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Existence and analytic regularity of certain solutions for the generalized BBM-Burgers equation in $\mathbb{R}^n$

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Abstract. We study the existence and analytic regularity of local solutions for the generalized Benjamin–Bona–Mahony–Burgers equation in $\mathbb{R}^n$. The results are obtained by the convolution method and the Fourier transformation method combined with the fixed point method.

1. Introduction. In this paper, we study analytic regularity of the solutions for the following generalized BBM-Burgers equation (Benjamin–Bona–Mahony–Burgers equation) in the whole space $\mathbb{R}^n$:

$$\begin{cases} u_t - \Delta u - \Delta u_t + \Delta^2 u = u^k & \text{in } \mathbb{R}^n \times (0,T), \\ u(x,0) = u_0(x) & \text{in } \mathbb{R}^n, \end{cases} \tag{1.1}$$

where $k \geq 2$ is an integer.

Equation (1.1) is motivated by physical considerations from fluid dynamics and is closely related to the well known BBM equation

$$u_t - u_{xxt} + u_x + uu_x = 0, \quad x \in \mathbb{R}^1,\ t \geq 0,$$

studied by Benjamin–Bona–Mahony [3] as a refinement of the KdV equation [2, 3]. As is well known, the KdV equation was originally derived for water waves and it is similarly justifiable as a model for long waves in many other physical systems. Since then, the asymptotic behavior of solutions to the Cauchy problem for various generalized BBM (BBM-Burgers) equations

$$u_t - u_{xxt} - \alpha u_{xx} + \beta u_x + \phi(u)_x = 0, \quad x \in \mathbb{R}^1,\ t \geq 0,$$

has been studied in [1, 11], and their stability in [12, 13]. Here α is a positive constant, β is any given constant in $\mathbb{R}^1$, and $\phi(u)$ is a C^2 smooth nonlinear function. Also, H. J. Zhao [15, 16] considered the following generalized BBM

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(BBM-Burgers) equations:

$$u_t - \Delta u_t - \Delta u + \Delta^2 u = \sum_{j=1}^{n} \phi_j(u)_{x_j}, \qquad x \in \mathbb{R}^n,\ t \geq 0, \tag{1.2}$$

and obtained decay estimates, existence and convergence of solutions. Here (1.1) is a special case of (1.2). Also equation (1.2) arises in the phenomena of water waves with dissipative term [14–16]. On the other hand, (1.1) can also be viewed as a generalized Kuramoto–Sivashinsky equation [14, 17] which represents unstable flame fronts and thin hydrodynamic films.

Starting with the seminal work of C. Foiaş and R. Temam on the Navier–Stokes equations (NSE), the use of so-called Gevrey norms has become standard in estimating the time evolution of the spatial radius of analyticity of solutions to nonlinear partial differential equations. Recently A. Biswas and D. Swanson [4–7] proved the Gevrey regularity for the Navier–Stokes equation, Kuramoto–Sivashinsky equation and other equations. In fact, the Gevrey regularity in [4–7] includes the classical analytic regularity. A. Biswas and D. Swanson considered the equations

$$u_t + \Lambda^m u = G(u), \tag{1.3}$$

where $\Lambda = (-\Delta)^{1/2}$ and $m > 1$. For more on Gevrey regularity, see [8–10].

As for the generalized BBM-Burgers equation, the main source of difficulty is the term Δu_t. Fortunately, we can use the special structure of the equation. Motivated by the work of A. Biswas and D. Swanson [5, 6], we obtain the existence and analytic regularity of local solutions to problem (1.1).

2. Preliminaries and main results. Throughout the paper we use the usual convention that $C_{k,\alpha,\dots}$ indicates a strictly positive real number whose value may change from line to line, but depends only on $k, \alpha, \dots$.

For convenience, we write the generalized BBM-Burgers equation in the following form:

$$\begin{cases} \check{u}_t - \Delta \check{u} - \Delta \check{u}_t + \Delta^2 \check{u} = \check{u}^k & \text{in } \mathbb{R}^n \times (0,T), \\ \check{u}(\xi, 0) = \check{u}_0(\xi) & \text{in } \mathbb{R}^n. \end{cases} \tag{2.1}$$

Applying the Fourier transform in ξ to (2.1), we obtain

$$\frac{\partial u(x,t)}{\partial t} + |x|^2 u(x,t) = \frac{\widehat{\check{u}^k}(x,t)}{1+|x|^2}, \tag{2.2}$$

where $u(x,t)$ is the Fourier transform of $\check{u}(\xi,t)$ in variable ξ. By direct calculation, we see that

$$\widehat{\check{u}^k}(x,t) = u * \cdots * u(x).$$

Now we give some definitions and notation.

