

Integers divisible by a large shifted prime

by

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1. Introduction. Let $N(x, y)$ be the number of integers $n \leq x$ divisible by some number $p - 1$, where $p > y$ is prime. The problem of bounding $N(x, y)$ originated in 1980 with Erdős and Wagstaff [2], who proved the upper bound

$$(1.1) \quad N(x, y) \ll \frac{x}{(\log y)^c} \quad \text{for some constant } c > 0,$$

uniformly for $x > y > 10$, and applied this estimate to the study of denominators of Bernoulli numbers.

In [7], the following improved estimates were shown. Here $\log_2 x = \log \log x$, $\log_3 x = \log \log \log x$ and $\delta = 1 - \frac{1 + \log \log 2}{\log 2} = 0.08607 \dots$

THEOREM A ([7]).

(i) *If $3 \leq y \leq x$, then*

$$N(x, y) \ll \frac{x}{(\log y)^\delta (\log_2 y)^{1/2}},$$

and for every $\varepsilon > 0$ there is an $\eta > 0$ such that for $3 \leq y \leq x \exp\{-(\log x)^{1-\eta}\}$,

$$N(x, y) \gg \frac{x}{(\log y)^{\delta+\varepsilon}}.$$

(ii) *If $y = x/\exp\{(\log x)^\alpha\}$ and $1/\log 4 \leq \alpha \leq 1 - (\log_3 x)/\log_2 x$, then*

$$N(x, y) = \frac{x(\log_2 x)^{O(1)}}{(\log x)^{\delta+\alpha-1-(\log \alpha)/\log 2}}.$$

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(iii) If $y = x/\exp\{(\log x)^\alpha\}$ and $0 \leq \alpha \leq 1/\log 4$, then

$$N(x, y) = \frac{x \log(x/y)(\log_2(x/y))^{O(1)}}{\log x}.$$

The authors remark [7, Remark 2.12] that they can very easily establish the following with their methods: For any $\varepsilon > 0$, if $y > x/\exp\{(\log x)^{1/2-\varepsilon}\}$ and $x/y \rightarrow \infty$, then

$$N(x, y) \sim \frac{x \log(x/y)}{\log x}.$$

They also claim [7, Remarks 2.11] that they can sharpen (iii) to $N(x, y) \asymp x \log(x/y)/\log x$ by taking more care of the “singular series” factor coming from a sieve estimate. As we shall see below, this is a delicate matter.

In this paper, we determine the correct order of magnitude for $N(x, y)$ uniformly for all x, y and show an asymptotic for $N(x, y)$ in most of the range (iii) of Theorem A. As in [7], define α implicitly by $y = x/\exp\{(\log x)^\alpha\}$, so that $0 \leq \alpha \leq 1$ in the range $1 \leq y \leq x/e$. Near the threshold value $\alpha = 1/\log 4$, define θ by

$$\alpha = \frac{1}{\log 4} + \frac{\theta}{\sqrt{\log_2 x}}.$$

THEOREM 1.

(i) If $3 \leq y \leq x^{1-c}$, where $c > 0$ is an arbitrary fixed constant, then

$$N(x, y) \asymp_c \frac{x}{(\log y)^\delta (\log_2 y)^{1/2}}.$$

(ii) If $1/\log 4 \leq \alpha \leq 1 - (\log 2)/\log_2 x$ (the upper bound is equivalent to $y \geq x^{1/2}$), then

$$N(x, y) \asymp \frac{x}{\max(1, \theta)(\log x)^{\delta+\alpha-1-(\log \alpha)/\log 2}}.$$

(iii) If $x/y \rightarrow \infty$ and $\theta \rightarrow -\infty$ (in particular, $0 < \alpha < 1/\log 4$), then

$$N(x, y) \sim \frac{x \log(x/y)}{\log x}.$$

Uniformly in the slightly larger range $x/\exp\{(\log x)^{1/\log 4}\} \leq y \leq x/2$, $x \geq 10$, we have

$$N(x, y) \asymp \frac{x \log(x/y)}{\log x}.$$

REMARKS. In part (ii) of Theorem 1, if $1/\log 4 < \alpha < 1$ is fixed, then $\theta \asymp \sqrt{\log_2 x}$.

Our proof of parts (ii) and (iii) refines the method offered in [7]. To prove the lower bound for part (i), we do not follow the method from [7] (which is based on the theory of the Carmichael λ -function), but rather use

a technique which is similar to that used in part (ii). The reason that this works is described in the next section.

NOTATION. $\omega(n)$ and $\Omega(n)$ denote the number of prime factors of n and the number of prime power factors of n , respectively; $\Omega(n, t)$ is the number of prime power divisors p^a of n with $p \leq t$; $\Omega^*(n, t)$ is the number of prime power divisors p^a of n with $2 < p \leq t$; $P^+(n)$ and $P^-(n)$ denote the largest and smallest prime factors of n , respectively.

2. Heuristic discussion. The quantity $N(x, y)$ counts integers with a particular type of divisor, thus results about the distribution of divisors of integers, say from [6, Ch. 2] or [3], may be relevant to the problem. To bound the density of integers possessing a divisor in an interval $(y, z]$, the right “measuring stick” for the problem is the sum of the densities of the integers which are divisible by each candidate divisor, namely the quantity $\eta := \sum_{y < d \leq z} 1/d \sim \log(z/y)$. When η is very small, the “events” $d|n$ for the various d are essentially independent and the likelihood of an integer having such a divisor is about η ; this independence persists below a threshold value of about $\eta = (\log y)^{1-\log 4}$. As η grows, however, these events become more and more dependent, and when $\eta \approx 1$, the likelihood that an integer has a divisor in $(y, z]$ has dropped to about $(\log y)^{-\delta} (\log_2 y)^{-3/2}$; moreover, the most likely integers to have such a divisor are those with $(\log_2 y)/\log 2 + O(1)$ prime divisors $\leq y$, and also these prime factors must be “nicely” distributed (if not, then the divisors of n are highly clustered and there is a much lower probability of having a divisor in $(y, x]$). When $\eta = (\log y)^{-\beta}$, with $0 \leq \beta \leq \log 4 - 1$, most integers with a divisor in $(y, z]$ have $\Omega(n, y) = \frac{1+\beta}{\log 2} \log_2 y + O(1)$; a heuristic explaining this may be found in [3, §1.5].

