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A uniform bound for the Lagrange polynomials of Leja points for the unit disk

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Abstract. We study uniform estimates for the family of fundamental Lagrange polynomials associated with any Leja sequence for the complex unit disk. The main result states that all these polynomials are uniformly bounded on the disk, i.e. independently of the length N of the associated N-Leja section. As an application, we get a new estimate for any compact subset whose boundary is an Alper-smooth Jordan curve.

1. Introduction

1.1. Definition of Leja points. In this paper we deal with estimates of the Lagrange polynomials of Leja points for some compact sets. For $N \geq 1$, for $\eta_1, \ldots, \eta_N$ different complex numbers and $z \in \mathbb{C}$, we define the *fundamental Lagrange interpolation polynomial* (FLIP) by

$$l_k^{(N)}(z) = \prod_{j=1,\, j\neq k}^{N} \frac{z-\eta_j}{\eta_k-\eta_j}, \qquad k = 1, \ldots, N. \tag{1.1}$$

Finding *good* sets $\{\eta_k\}_{k\geq 1}$ for Lagrange interpolation (i.e. for which we can have some control on the associated FLIPs) is of great interest. One such set, called a Leja sequence, will be considered in this paper.

Definition 1.1. A *Leja sequence* $\mathcal{L}$ for a compact set $K \subset \mathbb{C}$ is a sequence $(\eta_1, \eta_2, \ldots)$ with the following properties: $\eta_1 \in \partial K$ and for all $k \geq 2$,

$$\sup_{z\in K}\Big[\prod_{j=1}^{k-1} |z-\eta_j|\Big] = \prod_{j=1}^{k-1} |\eta_k - \eta_j|. \tag{1.2}$$

For all $N \geq 1$, the *N-Leja section* $\mathcal{L}_N$ of a Leja sequence $\mathcal{L}$ is the finite sequence given by the first N points of $\mathcal{L}$.

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These sequences took their name from F. Leja [10] but they were first considered by A. Edrei [6, p. 78]. They are not necessarily unique: the first k points η_j being fixed, the choice for η_{k+1} can be multiple in general. On the other hand, by the Maximum Modulus Principle, all the η_j's lie on the boundary ∂K.

In the following, we will essentially deal with the special case $K = \overline{\mathbb{D}}$ where $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ is the unit disk, and for which all the Leja sequences are explicit. One can find in [3] their complete description with proof: if we fix $\eta_1 = 1$, then for all $k \geq 2$,

$$\eta_k = \exp\Big(i\pi \sum_{l=0}^{s} j_l 2^{-l}\Big) \quad \text{where} \quad k-1 = \sum_{l=0}^{s} j_l 2^l,\ j_l \in \{0,1\}. \tag{1.3}$$

In particular, the first 2^s points form a complete set of roots of unity of degree 2^s (after a rotation if necessary).

Finally, Leja sequences can be seen as an approximation to one-dimensional Fekete sets (see [7]): an N-*Fekete set* for a compact set K is a set of N elements $\zeta_1, \ldots, \zeta_N \in K$ which maximize (in modulus) the Vandermonde determinant, i.e.

$$\begin{aligned}|\mathrm{VDM}(\zeta_1, \ldots, \zeta_N)| &= \sup_{z_1,\ldots,z_N \in K} |\mathrm{VDM}(z_1, \ldots, z_N)| \\ &= \sup_{z_1,\ldots,z_N \in K} \prod_{1 \leq i < j \leq N} |z_j - z_i|.\end{aligned}$$

One of the essential differences is that determining Fekete sets is an N-dimensional (with respect to $\mathbb{C}$) optimization problem, while determining Leja sequences is just a 1-dimensional one. In addition, it follows from Definition 1.1 that the construction of Leja sequences is inductive (unlike that of any N-Fekete set, which requires the search for an N-tuple $(\zeta_1, \ldots, \zeta_N)$ for every $N \geq 1$).

1.2. Main result and a first application. We have the following result.

THEOREM 1.2. *Let $\mathcal{L} = (\eta_1, \eta_2, \ldots)$ be a Leja sequence for the unit disk (with $|\eta_1| = 1$). Then*

$$\sup_{N \geq k \geq 1} \Big(\sup_{z \in \overline{\mathbb{D}}} |l_k^{(N)}(z)|\Big) \leq \pi \exp(3\pi).$$

Some remarks are in order. First, the explicit bound may not be optimal. Indeed, this can be seen along the whole proof. On the other hand, an improvement for the bound is handled in [8] in a special case for $N = 2^p - 1$: we prove that, except for an asymptotically negligible number of values for $k = 1, \ldots, N$, the FLIPs are asymptotically bounded by 1.

Next, an important interpretation of this result is that any N-Leja section for the disk has essentially the same property as any N-Fekete set. Indeed, it is known that the FLIPs associated with any N-Fekete set are always bounded by 1. Thus, the Fekete sets are essentially the best ones for Lagrange interpolation and uniform stability of the associated FLIPs. Nevertheless, constructing them is generally a hard task. Therefore, a natural question is if there exist *simpler* sets with the same property. Theorem 1.2 gives an affirmative answer with the Leja sequences for the unit disk (with a slightly bigger bound but still universal).

A direct consequence of Theorem 1.2 is an estimate by $N\pi\exp(3\pi)$ of the Nth Lebesgue constants of $\overline{\mathbb{D}}$ defined by

$$\Lambda_N(\overline{\mathbb{D}}) = \sup_{z\in\overline{\mathbb{D}}}\Big(\sum_{k=1}^{N}|l_k^{(N)}(z)|\Big).$$

However, the inequality $\Lambda_N(\overline{\mathbb{D}}) \le 2N$, proved recently in [5, (1.15)], gives a better estimate (see also [4, Corollary 7]). Finally, J.-P. Calvi and V. M. Phung [4] conjectured that $\Lambda_N(\overline{\mathbb{D}}) \le N$. This conjecture has just been confirmed by M. Ounaïes [11].

1.3. Compact sets with Alper-smooth Jordan boundary. Another application of Theorem 1.2 is an estimate of the FLIPs for compact sets whose boundary is an Alper-smooth Jordan curve. It is a special class of compact sets; for example, twice continuously differentiable Jordan curves are Alper-smooth. We denote by Φ the conformal mapping from $\overline{\mathbb{C}}\setminus\mathbb{D}$ onto $\overline{\mathbb{C}}\setminus K$.

If $\mathcal{L}$ is a Leja sequence for the unit disk, the image $\Phi(\mathcal{L}) = (\Phi(\eta_j))_{j\ge1}$ (that is well-defined since the η_j's belong to the unit circle) will not necessarily be a Leja sequence for K. Nevertheless, we can give good estimates for it, as specified by the following result.

THEOREM 1.3. *Let $\mathcal{L} = (\eta_j)_{j\ge1}$ be a Leja sequence for the unit disk with $|\eta_1| = 1$, and let $\Phi(\mathcal{L})$ be its image under the conformal mapping Φ. Then for all $N \ge 1$,*

$$\max_{1\le p\le N}\bigg[\sup_{z\in K}\bigg|\prod_{j=1,\,j\ne p}^{N}\frac{z-\Phi(\eta_j)}{\Phi(\eta_p)-\Phi(\eta_j)}\bigg|\bigg] = O(N^{2A/\ln(2)}),$$

where A is a positive constant depending only on K.

Similarly, we immediately get an estimate of the Lebesgue constant for K.

COROLLARY 1.4. *For all $N \ge 1$,*

$$\Lambda_N(K) = O(N^{1+2A/\ln(2)}),$$

where A is a positive constant depending only on K.

REMARK 1.5. As noticed by J. Ortega-Cerdà, one can make the constant A explicit (see [3, Lemmas 2, 3 and their proofs]):

$$A = \sup_{|w|=1} \int_0^{2\pi} \left| \frac{\Phi'(e^{it})}{\Phi(e^{it}) - \Phi(w)} - \frac{1}{e^{it} - w} \right| dt.$$

2. A couple of reminders and preliminary results

2.1. Reminders. First, for any $m \geq 1$ let

$$\mathcal{U}_m = \{\exp(2iu\pi/m) : u = 0, \dots, m-1\} \tag{2.1}$$

be the set of mth roots of unity. One has the following classical result.

LEMMA 2.1. *For all $m \geq 1$, $\eta_k \in \mathcal{U}_m$ and $z \in \overline{\mathbb{D}}$ with $z \neq \eta_k$,*

$$\prod_{\eta_j \in \mathcal{U}_m,\, \eta_j \neq \eta_k} (z - \eta_j) = \frac{z^m - 1}{z - \eta_k} = \sum_{j=0}^{m-1} \eta_k^{m-j-1} z^j.$$

It follows that $\prod_{\eta_j \in \mathcal{U}_m,\, \eta_j \neq \eta_k} |\eta_k - \eta_j| = m$, and for all $z \neq \eta_k$,

$$\begin{aligned} |l_k^{(m)}(z)| &= \prod_{\eta_j \in \mathcal{U}_m,\, \eta_j \neq \eta_k} \frac{|z - \eta_j|}{|\eta_k - \eta_j|} \\ &= \frac{1}{m} \frac{|z^m - 1|}{|z - \eta_k|} = \frac{1}{m} \left| \sum_{j=0}^{m-1} \eta_k^{m-j-1} z^j \right| \\ &\leq \frac{1}{m} \sum_{j=0}^{m-1} |\eta_k|^{m-j-1} |z|^j \leq 1. \end{aligned}$$

In addition,

$$\sup_{z \in \overline{\mathbb{D}}} |l_k^{(m)}(z)| = |l_k^{(m)}(\eta_k)| = 1. \tag{2.2}$$

Next, we will assume that any Leja sequence $\mathcal{L} = (\eta_1, \eta_2, \dots)$ starts at $\eta_1 = 1$. We will consider the binary decomposition of N,

$$N = 2^{p_1} + \dots + 2^{p_n}, \tag{2.3}$$

where

$$n \geq 1 \quad \text{and} \quad p_1 > \dots > p_n \geq 0 \tag{2.4}$$

(n is the number of ones in the decomposition).

2.2. Preliminary results. We begin with the following lemma that is a rewriting of the FLIPs by using the binary decomposition of N.