DEFINITION 2.1. Let $\alpha \in \mathbb{R}$. Define $\omega_\alpha : \mathbb{R}^n \to \mathbb{R}$ by

$$\omega_\alpha(x) = (1+|x|)^\alpha.$$

For any measurable function $u : \mathbb{R}^n \to \mathbb{R}$ we define

$$\|u\|_{\alpha,p} = \Big(\int_{\mathbb{R}^n} \omega_\alpha^p(x)|u(x)|^p\,dx\Big)^{1/p}$$

and

$$L^p_\alpha(\mathbb{R}^n) = \{u : \|u\|_{\alpha,p} < \infty\}.$$

We also define the corresponding analytic class

$$L^p_{\lambda,\alpha}(\mathbb{R}^n) = \{u : \|u\|_{\lambda,\alpha,p} < \infty\},$$

with the norm

$$\|u\|_{\lambda,\alpha,p} = \Big(\int_{\mathbb{R}^n} e^{p\lambda|x|}\omega_\alpha^p(x)|u(x)|^p\,dx\Big)^{1/p}.$$

Let

$$Au(x) = |x|^2u(x),$$
$$B[u_1,\ldots,u_k](x) = u_1 * \cdots * u_k(x)/(1+|x|^2).$$

Then equation (2.2) changes into

$$\frac{\partial u(x,t)}{\partial t} + Au(x,t) - B[\underbrace{u,\ldots,u}_{k}](x,t) = 0. \tag{2.3}$$

DEFINITION 2.2. Let $\alpha \in \mathbb{R}$, $0 < T \le \infty$, and $u_0 \in L^p_\alpha(\mathbb{R}^n)$. A *mild solution* of (2.3) with initial data u_0 is a function $u(x,t) \in C([0,T]; L^p_\alpha(\mathbb{R}^n))$ satisfying

$$u(x,t) = e^{-tA}u_0 + \int_0^t e^{-(t-s)A}B[\underbrace{u,\ldots,u}_{k}](x,s)\,ds, \qquad t \in [0,T]. \tag{2.4}$$

A mild solution $u(x,t)$ is said to be *analytic regular* if, in addition to satisfying (2.4), there exists $\lambda > 0$ such that

$$\sup_{0<t\le T} \|u(x,t)\|_{\lambda t,\alpha,p} < \infty. \tag{2.5}$$

Assume $1 < p < \infty$ and

$$\max\Big\{\frac{n}{p'} - \frac{4}{k-1}, \frac{kn}{(k+1)p'}\Big\} < \alpha < \frac{n}{p'}, \tag{2.6}$$

where p' denotes the Hölder conjugate of p.

For $0 < T \leq \infty$, $\lambda > 0$, $u(x,t) \in C([0,T]; L^p_\alpha(\mathbb{R}^n))$, we define

$$\|u(x,t)\|_E = \sup_{0\leq t\leq T} \|u(x,t)\|_{\lambda t,\alpha,p},$$
$$E = E_T = \{u(x,t) \in C([0,T]; L^p_\alpha(\mathbb{R}^n)) : \|u(x,t)\|_E < \infty\}.$$

Using fixed point theory, we will prove the following result.

THEOREM 2.3. *For equation* (2.1), *suppose that* $\alpha \in \mathbb{R}$ *satisfies* (2.6), *and* $u_0 \in L^p_\alpha(\mathbb{R}^n)$. *Then for any* $\lambda > 0$, *there exist* $T > 0$ *and a mild solution* $u(x,t) \in C([0,T]; L^p_\alpha(\mathbb{R}^n))$ *which satisfies* (2.4) *and the analytic regularity condition* (2.5). *Moreover, if* $\alpha < n/p' - 2/(k-1)$, *then one may take* $T > 0$ *satisfying*

$$T = C_{\lambda,k,\alpha,n,p}\|u_0\|_{\alpha,p}^{-\frac{2(k-1)}{(k-1)\alpha+4-(k-1)n/p'}}, \tag{2.7}$$

while if $\alpha \geq n/p' - 2/(k-1)$, *then one may take* $T > 0$ *satisfying*

$$T = \bar{C}_{\lambda,k,\alpha,n,p}\|u_0\|_{\alpha,p}^{-k+1}. \tag{2.8}$$

REMARK 2.4. From (2.6), if $n/p' \leq 2k/(k-1)$, then $\alpha \geq n/p' - 2/(k-1)$ and we can take T as in (2.8).