For $N(x, y)$ the analogous “measuring stick” is the quantity $\nu = \sum_{y < p \leq x} \frac{1}{p-1}$. When $x^{1/2} < y < x/\exp\{(\log x)^{1/\log 4}\}$, we have $(\log(x/y))^{1-\log 4} \ll \nu \ll 1$ and this roughly corresponds to the “short but not too short interval” case for unrestricted divisors, with ν replacing η and x/y replacing z (because x/y is roughly the size of the smaller factor of n in this case, and that is the deciding quantity). One might guess that the density is then given by [3, Theorem 1], but this is not quite the case. Because the interval $(y, z]$ in the unrestricted case is genuinely very short, integers with typical distribution of their prime factors have very non-uniform divisor distribution (lots of tight clusters), and this makes it highly unlikely to have a divisor in $(y, z]$. Thus, most integers with a divisor in $(y, z]$ have an atypical prime factor distribution. In the case of shifted prime factor divisors, the interval $(y, x]$ is very long, and this issue does not affect whether $(y, x]$ has a shifted prime divisor; the actual likelihood is therefore a bit larger (by roughly a

factor of $\log_2 y$). This also makes it much easier to obtain sharper bounds for $N(x, y)$, as delicate divisor distribution issues do not need to be dealt with.

The techniques of this paper may be easily adapted to obtain sharp estimates for the number of integers $n \leq x$ divisible by an integer $k > y$ which comes from an arbitrary set S which is “thin”, in the sense that that sum of reciprocals of elements of S diverges very slowly like that of primes, and for which the set has nice distribution in arithmetic progressions (in order to apply sieve methods and obtain, e.g., analogs of the Timofeev bounds from the next section).

3. Tools from the anatomy of integers. Beginning with the work of Hardy–Ramanujan (1917), and continuing with work of Erdős and others in the 1930s and beyond, it is now well-known that the prime factors of integers, viewed on a $\log \log$ -scale, behave like a Poisson process. In particular, the number of prime factors which are $\leq z$ behaves roughly like a Poisson random variable with parameter $\sim \log_2 z$ as $z \rightarrow \infty$.

LEMMA 3.1. *Fix $\delta > 0$. Uniformly for $x \geq 4$ and $0 \leq k \leq (2 - \delta) \log_2 x$, we have*

$$\sum_{\substack{n \leq x \\ \Omega(n)=k}} \frac{1}{n} \underset{\sim}{\asymp}_{\delta} \frac{(\log_2 x)^k}{k!}.$$

Proof. This is a corollary of a classical result of Selberg (see [9, Theorem II.6.5]) about the distribution of $\Omega(n)$. ■

The next two lemmas are due to Halász [4], with an extension of Hall and Tenenbaum [6, Theorem 08].

LEMMA 3.2. *Fix $\delta > 0$. Uniformly for $x \geq z \geq 3$ and $0 \leq m \leq (2 - \delta) \log_2 z$, we have*

$$\begin{aligned} \#\{n \leq x : \Omega(n, z) = m\} &\ll_{\delta} \frac{x(\log_2 z)^m}{m! \log z}, \\ \sum_{\substack{n \leq x \\ \Omega(n, z)=m}} \frac{1}{n} &\ll_{\delta} \frac{\log x}{\log z} \frac{(\log_2 z)^m}{m!}. \end{aligned}$$

Uniformly for $x \geq z \geq 3$ and $0 \leq m \leq (3 - \delta) \log_2 z$,

$$\begin{aligned} \#\{n \leq x : \Omega^*(n, z) = m\} &\ll_{\delta} \frac{x(\log_2 z)^m}{m! \log z}, \\ \sum_{\substack{n \leq x \\ \Omega^*(n, z)=m}} \frac{1}{n} &\ll_{\delta} \frac{\log x}{\log z} \frac{(\log_2 z)^m}{m!}. \end{aligned}$$

LEMMA 3.3. Fix $\delta > 0$. Uniformly for $x \geq z \geq 3$ and with m in the range $\delta \log_2 z \leq m \leq (2 - \delta) \log_2 z$, we have

$$\#\{n \leq x : \Omega(n, z) \in \{m, m + 1\}\} \gg_{\delta} \frac{x(\log_2 z)^m}{m! \log z}.$$

The next two lemmas, due to Timofeev [10], state that the prime factors of shifted primes have roughly the same distribution as prime factors of integers taken as a whole.

LEMMA 3.4. Fix $\delta > 0$. There is a constant $c_1(\delta)$ such that uniformly for $x \geq z \geq c_1(\delta)$ and $0 \leq m \leq (2 - \delta) \log_2 z$, we have

$$\begin{aligned} \#\{p \leq x : \Omega(p - 1, z) = m\} &\ll_{\delta} \frac{x(\log_2 z)^m}{m!(\log x)(\log z)}, \\ \#\{p \leq x : \Omega^*(p - 1, z) = m\} &\ll_{\delta} \frac{x(\log_2 z)^m}{m!(\log x)(\log z)}. \end{aligned}$$

Proof. This is a special case of [10, Theorem 2]. ■

LEMMA 3.5. Fix $\delta > 0$. There is a constant $c_2(\delta)$ such that uniformly for $x \geq z \geq c_2(\delta)$ and $\delta \log_2 z \leq m \leq (2 - \delta) \log_2 z$,

$$\#\{p \leq x : \Omega(p - 1, z) \in \{m, m + 1, m + 2\}\} \gg_{\delta} \frac{x(\log_2 z)^m}{m!(\log x)(\log z)}.$$

Proof. This is essentially a special case of part of [10, Theorem 3], except that in the cited theorem it is stated that we must have $z \rightarrow \infty$ as $x \rightarrow \infty$. This condition does not make sense in light of the uniformity claimed in the theorem, and in fact this stronger hypothesis on z (which comes into play when dealing with a set E of primes, which in our application is taken to be the set of primes in $[2, z]$) is never used in the proof. Indeed, in the place where it is claimed to be needed, prior to [10, (18)], no hypothesis is needed at all on the set E , since $E(x/t) \leq E(x)$ for any set E and (18) follows immediately. ■

LEMMA 3.6. Uniformly for $e^2 \leq z \leq x$, $k \leq 1.8 \log_2 z$, $0 \leq \xi \leq 1/(5 \log x)$ and $0 \leq c \leq 10$, we have

$$\sum_{\substack{P^+(n) \leq x \\ \Omega(n, z) = k}} \frac{1}{n^{1-\xi}} \left(\frac{n}{\phi(n)} \right)^c \ll \frac{\log x}{\log z} \frac{(\log_2 z)^k}{k!}.$$

Proof. We follow the proof of [6, Theorem 08] with small modifications. Note that $\xi \leq 0.1$. Thus, since $2^{0.9} > 1.86$, for any complex v with $|v| \leq 1.8$

we have

$$\begin{aligned} \sum_{P^+(n) \leq x} \frac{v^{\Omega(n,z)}}{n^{1-\xi}} \left(\frac{n}{\phi(n)}\right)^c &= \prod_{p \leq z} \left(1 + \frac{v}{p^{1-\xi}} \left(\frac{p}{p-1}\right)^c + O\left(\frac{1}{p^{1.8}}\right)\right) \\ &\times \prod_{z < p \leq x} \left(1 + \frac{1}{p^{1-\xi}} \left(\frac{p}{p-1}\right)^c + O\left(\frac{1}{p^{1.8}}\right)\right). \end{aligned}$$