LEMMA 2.2. *Let N and $n \geq 2$ be as in* (2.3) *and* (2.4). *Then for all $k = 1, \ldots, 2^{p_1}$ (i.e., $\eta_k \in \mathcal{U}_{2^{p_1}}$) and $z \in \mathbb{C}$ with $z \neq \eta_k$,*

$$(2.5) \qquad |l_k^{(N)}(z)| = \frac{1}{2^{p_1}} \frac{|z^{2^{p_1}} - 1|}{|z - \eta_k|} \times \prod_{q=2}^{n} \frac{|z^{2^{p_q}} + \omega_0^{2^{p_q}}|}{|\eta_k^{2^{p_q}} + \omega_0^{2^{p_q}}|}$$
$$= \frac{1}{2^{p_1}} \Big| \sum_{j=0}^{2^{p_1}-1} \eta_k^{2^{p_1}-j-1} z^j \Big| \times \prod_{q=2}^{n} \frac{|z^{2^{p_q}} + \omega_0^{2^{p_q}}|}{|\eta_k^{2^{p_q}} + \omega_0^{2^{p_q}}|},$$

where ω_0 is a 2^{p_1}th root of -1. In fact, $\omega_0 = \eta_{N+1}$.

Proof. We could prove the lemma by using [3, Theorem 5] and the relation

$$(2.6) \qquad \mathcal{L}_N = (\mathcal{U}_{2^{p_1}}, \rho_1 \widetilde{\mathcal{L}}_{N-2^{p_1}}),$$

where ρ_1 is a 2^{p_1}th root of -1 and $\widetilde{\mathcal{L}}_{N-2^{p_1}}$ is (maybe another) $(N-2^{p_1})$-Leja section for the disk (that also starts at 1). Here we will apply [5, (2.2)]:

$$(2.7) \qquad \prod_{\eta_j \in \mathcal{L}_N} (z - \eta_j) = \prod_{q=1}^{n} (z^{2^{p_q}} + \omega_0^{2^{p_q}}).$$

It follows (since $\omega_0^{2^{p_1}} = -1$) that

$$\prod_{\eta_j \in \mathcal{L}_N,\, \eta_j \neq \eta_k} (z - \eta_j) = \frac{\prod_{q=1}^{n} (z^{2^{p_q}} + \omega_0^{2^{p_q}})}{z - \eta_k} = \frac{z^{2^{p_1}} - 1}{z - \eta_k} \prod_{q=2}^{n} (z^{2^{p_q}} + \omega_0^{2^{p_q}}).$$

In particular (by taking the limit as $z \to \eta_k$),

$$\prod_{\eta_j \in \mathcal{L}_N,\, \eta_j \neq \eta_k} (\eta_k - \eta_j) = 2^{p_1} \eta_k^{2^{p_1}-1} \prod_{q=2}^{n} (\eta_k^{2^{p_q}} + \omega_0^{2^{p_q}}),$$

thus formula (1.1) yields (2.5). ∎

REMARK 2.3. We know [3, Theorem 5] that

$$(2.8) \qquad \mathcal{L}_N = (\mathcal{U}_{2^{p_1}}, \rho_1 \mathcal{U}_{2^{p_2}}, \rho_1 \rho_2 \mathcal{U}_{2^{p_3}}, \ldots, \rho_1 \cdots \rho_{n-1} \mathcal{U}_{2^{p_n}}),$$

where ρ_q is some 2^{p_q}th root of -1 for all $q = 1, \ldots, n-1$, and the above equality is meant as equality of sets. In addition, the number ω_0 considered in the above lemma is any of the 2^{p_n} choices for the $(N+1)$st Leja point η_{N+1} (as can be seen in (2.7)).

Conversely, the data of ω_0 as any 2^{p_1}th root of -1 gives an N-Leja section (that starts at $\eta_1 = 1$) and for which ω_0 is the $(N+1)$st Leja point. Indeed, it suffices to consider the N-section (2.8) where ρ_q is any 2^{p_q}th root of -1 for all $q = 2, \ldots, n$, and $\rho_1 = \omega_0/(\rho_2 \ldots \rho_n)$. In particular, for all $k = 1, \ldots, 2^{p_1}$, the function defined by (2.5) is the FLIP (in modulus) associated with the kth point $\eta_k \in \mathcal{U}_{2^{p_1}} \subset \mathcal{L}_N$.

We have the following trigonometric formula (whose proof is by induction on $m \geq 0$).

LEMMA 2.4. *For all $m \geq 0$ and $\alpha \notin \pi\mathbb{Z}$,*

$$\prod_{j=1}^{m} \cos(\alpha/2^j) = \frac{\sin(\alpha)}{2^m \sin(\alpha/2^m)}. \tag{2.9}$$

In the next section, we will not use the binary decomposition of l, but the alternating binary decomposition as specified by the following result.

LEMMA 2.5. *For all integer $l \geq 1$, we have*

$$l = \sum_{i=1}^{2L} (-1)^{i-1} 2^{s_i}, \qquad \text{where } L \geq 1 \text{ and } s_1 > \cdots > s_{2L} \geq 0. \tag{2.10}$$

Proof. We apply the binary decomposition (2.3) of l and write every 2^{p_q} as $2^{p_q+1} - 2^{p_q}$, i.e. $l = \sum_{q=1}^{n} (2^{p_q+1} - 2^{p_q})$. The simplification of the possible intermediary terms yields the required decomposition. ∎

Since in the next section we will deal with alternating sums, the following result will be useful (its proof is an application of the Leibniz Criterion).

LEMMA 2.6. *Let $a_1 \geq \cdots \geq a_M > 0$. Then for all $J = 1, \ldots, M$,*

$$\Big|\sum_{i=J}^{M} (-1)^i a_i\Big| = \sum_{i=J}^{M} (-1)^{i-J} a_i \leq a_J. \tag{2.11}$$

3. Proof of Theorem 1.2 in a special case. In this section, we will deal with the FLIP $l_1^{(N)}$ (associated with $\eta_1 = 1$) under the following hypotheses:

- $N \geq 1$ and N is not a pure power of 2, i.e. there is $p_1 \geq 1$ such that $2^{p_1} < N < 2^{p_1+1}$;
- as in Lemma 2.2, ω_0 is a 2^{p_1}th root of -1; since we will omit the case of $\exp(i\pi/2^{p_1})$ in this section, we can write

$$\omega_0 = \exp((2l+1)i\pi/2^{p_1}) \quad \text{with } 1 \leq l \leq 2^{p_1} - 1; \tag{3.1}$$

- we will also assume that $|z| = 1$, i.e.

$$z = \exp(i\pi\theta) \quad \text{with } \theta \in \,]-1, 1]. \tag{3.2}$$

The goal of this section is to prove Theorem 1.2 for the FLIP $l_1^{(N)}$ under the above restrictions:

$$|l_1^{(N)}(z)| \leq \pi \exp(3\pi) \quad \text{for all } |z| = 1 \text{ and } l = 1, \ldots, 2^{p_1} - 1. \tag{3.3}$$

We begin by noticing that since $|\theta| \leq 1$, one has either

(1) $|\theta| \leq 1/2^{p_1}$, and this case will be handled in Subsection 3.2; or

(2) $1/2^{p_1} < |\theta| \le 1/2^{p_n}$, and then since $0 < 1/2^{p_1} < \dots < 1/2^{p_n} \le 1$ by (2.4), one has

$$1/2^{p_{q_\theta - 1}} < |\theta| \le 1/2^{p_{q_\theta}} \quad \text{with } q_\theta = 2, \dots, n \tag{3.4}$$

(we will deal with this case in Subsection 3.3); or

(3) $1/2^{p_n} < |\theta| \le 1$, and this case will be handled in Subsection 3.4.

3.1. Some preliminary results

LEMMA 3.1. *Let $l \ge 1$ be defined by* (3.1) *and consider its alternating binary decomposition* (2.10). *Then $s_1 \le p_1$. In addition, if we set*

$$s_{2L+1} := -1 \quad \text{and} \quad s_0 := p_1 + 1, \tag{3.5}$$

then one still has $s_0 > s_1 > \dots > s_{2L} > s_{2L+1}$, and for all $q = 2, \dots, n$, either

$$p_q \in T := \bigcup_{j=0}^{2L} [p_1 - s_j, p_1 - s_{j+1} - 2] \cap \mathbb{Z} \tag{3.6}$$

(*with the convention that the subsets for which $s_{j+1} = s_j - 1$ are empty*), *or*

$$p_q \in S := \{p_1 - s_j - 1 : j = 1, \dots, 2L\}. \tag{3.7}$$

Proof. First, decomposition (2.10) yields

$$l = \sum_{i=1}^{2L} (-1)^{i-1} 2^{s_i} \ge 2^{s_1} - \Big| \sum_{i=2}^{2L} (-1)^{i-1} 2^{s_i} \Big| \ge 2^{s_1} - 2^{s_2},$$

the last estimate being justified by Lemma 2.6 (because s_j is non-increasing). Since $s_2 \le s_1 - 1$ by (2.10), it follows by (3.1) that $2^{p_1} - 1 \ge l \ge 2^{s_1} - 2^{s_1 - 1} = 2^{s_1 - 1}$. Hence $s_1 \le p_1$.

Now fix $q = 2, \dots, n$. We know by (2.4) and (3.5) that $p_q \le p_2 \le p_1 - 1 = p_1 - s_{2L+1} - 2$. Similarly, by (2.4) and (3.5) we have $p_q \ge p_n \ge 0 > p_1 - s_0$. It follows that $p_1 - s_0 \le p_q \le p_1 - s_{2L+1} - 2$, i.e. $p_q \in T \cup S$, and the lemma is proved (since T and S are separable). ∎

Now we will give auxiliary results for the sets T, S.

