le c_1/2$ we get

$$\frac{dX}{dt} + \frac{c_1}{2}X \le c_2(X^3 + G) \quad \text{for all } t\in(kT,(k+1)T],\ k\in\mathbb{N}_0.$$

By the assumption, $X(0) \le \gamma$. Now, assume that for some $k\in\mathbb{N}_0$,

$$X(kT) \le \gamma.$$

Denote

$$t_* = \inf\{t\in(kT,(k+1)T] : X(t) > \gamma\}. \tag{4.7}$$

Then

$$\frac{dX}{dt}(t_*) + \frac{c_1}{2}\gamma \le c_2(\gamma^3 + \delta_2\gamma).$$

Let δ_2 and γ be so small that

$$c_2(\gamma^2 + \delta_2) \le c_1/4.$$

Then

$$\frac{dX}{dt}(t_*) < 0$$

and we get a contradiction with (4.7). Therefore $X(t) \le \gamma$ for $t\in[kT,(k+1)T]$ and (4.6) follows. ∎

LEMMA 4.3. *Let the assumptions of Lemma* 4.2 *hold. Then*

$$\|u\|^2_{H^{2,1}(\Omega\times(kT,(k+1)T))} + \|\chi\|^2_{L_2(kT,(k+1)T;H^1(\Omega))} + \|\chi_t\|^2_{L_2(kT,(k+1)T;H^{-1}(\Omega))} + \|\nabla\eta\|^2_{L_2(kT,(k+1)T;L_2(\Omega))} \le c(T)\gamma. \tag{4.8}$$

Proof. Integrating (4.1) with respect to time from kT to $(k+1)T$ we obtain

$$\|u\|^2_{L_2(kT,(k+1)T;H^2(\Omega))} + \|\chi\|^2_{L_2(kT,(k+1)T;H^1(\Omega))} \le c(T)\gamma. \tag{4.9}$$

Next, (4.9) and equations $(1.3)_1$, $(1.3)_3$ yield

$$\|u_t\|^2_{L_2(kT,(k+1)T;L_2(\Omega))} + \|\nabla\eta\|^2_{L_2(kT,(k+1)T;L_2(\Omega))} + \|\chi_t\|^2_{L_2(kT,(k+1)T;H^{-1}(\Omega))} \le c(T)\gamma \quad \text{for all } k\in\mathbb{N}\cup\{0\}. \tag{4.10}$$

By (4.9)–(4.10) estimate (4.8) follows. ∎

Proof of Theorems 1.4 and 1.5. Theorem 1.4 follows immediately from Lemmas 4.1–4.3. Theorem 1.5 is a consequence of Lemmas 4.2–4.3 combined with the Faedo–Galerkin method. ∎

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Ewa Zadrzyńska
Faculty of Mathematics and Information Sciences
Warsaw University of Technology
Koszykowa 75
00-662 Warszawa, Poland
E-mail: emzad@mini.pw.edu.pl

Wojciech M. Zajączkowski
Institute of Mathematics
Polish Academy of Sciences
Śniadeckich 8
00-656 Warszawa, Poland
and
Institute of Mathematics and Cryptology
Cybernetics Faculty
Military University of Technology
Kaliskiego 2
00-908 Warszawa, Poland
E-mail: wz@impan.pl