Now $\left(\frac{p}{p-1}\right)^c = 1 + O(1/p)$ and $p^\xi = 1 + O(\xi \log p)$ since $\xi \leq \frac{1}{5 \log x} \leq \frac{1}{5 \log p}$. So

$$\begin{aligned} \sum_{P^+(n) \leq x} \frac{v^{\Omega(n,z)}}{n^{1-\xi}} \left(\frac{n}{\phi(n)}\right)^c &= \prod_{p \leq z} \left(1 + \frac{v}{p} + O\left(\frac{\xi \log p}{p}\right)\right) \\ &\times \prod_{z < p \leq x} \left(1 + \frac{1}{p} + O\left(\frac{\xi \log p}{p}\right)\right) \\ &\ll (\log z)^{\Re v} \frac{\log x}{\log z}. \end{aligned}$$

Let $r = k/\log_2 z$ and $v = re^{i\theta}$ where $0 \leq \theta \leq 2\pi$. Then, as in [6],

$$\begin{aligned} \sum_{\substack{P^+(n) \leq x \\ \Omega(n)=k}} \frac{1}{n^{1-\xi}} \left(\frac{n}{\phi(n)}\right)^c &= \frac{1}{2\pi r^k} \int_0^{2\pi} e^{-ik\theta} \sum_{P^+(n) \leq x} \frac{(re^{i\theta})^{\Omega(n,z)}}{n^{1-\xi}} \left(\frac{n}{\phi(n)}\right)^c d\theta \\ &\ll \frac{\log x}{\log z} \frac{(\log_2 z)^k}{k^k} \int_0^{2\pi} e^{k \cos \theta} d\theta \ll \frac{\log x}{\log z} \frac{(\log_2 z)^k}{k!}. \blacksquare \end{aligned}$$

Our next tool is a hybrid of the classical theorem of Hardy–Ramanujan and the Brun–Titchmarsh inequality.

LEMMA 3.7 ([1, Theorem 1.1]). *Uniformly for $x > 1$, $k \geq 0$, $q \in \mathbb{N}$ and $(a, q) = 1$ with $1 \leq q < x$, we have*

$$\sum_{\substack{n \leq x \\ n \equiv a \pmod{q} \\ \omega(n) \leq k}} 1 \ll \frac{x}{\phi(q) \log(10x/q)} \sum_{j=0}^{k-1} \frac{(\log_2(10x/q))^j}{j!}.$$

Finally, we need crude estimates for partial sums of the Poisson distribution.

LEMMA 3.8. *Let $v > 0$ and $v^{-1/2} \leq \lambda \leq 1/2$. Then*

$$\frac{e^{-vQ(1-\lambda)}}{\lambda\sqrt{v}} \ll \sum_{(1-\lambda)v-1/\lambda \leq k \leq (1-\lambda)v} e^{-v} \frac{v^k}{k!} \leq \sum_{k \leq (1-\lambda)v} e^{-v} \frac{v^k}{k!} \ll \frac{e^{-vQ(1-\lambda)}}{\lambda\sqrt{v}},$$

$$\frac{e^{-vQ(1+\lambda)}}{\lambda\sqrt{v}} \ll \sum_{(1+\lambda)v \leq k \leq (1+\lambda)v+1/\lambda} e^{-v} \frac{v^k}{k!} \leq \sum_{k \geq (1+\lambda)v} e^{-v} \frac{v^k}{k!} \ll \frac{e^{-vQ(1+\lambda)}}{\lambda\sqrt{v}},$$

where $Q(y) = y \log y - y + 1$.

Proof. These may be found, e.g., in Norton [8, §4]. ■

Useful corollaries of these bounds include bounds on the “tails” of the distribution of $\Omega(n, z)$ and $\Omega^*(n, z)$.

LEMMA 3.9. Fix $\delta > 0$ and suppose z is sufficiently large in terms of δ .

(i) Uniformly for $x \geq z$ and $1 \leq \lambda \leq 1 - \delta$, we have

$$\#\{m \leq x : \Omega(m, z) \geq (1 + \lambda) \log_2 z\} \ll_{\delta} \frac{x}{(\log z)^{Q(1+\lambda)} \max(1, \lambda\sqrt{\log_2 z})}.$$

(ii) Uniformly for $x \geq z$ and $1 \leq w \leq 2 - \delta$, we have

$$\#\{m \leq x : \Omega^*(m, z) \geq (1 + \lambda) \log_2 z\} \ll_{\delta} \frac{x}{(\log z)^{Q(1+\lambda)} \max(1, \lambda\sqrt{\log_2 z})}.$$

Proof. Without loss of generality, suppose $\delta < 1/10$. First, the number of integers $m \leq x$ satisfying $\Omega(m, z) \geq (1 + \lambda) \log_2 z$ is bounded above by

$$\sum_{(1+\lambda) \log_2 z \leq j \leq (2-\delta/2) \log_2 z} \#\{m \leq x : \Omega(m, z) = j\} + \sum_{m \leq x} (2 - \delta/2)^{\Omega(m, z) - (2-\delta/2) \log_2 z},$$

the second sum being an upper bound for the number of $m \leq x$ with $\Omega(m, z) > (2 - \delta/2) \log_2 z$. The terms in the first sum are estimated with the first part of Lemma 3.2 together with Lemma 3.8. The second sum is dealt with using standard estimates for sums of multiplicative functions (e.g., [6, Theorem 01]), and one obtains

$$\sum_{m \leq x} (2 - \delta/2)^{\Omega(m, z) - (2-\delta/2) \log_2 z} \ll x(\log z)^{-Q(2-\delta/2)},$$

which is smaller than the other term, since $Q(u)$ is an increasing function for $u > 1$.

Part (ii) is proved similarly, using the second part of Lemma 3.2, and by breaking up the sum at $j = (3 - \delta/2) \log_2 z$. ■

4. Tools from sieve methods

LEMMA 4.1. We have $\#\{n \leq x : P^+(n) \leq y\} \ll xe^{-0.5(\log x)/\log y}$ uniformly for $x \geq y \geq 2$.