LEMMA 3.2. *Let $q = 2, \dots, n$ and $l = 1, \dots, 2^{p_1} - 1$. If $p_q \in T$ then for all $|\theta| \le 1$,*

$$\frac{\left|\cos\left(\frac{2l+1}{2^{p_1 - p_q + 1}}\pi - \frac{2^{p_q}\pi\theta}{2}\right)\right|}{\left|\cos\left(\frac{2l+1}{2^{p_1 - p_q + 1}}\pi\right)\right|} \le 1 + \frac{2^{p_q}\pi|\theta|}{2}.$$

Proof. First, since $p_q \in T$, by (3.6) there is j_q with $0 \le j_q \le 2L$ such that

$$p_1 - s_{j_q} \le p_q \le p_1 - s_{j_q + 1} - 2. \tag{3.8}$$

Then (by (2.10) and convention (3.5))

$$\frac{2l+1}{2^{p_1-p_q+1}}\pi = \frac{\pi}{2^{p_1-p_q}}(l+2^{s_{2L+1}}) = \frac{\pi}{2^{p_1-p_q}}\sum_{i=1}^{2L+1}(-1)^{i-1}2^{s_i}$$
$$= \pi\sum_{i=1}^{j_q}(-1)^{i-1}2^{s_i-p_1+p_q} + \pi\sum_{i=j_q+1}^{2L+1}\frac{(-1)^{i-1}}{2^{p_1-p_q-s_i}}.$$

In the first sum, one has $s_i - p_1 + p_q \geq s_{j_q} - p_1 + p_q \geq 0$ by (2.10) and (3.8) for all $i = 1, \dots, j_q$, so $b_q := \sum_{i=1}^{j_q}(-1)^{i-1}2^{s_i-p_1+p_q} \in \mathbb{Z}$. In the second sum, one has $\sum_{i=j_q+1}^{2L+1}(-1)^{i-1}/2^{p_1-p_q-s_i} = (-1)^{j_q}/2^{p_1-p_q-s_{j_q+1}} \times \sum_{i=j_q+1}^{2L+1}(-1)^{i-j_q-1}/2^{s_{j_q+1}-s_i}$, and it follows that

$$\left|\tan\left(\frac{2l+1}{2^{p_1-p_q+1}}\pi\right)\right| = \left|\tan\left(b_q\pi + \frac{(-1)^{j_q}\pi}{2^{p_1-p_q-s_{j_q+1}}}\sum_{i=j_q+1}^{2L+1}\frac{(-1)^{i-j_q-1}}{2^{s_{j_q+1}-s_i}}\right)\right|$$
$$= \left|\tan\left(\frac{\pi}{2^{p_1-p_q-s_{j_q+1}}}\sum_{i=j_q+1}^{2L+1}\frac{(-1)^{i-j_q-1}}{2^{s_{j_q+1}-s_i}}\right)\right|. \tag{3.9}$$

Next, we claim that

$$0 < \frac{\pi}{2^{p_1-p_q-s_{j_q+1}}}\sum_{i=j_q+1}^{2L+1}\frac{(-1)^{i-j_q-1}}{2^{s_{j_q+1}-s_i}} \leq \frac{\pi}{4}. \tag{3.10}$$

Indeed, on the one hand, $p_1 - p_q - s_{j_q+1} \geq 2$ by (3.8), so

$$0 < \pi/2^{p_1-p_q-s_{j_q+1}} \leq \pi/4. \tag{3.11}$$

On the other hand, by Lemma 2.6,

$$\sum_{i=j_q+1}^{2L+1}(-1)^{i-j_q-1}/2^{s_{j_q+1}-s_i} \leq 1/2^{s_{j_q+1}-s_{j_q+1}} = 1, \tag{3.12}$$

and

$$\sum_{i=j_q+1}^{2L+1}(-1)^{i-j_q-1}/2^{s_{j_q+1}-s_i} \geq 1 - \left|\sum_{i=j_q+2}^{2L+1}(-1)^{i-j_q}/2^{s_{j_q+1}-s_i}\right|$$
$$\geq 1 - 1/2^{s_{j_q+1}-s_{j_q+2}} \geq 1/2, \tag{3.13}$$

the last estimate coming from (2.10) (notice that the last sum may be empty if $j_q = 2L$; even in this case, the above estimate holds). Hence $1/2 \leq \sum_{i=j_q+1}^{2L+1}(-1)^{i-j_q-1}/2^{s_{j_q+1}-s_i} \leq 1$, and the claim follows by applying (3.11).

It follows by (3.9) and (3.11) that

$$\left|\tan\left(\frac{2l+1}{2^{p_1-p_q+1}}\pi\right)\right| = \tan\left(\frac{\pi}{2^{p_1-p_q-s_{j_q+1}}}\sum_{i=j_q+1}^{2L+1}\frac{(-1)^{i-j_q-1}}{2^{s_{j_q+1}-s_i}}\right) \le \tan\left(\frac{\pi}{4}\right) = 1.$$

Since $|\sin(x)| \le |x|$ for $x \in \mathbb{R}$ and $|\cos(\alpha-\beta)|/|\cos(\alpha)| \le |\cos(\beta)| + |\tan(\alpha)|\,|\sin(\beta)| \le 1 + |\beta|\,|\tan(\alpha)|$ for $\alpha, \beta \in \mathbb{R}$, $\alpha \notin \pi/2 + \pi\mathbb{Z}$, we have

$$\frac{\left|\cos\left(\frac{2l+1}{2^{p_1-p_q+1}}\pi - \frac{2^{p_q}\pi\theta}{2}\right)\right|}{\left|\cos\left(\frac{2l+1}{2^{p_1-p_q+1}}\pi\right)\right|} \le 1 + \frac{2^{p_q}\pi|\theta|}{2}\left|\tan\left(\frac{2l+1}{2^{p_1-p_q+1}}\pi\right)\right|$$

$$\le 1 + \frac{2^{p_q}\pi|\theta|}{2}. \blacksquare$$

LEMMA 3.3. *Let $q = 2, \ldots, n$ and $l = 1, \ldots, 2^{p_1} - 1$. Assume that $p_q \in S$, i.e. there is j_q with $1 \le j_q \le 2L$ such that*

$$p_q = p_1 - s_{j_q} - 1. \tag{3.14}$$

Then for all $|\theta| \le 1$,

$$\frac{\left|\cos\left(\frac{2l+1}{2^{p_1-p_q+1}}\pi - \frac{2^{p_q}\pi\theta}{2}\right)\right|}{\left|\cos\left(\frac{2l+1}{2^{p_1-p_q+1}}\pi\right)\right|} \le 1 + \frac{2^{p_1}|\theta|\pi}{2^{1+s_{j_q+1}}}.$$

Proof. First, q and the associated j_q being fixed, by (2.10) and (3.5) one has

$$\frac{2l+1}{2^{p_1-p_q+1}}\pi = \frac{\pi}{2^{p_1-p_q}}\sum_{i=1}^{2L+1}(-1)^{i-1}2^{s_i}$$

$$= \pi\sum_{i=1}^{j_q-1}(-1)^{i-1}2^{s_i-p_1+p_q} + \frac{(-1)^{j_q-1}\pi}{2^{p_1-p_q-s_{j_q}}} + \pi\sum_{i=j_q+1}^{2L+1}\frac{(-1)^{i-1}}{2^{p_1-p_q-s_i}}.$$

As before, if the first sum is not empty (otherwise, we get $0 \in \pi\mathbb{Z}$), by (2.10) and (3.14) for all $i = 1, \ldots, j_q - 1$ one has $s_i - p_1 + p_q \ge s_{j_q-1} - p_1 + p_q \ge s_{j_q} + 1 - p_1 + p_q = -1 + 1 = 0$, so

$$\widetilde{b}_q := \sum_{i=1}^{j_q-1}(-1)^{i-1}2^{s_i-p_1+p_q} \in \mathbb{Z}.$$

Similarly, for all $i = j_q + 1, \ldots, 2L + 1$,

$$\sum_{i=j_q+1}^{2L+1}\frac{(-1)^{i-1}}{2^{p_1-p_q-s_i}} = \frac{(-1)^{j_q}}{2^{p_1-p_q-s_{j_q+1}}}\sum_{i=j_q+1}^{2L+1}\frac{(-1)^{i-j_q-1}}{2^{s_{j_q+1}-s_i}}.$$

Since by (3.14) again, $p_1 - p_q - s_{j_q} = 1$, it follows that

$$\begin{aligned}
(3.15)\qquad &\left|\tan\left(\frac{2l+1}{2^{p_1-p_q+1}}\pi\right)\right| \\
&= \left|\tan\left(\widetilde{b}_q\pi + (-1)^{j_q-1}\frac{\pi}{2} + \frac{(-1)^{j_q}\pi}{2^{p_1-p_q-s_{j_q+1}}}\sum_{i=j_q+1}^{2L+1}\frac{(-1)^{i-j_q-1}}{2^{s_{j_q+1}-s_i}}\right)\right| \\
&= \left|\cot\left(\frac{\pi}{2^{p_1-p_q-s_{j_q+1}}}\sum_{i=j_q+1}^{2L+1}\frac{(-1)^{i-j_q-1}}{2^{s_{j_q+1}-s_i}}\right)\right| \\
&\leq \frac{1}{\left|\sin\left(\frac{\pi}{2^{p_1-p_q-s_{j_q+1}}}\sum_{i=j_q+1}^{2L+1}\frac{(-1)^{i-j_q-1}}{2^{s_{j_q+1}-s_i}}\right)\right|}.
\end{aligned}$$

Next, as in the proof of Lemma 3.2, we claim that

$$(3.16)\qquad \frac{\pi/2}{2^{p_1-p_q-s_{j_q+1}}} \leq \frac{\pi}{2^{p_1-p_q-s_{j_q+1}}}\sum_{i=j_q+1}^{2L+1}\frac{(-1)^{i-j_q-1}}{2^{s_{j_q+1}-s_i}} \leq \frac{\pi}{4}.$$

Indeed, on the one hand,

$$p_1 - p_q - s_{j_q+1} \geq p_1 - p_q - s_{j_q} + 1 = 1 + 1 = 2$$

by (2.10) and (3.14), so that

$$(3.17)\qquad \pi/2^{p_1-p_q-s_{j_q+1}} \leq \pi/4.$$

On the other hand, as in the proof of Lemma 3.2, we obtain (3.12) and (3.13). Consequently, the claim follows by applying (3.17).

It follows from (3.15) and (3.16) that

$$\begin{aligned}
\left|\tan\left(\frac{2l+1}{2^{p_1-p_q+1}}\pi\right)\right| &\leq \frac{1}{\sin\left(\frac{\pi}{2^{p_1-p_q-s_{j_q+1}}}\sum_{i=j_q+1}^{2L+1}(-1)^{i-j_q-1}/2^{s_{j_q+1}-s_i}\right)} \\
&\leq \frac{1}{\sin\left(\frac{\pi/2}{2^{p_1-p_q-s_{j_q+1}}}\right)} \\
&\leq \frac{1}{\frac{2}{\pi}\times\frac{\pi/2}{2^{p_1-p_q-s_{j_q+1}}}} = 2^{p_1-p_q-s_{j_q+1}},
\end{aligned}$$

because $\sin(x) \geq 2x/\pi$ for $x \in [0, \pi/2]$.

