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A non-uniform distribution property of most orbits, in case the $3x+1$ conjecture is true

by

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Dedicated to the memory of Pierre Liardet

1. Introduction. As can be seen in one example given by Lagarias, if we choose a large integer (for instance the one of Figure 2 in http://www.ams.org/bookstore/pspdf/mbk-78-prev.pdf), in general its orbit under the transform

$$T := n \mapsto \begin{cases} 3n+1 & (n \text{ odd}), \\ n/2 & (n \text{ even}) \end{cases}$$

contains about twice fewer odd numbers than even numbers, due to the fact that $3n+1$ is even for any odd n. This figure shows that the orbit of the integer $100\lfloor \pi \cdot 10^{35} \rfloor$ has length about 900 and, as expected, about 300 odd and 600 even elements, because $100\lfloor \pi \cdot 10^{35} \rfloor \cdot \frac{3^{300}}{2^{600}} \approx 1$. Fortunately, the method we use to prove the following theorem cannot be used to contradict this property.

THEOREM 1. *Set*

$$\mathcal{O}_n := \{m \in \mathbb{Z} : \exists k \geq 0,\, m = T^k(n)\} \qquad (\textit{orbit of the integer } n),$$
$$c_i(n) := \#\{m \in \mathcal{O}_n : m \equiv i \bmod 18\} \qquad (\textit{finite or infinite}),$$
$$I := \{1, 5, 7, 11, 13, 17\},$$
$$W := \left\{ n \in \mathbb{Z} : \exists k \geq 0,\, T^k(n) = 1 \text{ and } \forall i \in I,\, \frac{c_i(n)}{\sum_{i\in I} c_i(n)} \leq \frac{1}{6} + 0.0215 \right\}.$$

Then for any N large enough

$$\# W \cap \{1, \ldots, N\} \leq N^{0.9999}.$$

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Of course this theorem remains true if we replace the condition $T^k(n) = 1$ by $T^k(n) = n_0$, where $n_0 \in \mathbb{Z} \setminus \{0\}$ is fixed. In case $n_0 < 0$, we replace the interval $\{1, \dots, N\}$ by $\{-N, \dots, -1\}$.

To prove this theorem we use the same method as used by Krasikov and Lagarias [4]; it consists in describing the set of antecedents of 1 under powers of T (see also [1–3, 5–8]).

Remark 2. To give a numerical example we consider the orbit of each of the integers $n \in \{1, \dots, 26\}$ and we compute $c_i := \sum_{n=1}^{26} c_i(n)$:

$$(c_1, \dots, c_{18}) = (28, 41, 5, 49, 22, 4, 5, 37, 2, 23, 10, 2, 11, 4, 1, 47, 13, 1).$$

As expected, $\sum_{i\,\mathrm{odd}} c_i = 97$ is close to the half of $\sum_{i\,\mathrm{even}} c_i = 208$. Among the c_i with i odd, $c_7 = 5$ is smaller than $c_1 = 28$, $c_5 = 22$, $c_{11} = 10$, $c_{13} = 11$ and $c_{17} = 13$. The proof of the theorem allows one to see, in the general case when $n \in \{1, \dots, N\}$, why c_7 is smaller than c_1, c_5, c_{11}, c_{13}, c_{17}. On the other hand, the c_i for i a multiple of 3 are small for an obvious reason: $(3n+1)/2^k$ is never a multiple of 3.

2. Notation used to describe the set of antecedents of 1. Instead of T we use the transform defined by Sinai [9], which we call S:

$$\begin{aligned} &S : \Pi \to \Pi, \\ &\Pi := \{n \in \mathbb{Z} : n \text{ odd and } n \notin 3\mathbb{Z}\} = \{1, 5, 7, 11, 13, 17\} + 18\mathbb{Z}, \\ &S(n) := (3n+1)/2^k, \qquad k \in \mathbb{N}. \end{aligned}$$