Proof. Standard. See, e.g., [9, Theorem III.5.1]. ■

LEMMA 4.2. *We have $\#\{n \leq x : P^-(n) > z\} \asymp x/\log z$ uniformly for $x \geq 2z \geq 4$. The upper bound holds uniformly for $x \geq z \geq 2$.*

Proof. Standard. Use the asymptotic formula [9, Theorem II.6.3 and (22)] when x is large and x/z is large, the prime number theorem for x/z bounded, and Bertrand’s postulate for small x . ■

LEMMA 4.3. *Suppose that ρ is a non-negative integer-valued multiplicative function with $\rho(p) \leq \min(\kappa, p - 1)$ for every prime p , and for any prime p , Ω_p is some set of $\rho(p)$ residue classes modulo p . Then*

$$\#\{1 \leq n \leq x : \forall p, n \pmod{p} \notin \Omega_p\} \ll_{\kappa} x \prod_{p \leq x} \left(1 - \frac{\rho(p)}{p}\right).$$

Proof. This is a standard application of Montgomery’s Large Sieve (see, e.g., [9, Corollary I.4.6.1]), together with an estimate for the denominator in the sieve bound (e.g., [5, Lemma 4.1]). It does not seem to appear explicitly in the literature anywhere, to the author’s knowledge. ■

LEMMA 4.4. *Let $z \geq 2$, $x > 2z$, and suppose B and C are distinct, even, positive integers. Then*

$$\begin{aligned} \#\{h \leq x : P^-(h) \geq z, Bh + 1 \text{ prime}\} \\ \ll \frac{x}{(\log z)(\log x)} \frac{B}{\phi(B)} \ll \frac{x \log_2(2B)}{(\log z)(\log x)}, \end{aligned}$$

and the number of positive integers $h \leq x$ with $P^-(h) \geq z$, $Bh + 1$ prime and $Ch + 1$ prime is

$$\ll \frac{x}{(\log z)(\log^2 x)} \prod_{p|BC(B-C)} \frac{p}{p-1} \prod_{p|(B,C)} \frac{p}{p-1} \ll \frac{x(\log_2 BC)^2}{(\log z)(\log^2 x)}.$$

Proof. Completely routine exercise using Lemma 4.3. If $z = 2$ or $\log z \gg \log x$, the estimates follow from classical literature, e.g., [5, Theorem 2.2]. ■

5. Proof of Theorem 1: upper bounds. The upper bound in (i) is proven in [7, Theorem 1.2].

The upper bound in (iii) is very easy. Mertens’ theorem implies that

$$N(x, y) \leq \sum_{y < p \leq x} \frac{x}{p-1} = x(\log_2 x - \log_2 y + O(1/\log y)).$$

If $x/y \rightarrow \infty$, the right side is $\sim x \log(x/y)/\log x$. In the larger range $y \leq x/2$, the right side is $O(x \log(x/y)/\log x)$.

Finally, we prove (ii). Let $z = x/y$, $\gamma = 1/(\alpha \log 4)$ and $w = \lfloor \gamma \log_2 z \rfloor$. The hypotheses on α imply that $w \leq \log_2 z$. Consider first integers $n \leq x$

with $\Omega^*(n, z) > 2w$. By Lemma 3.9, the number of such n is

$$\ll \frac{x}{(\log z)^{Q(2\gamma)} \sqrt{\log_2 z}} = \frac{x}{(\log x)^{\delta+\alpha-1-\log \alpha/\log 2} \sqrt{\log_2 z}}.$$

Next, consider integers of the form $n = (p-1)m$ with $m \leq z$, $\Omega^*(p-1, z) = i$ and $\Omega^*(m) = j$, where $i+j \leq 2w$. With i and j fixed, we may use Lemma 3.4, provided that $i \leq 1.99 \log_2 z$, together with Lemma 3.2, to bound the number of such n by

$$\ll \sum_{\substack{m \leq z \\ \Omega^*(m)=j}} \frac{x(\log_2 z)^i}{i!m(\log x)(\log z)} \ll \frac{x(\log_2 z)^{i+j}}{i!j!(\log x)(\log z)}.$$

Summing over all possible i and j , and using Lemma 3.8, we see that the total number of integers counted is

$$\begin{aligned} &\ll \frac{x}{(\log x)(\log z)} \sum_{i+j \leq 2w} \frac{(\log_2 z)^{i+j}}{i!j!} \ll \frac{x}{(\log x)(\log z)} \sum_{h \leq 2w} \frac{(2 \log_2 z)^h}{h!} \\ &\ll \frac{x}{(\log x)^{\delta+\alpha-1-\log \alpha/\log 2} \max(1, \theta)}. \end{aligned}$$

If $i \geq 1.99 \log_2 z$, then $j \leq 0.01 \log_2 z$. So n has the form $n = (p-1)m$ with $\Omega^*(m, z) \leq 0.01 \log_2 z$. The number of such integers n is bounded above by

$$\begin{aligned} \sum_{\substack{m \leq z \\ \Omega^*(m,z) \leq 0.01 \log_2 z}} \pi(x/m) &\ll \frac{x}{\log x} \sum_{\substack{m \leq z \\ \Omega^*(m,z) \leq 0.01 \log_2 z}} \frac{1}{m} \\ &\ll \frac{x}{\log x} (\log z)^{0.01+0.01 \log 100} \ll \frac{x}{(\log x)^{0.9}} \end{aligned}$$

(using Lemma 3.2), which is much smaller than the bound for the other cases. This completes the proof of the upper bound in (ii).

6. Proof of Theorem 1(iii): lower bound when $\theta \rightarrow -\infty$. Here we prove the lower bound claim in part (iii) of the theorem, except in the case where θ is positive and bounded. We begin with a lemma, which is similar to Lemma 3.9.

LEMMA 6.1. *Uniformly for z sufficiently large and $0 \leq \lambda \leq 0.7$, we have*

$$\sum_{\substack{P^+(m) \leq z \\ \Omega(m) > (1+\lambda) \log_2 z}} \frac{1}{\phi(m)} \ll \frac{(\log z)^{1-Q(1+\lambda)}}{\max(1, \lambda \sqrt{\log_2 z})}.$$

Proof. Let $w = (1 + \lambda) \log_2 z$. We use Lemma 3.6 to take care of the summands with $w \leq \Omega(m) \leq 1.8 \log_2 z$, and a simple ‘‘Rankin trick’’ for the

rest, as in the proof of that lemma. We obtain

$$\sum_{\substack{P^+(m) \leq z \\ \Omega(m) > w}} \frac{1}{\phi(m)} \ll \sum_{w \leq j \leq 1.8 \log_2 z} \frac{(\log_2 z)^j}{j!} + \sum_{P^+(m) \leq z} \frac{1.8^{\Omega(m) - 1.8 \log_2 z}}{\phi(m)}.$$