Finally, as in the proof of Lemma 3.2, we obtain

$$\frac{\left|\cos\left(\frac{2l+1}{2^{p_1-p_q+1}}\pi-\frac{2^{p_q}\pi\theta}{2}\right)\right|}{\left|\cos\left(\frac{2l+1}{2^{p_1-p_q+1}}\pi\right)\right|}\le 1+\frac{2^{p_q}\pi|\theta|}{2}\left|\tan\left(\frac{2l+1}{2^{p_1-p_q+1}}\pi\right)\right|$$
$$\le 1+\frac{2^{p_q}\pi|\theta|}{2}\times 2^{p_1-p_q-s_{j_q+1}}=1+\frac{2^{p_1}\pi|\theta|}{2^{1+s_{j_q+1}}}. \blacksquare$$

We close the subsection with a result about the map $q\mapsto j_q$ that is an immediate consequence of (2.4) and (2.10).

LEMMA 3.4. *The map*

$$\big\{q\in\{2,\dots,n\}:p_q\in S\big\}\ni q\mapsto j_q\in\{1,\dots,2L\} \tag{3.18}$$

such that $p_q=p_1-s_{j_q}-1$, is decreasing. It is in particular injective.

3.2. First case. In this subsection, we deal with the case $|\theta|\le 1/2^{p_1}$. We can assume that $\theta\ne 0$ since $l_1^{(N)}(1)=1$. First, by (2.5) with $\eta_k=\eta_1=1$, (3.1) and (3.2) one has

$$|l_1^{(N)}(\exp(i\pi\theta))|$$
$$=\frac{1}{2^{p_1}}\left|\sum_{j=0}^{2^{p_1}-1}\exp(ji\pi\theta)\right|\prod_{q=2}^{n}\frac{\left|\exp(2^{p_q}i\pi\theta)+\exp\left(\frac{2l+1}{2^{p_1-p_q}}i\pi\right)\right|}{\left|1+\exp\left(\frac{2l+1}{2^{p_1-p_q}}i\pi\right)\right|}$$
$$\le\frac{2^{p_1}}{2^{p_1}}\prod_{q=2}^{n}\frac{2\left|\cos\left(\frac{2l+1}{2^{p_1-p_q}}\frac{\pi}{2}-\frac{2^{p_q}\pi\theta}{2}\right)\right|}{2\left|\cos\left(\frac{2l+1}{2^{p_1-p_q}}\frac{\pi}{2}\right)\right|}=\prod_{q=2}^{n}\frac{\left|\cos\left(\frac{2l+1}{2^{p_1-p_q+1}}\pi-\frac{2^{p_q}\pi\theta}{2}\right)\right|}{\left|\cos\left(\frac{2l+1}{2^{p_1-p_q+1}}\pi\right)\right|}.$$

Next, an application of Lemma 3.1 yields

$$|l_1^{(N)}(z)| \le\Big[\prod_{2\le q\le n,\,p_q\in T}\times\prod_{2\le q\le n,\,p_q\in S}\Big]\frac{\left|\cos\left(\frac{2l+1}{2^{p_1-p_q+1}}\pi-\frac{2^{p_q}\pi\theta}{2}\right)\right|}{\left|\cos\left(\frac{2l+1}{2^{p_1-p_q+1}}\pi\right)\right|}=:L_T\times L_S. \tag{3.19}$$

We first deal with the product associated with T, denoted by L_T. By Lemma 3.2,

$$L_T\le\prod_{2\le q\le n,\,p_q\in T}\left(1+\frac{2^{p_q}\pi|\theta|}{2}\right)\le\prod_{q=2}^{n}\left(1+\frac{2^{p_q}\pi|\theta|}{2}\right)\le\prod_{q=2}^{n}\left(1+\frac{\pi/2}{2^{p_1-p_q}}\right),$$

the last estimate following from $|\theta| \leq 1/2^{p_1}$. On the other hand, by (2.4) we get $p_1 - p_q \geq q - 1$. Since $1 + x \leq \exp(x)$ for all $x \in \mathbb{R}$, we have

$$L_T \leq \prod_{q=2}^{n}\Big(1 + \frac{\pi/2}{2^{q-1}}\Big) \leq \prod_{q=2}^{n} \exp\Big(\frac{\pi/2}{2^{q-1}}\Big) \leq \exp\Big(\sum_{q\geq 1} \pi/2^{q+1}\Big) = \exp(\pi/2). \tag{3.20}$$

Now we deal with the product L_S. Lemma 3.3 leads to

$$L_S \leq \prod_{2\leq q\leq n,\, p_q\in S}\Big(1 + \frac{2^{p_1}|\theta|\pi}{2^{1+s_{j_q+1}}}\Big) \leq \prod_{2\leq q\leq n,\, p_q\in S}\Big(1 + \frac{\pi}{2^{1+s_{j_q+1}}}\Big),$$

the last estimate following from $|\theta| \leq 1/2^{p_1}$. On the other hand, the map $q \mapsto 1 + j_q \in \{2, \ldots, 2L+1\}$ is injective by Lemma 3.4. It follows that

$$L_S \leq \prod_{j=2}^{2L+1}\Big(1 + \frac{\pi}{2^{1+s_j}}\Big).$$

From (2.10) and (3.5) we have $1 + s_j \geq 2L + 1 - j$. Since $1 + x \leq \exp(x)$ for all $x \in \mathbb{R}$, we get

$$L_S \leq \prod_{j=2}^{2L+1}\Big(1 + \frac{\pi}{2^{2L+1-j}}\Big) \leq \prod_{j=2}^{2L+1} \exp\Big(\frac{\pi}{2^{2L+1-j}}\Big) \leq \exp\Big(\sum_{j\geq 0} \pi/2^j\Big) = \exp(2\pi). \tag{3.21}$$

Finally, estimates (3.19)–(3.21) together yield

$$|l_1^{(N)}(z)| \leq \exp(\pi/2) \times \exp(2\pi) \leq \exp(3\pi),$$

and this proves the required estimate (3.3) in the case $|\theta| \leq 1/2^{p_1}$.

3.3. Second case. In this subsection, we assume that $1/2^{p_1} < |\theta| \leq 1/2^{p_n}$ and we fix $q = q_\theta$ such that (3.4) holds. Set

$$S_\theta := \big\{q \in \{q_\theta, \ldots, n\},\, p_q \in S\big\} = \{q_i\}_{1\leq i\leq m}, \tag{3.22}$$

where (if S_θ is nonempty) $m = \operatorname{card}(S_\theta) \geq 1$ and the numbering $(q_i)_{1\leq i\leq m}$ is such that

$$q_\theta \leq q_1 < \cdots < q_m \leq n. \tag{3.23}$$

Before dealing with this case, we give a couple of auxiliary results. The first one deals with the q's for which $p_q \in T$ (see (3.6)).

LEMMA 3.5. *For all* $\widetilde{q}$ *with* $q_\theta \le \widetilde{q} \le n$, *and* $|\theta| \le 1/2^{p_{q_\theta}}$,

$$\prod_{\widetilde{q} \le q \le n,\, p_q \in T} \frac{\left|\cos\left(\frac{2l+1}{2^{p_1-p_q+1}}\pi - \frac{2^{p_q}\pi\theta}{2}\right)\right|}{\left|\cos\left(\frac{2l+1}{2^{p_1-p_q+1}}\pi\right)\right|} \le \exp(\pi).$$

Proof. Denote by L the left-hand side of the above inequality. For all $q = \widetilde{q}, \dots, n$ such that $p_q \in T$, one can apply Lemma 3.2 and the assumption on θ to get

$$\begin{aligned} L &\le \prod_{\widetilde{q} \le q \le n,\, p_q \in T} \left(1 + \frac{2^{p_q}\pi|\theta|}{2}\right) \le \prod_{q_\theta \le q \le n,\, p_q \in T} \left(1 + \frac{2^{p_q}\pi|\theta|}{2}\right) \\ &\le \prod_{q=q_\theta}^{n} \left(1 + \frac{2^{p_q}\pi|\theta|}{2}\right) \le \prod_{q=q_\theta}^{n} \left(1 + \frac{\pi/2}{2^{p_{q_\theta}-p_q}}\right). \end{aligned}$$

On the other hand, from (2.4) we have $p_{q_\theta} - p_q \ge q - q_\theta$. Consequently,

$$\begin{aligned} L &\le \prod_{q=q_\theta}^{n} \left(1 + \frac{\pi/2}{2^{q-q_\theta}}\right) \le \prod_{q=q_\theta}^{n} \exp\left(\frac{\pi/2}{2^{q-q_\theta}}\right) \\ &\le \exp\Big(\sum_{j\ge 0} \pi/2^{j+1}\Big) = \exp(\pi). \quad \blacksquare \end{aligned}$$

The next lemma deals with the set S_θ.

LEMMA 3.6. *Fix* $q_\theta \in \{2, \dots, n\}$ *and let* q_1 *be defined by* (3.22). *Then for all* $|\theta| \le 1/2^{p_{q_\theta}}$,

$$\prod_{q \in S_\theta,\, q \ne q_1} \frac{\left|\cos\left(\frac{2l+1}{2^{p_1-p_q+1}}\pi - \frac{2^{p_q}\pi\theta}{2}\right)\right|}{\left|\cos\left(\frac{2l+1}{2^{p_1-p_q+1}}\pi\right)\right|} \le \exp(2\pi).$$

Proof. First, one can assume that neither S_θ nor $S_\theta \setminus \{q_1\}$ is empty, i.e. $m \ge 2$ in (3.22), otherwise there is nothing to prove. Then $S_\theta \setminus \{q_1\} = \{q_i : i = 2, \dots, m\}$. Denote by L_{S_θ} the left-hand side of the above inequality. By (3.14), $p_{q_i} = p_1 - s_{j_{q_i}} - 1$ for all $i = 2, \dots, m$. It follows from Lemma 3.3 that

$$L_{S_\theta} \le \prod_{i=2}^{m} \frac{\left|\cos\left(\frac{2l+1}{2^{p_1-p_{q_i}+1}}\pi - \frac{2^{p_{q_i}}\pi\theta}{2}\right)\right|}{\left|\cos\left(\frac{2l+1}{2^{p_1-p_{q_i}+1}}\pi\right)\right|} \le \prod_{i=2}^{m} \left(1 + \frac{2^{p_1}|\theta|\pi}{2^{1+s_{j_{q_i}+1}}}\right).$$

On the other hand, by Lemma 3.4 the map $q \mapsto j_q$ is decreasing. Since by (3.23), $q_{i-1} < q_i$ for all $i = 2, \dots, m$, it follows that $j_{q_{i-1}} > j_{q_i}$, i.e.