The antecedents of 1 under S are the integers

$$n_1 = \tfrac{1}{3}(2^{\varepsilon_1} - 1) \tag{1}$$

that belong to Π; this is equivalent to $\varepsilon_1 \in \{2, 4\} \bmod 6$. Let now $n_\alpha, n_{\alpha-1}, \dots, n_1$ be some integers such that

$$n_\alpha \xrightarrow{S} n_{\alpha-1} \xrightarrow{S} \cdots \xrightarrow{S} n_1 \xrightarrow{S} n_0 := 1.$$

For any $0 \le j < \alpha$ there exists $\varepsilon_{j+1} \in \mathbb{N}$ such that

$$n_{j+1} = \tfrac{1}{3}(2^{\varepsilon_{j+1}} n_j - 1). \tag{2}$$

One has $n_{j+1} \in \Pi$, and this is equivalent to $2^{\varepsilon_{j+1}} n_j - 1 \in 3\mathbb{N} \setminus 9\mathbb{N}$. This means that when we know the value of n_j, or equivalently when we know $\varepsilon_1, \dots, \varepsilon_j$, the positive integer ε_{j+1} must satisfy the conditions:

$$(3)\qquad \begin{aligned} n_j &\equiv 1 \bmod 18 \Rightarrow \varepsilon_{j+1} \in \{2,4\} \bmod 6,\\ n_j &\equiv 5 \bmod 18 \Rightarrow \varepsilon_{j+1} \in \{3,5\} \bmod 6,\\ n_j &\equiv 7 \bmod 18 \Rightarrow \varepsilon_{j+1} \in \{4,6\} \bmod 6,\\ n_j &\equiv 11 \bmod 18 \Rightarrow \varepsilon_{j+1} \in \{1,3\} \bmod 6,\\ n_j &\equiv 13 \bmod 18 \Rightarrow \varepsilon_{j+1} \in \{2,6\} \bmod 6,\\ n_j &\equiv 17 \bmod 18 \Rightarrow \varepsilon_{j+1} \in \{1,5\} \bmod 6 \end{aligned}$$

(notice that the case $n_j \equiv 7 \bmod 18$ gives the largest values: $\varepsilon_{j+1} \geq 4$ and $n_{j+1} \geq \frac{1}{3}(16n_j - 1)$). So all the antecedents of 1 under S^α are obtained by the following formula, subject to the conditions (3):

$$(4)\qquad n_\alpha = \frac{1}{3^\alpha}(2^{\varepsilon_1+\dots+\varepsilon_\alpha} - 2^{\varepsilon_2+\dots+\varepsilon_\alpha}3^0 - \dots - 2^{\varepsilon_\alpha}3^{\alpha-2} - 2^0 3^{\alpha-1}).$$

We give a first estimation of n_α:

Lemma 3. *If $\alpha \geq 2$ and $\varepsilon_1 \neq 2$, then*

$$\frac{2^{\varepsilon_1+\dots+\varepsilon_\alpha}}{\alpha 3^\alpha} \leq n_\alpha \leq \frac{2^{\varepsilon_1+\dots+\varepsilon_\alpha}}{3^\alpha}.$$

Proof. The upper bound is an immediate consequence of (4). The lower bound can be deduced from the straightforward equality

$$(5)\qquad 3n+1 = 3^{1+\alpha_n} n \quad \text{with} \quad \alpha_n = \frac{1}{\log 3}\log\left(1+\frac{1}{3n}\right) \leq \frac{1}{3n}.$$

Indeed, (2) and (5) imply

$$n_j \leq \frac{3^{1+\frac{1}{3n_{j+1}}}}{2^{\varepsilon_{j+1}}} n_{j+1},$$

hence

$$n_0 \leq \frac{3^{\alpha+\frac{1}{3n_1}+\dots+\frac{1}{3n_\alpha}}}{2^{\varepsilon_1+\dots+\varepsilon_\alpha}} n_\alpha.$$

Now the n_j are distinct (no cycle between n_α and 1, because (1) and the hypothesis $\varepsilon_1 \neq 2$ imply $n_1 \neq 1$), and consequently $\frac{1}{n_1} + \dots + \frac{1}{n_\alpha} \leq \frac{1}{1} + \dots + \frac{1}{\alpha} \leq 1 + \log\alpha$. If $\alpha \geq 2$, we have the inequality $3^{\frac{1}{3n_1}+\dots+\frac{1}{3n_\alpha}} \leq \alpha$, and the lemma follows. ∎