Use Lemma 3.8 for the first sum. The second sum equals

$$(\log z)^{-1.8 \log 1.8} \prod_{p \leq z} \left(1 + \frac{1.8}{p-1} + \frac{1.8^2}{p(p-1)} + \dots \right) \ll (\log z)^{1-Q(1.8)},$$

which is smaller than the bound claimed. ■

Let $z = x/y$. First assume that $\alpha \leq 1/3$. Let $r(n)$ be the number of ways to write $n = (p-1)m$ where $y < p \leq x$ is prime and m is any integer. Note that $m \leq z$ is very small. By the upper bound calculation, $M_1 := \sum_{n \leq x} r(n) \sim x \frac{\log z}{\log x}$ if $z \rightarrow \infty$ (and $M_1 \gg x(\log z)/\log x$ in the larger range $y \leq x/2$). The quantity $M'_2 = \sum_{n \leq x} r(n)^2 - r(n)$ counts solutions of $(p_1 - 1)m_1 = (p_2 - 1)m_2$ with $p_1 \neq p_2$. Let $a = (m_1, m_2)$, $m_1 = ab$, $m_2 = ac$, $g = (p_1 - 1, p_2 - 1)$, so that $p_1 - 1 = gc$ and $p_2 - 1 = gb$. Note that $abc \leq z^2 \ll x^{1/10}$. By Lemma 4.4, given a, b, c , the number of choices for g is $O\left(\frac{x(\log_2 x)^2}{abc \log^2 x}\right)$. Hence

$$M'_2 \ll \frac{x(\log_2 x)^2}{\log^2 x} \sum_{a, b, c \leq z} \frac{1}{abc} \ll \frac{x(\log_2 x)^2}{(\log x)^{2-3\alpha}} = o(M_1).$$

By simple inclusion-exclusion, $N(x, y) \geq M_1 - M'_2 \sim x(\log z)/\log x$ if $z \rightarrow \infty$, and in the larger range $y \leq x/2$ we have $N(x, y) \geq M_1 - M'_2 \gg x(\log z)/\log x$.

Now assume that $\alpha \geq 1/3$. It follows easily from $\theta \rightarrow -\infty$ that $z \rightarrow \infty$ as well. Let $r(n)$ be the number of ways to write $n = (p-1)m$ with p prime, $p > y$, and $\max(\Omega(m), \Omega(p-1, z)) \leq w$, where $w = \lfloor \log_2 z - (\theta/2)\sqrt{\log_2 z} \rfloor$. The hypotheses on α imply that $\log_2 z \leq w \leq 1.7 \log_2 z$. We have

$$(6.1) \quad M_1 = \sum_{m \leq z} (\pi(x/m) - \#\{p \leq x/m : \Omega(p-1, z) > w\}) - O\left(\sum_{\substack{m \leq z \\ \Omega(m) > w}} \pi(x/m)\right).$$

Applying Lemma 6.1, we quickly find that the O -term in (6.1) is

$$\ll \frac{x}{\log x} \sum_{\substack{m \leq z \\ \Omega(m) > w}} \frac{1}{m} \ll \frac{x(\log z)^{1-Q(w/\log_2 z)}}{(-\theta) \log x} = o\left(\frac{x \log z}{\log x}\right).$$

Next, consider a prime $p \leq x/m$ with $\Omega(p-1, z) > w$. The number of primes with $P^+(p-1) \leq z$ is, by Lemma 4.1, $O(x/(m \log^{10} x))$. If $P^+(p-1) > z$, let k be the largest factor of $p-1$ which is composed only of primes $\leq z$, so that $k \leq x/(mz)$ and $\Omega(k) > w$. By Lemma 4.4, for a given k , the number

of such primes p is $O\left(\frac{x}{m\phi(k)\log x \log z}\right)$. Thus the total number of such primes is, in view of Lemma 6.1, bounded above by

$$\ll \frac{x}{m(\log x)(\log z)} \sum_{\substack{P^+(k) \leq z \\ \Omega(k) > w}} \frac{1}{\phi(k)} \ll \frac{x(\log z)^{-Q(w/\log_2 z)}}{(-\theta)m \log x} = o\left(\frac{x}{m \log x}\right).$$

We also have

$$\sum_{m \leq z} \pi(x/m) \sim \frac{x}{\log x} \sum_{m \leq z} \frac{1}{m} \sim \frac{x \log z}{\log x},$$

and therefore conclude from (6.1) that

$$(6.2) \quad M_1 \sim \frac{x \log z}{\log x} \quad (\theta \rightarrow -\infty).$$

Arguing as in [7], we note that $M'_2 := \sum_{n \leq x} r(n)^2 - r(n)$ counts the number of solutions of $(p_1 - 1)m_1 = (p_2 - 1)m_2$ with $\Omega(p_i - 1, z) \leq w$, $\Omega(m_i) \leq w$ for $i = 1, 2$, and $p_1 \neq p_2$. We may assume $p_1 < p_2$. Again let $a = (m_1, m_2)$, $m_1 = ab$, $m_2 = ac$, $g = (p_1 - 1, p_2 - 1)$, so that $p_1 - 1 = gc$ and $p_2 - 1 = gb$. Let $g = dh$, where $P^+(d) \leq z < P^-(h)$. Observe that $d \leq z^{\Omega(d)} \leq z^w \leq x^{1/10}$. Given a, b, c, d , we bound the number of h with $hcd + 1$ and $hbd + 1$ both prime using Lemma 4.4, and get

$$M'_2 \ll \frac{x(\log_2 z)^2}{(\log^2 x) \log z} \sum_{\substack{a, b, c, d \leq z \\ \Omega(abcd) \leq 2w}} \frac{1}{abcd}.$$

We bound the quadruple sum over a, b, c, d with another application of the Rankin trick. Since

$$\begin{aligned} \sum_{\substack{a, b, c, d \leq z \\ \Omega(abcd) \leq 2w}} \frac{1}{abcd} &\leq \sum_{a, b, c, d \leq z} \frac{2^{2w - \Omega(abcd)}}{abcd} \leq 2^{2w} \sum_{P^+(abcd) \leq z} \frac{2^{-\Omega(abcd)}}{abcd} \\ &= 2^{2w} \prod_{p \leq z} \left(1 - \frac{1}{2p}\right)^{-4} \ll 2^{2w} (\log z)^2 \\ &\ll (\log z)^{2 + \log 4} \exp\{(\log 4)(-\theta/2)\sqrt{\log_2 z}\}, \end{aligned}$$

we see that

$$\begin{aligned} M'_2 &\ll \frac{x \log z}{\log x} \frac{(\log z)^{\log 4}}{\log x} (\log_2 z)^2 \exp\{(\log 4)(-\theta/2)\sqrt{\log_2 z}\} \\ &= \frac{x \log z}{\log x} (\log_2 z)^2 \exp\{-\theta(\log 4)(-\sqrt{\log_2 x} + \frac{1}{2}\sqrt{\log_2 z})\} = o(M_1) \end{aligned}$$

as $\theta \rightarrow -\infty$. The theorem now follows upon comparing with (6.2).