$j_{q_{i-1}} \geq j_{q_i} + 1$. As $j \mapsto s_j$ is decreasing by (2.10), this yields $s_{j_{q_i}+1} \geq s_{j_{q_{i-1}}}$, and so the above inequality leads to

$$L_{S_\theta} \leq \prod_{i=2}^{m} \left(1 + \frac{2^{p_1}|\theta|\pi}{2^{1+s_{j_{q_i}+1}}}\right) \leq \prod_{i=2}^{m} \left(1 + \frac{2^{p_1}|\theta|\pi}{2^{1+s_{j_{q_{i-1}}}}}\right) = \prod_{i=1}^{m-1} \left(1 + \frac{2^{p_1}|\theta|\pi}{2^{1+s_{j_{q_i}}}}\right). \tag{3.24}$$

Since q_i satisfies (3.14) for all $i = 1, \ldots, m-1$, we have $s_{j_{q_i}} + 1 = p_1 - p_{q_i}$, and by (3.24) it follows that

$$L_{S_\theta} \leq \prod_{i=1}^{m-1} \left(1 + \frac{2^{p_1}|\theta|\pi}{2^{p_1 - p_{q_i}}}\right) \leq \prod_{i=1}^{m-1} \exp(2^{p_{q_i}}|\theta|\pi) \leq \exp\left(\sum_{q=q_\theta}^{n} 2^{p_q}|\theta|\pi\right) \leq \exp\left(\sum_{q=q_\theta}^{n} \frac{\pi}{2^{p_{q_\theta} - p_q}}\right), \tag{3.25}$$

the third estimate coming from

$$\{q_i : 1 \leq i \leq m-1\} \subset S_\theta \subset \{q_\theta \leq q \leq n\},$$

and the last one from $|\theta| \leq 1/2^{p_{q_\theta}}$. By (2.4) we get $p_{q_\theta} - p_q \geq q - q_\theta$. Estimate (3.25) then becomes

$$L_{S_\theta} \leq \exp\left(\sum_{q=q_\theta}^{n} \frac{\pi}{2^{q-q_\theta}}\right) \leq \exp\left(\sum_{j\geq 0} \frac{\pi}{2^j}\right) = \exp(2\pi). \quad \blacksquare$$

Now we can deal with required estimate (3.3) after fixing $q_\theta = 2, \ldots, n$ and θ with $1/2^{p_{q_\theta - 1}} \leq |\theta| \leq 1/2^{p_{q_\theta}}$. First (since $\theta \neq 0$ then $z \neq 1$), by (2.5) with $\eta_k = \eta_1 = 1$, (3.1) and (3.2) one has

$$|l_1^{(N)}(z)| = \frac{1}{2^{p_1}} \frac{|z^{2^{p_1}} - 1|}{|z-1|} \times \prod_{q=2}^{n} \frac{|z^{2^{p_q}} + \omega_0^{2^{p_q}}|}{|1 + \omega_0^{2^{p_q}}|}$$
$$\leq \frac{1}{2^{p_1}} \frac{2}{|\exp(i\pi\theta) - 1|} \prod_{q=2}^{q_\theta - 1} \frac{2}{\left|1 + \exp\left(\frac{2l+1}{2^{p_1 - p_q}} i\pi\right)\right|} \times \prod_{q=q_\theta}^{n} \frac{\left|\exp(2^{p_q} i\pi\theta) + \exp\left(\frac{2l+1}{2^{p_1-p_q}} i\pi\right)\right|}{\left|1 + \exp\left(\frac{2l+1}{2^{p_1-p_q}} i\pi\right)\right|},$$

and hence

$$(3.26)\quad |l_1^{(N)}(z)|$$
$$\leq \frac{1/2^{p_1}}{|\sin(\pi\theta/2)|}\prod_{q=2}^{q_\theta-1}\frac{1}{\left|\cos\left(\frac{2l+1}{2^{p_1-p_q}}\frac{\pi}{2}\right)\right|}\prod_{q=q_\theta}^{n}\frac{\left|\cos\left(\frac{2l+1}{2^{p_1-p_q}}\frac{\pi}{2}-\frac{2^{p_q}\pi\theta}{2}\right)\right|}{\left|\cos\left(\frac{2l+1}{2^{p_1-p_q}}\frac{\pi}{2}\right)\right|}$$
$$\leq \frac{1}{2^{p_1}|\theta|}\prod_{q=2}^{q_\theta-1}\frac{1}{\left|\cos\left(\frac{2l+1}{2^{p_1-p_q}}\frac{\pi}{2}\right)\right|}\prod_{q=q_\theta}^{n}\frac{\left|\cos\left(\frac{2l+1}{2^{p_1-p_q}}\frac{\pi}{2}-\frac{2^{p_q}\pi\theta}{2}\right)\right|}{\left|\cos\left(\frac{2l+1}{2^{p_1-p_q}}\frac{\pi}{2}\right)\right|},$$

the last estimate being a consequence of the inequality

$$(3.27)\quad \left|\sin\left(\frac{\pi\theta}{2}\right)\right| = \left|\sin\left(\frac{\pi|\theta|}{2}\right)\right| = \sin\left(\frac{\pi|\theta|}{2}\right) \geq \frac{2}{\pi}\times\frac{\pi|\theta|}{2} = |\theta|.$$

Now if we assume that $S_\theta = \emptyset$, then $\{p_q : q_\theta \leq q \leq n\} \subset T$ by Lemma 3.1 and

$$(3.28)\quad |l_1^{(N)}(z)| \leq \frac{1}{2^{p_1}|\theta|}\frac{1}{\prod_{q=2}^{q_\theta-1}\left|\cos\left(\frac{2l+1}{2^{p_1-p_q+1}}\pi\right)\right|}$$
$$\times \prod_{q_\theta\leq q\leq n,\, p_q\in T}\frac{\left|\cos\left(\frac{2l+1}{2^{p_1-p_q+1}}\pi-\frac{2^{p_q}\pi\theta}{2}\right)\right|}{\left|\cos\left(\frac{2l+1}{2^{p_1-p_q+1}}\pi\right)\right|}.$$

On the other hand,

$$\prod_{q=2}^{q_\theta-1}\left|\cos\left(\frac{2l+1}{2^{p_1-p_q+1}}\pi\right)\right| \geq \prod_{j=p_1-p_2+1}^{p_1-p_{q_\theta-1}+1}\left|\cos\left(\frac{2l+1}{2^j}\pi\right)\right|$$
$$\geq \prod_{j=1}^{p_1-p_{q_\theta-1}}\left|\cos\left(\frac{(2l+1)\pi/2}{2^j}\right)\right|,$$

since $p_1 - p_2 + 1 \geq 2$ and any term of the products involved is no greater than 1. An application of Lemma 2.4 (with $\alpha = (2l+1)\pi/2 \notin \pi\mathbb{Z}$ and $m = p_1 - p_{q_\theta-1} \geq p_1 - p_1 = 0$ since $q_\theta \geq 2$) leads to

$$(3.29)\quad \prod_{q=2}^{q_\theta-1}\left|\cos\left(\frac{2l+1}{2^{p_1-p_q+1}}\pi\right)\right| \geq \frac{|\sin((2l+1)\pi/2)|}{2^{p_1-p_{q_\theta-1}}\left|\sin\left(\frac{(2l+1)\pi/2}{2^{p_1-p_{q_\theta-1}}}\right)\right|}$$
$$\geq \frac{1}{2^{p_1-p_{q_\theta-1}}}.$$

Thus (3.28), (3.29) and Lemma 3.5 (with $\widetilde{q} = q_\theta$) yield

$$|l_1^{(N)}(z)| \leq \frac{1}{2^{p_1}|\theta|} \times 2^{p_1 - p_{q_\theta} - 1} \times \exp(\pi) = \frac{\exp(\pi)}{2^{p_{q_\theta}-1}|\theta|} \leq \exp(\pi),$$

the last estimate following from $|\theta| \geq 1/2^{p_{q_\theta}-1}$, and this proves the required assertion in the case $S_\theta = \emptyset$.

Now assume that S_θ is nonempty. In particular, we can deal with q_1, i.e.

(3.30)
$$\frac{\left|\cos\left(\frac{2l+1}{2^{p_1-p_{q_1}+1}}\pi - \frac{2^{p_{q_1}}\pi\theta}{2}\right)\right|}{\left|\cos\left(\frac{2l+1}{2^{p_1-p_{q_1}+1}}\pi\right)\right|} \leq \frac{\left|\cos\left(\frac{2l+1}{2^{p_1-p_{q_1}+1}}\pi\right)\right| + \left|\sin\left(\frac{2^{p_{q_1}}\pi\theta}{2}\right)\right|}{\left|\cos\left(\frac{2l+1}{2^{p_1-p_{q_1}+1}}\pi\right)\right|}$$
$$\leq \frac{2\max\left[\left|\cos\left(\frac{2l+1}{2^{p_1-p_{q_1}+1}}\pi\right)\right|, \frac{2^{p_{q_1}}\pi|\theta|}{2}\right]}{\left|\cos\left(\frac{2l+1}{2^{p_1-p_{q_1}+1}}\pi\right)\right|},$$

since $|\sin(2^{p_{q_1}}\pi\theta/2)| \leq 2^{p_{q_1}}\pi|\theta|/2$.

Now assume that

$$\max\left[\left|\cos\left(\frac{2l+1}{2^{p_1-p_{q_1}+1}}\pi\right)\right|, \frac{2^{p_{q_1}}\pi|\theta|}{2}\right] = \left|\cos\left(\frac{2l+1}{2^{p_1-p_{q_1}+1}}\pi\right)\right|.$$

Then (3.30) becomes

$$\frac{\left|\cos\left(\frac{2l+1}{2^{p_1-p_{q_1}+1}}\pi - \frac{2^{p_{q_1}}\pi\theta}{2}\right)\right|}{\left|\cos\left(\frac{2l+1}{2^{p_1-p_{q_1}+1}}\pi\right)\right|} \leq 2. \tag{3.31}$$

It follows by Lemma 3.5 (with the choice of $\widetilde{q} = q_\theta$) and Lemma 3.6 that

$$\prod_{q=q_\theta}^{n} \frac{\left|\cos\left(\frac{2l+1}{2^{p_1-p_q+1}}\pi - \frac{2^{p_q}\pi\theta}{2}\right)\right|}{\left|\cos\left(\frac{2l+1}{2^{p_1-p_q+1}}\pi\right)\right|} = \frac{\left|\cos\left(\frac{2l+1}{2^{p_1-p_{q_1}+1}}\pi - \frac{2^{p_{q_1}}\pi\theta}{2}\right)\right|}{\left|\cos\left(\frac{2l+1}{2^{p_1-p_{q_1}+1}}\pi\right)\right|} \tag{3.32}$$
$$\times \left[\prod_{q_\theta \leq q \leq n,\, p_q \in T} \times \prod_{q \in S_\theta, q \neq q_1}\right] \frac{\left|\cos\left(\frac{2l+1}{2^{p_1-p_q+1}}\pi - \frac{2^{p_q}\pi\theta}{2}\right)\right|}{\left|\cos\left(\frac{2l+1}{2^{p_1-p_q+1}}\pi\right)\right|}$$
$$\leq 2 \times \exp(\pi) \times \exp(2\pi) \leq \pi\exp(3\pi).$$

Then (3.26), (3.29) and (3.32) yield

$$|l_1^{(N)}(z)| \le \frac{1}{2^{p_1}|\theta|} \times 2^{p_1 - p_{q_\theta - 1}} \times \pi \exp(3\pi)$$
$$= \frac{\pi \exp(3\pi)}{2^{p_{q_\theta - 1}}|\theta|} \le \pi \exp(3\pi),$$

the last estimate following from $|\theta| \ge 1/2^{p_{q_\theta - 1}}$. This proves the required assertion in this case.