Here we give an indexation and a new lower bound for n_α:

Lemma 4. *There exists a one-to-one map*

$$\mathfrak{n} : (\mathbb{N}\setminus\{1\}) \times \mathbb{N}^{\alpha-1} \leftrightarrow \Pi_\alpha := \{n \in \Pi : S^\alpha(n) = 1 \neq S^{\alpha-1}(n)\}$$

such that, for any $(i_1,\dots,i_\alpha) \in (\mathbb{N}\setminus\{1\}) \times \mathbb{N}^{\alpha-1}$,

$$\mathfrak{n}(i_1,\dots,i_\alpha) \ge \frac{2^{3(i_1+\dots+i_\alpha)-c(\mathfrak{n}(i_1,\dots,i_{\alpha-1}))+\alpha'(i_1,\dots,i_\alpha)}}{\alpha 3^\alpha}, \quad \textit{where}$$

(6) $$c(n) := 2c_1(n) + c_5(n) + 3c_{11}(n) + 2c_{13}(n) + 3c_{17}(n) \qquad (n \in \mathbb{Z}),$$
$$\alpha'(i_1,\dots,i_\alpha) := \#\{1 \le j \le \alpha : i_j \ \textit{odd}\}.$$

Proof. We define $\mathfrak{n}(i_1,\dots,i_\alpha)$ by induction on α. When $\alpha = 1$, according to (1) the antecedents of 1 under S, distinct from 1, are the following integers indexed by $i_1 \ge 2$:

(7) $$\mathfrak{n}(i_1) := \tfrac{1}{3}(2^{\varepsilon_1(i_1)} - 1) \quad \text{where} \quad \varepsilon_1(i) := \begin{cases} 3i-1 & (i \text{ odd}), \\ 3i-2 & (i \text{ even}). \end{cases}$$

Suppose now that $\mathfrak{n}(i_1,\dots,i_j)$ (antecedent of 1 under S^j and not under S^{j-1}) is already defined for any $(i_1,\dots,i_j) \in (\mathbb{N}\setminus\{1\}) \times \mathbb{N}^{j-1}$. We denote by $0 < \varepsilon(i_1,\dots,i_j,1) < \varepsilon(i_1,\dots,i_j,2) < \cdots$ $(i \in \mathbb{N})$ the possible values of ε_{j+1} in (3); the antecedents of $\mathfrak{n}(i_1,\dots,i_j)$ under S are

$$\mathfrak{n}(i_1,\dots,i_j,i) := \tfrac{1}{3}(2^{\varepsilon(i_1,\dots,i_j,i)}\mathfrak{n}(i_1,\dots,i_j) - 1) \qquad (i \in \mathbb{N}).$$

In this way we obtain all the antecedents of 1 under S^{j+1} that are not antecedents of 1 under S^j. With this notation, the conditions in (3) are equivalent to

$$\begin{aligned}
\mathfrak{n}(i_1,\dots,i_j) \equiv 1 &\Rightarrow \varepsilon(i_1,\dots,i_j,i) = 3i-1 \ (i \text{ odd}) \ \text{ or } \ 3i-2 \ (i \text{ even}),\\
\mathfrak{n}(i_1,\dots,i_j) \equiv 5 &\Rightarrow \varepsilon(i_1,\dots,i_j,i) = 3i \ (i \text{ odd}) \ \text{ or } \ 3i-1 \ (i \text{ even}),\\
\mathfrak{n}(i_1,\dots,i_j) \equiv 7 &\Rightarrow \varepsilon(i_1,\dots,i_j,i) = 3i+1 \ (i \text{ odd}) \ \text{ or } \ 3i \ (i \text{ even}),\\
\mathfrak{n}(i_1,\dots,i_j) \equiv 11 &\Rightarrow \varepsilon(i_1,\dots,i_j,i) = 3i-2 \ (i \text{ odd}) \ \text{ or } \ 3i-3 \ (i \text{ even}),\\
\mathfrak{n}(i_1,\dots,i_j) \equiv 13 &\Rightarrow \varepsilon(i_1,\dots,i_j,i) = 3i-1 \ (i \text{ odd}) \ \text{ or } \ 3i \ (i \text{ even}),\\
\mathfrak{n}(i_1,\dots,i_j) \equiv 17 &\Rightarrow \varepsilon(i_1,\dots,i_j,i) = 3i-2 \ (i \text{ odd}) \ \text{ or } \ 3i-1 \ (i \text{ even}).
\end{aligned}$$