7. Proof of Theorem 1(i)–(iii): lower bounds when $-\theta$ is bounded.

This portion of the argument utilizes the second part of the sieve bound in Lemma 4.4. Complicating the analysis are the product factors in the conclusion of the lemma, which have the shape $\prod_j C_j/\phi(C_j)$ for certain numbers C_j . We need to estimate accurately averages of this type of products over particular sets of integers, which have the shape

$$S(z, Y; w; \xi) := \sum_{\substack{a, b, c \leq z \\ P^+(d) \leq Y \\ \Omega(abcd) \leq w \\ b > c}} \frac{1}{abcd^{1-\xi}} \left(\frac{d}{\phi(d)} \right)^2 \frac{b}{\phi(b)} \cdot \frac{c}{\phi(c)} \cdot \frac{b-c}{\phi(b-c)}.$$

The only messy part to take care of is the fraction $(b-c)/\phi(b-c)$.

LEMMA 7.1. *Suppose that $z \geq e^3$, $2 \leq Y \leq z$, $1 \leq w \leq 1.5 \log_2 z$ and $0 \leq \xi \leq 1/(10 \log Y)$.*

(i) *If $Y \leq \exp\{(\log z)^{0.99}\}$, then*

$$S(z, Y; w; \xi) \ll (\log z)^5.$$

(ii) *If $Y \geq \exp\{(\log z)^{0.99}\}$, then*

$$S(z, Y; w; \xi) \ll \frac{(4 \log_2 z)^w}{w!}.$$

Proof. Part (i) follows immediately from the elementary bounds $n/\phi(n) \ll \log_2 P^+(n)$, $\sum_{n \leq z} 1/n \ll \log z$ and

$$\sum_{P^+(d) \leq Y} \frac{1}{d^{1-\xi}} \ll \prod_{p \leq Y} \left(1 + \frac{p^\xi}{p} \right) \leq \exp \left\{ \sum_{p \leq Y} \frac{1 + O(\xi \log p)}{p} \right\} \ll \log Y,$$

since $\xi \leq 1/(10 \log Y) \leq 1/(10 \log p)$ for all $p \leq Y$.

For (ii), first apply Cauchy’s inequality to get $S(z, Y; w; \xi) \leq S_1^{1/2} S_2^{1/2}$, where

$$S_1 = \sum_{a, b, c, d} \frac{1}{abcd^{1-2\xi}} \left(\frac{d}{\phi(d)} \right)^4 \left(\frac{b}{\phi(b)} \right)^2 \left(\frac{c}{\phi(c)} \right)^2,$$

$$S_2 = \sum_{a, b, c, d} \frac{1}{abcd} \left(\frac{b-c}{\phi(b-c)} \right)^2,$$

and in each sum we have the same conditions on a, b, c, d as in the definition of $S(z, Y; w; \xi)$. We may quickly deal with S_1 using Lemma 3.6 repeatedly. We obtain

(7.1)

$$S_1 \ll \sum_{r+s+t+u \leq w} \frac{(\log_2 z)^{r+s+t} (\log_2 Y)^u}{r!s!t!u!} \leq \sum_{j \leq w} \frac{(4 \log_2 z)^j}{j!} \ll \frac{(4 \log_2 z)^w}{w!},$$

where we have used the lower bound on Y , which implies that $u \leq w \leq 1.5 \log_2 z \leq 1.8 \log_2 Y$ and $Y \geq e^2$.

For S_2 , write $(f/\phi(f))^2 = \sum_{l|f} g(l)$, where g is multiplicative, supported on squarefree numbers and $g(p) = (2p - 1)/(p - 1)^2$ for primes p . Let $l_0 = \lceil \log^5 z \rceil$. Recalling that $(b, c) = 1$, we then have

$$\begin{aligned} S_2 &= \sum_l g(l) \sum_{\substack{a,b,c,d \\ b>c, l|(b-c)}} \frac{1}{abcd} \\ &= \sum_{l \leq l_0} g(l) \sum_{r+s+t+u \leq w} \sum_{\substack{a \leq z \\ \Omega(a)=r}} \frac{1}{a} \sum_{\substack{P^+(d) \leq Y \\ \Omega(d)=s}} \frac{1}{d} \sum_{\substack{c \leq z \\ \Omega(c)=u \\ (c,l)=1}} \frac{1}{c} \sum_{\substack{c < b \leq z \\ \Omega(b)=t \\ b \equiv c \pmod{l}}} \frac{1}{b} + E, \end{aligned}$$

where the “error term” E satisfies

$$\begin{aligned} E &\leq \sum_{l > l_0} g(l) \sum_{\substack{a,b,c \leq z, P^+(d) \leq z \\ b > c, l|(b-c)}} \frac{1}{abcd} \ll \sum_{l > l_0} \frac{3^{\omega(l)}}{l} (\log z)^2 \sum_{c \leq z} \frac{1}{c} \sum_{\substack{c < b \leq z \\ b \equiv c \pmod{l}}} \frac{1}{b} \\ &\ll \sum_{l > l_0} \frac{3^{\omega(l)}}{l} \frac{(\log z)^4}{l} \ll \frac{(\log z)^4}{l_0^{0.99}} \ll 1. \end{aligned}$$

By Lemma 3.7 and partial summation (since $\Omega(c) = u$ implies that $\omega(c) \leq u$),

$$\begin{aligned} \sum_{\substack{c < b \leq z \\ \Omega(b)=t \\ b \equiv c \pmod{l}}} \frac{1}{b} &\ll \int_{c+l}^z \frac{1}{\phi(l)s \log(10s/l)} \sum_{j=0}^{t-1} \frac{(\log_2(10s/l))^j}{j!} ds \\ &\ll \frac{1}{\phi(l)} \sum_{j=0}^t \frac{(\log_2 y)^j}{j!}. \end{aligned}$$

Applying Lemma 3.2 to the sums over a, b and d in S_2 , we obtain

$$\begin{aligned} S_2 &\ll \sum_{l \leq l_0} \frac{g(l)}{\phi(l)} \sum_{r+s+t+u \leq w} \sum_{j=0}^t \frac{(\log_2 y)^{r+s+u+j}}{r!s!u!j!} \\ &\ll \sum_{r+s+u+j \leq w} \frac{(\log_2 y)^{r+s+u+j}}{r!s!u!j!} (w - r - s - u - j + 1) \\ &= \sum_{v \leq w} (w - v + 1) \frac{(4 \log_2 y)^v}{v!}. \end{aligned}$$

Since $w - v + 1 \ll 2^{w-v}$, we quickly arrive at

$$(7.2) \quad S_2 \ll \frac{(4 \log_2 z)^w}{w!} + 1 \ll \frac{(4 \log_2 z)^w}{w!}.$$

Combining (7.1) and (7.2) gives the assertion. ■

Now we prove the lower bounds in Theorem 1. Write $z = \min(y, x/y)$, $\gamma = \min(1, 1/(\alpha \log 4))$ and let $w = 2\lceil \gamma \log_2 z \rceil$. Notice that $\gamma = 1/(\alpha \log 4)$ unless $\theta \leq 0$ and we are in case (iii).