The remaining case is the one for which

$$\max\left[\left|\cos\left((2l+1)\pi/2^{p_1 - p_{q_1}+1}\right)\right|, 2^{p_{q_1}}\pi|\theta|/2\right] = 2^{p_{q_1}}\pi|\theta|/2. \tag{3.33}$$

We prove an estimate similar to (3.26) with q_θ replaced by q_1. Since $z \ne 1$, by (2.5) with $\eta_k = \eta_1 = 1$, (3.1) and (3.2) one still has

$$\begin{aligned}
|l_1^{(N)}(z)| &= \frac{|z^{2^{p_1}} - 1|}{2^{p_1}|z-1|} \prod_{q=2}^{n} \frac{|z^{2^{p_q}} + \omega_0^{2^{p_q}}|}{|1 + \omega_0^{2^{p_q}}|} \\
&\le \frac{2}{2^{p_1}|z-1|} \prod_{q=2}^{q_1 - 1} \frac{2}{|1 + \omega_0^{2^{p_q}}|} \prod_{q=q_1}^{n} \frac{|z^{2^{p_q}} + \omega_0^{2^{p_q}}|}{|1 + \omega_0^{2^{p_q}}|} \\
&\le \frac{1}{2^{p_1}} \frac{1}{|\sin(\pi\theta/2)|} \prod_{q=2}^{q_1 - 1} \frac{1}{\left|\cos\left(\frac{2l+1}{2^{p_1 - p_q}}\frac{\pi}{2}\right)\right|} \\
&\quad \times \frac{\left|\cos\left(\frac{(2l+1)\pi}{2^{p_1 - p_{q_1}+1}} - \frac{2^{p_{q_1}}\pi\theta}{2}\right)\right|}{\left|\cos\left(\frac{2l+1}{2^{p_1 - p_{q_1}}}\frac{\pi}{2}\right)\right|} \prod_{q=q_1+1}^{n} \frac{\left|\cos\left(\frac{2l+1}{2^{p_1 - p_q}}\frac{\pi}{2} - \frac{2^{p_q}\pi\theta}{2}\right)\right|}{\left|\cos\left(\frac{2l+1}{2^{p_1 - p_q}}\frac{\pi}{2}\right)\right|}.
\end{aligned}$$

On the other hand, by (3.30) and (3.33),

$$\begin{aligned}
\frac{\left|\cos\left(\frac{2l+1}{2^{p_1 - p_{q_1}+1}}\pi - \frac{2^{p_{q_1}}\pi\theta}{2}\right)\right|}{\left|\cos\left(\frac{2l+1}{2^{p_1 - p_{q_1}+1}}\pi\right)\right|} &\le \frac{2\max\left[\left|\cos\left(\frac{2l+1}{2^{p_1 - p_{q_1}+1}}\pi\right)\right|, \frac{2^{p_{q_1}}\pi|\theta|}{2}\right]}{\left|\cos\left(\frac{2l+1}{2^{p_1 - p_{q_1}+1}}\pi\right)\right|} \\
&\le \frac{2^{p_{q_1}}\pi|\theta|}{\left|\cos\left(\frac{2l+1}{2^{p_1 - p_{q_1}+1}}\pi\right)\right|}.
\end{aligned}$$

It follows by also applying (3.27) that

$$(3.34)\qquad |l_1^{(N)}(z)| \le \frac{1}{2^{p_1}}\,\frac{1}{|\theta|}\,\frac{1}{\prod_{q=2}^{q_1}\left|\cos\left(\frac{2l+1}{2^{p_1-p_q+1}}\pi\right)\right|} \times 2^{p_{q_1}}\pi|\theta| \prod_{q=q_1+1}^{n} \frac{\left|\cos\left(\frac{2l+1}{2^{p_1-p_q+1}}\pi - \frac{2^{p_q}\pi\theta}{2}\right)\right|}{\left|\cos\left(\frac{2l+1}{2^{p_1-p_q+1}}\pi\right)\right|}.$$

Now since $p_1 - p_2 + 1 \ge 2$ and any term of the following products is no greater than 1, one has

$$(3.35)\qquad \prod_{q=2}^{q_1}\left|\cos\left(\frac{2l+1}{2^{p_1-p_q+1}}\pi\right)\right| \ge \prod_{j=p_1-p_2+1}^{p_1-p_{q_1}+1}\left|\cos\left(\frac{2l+1}{2^j}\pi\right)\right| \ge \left|\prod_{j=1}^{p_1-p_{q_1}}\cos\left(\frac{(2l+1)\pi/2}{2^j}\right)\right| = \frac{|\sin((2l+1)\pi/2)|}{2^{p_1-p_{q_1}}\left|\sin\left(\frac{(2l+1)\pi/2}{2^{p_1-p_{q_1}}}\right)\right|} \ge \frac{1}{2^{p_1-p_{q_1}}},$$

the equality being an application of Lemma 2.4 (with $\alpha = (2l+1)\pi/2 \notin \pi\mathbb{Z}$ and $m = p_1 - p_{q_1} \ge 0$). Next, since by (3.22), $\{q_1 + 1 \le q \le n : p_q \in S\} = \{q_i : i = 2, \ldots, m\} = S_\theta \setminus \{q_1\}$, one can apply Lemma 3.1 (with the subset $\{q : q_1 + 1 \le q \le n\}$), Lemma 3.5 (with $\widetilde{q} = q_1 + 1 > q_1 \ge q_\theta$) and Lemma 3.6 to get

$$(3.36)\qquad \prod_{q=q_1+1}^{n} \frac{\left|\cos\left(\frac{2l+1}{2^{p_1-p_q+1}}\pi - \frac{2^{p_q}\pi\theta}{2}\right)\right|}{\left|\cos\left(\frac{2l+1}{2^{p_1-p_q+1}}\pi\right)\right|} = \left[\prod_{q_1+1\le q\le n,\, p_q\in T} \times \prod_{q\in S_\theta,\, q\ne q_1}\right] \frac{\left|\cos\left(\frac{2l+1}{2^{p_1-p_q+1}}\pi - \frac{2^{p_q}\pi\theta}{2}\right)\right|}{\left|\cos\left(\frac{2l+1}{2^{p_1-p_q+1}}\pi\right)\right|} \le \exp(\pi) \times \exp(2\pi) = \exp(3\pi).$$

Finally, estimates (3.34)–(3.36) together yield

$$|l_1^{(N)}(z)| \le \frac{1}{2^{p_1}|\theta|} \times 2^{p_1-p_{q_1}} \times 2^{p_{q_1}}\pi|\theta| \times \exp(3\pi) = \pi\exp(3\pi),$$

and this proves the required estimate (3.3) in the case $1/2^{p_{q_\theta}-1} \le |\theta| \le 1/2^{p_{q_\theta}}$.

3.4. Third case. Now we fix θ with $1/2^{p_n} \le |\theta| \le 1$. In particular, $z \neq 1$. Then by (2.5) with $\eta_k = \eta_1 = 1$, (3.1) and (3.2),

$$\begin{aligned}
|l_1^{(N)}(z)| &= \frac{1}{2^{p_1}} \frac{|z^{2^{p_1}} - 1|}{|z-1|} \prod_{q=2}^{n} \frac{|z^{2^{p_q}} + \omega_0^{2^{p_q}}|}{|1 + \omega_0^{2^{p_q}}|} \\
&\le \frac{1}{2^{p_1}} \frac{2}{|\exp(i\pi\theta) - 1|} \prod_{q=2}^{n} \frac{2}{\left|1 + \exp\left(\frac{2l+1}{2^{p_1-p_q}} i\pi\right)\right|} \\
&= \frac{1}{2^{p_1}|\sin(\pi\theta/2)|} \prod_{q=2}^{n} \frac{1}{\left|\cos\left(\frac{2l+1}{2^{p_1-p_q}} \frac{\pi}{2}\right)\right|} \\
&\le \frac{1}{2^{p_1}} \frac{1}{|\theta|} \prod_{q=2}^{n} \frac{1}{\left|\cos\left(\frac{2l+1}{2^{p_1-p_q+1}} \pi\right)\right|},
\end{aligned}$$

the last estimate being an application of (3.27). On the other hand, for all $q = 2, \dots, n$, by (2.4) one has $2 \le p_1 - p_2 + 1 \le p_1 - p_q + 1 \le p_1 - p_n + 1$. Since any term of the following products is no greater than 1, we get

$$\begin{aligned}
\prod_{q=2}^{n} \left|\cos\left(\frac{2l+1}{2^{p_1-p_q+1}} \pi\right)\right| &\ge \prod_{j=p_1-p_2+1}^{p_1-p_n+1} \left|\cos\left(\frac{2l+1}{2^j} \pi\right)\right| \\
&\ge \left|\prod_{j=1}^{p_1-p_n} \cos\left(\frac{(2l+1)\pi/2}{2^j}\right)\right| \\
&= \frac{|\sin((2l+1)\pi/2)|}{2^{p_1-p_n}\left|\sin\left(\frac{(2l+1)\pi/2}{2^{p_1-p_n}}\right)\right|} \ge \frac{1}{2^{p_1-p_n}},
\end{aligned}$$

the equality being an application of Lemma 2.4 (with $\alpha = (2l+1)\pi/2 \notin \pi\mathbb{Z}$ and $m = p_1 - p_n \ge 0$). Since $|\theta| \ge 1/2^{p_n}$, the above estimate of $l_1^{(N)}(z)$ becomes

$$|l_1^{(N)}(z)| \le \frac{1}{2^{p_1-p_n}} \times 2^{p_1-p_n} = 1,$$

and this proves (3.3) in this last case, completing the proof.