Setting

$$r(i) := \begin{cases} 1 & (i \text{ odd}), \\ 0 & (i \text{ even}) \end{cases}$$

(remainder of n modulo 2), we have, for any $i \in \mathbb{N}$,

(8) $$\begin{aligned}
\mathfrak{n}(i_1,\dots,i_j) \equiv 1 &\Rightarrow \varepsilon(i_1,\dots,i_j,i) \ge 3i-2+r(i),\\
\mathfrak{n}(i_1,\dots,i_j) \equiv 5 &\Rightarrow \varepsilon(i_1,\dots,i_j,i) \ge 3i-1+r(i),\\
\mathfrak{n}(i_1,\dots,i_j) \equiv 7 &\Rightarrow \varepsilon(i_1,\dots,i_j,i) \ge 3i-0+r(i),\\
\mathfrak{n}(i_1,\dots,i_j) \equiv 11 &\Rightarrow \varepsilon(i_1,\dots,i_j,i) \ge 3i-3+r(i),\\
\mathfrak{n}(i_1,\dots,i_j) \equiv 13 &\Rightarrow \varepsilon(i_1,\dots,i_j,i) \ge 3i-2+r(i),\\
\mathfrak{n}(i_1,\dots,i_j) \equiv 17 &\Rightarrow \varepsilon(i_1,\dots,i_j,i) \ge 3i-3+r(i).
\end{aligned}$$

We consider the formula (7), and the formulas (8) for $j = 1,\dots,\alpha-1$; they depend on the value modulo 18 of the integers $1, \mathfrak{n}(i_1),\dots,\mathfrak{n}(i_1,\dots,i_{\alpha-1})$ respectively. In other words, these formulas depend on the orbit of $\mathfrak{n}(i_1,\ldots,i_{\alpha-1})$ under S. Using Lemma 3 and the definitions of $c(n)$ and $\alpha'(i_1,\ldots,i_\alpha)=\sum_{j=0}^{\alpha} r(i_j)$ we deduce

$$\mathfrak{n}(i_1,\ldots,i_\alpha)\ge\frac{2^{\sum_{j=1}^{\alpha}\varepsilon(i_1,\ldots,i_j)}}{\alpha 3^\alpha}\ge\frac{2^{3(i_1+\cdots+i_\alpha)-c(\mathfrak{n}(i_1,\ldots,i_{\alpha-1}))+\alpha'(i_1,\ldots,i_\alpha)}}{\alpha 3^\alpha}.\ \blacksquare$$

3. A first bound for $\#W\cap\{1,\ldots,N\}$. In the following lemma we specify how to obtain all the antecedents of 1 under the powers of S or T.

LEMMA 5. (i) *The set of the antecedents of* 1 *under the powers of* S (*resp. under the powers of* T), *namely*

$$\mathcal{S}:=\{n\in\Pi:\exists\alpha\ge 0,\ S^\alpha(n)=1\}\quad(\textit{resp. }\mathcal{T}:=\{n\in\mathbb{N}:\exists k\ge 0,\ T^k(n)=1\}),$$

can also be defined by

$$\mathcal{S}=\bigcup_{\alpha\ge 0}\Pi_\alpha\ (\textit{where }\Pi_0:=\{1\})\quad\textit{and}\quad\mathcal{T}=\mathbb{N}\cap\bigcup_{i\ge 0}\bigcup_{j\ge 1}\frac{2^i}{3}(2^j\mathcal{S}-1).$$