We may assume without loss of generality that (a) $y \leq x^{1/11}$, or that (b) $y \geq x^{10/11}$, for if $x^{1/11} < y < x^{10/11}$, we have $N(x, y) \geq N(x, x^{10/11})$ and the result follows from the lower bound for the case $y = x^{10/11}$. Consequently, in either case we have $z \leq x^{1/11}$. We may also assume that x and y are sufficiently large so that $z \geq e^3$.

Let $r(n)$ denote the number of ways to factor n as $n = (p - 1)m$, where p is prime, $y < p < y^{1.1}$, $\Omega((p - 1)m, z) \leq w$, and furthermore in case (a) we have $m = kh$ with $k \leq y^{1/10}$ and $P^-(h) > y^{1.1}$.

By Cauchy’s inequality,

(7.3)

$$N(x, y) \geq \#\{n \leq x : r(n) > 0\} \geq \frac{M_1^2}{M_2}, \quad M_1 := \sum_{n \leq x} r(n), \quad M_2 := \sum_{n \leq x} r(n)^2.$$

First we bound M_1 from below. Start with case (a). Given p and k , we have $(p - 1)k \leq y^{1.2} \leq x^{0.2}$, so by Lemma 4.2, the number of possible choices for h is $\gg x/(pk \log y)$. Hence

$$M_1 \gg \frac{x}{\log y} \sum_{\substack{y < p \leq y^{1.1} \\ k \leq y^{1/10} \\ \Omega(k(p-1)) \leq w}} \frac{1}{kp}.$$

We note that $\alpha = 1 - O(1/\log_2 x)$ and $w = (\log_2 y)/\log 2 + O(1)$. Consider numbers with $\Omega(k) = m_1$, $\Omega(p - 1) = m_2$ and $m_1 + m_2 \in \{w - 2, w - 1, w\}$. With $m_1 \leq 0.9w - 2$ fixed, Lemma 3.5 implies that

$$\sum_{\substack{y < p \leq y^{1.1} \\ w-2-m_1 \leq \Omega(p-1) \leq w-m_1}} \frac{1}{p} \gg \frac{(\log_2 y)^{w-2-m_1}}{(w - 2 - m_1)! \log y}.$$

By Lemma 3.1,

$$\sum_{\substack{k \leq y^{1/10} \\ \Omega(k) = m_1}} \frac{1}{k} \gg \frac{(\log_2 y)^{m_1}}{m_1!}$$

uniformly in m_1 . Putting these bounds together and summing on m_1 , we obtain

$$\begin{aligned} (7.4) \quad M_1 &\gg \frac{x(\log_2 y)^{w-2}}{\log^2 y} \sum_{1 \leq m_1 \leq 0.9w-2} \frac{1}{m_1!(w - 2 - m_1)!} \\ &\gg \frac{x(\log_2 y)^{w-2} 2^{w-2}}{(\log^2 y)(w - 2)!} \gg \frac{x}{(\log y)^\delta \sqrt{\log_2 y}}. \end{aligned}$$

In case (b), we similarly use Lemmas 3.1 and 3.5 to bound separately the number of n with $\Omega(m) = j$ and $\Omega(p - 1, z) \in \{k - 2, k - 1, k\}$ such that $j + k \leq w$. This gives

$$\begin{aligned}
 M_1 &\geq \frac{1}{3} \sum_{\substack{j+k \leq w \\ 0.1w \leq k \leq 0.9w}} \sum_{\substack{m \leq z \\ \Omega(m)=j}} \#\{p \leq x/m : \Omega(p - 1, z) \in \{k - 2, k - 1, k\}\} \\
 &\gg \sum_{\substack{j+k \leq w \\ 0.1w \leq k \leq 0.9w}} \frac{x(\log_2 z)^{j+k-2}}{(\log x)(\log z)j!(k - 2)!}.
 \end{aligned}$$

Next, gather together the summands with $j + k = l$ for fixed $l \geq w/2$. We obtain

$$\begin{aligned}
 (7.5) \quad M_1 &\gg \frac{x}{(\log x)(\log z)} \sum_{w/2 \leq l \leq w-2} \frac{(\log_2 z)^l}{l!} \sum_{\substack{k \leq l \\ 0.1w \leq k \leq 0.9w}} \binom{l}{k - 2} \\
 &\gg \frac{x}{(\log x)(\log z)} \sum_{w/2 \leq l \leq w-2} \frac{(2 \log_2 z)^l}{l!} \\
 &\gg \frac{x}{(\log x)(\log z)} \frac{e^{(2 \log_2 z)(1-Q(\gamma))}}{\max(1, \theta)} \\
 &= \frac{x}{\max(1, \theta)(\log x)(\log z)^{1+2\gamma \log \gamma - 2\gamma}}.
 \end{aligned}$$

We next bound from above the quantity $M'_2 = M_2 - M_1$. In case (a), M'_2 counts the number of solutions of

$$(p_1 - 1)k_1 h = (p_2 - 1)k_2 h \leq x,$$

with p_j prime, $y < p_j < y^{1.1}$, $k_j < y^{1/10}$, $\Omega(k_j) + \Omega(p_j - 1) \leq w$ ($j = 1, 2$), $P^-(h) > y^{1.1}$ and $p_1 \neq p_2$. We may assume that $p_1 < p_2$. Given p_1, p_2, k_1, k_2 , there are

$$O\left(\frac{x}{(p_1 - 1)k_1 \log y}\right)$$

choices for h by Lemma 4.2. Write $a = (k_1, k_2)$, let $k_1 = ab$, $k_2 = ac$, $g = (p_1 - 1, p_2 - 1)$ so that $p_1 - 1 = cg$, $p_2 - 1 = bg$. Note that $g > y^{9/10}$ and $b > c$. Let $t = P^+(g)$, $g = td$. Then $(p_1 - 1)k_1 = abcdt$. Suppose that $T \leq t < 2T$, where T is a power of 2. By Lemma 4.4, if a, b, c, d, T are fixed, the number of t such that t , $cdt + 1$ and $bdt + 1$ are all prime is bounded above by

$$\ll \frac{T}{(\log T)^3} \left(\frac{d}{\phi(d)}\right)^2 \frac{b}{\phi(b)} \cdot \frac{c}{\phi(c)} \cdot \frac{b - c}{\phi(b - c)}.$$