4. Proof of Theorem 1.2 in the general case

4.1. Auxiliary results. In the previous section, we have considered the special case of $l_1^{(N)}$ (i.e. the FLIP associated with $\eta_1 = 1$) and $l = 1, \dots, 2^{p_1} - 1$ for ω_0. The following result gives a way to extend (3.3) to every FLIP associated with η_k where $k = 2, \dots, 2^{p_1}$. We will also use the notation (for any function f defined on $\overline{\mathbb{D}}$)

$$\|f\|_{\overline{\mathbb{D}}} := \sup_{z\in\overline{\mathbb{D}}} |f(z)|.$$

LEMMA 4.1. *Let $\mathcal{L}_N$ be the N-Leja section of any fixed Leja sequence $\mathcal{L}$ (that starts at $\eta_1 = 1$). For all $k = 1, \ldots, 2^{p_1}$, consider $\eta_k \in \mathcal{L}$ (i.e. $\eta_k \in \mathcal{U}_{2^{p_1}}$ is any 2^{p_1}th root of the unity) and the associated FLIP $l_k^{(N)}$. Then for all $z \in \mathbb{C}$,*

$$|l_k^{(N)}(\eta_k z)| = |\widetilde{l}_1^{(N)}(z)|, \tag{4.1}$$

where $\widetilde{l}_1^{(N)}$ is the FLIP associated with $\widetilde{\eta}_1 = 1$ of (possibly another) N-Leja section $\widetilde{\mathcal{L}}_N$ (that also starts at 1). In particular,

$$\|l_k^{(N)}\|_{\overline{\mathbb{D}}} = \|\widetilde{l}_1^{(N)}\|_{\overline{\mathbb{D}}}. \tag{4.2}$$

Proof. Fix $\eta_k \in \mathcal{U}_{2^{p_1}}$. By (2.5) there is a 2^{p_1}th root ω_0 of -1 such that, for all $z \in \mathbb{C}$ with $z \neq 1$ (so that $\eta_k z \neq \eta_k$),

$$\begin{aligned} |l_k^{(N)}(\eta_k z)| &= \frac{1}{2^{p_1}} \frac{|\eta_k^{2^{p_1}} z^{2^{p_1}} - 1|}{|z\eta_k - \eta_k|} \prod_{q=2}^{n} \frac{|\eta_k^{2^{p_q}} z^{2^{p_q}} + \omega_0^{2^{p_q}}|}{|\eta_k^{2^{p_q}} + \omega_0^{2^{p_q}}|} \\ &= \frac{1}{2^{p_1}} \frac{|z^{2^{p_1}} - 1|}{|z-1|} \prod_{q=2}^{n} \frac{|z^{2^{p_q}} + (\omega_0/\eta_k)^{2^{p_q}}|}{|1 + (\omega_0/\eta_k)^{2^{p_q}}|}. \end{aligned}$$

It follows that for all $z \neq 1$ we have

$$|l_k^{(N,\omega_0)}(\eta_k z)| = |l_1^{(N,\omega_1)}(z)|,$$

where $\omega_1 := \omega_0/\eta_k$ is still a 2^{p_1}th root of -1 and $l_k^{(N,\omega_0)}$ (resp., $l_1^{(N,\omega_1)}$) is the FLIP associated with η_k (resp., $\widetilde{\eta}_1 = 1$) and the 2^{p_1}th root ω_0 (resp., ω_1). We recall that, as specified by (2.8), the data of ω_1 conversely gives an N-Leja section (that starts at $\widetilde{\eta}_1 = 1$) and whose first FLIP is exactly $l_1^{(N,\omega_1)}$. This proves (4.1) by setting $\widetilde{l}_1^{(N)} := l_1^{(N,\omega_1)}$, and (4.2) follows since $|\eta_k| = 1$. ■

We close this subsection with the following result that is the proof of (3.3) for $l_1^{(N)}$ and the special case of ω_0 that was not considered in the previous section.

LEMMA 4.2. *Fix $k = 1$ (i.e. $\eta_k = \eta_1 = 1$) and $l = 0$ in (3.1), i.e. $\omega_0 = \exp(i\pi/2^{p_1})$. Then*

$$\|l_1^{(N)}\|_{\overline{\mathbb{D}}} \leq \pi/2.$$

Proof. For all $z \in \mathbb{C}$ with $|z| \le 1$ one has, by (2.5) with $\eta_k = \eta_1 = 1$,

$$\sup_{|z|\le 1} |l_1^{(N)}(z)| = \sup_{|z|\le 1}\left[\frac{1}{2^{p_1}}\Big|\sum_{j=0}^{2^{p_1}-1} z^j\Big| \times \prod_{q=2}^{n} \frac{|z^{2^{p_q}} + \omega_0^{2^{p_q}}|}{|1+\omega_0^{2^{p_q}}|}\right] \tag{4.3}$$

$$\le 1 \times \prod_{q=2}^{n} \frac{2}{|1+\exp(i\pi/2^{p_1-p_q})|} = \frac{1}{\prod_{q=2}^{n}\Big|\cos\Big(\frac{1}{2^{p_1-p_q}}\frac{\pi}{2}\Big)\Big|}.$$

On the other hand, for all $q = 2, \dots, n$, by (2.4) one has $2 \le p_1 - p_2 + 1 \le p_1 - p_q + 1 \le p_1 - p_n + 1 \le p_1 + 1$, and so (since any term in the following products is no greater than 1)

$$\prod_{q=2}^{n}\Big|\cos\Big(\frac{\pi}{2^{p_1-p_q+1}}\Big)\Big| \ge \prod_{j=p_1-p_2+1}^{p_1-p_n+1}\Big|\cos\Big(\frac{\pi}{2^j}\Big)\Big| \ge \Big|\prod_{j=1}^{p_1}\cos\Big(\frac{\pi/2}{2^j}\Big)\Big|$$

$$= \frac{|\sin(\pi/2)|}{2^{p_1}\Big|\sin\Big(\frac{\pi/2}{2^{p_1}}\Big)\Big|} \ge \frac{1}{2^{p_1}\times\frac{\pi/2}{2^{p_1}}} = \frac{2}{\pi},$$

where the first equality is an application of Lemma 2.4 (with $\alpha = \pi/2$ and $m = p_1$) and the last estimate follows from $|\sin(x)| \le |x|$ for all $x \in \mathbb{R}$. By (4.3), the proof is complete. ∎

4.2. Proof of Theorem 1.2. We need the following result in which we deal with FLIPs associated with $\eta_k \in \mathcal{L}_N \setminus \mathcal{U}_{2^{p_1}}$.

LEMMA 4.3. *Let $\mathcal{L}_N$ be an N-Leja section with $\eta_1 = 1$. There is an $(N-2^{p_1})$-Leja section $\widetilde{\mathcal{L}}_{N-2^{p_1}}$ that also starts at $\widetilde{\eta}_1 = 1$, with the following property: for all $k = 2^{p_1}+1, \dots, N$, there exists a unique k' with $1 \le k' \le N - 2^{p_1}$ such that*

$$\|l_k^{(N)}\|_{\overline{\mathbb{D}}} \le \|\widetilde{l}_{k'}^{(N-2^{p_1})}\|_{\overline{\mathbb{D}}},$$

where $\widetilde{l}_{k'}^{(N-2^{p_1})}$ is the FLIP associated with $\widetilde{\eta}_{k'} \in \widetilde{\mathcal{L}}_{N-2^{p_1}}$.

Proof. First, let $k = 2^{p_1}+1, \dots, N$ and consider the FLIP $l_k^{(N)}$ associated with η_k. Since $\mathcal{L}_N$ is an N-Leja section that starts at $\eta_1 = 1$, by [3, Theorem 5] (or (2.6)) one has $\eta_k \notin \mathcal{U}_{2^{p_1}}$, i.e. η_k is a 2^{p_1}th root of -1. It follows that

$$|l_k^{(N)}(z)| = \Big|\prod_{\eta_j\in\mathcal{U}_{2^{p_1}}}\frac{z-\eta_j}{\eta_k-\eta_j}\Big| \times \Big|\prod_{\eta_j\in\mathcal{L}_N\setminus\mathcal{U}_{2^{p_1}},\,\eta_j\ne\eta_k}\frac{z-\eta_j}{\eta_k-\eta_j}\Big|. \tag{4.4}$$

On the one hand, for all $|z| \le 1$ (since $\eta_k^{2^{p_1}} = -1$) one has

$$\Big|\prod_{\eta_j\in\mathcal{U}_{2^{p_1}}}\frac{z-\eta_j}{\eta_k-\eta_j}\Big| = \frac{|z^{2^{p_1}}-1|}{|\eta_k^{2^{p_1}}-1|} \le \frac{|z|^{2^{p_1}}+1}{|-1-1|} = \frac{2}{2} = 1. \tag{4.5}$$