(ii) *If $n\in\mathcal{S}$, there exist $\alpha\ge 0$, $i'_1,\ldots,i'_\alpha\ge 1$ and $A\subset\{1,\ldots,\alpha\}$ such that*

$$n=\mathfrak{n}(i_1,\ldots,i_\alpha)\quad\textit{with}\quad i_j=\begin{cases}2i'_j-1 & \textit{if } j\in A,\\ 2i'_j & \textit{otherwise}\end{cases}\tag{9}$$

and $i'_1\ne 1$ if $1\in A$. If $n=\mathfrak{n}(i_1,\ldots,i_\alpha)$ belongs to W, then

$$i'_1+\cdots+i'_\alpha-\frac{11}{6}\left(\frac{1}{6}+0.0215\right)(\alpha+1)-\frac{1}{3}\#A-\frac{\log\alpha}{6\log 2}-\alpha\frac{\log 3}{6\log 2}\le\frac{\log n}{6\log 2}.\tag{10}$$

Proof. (i) The first relation is obvious, and the second follows from the fact that the orbit of any $n\in\mathbb{N}$ under the transformation T begins with

$$n\xrightarrow{T^i\,(i\ge 0)}\frac{n}{2^i}\xrightarrow{T}3\frac{n}{2^i}+1\xrightarrow{T^j\,(j\ge 1)}\frac{3\frac{n}{2^i}+1}{2^j}\in\Pi\quad(\text{or }\in\mathcal{S},\text{ if }n\in\mathcal{T}).$$

(ii) (9) is a consequence of Lemma 4. For any $n\in W$,

$$c(n)\le 11\left(\frac{1}{6}+0.0215\right)\sum_{i\in I}c_i(n)=11\left(\frac{1}{6}+0.0215\right)\#\{m\in\mathcal{O}_n:m\text{ odd}\}.\tag{11}$$

Now if $n=\mathfrak{n}(i_1,\ldots,i_\alpha)$, there are $\alpha+1$ odd integers in $\mathcal{O}_n$, so (10) follows from (6) and (11). ∎

LEMMA 6. *There exists a constant K such that*

$$\#W\cap\{1,\ldots,N\}\le K(\log N)^K\max_{(\alpha',\alpha'')\in A(N)}\frac{(\alpha'+\alpha''+\alpha''')!}{\alpha'!\alpha''!\alpha'''!}$$

where $\alpha''' = \alpha'''(N, \alpha', \alpha'')$ *is defined by*

$$\alpha'''(N, \alpha', \alpha'') := \lfloor 0.2405 \log N + 0.345 - 0.05749\alpha' - 0.39083\alpha'' \rfloor$$

and $A(N) := \{(\alpha', \alpha'') \in (\mathbb{N} \cup \{0\})^2 : \alpha'''(N, \alpha', \alpha'') \geq 0\}$.

Proof. We use the one-to-one map $\mathfrak{n} : (\mathbb{N} \setminus \{1\}) \times \mathbb{N}^{\alpha-1} \leftrightarrow \Pi_\alpha$ defined in Lemma 4, and the notation

$$A(i_1, \ldots, i_\alpha) := \{1 \leq j \leq \alpha : i_j \text{ odd}\},$$
$$\Pi_{\alpha,\alpha'} := \{n = \mathfrak{n}(i_1, \ldots, i_\alpha) \in \Pi_\alpha : \#A(i_1, \ldots, i_\alpha) = \alpha'\}.$$

Now the integers $N \geq 1$ and $\alpha, \alpha', \alpha'' \geq 0$ are fixed with $\alpha = \alpha' + \alpha''$. Assume for instance that $\alpha \geq 10^{10}$; then $(\log \alpha)/\alpha \leq 10^{-8}$. We use the notation of Lemma 5(ii); we deduce from (10) that if there exists at least one element $n = \mathfrak{n}(i_1, \ldots, i_\alpha) \in \Pi_{\alpha,\alpha'} \cap W \cap \{1, \ldots, N\}$, then

$$i'_1 + \cdots + i'_\alpha - 0.345(\alpha + 1) - 0.33334\alpha' - 0.26417\alpha \leq 0.2405 \log n. \tag{12}$$

This inequality is equivalent to

$$i'_1 + \cdots + i'_\alpha \leq \alpha + \alpha'''(n, \alpha', \alpha'').$$

This last inequality with $\alpha \leq i'_1 + \cdots + i'_\alpha$ implies $\alpha'''(n, \alpha', \alpha'') \geq 0$ and a fortiori $\alpha'''(N, \alpha', \alpha'') \geq 0$. So we have proved that if the set $\Pi_{\alpha,\alpha'} \cap W \cap \{1, \ldots, N\}$ is not empty, then (α', α'') belongs to $A(N)$.