If $T \geq y^{0.1}$, set $\xi = 0$; otherwise let $\xi = \frac{1}{10 \log(2T)}$. In the latter case $d \geq y^{0.8}$, and thus in either case we have

$$\frac{1}{d} \leq \frac{1}{y^{0.8\xi}} \cdot \frac{1}{d^{1-\xi}}.$$

Hence,

$$M'_2 \ll \frac{x}{\log y} \sum_{T=2^j \leq y^{1.1}} \frac{1}{y^{0.8\xi} (\log T)^3} S(y^{1.1}, 2T; w; \xi).$$

If $T < \exp\{(\log y)^{0.99}\}$, then $y^{0.8\xi} > (\log y)^{100}$, hence by Lemma 7.1, these summands contribute $O(x/\log^{80} y)$ to the above right side. For each T satisfying $T \geq \exp\{(\log y)^{0.99}\}$, we see from Lemma 7.1 that

$$\frac{S(y^{1.1}, 2T; w; \xi)}{y^{0.8\xi} (\log T)^3} \ll \frac{x \log^{2-\delta} y}{\sqrt{\log_2 y}} \frac{1}{(\log T)^3 e^{0.08(\log y)/\log(2T)}}.$$

Summing over T which are powers of two, we get

$$(7.6) \quad M'_2 \ll \frac{x}{(\log y)^\delta \sqrt{\log_2 y}}.$$

In case (b), M'_2 equals twice the number of solutions of the equation

$$(p_1 - 1)m_1 = (p_2 - 1)m_2 \leq x,$$

with $p_i > y$, $p_1 < p_2$, $m_i \leq z = x/y$, and $\Omega((p_i - 1)m_i, z) \leq w$ for $i = 1, 2$. As in case (a), we write $a = (m_1, m_2)$, $m_1 = ab$, $m_2 = ac$, and note that $b > c$. Also write $(p_1 - 1, p_2 - 1) = dh$, where $P^+(d) \leq z < P^-(h)$. Then $p_1 - 1 = cdh$ and $p_2 - 1 = bdh$. There are two cases to consider: $d \leq \sqrt{x}$ and $d > \sqrt{x}$. If $d \leq \sqrt{x}$, we have $abcd \leq x^{1/2} z^2 \leq x^{15/22}$. Using Lemma 4.4 to bound the number of possible h for a given quadruple (a, b, c, d) , and then applying Lemma 7.1 and Stirling's formula, we find that the number of solutions in this case is bounded by

$$(7.7) \quad \ll \frac{x}{(\log z)(\log x)^2} S(z, z; w; 0) \ll \frac{x}{(\log z)(\log x)^2} \frac{(4 \log_2 z)^w}{w!} \\ \ll \frac{x}{(\log x)^2 (\log z)^{1+2\gamma \log \gamma - 2\gamma - 2\gamma \log 2} \sqrt{\log_2 z}}.$$

Now assume $d > \sqrt{x}$ and let $t = P^+(d)$, $d = d't$. We further subdivide into two subcases. If $abcd'h \leq x^{3/4}$, then by Lemma 4.4, for each quintuple (a, b, c, d', h) the number of possible $t \leq \frac{x}{abcd'h}$ with $t, tbd'h + 1$ and $tcd'h + 1$ all prime (with $b > c$) is

$$\ll \frac{x}{abcd'h(\log x)^3} \left(\frac{d'}{\phi(d')} \right)^2 \frac{b}{\phi(b)} \cdot \frac{c}{\phi(c)} \cdot \frac{b-c}{\phi(b-c)}.$$

Summing over all possible a, b, c, d', h , we see that the above is

$$\ll \frac{x}{(\log x)^2 (\log z)} S(z, z; w; 0)$$

and we get the same upper bound as in (7.7) for the number of solutions. When $abcd'h > x^{3/4}$, we note that $t \geq P^+(d')$ and thus $abcd'h \leq x/P^+(d')$. By Lemma 4.4, for each quintuple (a, b, c, d', h) the number of possible t is

$$\ll \frac{x}{abcd'h(\log P)^{3+}(d')} \left(\frac{d'}{\phi(d')} \right)^2 \frac{b}{\phi(b)} \cdot \frac{c}{\phi(c)} \cdot \frac{b-c}{\phi(b-c)}.$$

Suppose that $V < P^+(d') \leq V^2$, where V is of the form $V = x^{1/2^l}$ for some positive integer l . Set $\xi = 1/(10 \log(V^2))$. Since $t < x^{1/4}$, we have $d' > x^{1/4}$ and it follows that

$$\frac{1}{d'} \leq \frac{1}{x^{\xi/4}(d')^{1-\xi}}.$$

By Mertens' estimate,

$$\sum_{\substack{h \leq x \\ P^-(h) > z}} \frac{1}{h} \ll \frac{\log x}{\log z}.$$

Summing over all possible a, b, c, d' , we find that the total number of solutions counted in this subcase is at most

$$\ll \frac{x \log x}{\log z} \sum_{V=x^{1/2^l} \leq z^2} \frac{1}{x^{\xi/4}(\log V)^3} S(z, V^2, w; \xi).$$

When $V \leq \exp\{(\log z)^{0.99}\}$, we have $x^{\xi/4} > (\log x)^{100}$ and the number of solutions is $O(x/(\log x)^{80})$ by Lemma 7.1. Otherwise, again by Lemma 7.1, the number is bounded above by

$$\begin{aligned} &\ll \frac{x \log x}{\log z} \sum_{V=x^{1/2^l} \leq z^2} \frac{1}{x^{\xi/4}(\log V)^3} \frac{(4 \log_2 z)^w}{w!} \\ &\ll \frac{x}{(\log x)^2(\log z)^{1+2\gamma \log \gamma - 2\gamma - 2\gamma \log 2} \sqrt{\log_2 z}} \sum_{l \geq 1} \frac{2^{3l}}{\exp\{2^{l-7}\}} \\ &\ll \frac{x}{(\log x)^2(\log z)^{1+2\gamma \log \gamma - 2\gamma - 2\gamma \log 2} \sqrt{\log_2 z}}. \end{aligned}$$

Recalling the bound (7.7) for the number of solutions with $d \leq \sqrt{x}$, we find that

$$\begin{aligned} (7.8) \quad M'_2 &\ll \frac{x}{(\log x)^2(\log z)^{1+2\gamma \log \gamma - 2\gamma - 2\gamma \log 2} \sqrt{\log_2 z}} \\ &\leq \frac{x}{(\log x)(\log z)^{1+2\gamma \log \gamma - 2\gamma} \sqrt{\log_2 z}}, \end{aligned}$$

since $\gamma \leq 1/(\alpha \log 4)$.

Inserting (7.4) and (7.6) into (7.3) gives the desired bound for part (i). Inserting (7.5) and (7.8) into (7.3) gives the desired bound for part (ii), and also handles the case when $-\theta$ is bounded in part (iii).

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