On the other hand, (by [3, Theorem 5] again, or (2.6)) one has $\mathcal{L}_N \setminus \mathcal{U}_{2^{p_1}} = \omega_1 \widetilde{\mathcal{L}}_{N-2^{p_1}}$, where ω_1 is a 2^{p_1}th root of -1, $\widetilde{\mathcal{L}}_{N-2^{p_1}}$ is the $(N-2^{p_1})$-section of (maybe another) Leja sequence $\widetilde{\mathcal{L}} = \{\widetilde{\eta}_1, \widetilde{\eta}_2, \ldots\}$ with $\widetilde{\eta}_1 = 1$, and the above equality is meant as equality of sets. In particular, $\eta_k \in \omega_1 \widetilde{\mathcal{L}}_{N-2^{p_1}}$ can be written as $\eta_k = \omega_1 \widetilde{\eta}_{k'}$ with $1 \le k' \le N - 2^{p_1}$. This proves that the map

$$\eta_k, 2^{p_1} + 1 \le k \le N \;\mapsto\; \widetilde{\eta}_{k'}, 1 \le k' \le N - 2^{p_1},$$

is well-defined and injective. Since

$$\operatorname{card}(\mathcal{L}_N \setminus \mathcal{U}_{2^{p_1}}) = N - 2^{p_1} = \operatorname{card}(\omega_1 \widetilde{\mathcal{L}}_{N-2^{p_1}}),$$

it is also one-to-one. In particular, this leads to

$$(4.6)\quad \left| \prod_{\eta_j \in \mathcal{L}_N \setminus \mathcal{U}_{2^{p_1}},\, \eta_j \neq \eta_k} \frac{z - \eta_j}{\eta_k - \eta_j} \right| = \left| \prod_{\eta_j \in \omega_1 \widetilde{\mathcal{L}}_{N-2^{p_1}},\, \eta_j \neq \eta_k} \frac{z - \eta_j}{\eta_k - \eta_j} \right|$$
$$= \left| \prod_{\widetilde{\eta}_j \in \widetilde{\mathcal{L}}_{N-2^{p_1}},\, \widetilde{\eta}_j \neq \widetilde{\eta}_{k'}} \frac{z - \omega_1 \widetilde{\eta}_j}{\omega_1 \widetilde{\eta}_{k'} - \omega_1 \widetilde{\eta}_j} \right|$$
$$= \left| \prod_{\widetilde{\eta}_j \in \widetilde{\mathcal{L}}_{N-2^{p_1}},\, \widetilde{\eta}_j \neq \widetilde{\eta}_{k'}} \frac{z/\omega_1 - \widetilde{\eta}_j}{\widetilde{\eta}_{k'} - \widetilde{\eta}_j} \right| = \left| \widetilde{l}_{k'}^{(N-2^{p_1})}\left(\frac{z}{\omega_1}\right) \right|,$$

where $\widetilde{l}_{k'}^{(N-2^{p_1})}$ is the FLIP associated with $\widetilde{\eta}_{k'}$ from the $(N-2^{p_1})$-Leja section $\widetilde{\mathcal{L}}_{N-2^{p_1}}$.

Finally, estimates (4.4)–(4.6) together yield

$$\sup_{|z|\le 1} |l_k^{(N)}(z)| \le 1 \times \sup_{|z|\le 1} |\widetilde{l}_{k'}^{(N-2^{p_1})}(z/\omega_1)| = \|\widetilde{l}_{k'}^{(N-2^{p_1})}\|_{\overline{\mathbb{D}}}. \ \blacksquare$$

Now we can finally give the proof of Theorem 1.2. Consider the N-section $\mathcal{L}_N$ of any fixed Leja sequence, where

$$(4.7)\qquad N = 2^{p_1} + \cdots + 2^{p_n} \quad \text{with } p_1 > \cdots > p_n \ge 0 \text{ and } n \ge 1.$$

Proof of Theorem 1.2. First, by the symmetry of the disk, we can assume that $\eta_1 = 1$. Next, by the Maximum Modulus Principle, it suffices to prove the required estimate for all $|z| = 1$, i.e. $z = \exp(i\pi\theta)$ with $\theta \in\,]-1, 1]$. The proof is by induction on $n \ge 1$.

The special case of $n = 1$ means that $N = 2^{p_1}$ with $p_1 \ge 0$. Then $\mathcal{L}_{2^{p_1}} = \mathcal{U}_{2^{p_1}}$ by [3, Theorem 5], and (2.2) yields $\sup_{z\in\overline{\mathbb{D}}} |l_k^{(2^{p_1})}(z)| = 1$ for all $k = 1, \ldots, 2^{p_1}$.

Now consider N with $n \ge 2$ (i.e. $2^{p_1} < N < 2^{p_1+1}$) and let ω_0 be as defined in Lemma 2.2; we can write $\omega_0 = \exp((2l + 1)i\pi/2^{p_1})$ with $l = 0, \ldots, 2^{p_1} - 1$. Then the theorem is true for the FLIP $l_1^{(N)}$ associated with $\eta_1 = 1$. Indeed, if $l = 0$, this is a consequence of Lemma 4.2. Otherwise, $1 \le l \le 2^{p_1} - 1$ and we apply Section 3.

Next, if $2 \le k \le 2^{p_1}$, then $\eta_k \in \mathcal{U}_{2^{p_1}}$ (i.e. η_k is a 2^{p_1}th root of the unity, see [4, Theorem 1] or (2.6)). An application of (4.2) yields $\|l_k^{(N)}\|_{\overline{\mathbb{D}}} = \|\widetilde{l}_1^{(N)}\|_{\overline{\mathbb{D}}}$, where $\widetilde{l}_1^{(N)}$ is the FLIP associated with $\widetilde{\eta}_1 = 1$ from (maybe another) N-Leja section $\widetilde{\mathcal{L}}_N$. Since the theorem is valid for $l_1^{(N)}$ and any N-Leja section, it follows that it holds for $l_k^{(N)}$.

Lastly, let $k = 2^{p_1} + 1, \ldots, N$ and consider the FLIP $l_k^{(N)}$ associated with η_k. As $\mathcal{L}_N$ is an N-Leja section that starts at $\eta_1 = 1$, necessarily $\eta_k \notin \mathcal{U}_{2^{p_1}}$, i.e. η_k is a 2^{p_1}th root of -1 (see [3, Theorem 5]). An application of Lemma 4.3 leads to

$$\|l_k^{(N)}\|_{\overline{\mathbb{D}}} \le \|\widetilde{l}_{k'}^{(N-2^{p_1})}\|_{\overline{\mathbb{D}}},$$

where $\widetilde{l}_{k'}^{(N-2^{p_1})}$ is the FLIP associated with $\widetilde{\eta}_{k'} \in \widetilde{\mathcal{L}}_{N-2^{p_1}}$ and $\widetilde{\mathcal{L}}_{N-2^{p_1}}$ is an $(N-2^{p_1})$-Leja section that also starts at $\widetilde{\eta}_1 = 1$. Since by (4.7),

$$N - 2^{p_1} = 2^{p_2} + \cdots + 2^{p_n} = \sum_{q=1}^{n-1} 2^{p_{q+1}},$$

it follows that the induction hypothesis can be applied to $\widetilde{\mathcal{L}}_{N-2^{p_1}}$ with $n-1$, and the above inequality becomes $\|l_k^{(N)}\|_{\overline{\mathbb{D}}} \le \|\widetilde{l}_{k'}^{(N-2^{p_1})}\|_{\overline{\mathbb{D}}} \le \pi \exp(3\pi)$. This completes the induction and the whole proof of the theorem. ∎

5. Compact sets with Alper-smooth Jordan boundary. In this section, we deal with the case of a compact set K whose boundary is an Alper-smooth Jordan curve, and Φ denotes the exterior conformal mapping from $\overline{\mathbb{C}} \setminus \mathbb{D}$ onto $\overline{\mathbb{C}} \setminus K$. Recall that $\Gamma = \partial K$ is an *Alper-smooth Jordan curve* if the modulus of continuity ω of the angle $\theta(s)$ between the tangent at $\Gamma(s)$ and the positive real axis (where s is the arc-length parameter) satisfies (see [9])

$$\int_0^h \frac{\omega(x)}{x} |\ln x| \, dx < \infty.$$

In particular, twice continuously differentiable Jordan curves are Alper-smooth.

We first recall the following result.

Lemma 5.1 ([3, Lemma 3]). *Let K be a compact set whose boundary is an Alper-smooth Jordan curve. Let Φ be the conformal mapping of the exterior of the unit disk onto the exterior of K. Lastly, let $(a_j)_{j\ge 0}$ be a Leja sequence for the unit disk with $|a_0| = 1$. Then for any z on the unit disk and $n \in \mathbb{N}^*$,*

$$\frac{C(K)^n}{c_n} \le \left| \prod_{j=0}^{n-1} \frac{\Phi(z) - \Phi(a_j)}{z - a_j} \right| \le C(K)^n c_n, \tag{5.1}$$

where $C(K)$ is the logarithmic capacity of K, $c_n \leq (n+1)^{A/\ln(2)}$ and A is a positive constant depending only on K.

We also recall another property of Φ that can be found in [1, Sections 1 and 2] or [2, Eq. (3), p. 45]. There exist positive constants M_1 and M_2 such that for all z, w in the unit circle with $z \neq w$,

$$M_1 \leq \left| \frac{\Phi(z) - \Phi(w)}{z - w} \right| \leq M_2. \tag{5.2}$$

Proof of Theorem 1.3. First, by Theorem 1.2, it is sufficient to prove that for all $N \geq 1$, $p = 1, \dots, N$ and z on the unit circle,

$$\left| \prod_{j=1,\, j\neq p}^{N} \frac{\Phi(z) - \Phi(\eta_j)}{\Phi(\eta_p) - \Phi(\eta_j)} \right| \leq M(N+1)^{2A/\ln(2)} \left| \prod_{j=1,\, j\neq p}^{N} \frac{z - \eta_j}{\eta_p - \eta_j} \right|, \tag{5.3}$$

where M and A are positive constants depending only on K.

To prove (5.3), we use the same method as for [4, Theorem 13]. We can assume that $N \geq 2$ (otherwise the estimate is obvious), we fix an N-Leja section for the unit disk and consider z on the unit circle with $z \neq \eta_j$ for all $j = 1, \dots, N$. Then for all $p = 1, \dots, N$,

$$\left| \prod_{j=1,\, j\neq p}^{N} \frac{\Phi(z) - \Phi(\eta_j)}{z - \eta_j} \right| = \left| \frac{z - \eta_p}{\Phi(z) - \Phi(\eta_p)} \right| \left| \prod_{j=0}^{N-1} \frac{\Phi(z) - \Phi(\eta_{j+1})}{z - \eta_{j+1}} \right|.$$

On the one hand, an application of (5.2) with $w = \eta_p$, and on the other hand an application of (5.1) with the N-Leja section $\{a_j\}_{0\leq j\leq N-1} = \{\eta_{j+1}\}_{0\leq j\leq N-1}$, give

$$\frac{C(K)^N}{M_2 c_N} \leq \left| \prod_{j=1,\, j\neq p}^{N} \frac{\Phi(z) - \Phi(\eta_j)}{z - \eta_j} \right| \leq \frac{C(K)^N c_N}{M_1}.$$

In particular, these estimates are still valid for $z = \eta_p$ by continuity, and lead to

$$\left| \prod_{j=1,\, j\neq p}^{N} \frac{\Phi(z) - \Phi(\eta_j)}{\Phi(\eta_p) - \Phi(\eta_j)} \right| \leq \frac{M_2\, c_N^2}{M_1} \left| \prod_{j=1,\, j\neq p}^{N} \frac{z - \eta_j}{\eta_p - \eta_j} \right|.$$

Lastly, by continuity (and since $c_N \leq (N+1)^{A/\ln(2)}$), the above inequality holds for every z on the unit circle and yields the required estimate (5.3). ■

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