Let us bound the number of elements of $\Pi_{\alpha,\alpha'} \cap W \cap \{1, \ldots, N\}$. We can associate injectively to any n in this set some integers $i'_1, \ldots, i'_\alpha$ such that

$$1 \leq i'_1 < i'_1 + i'_2 < \cdots < i'_1 + \cdots + i'_\alpha \leq \alpha + \alpha''' \quad (\text{where } \alpha''' = \alpha'''(N, \alpha', \alpha'')),$$

and a subset $A \subset \{1, \ldots, \alpha\}$ of cardinality α' such that

$$A(i_1, \ldots, i_\alpha) = A, \quad \text{where} \quad i_j = \begin{cases} 2i'_j - 1 & \text{if } n \in A, \\ 2i'_j & \text{else.} \end{cases}$$

Consequently,

$$\#\Pi_{\alpha,\alpha'} \cap W \cap \{1, \ldots, N\} \leq \binom{\alpha + \alpha'''}{\alpha} \cdot \binom{\alpha}{\alpha'} = \frac{(\alpha' + \alpha'' + \alpha''')!}{\alpha'!\alpha''!\alpha'''!}.$$

The inequality $\alpha'''(N, \alpha', \alpha'') \geq 0$ implies $\alpha \leq K_1 \log N$ with K_1 constant, hence

$$\sum_{\alpha=10^{10}}^{\lfloor K_1 \log N \rfloor} \sum_{\alpha'=0}^{\alpha} \#\Pi_{\alpha,\alpha'} \cap W \cap \{1, \ldots, N\} \tag{13}$$
$$\leq (K_1 \log N)^2 \max_{(\alpha',\alpha'') \in A(N)} \frac{(\alpha' + \alpha'' + \alpha''')!}{\alpha'!\alpha''!\alpha'''!}.$$

It remains to bound $\sum_{\alpha=0}^{10^{10}-1} \#\Pi_\alpha \cap W \cap \{1, \ldots, N\}$. One can associate injectively to any $n \in \Pi_\alpha \cap W \cap \{1, \ldots, N\}$ some positive integers $\varepsilon_1, \ldots, \varepsilon_\alpha$

such that (4) holds. According to Lemma 3 one has $2^{\varepsilon_1+\dots+\varepsilon_\alpha} \le \alpha 3^\alpha N < 10^{10} 3^{10^{10}} N$, hence any ε_j is bounded by $K_2 \log N$ with K_2 constant. Consequently,

$$(14) \qquad \sum_{\alpha=0}^{10^{10}-1} \# \Pi_\alpha \cap W \cap \{1,\dots,N\} \le 10^{10} (K_2 \log N)^{10^{10}}.$$

From Lemma 5(i), $\mathcal{S}$ is the union of the Π_α, hence, from (13) and (14), there exists a constant K_3 such that

$$(15) \quad \#\mathcal{S} \cap W \cap \{1,\dots,N\} \le K_3 (\log N)^{K_3} \max_{(\alpha',\alpha'')\in A(N)} \frac{(\alpha'+\alpha''+\alpha''')!}{\alpha'!\,\alpha''!\,\alpha'''!}.$$

Let us now bound $\# W \cap \{1,\dots,N\}$. By Lemma 5(i) any $n \in \mathcal{T} \cap W \cap \{1,\dots,N\} = W \cap \{1,\dots,N\}$ can be written as

$$n = \frac{2^i}{3}(2^j s - 1) \qquad \text{with } i \ge 0,\ j \ge 1,\ s \in \mathcal{S} \cap W.$$

This implies $s \le 2N$, $2^i \le 3N$ and $2^j \le 3N+1$. So there are at most $K_4 (\log N)^2$ possible values for the couple (i,j), with K_4 constant, and

$$(16) \qquad \# W \cap \{1,\dots,N\} \le K_4 (\log N)^2 \# \mathcal{S} \cap W \cap \{1,\dots,2N\}.$$

The lemma follows from (15) and (16). ∎

4. Proof of the theorem

LEMMA 7. *Let $\ell(N) := 0.2405 \log N + 0.345$. With the notation of Lemma 6, one has*

$$\max_{(\alpha',\alpha'')\in A(N)} \frac{(\alpha'+\alpha''+\alpha''')!}{\alpha'!\alpha''!\alpha'''!} \le \max_{(x,y)\in T} \Big(\frac{(x+y+z)^{x+y+z}}{x^x y^y z^z}\Big)^{\ell(N)}$$

where $z = 1 - 0.05749x - 0.39083y$ and $\mathcal{T} := \{(x,y) : x,y,z \ge 0\}$.

Proof. Let $(\alpha',\alpha'') \in A(N)$. The reals

$$x = \alpha'/\ell(N) \quad \text{and} \quad y = \alpha''/\ell(N)$$

satisfy $(x,y) \in \mathcal{T}$. By Corollary 9 (see Appendix A) one has

$$(17) \qquad \frac{(\alpha'+\alpha''+\alpha''')!}{\alpha'!\alpha''!\alpha'''!} \le \frac{(\alpha'+\alpha''+\alpha''')^{\alpha'+\alpha''+\alpha'''}}{\alpha'^{\alpha'}\alpha''^{\alpha''}\alpha'''^{\alpha'''}}.$$

The map $t \mapsto \frac{(\alpha'+\alpha''+t)^{\alpha'+\alpha''+t}}{\alpha'^{\alpha'}\alpha''^{\alpha''}t^t}$ is non-decreasing because the derivative of its logarithm is $\log\big(1+\frac{\alpha'+\alpha''}{t}\big) \ge 0$. Recall that $\alpha''' = \alpha'''(N,\alpha',\alpha'')$ is the integral part of

$$\alpha'''' = \alpha''''(N,\alpha',\alpha'') := \ell(N) - 0.0575\alpha' - 0.39084\alpha'',$$

so one has

$$\frac{(\alpha'+\alpha''+\alpha''')^{\alpha'+\alpha''+\alpha'''}}{\alpha'^{\alpha'}\alpha''^{\alpha''}\alpha'''^{\alpha'''}} \le \frac{(\alpha'+\alpha''+\alpha'''')^{\alpha'+\alpha''+\alpha''''}}{\alpha'^{\alpha'}\alpha''^{\alpha''}\alpha''''^{\alpha''''}}. \tag{18}$$

Lemma 7 results from (17) and (18) because the real z, as defined in this lemma, is equal to $\alpha''''/\ell(N)$. ∎

End of the proof of the theorem. We apply Lemma 10 (Appendix B) to $a = 0.05749$ and $b = 0.39083$. The function φ attains its maximum in the interior of $\mathcal{T}$; let (x_0, y_0) be a point where φ is maximal and let $z_0 = 1 - ax_0 - by_0$. Since φ is differentiable on the interior of $\mathcal{T}$, the partial derivatives are null at (x_0, y_0):

$$\begin{cases}(1-a)\log(x_0+y_0+z_0) - \log x_0 + a\log z_0 = 0,\\(1-b)\log(x_0+y_0+z_0) - \log y_0 + b\log z_0 = 0.\end{cases}$$

Let $w_0 := 0.2405\varphi(x_0, y_0, z_0)$; we compute x_0, y_0, z_0, w_0 by approximation and obtain

$$w_0 - 0.9998 \in (0, 10^{-4}).$$

From Lemmas 6 and 7,

$$\begin{aligned}\#W \cap \{1,\dots,N\} &\le \text{const} \cdot (\log N)^K \cdot e^{\varphi(x_0,y_0,z_0)\ 0.2405\log N}\\&\le \text{const} \cdot (\log N)^K N^{w_0},\end{aligned}$$

hence $\#W \cap \{1,\dots,N\} \le N^{0.9999}$ for N large enough. ∎

Appendix A. Classical bound for the binomial coefficients

LEMMA 8. *For any $m, n \in \mathbb{N}$,*

$$\frac{(m+n)!}{m!n!} \le \frac{(m+n)^{m+n}}{m^m n^n}.$$

Proof. Let

$$f(m,n) = \frac{(m+n)!}{m!n!} \cdot \frac{m^m n^n}{(m+n)^{m+n}}.$$

Obviously $f(m, 1) \le 1$ and it remains to prove that $f(m, n+1) \le f(m, n)$.

Let $r = n + 1$ and $s = m + n + 1$. Then

$$\begin{aligned}\frac{f(m,n+1)}{f(m,n)} &= \frac{(m+n+1)!}{m!(n+1)!} \cdot \frac{m^m(n+1)^{n+1}}{(m+n+1)^{m+n+1}} \cdot \frac{m!n!}{(m+n)!} \cdot \frac{(m+n)^{m+n}}{m^m n^n}\\&= \frac{(n+1)^n}{(m+n+1)^{m+n}} \cdot \frac{(m+n)^{m+n}}{n^n}\\&= \left(\frac{r}{r-1}\right)^{r-1} \cdot \left(\frac{s-1}{s}\right)^{s-1}.\end{aligned}$$

Since $r < s$, it remains to prove that the function $g(x) = \left(\frac{x}{x-1}\right)^{x-1}$ is non-decreasing: this holds because

$$\begin{aligned}(\log g(x))' &= \log\left(\frac{x}{x-1}\right) - \frac{1}{x} \\ &= -\log(1+t) + t \quad \text{ with } t = -1/x \\ &\geq 0. \blacksquare\end{aligned}$$

COROLLARY 9. *For any* $m, n, p \in \mathbb{N}$,

$$\frac{(m+n+p)!}{m!n!p!} \leq \frac{(m+n+p)^{m+n+p}}{m^m n^n p^p}.$$

Proof. Using Lemma 8, we obtain

$$\begin{aligned}\frac{(m+n+p)!}{m!n!p!} &= \frac{(m+n+p)!}{(m+n)!p!}\,\frac{(m+n)!}{m!n!} \\ &\leq \frac{(m+n+p)^{m+n+p}}{(m+n)^{m+n}p^p}\,\frac{(m+n)^{m+n}}{m^m n^n} = \frac{(m+n+p)^{m+n+p}}{m^m n^n p^p}. \blacksquare\end{aligned}$$

Appendix B. Study of a function

LEMMA 10. *Let* $a, b \in (0,1]$. *The maximum of the function*

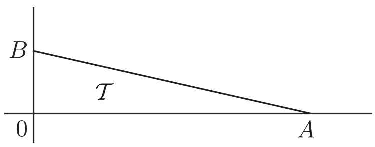
$$\varphi(x,y) := \log\left(\frac{(x+y+z)^{x+y+z}}{x^x y^y z^z}\right) \quad (\text{where } z = 1 - ax - by)$$

is attained in the interior of the triangle $\mathcal{T} := \{(x,y) : x, y, z \geq 0\}$.

Proof. The function

$$\varphi(x,y) = (x+y+z)\log(x+y+z) - x\log x - y\log y - z\log z$$

is continuous on the closed triangle $\mathcal{T}$ whose vertices are the origin, the point $A(1/a, 0)$ and the point $B(0, 1/b)$, hence it has a maximum on $\mathcal{T}$.



Notice that $x + y + z \neq 0$ on T. The partial derivative

$$\frac{\partial}{\partial x}(\varphi(x,y)) = (1-a)\log(x+y+z) - \log x + a\log z$$

has limits $+\infty$ when $x \to 0$ and $-\infty$ when $x \to x_1$, with x_1 such that $1 - ax_1 - by = 0$, hence the map $x \mapsto \varphi(x,y)$ increases in the neighborhood of 0 and decreases in the neighborhood of x_1, it has a maximum in the open interval $(0, x_1)$.

Similarly the map $y \mapsto \varphi(x,y)$ has a maximum in $(0, y_1)$, with y_1 such that $1 - ax - by_1 = 0$.

We deduce that the map $(x, y) \mapsto \varphi(x, y)$ cannot have a maximum on the boundary of T. ■

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