REMARK 2.5. Furthermore, if we define the corresponding Gevrey class

$$L^p_{\lambda,\alpha,s}(\mathbb{R}^n) = \{u : \|u\|_{\lambda,\alpha,p,s} < \infty\}$$

with the norm

$$\|u\|_{\lambda,\alpha,p,s} = \Big(\int_{\mathbb{R}^n} e^{p\lambda|x|^{1/s}}\omega^p_\alpha(x)|u(x)|^p\,dx\Big)^{1/p} \quad \text{for } s > 1/2,$$

then the result of Theorem 2.3 is also true. This implies the Gevrey smoothing effect for mild solutions to (2.1).

REMARK 2.6. For the generalized BBM-Burgers equation

$$\begin{cases} u_t - \gamma_1 \Delta u - \gamma_2 \Delta u_t + \gamma_3 \Lambda^m u = f(u, \nabla u) & \text{in } \mathbb{R}^n \times (0,T), \\ u(x,0) = u_0(x) & \text{in } \mathbb{R}^n, \end{cases}$$

where $\gamma_1, \gamma_2, \gamma_3 > 0$, $\Lambda = (-\Delta)^{1/2}$, $m > 2$, and $f(x_1, x_2)$ is a polynomial function of x_1 and x_2, we also have the corresponding analytic regularity.

3. Estimates on the linear and nonlinear term. First, we estimate the linear term $e^{-tA}u_0$.

PROPOSITION 3.1. *Let* $\alpha > 0$ *and* $u_0 \in L^p_\alpha(\mathbb{R}^n)$. *Then for* $0 \leq t \leq T$ *and* $\lambda > 0$, *we have* $e^{-tA}u_0 \in E$.

Proof. We have

$$\|e^{-tA}u_0\|_{\lambda t,\alpha,p} = \Big(\int_{\mathbb{R}^n} e^{p\lambda t|x|}e^{-pt|x|^2}\omega_\alpha^p(x)|u_0(x)|^p\,dx\Big)^{1/p} \tag{3.1}$$
$$\le e^{\lambda^2 t/4}\Big(\int_{\mathbb{R}^n} \omega_\alpha^p(x)|u_0(x)|^p\,dx\Big)^{1/p} = e^{\lambda^2 t/4}\|u_0\|_{\alpha,p}.$$

Thus $e^{-tA}u_0 \in E$, and

$$\|e^{-tA}u_0\|_E \le e^{\lambda^2 T/4}\|u_0\|_{\alpha,p}. \ \blacksquare \tag{3.2}$$

Next, we estimate the nonlinear term

$$b[u_1,\ldots,u_k] := \int_0^t e^{-(t-s)A}B[u_1,\ldots,u_k](x,s)\,ds.$$

LEMMA 3.2. *If $0 < a,b < n$, $a+b > n$, then*

$$\omega_{-a}*\omega_{-b}(x) \le C_{a,b,n}\omega_{n-a-b}(x) \quad \textit{for all } x\in\mathbb{R}^n.$$

Proof. See [6, Proposition 11]. ■

LEMMA 3.3. *For $kn/((k+1)p') < \gamma < n/p'$, if $u_1,\ldots,u_k \in L^p_\gamma(\mathbb{R}^n)$, then*

(1) *$u_1*\cdots*u_k \in L^p_{k\gamma-(k-1)n/p'}(\mathbb{R}^n)$, and*

$$\|u_1*\cdots*u_k\|_{k\gamma-(k-1)n/p',p} \le C_{\gamma,n}\prod_{i=1}^k\|u_i\|_{\gamma,p}; \tag{3.3}$$

(2) *$B[u_1,\ldots,u_k] \in L^p_{k\gamma+2-(k-1)n/p'}(\mathbb{R}^n)$, and*

$$\|B[u_1,\ldots,u_k]\|_{k\gamma+2-(k-1)n/p',p} \le C_{\gamma,n}\prod_{i=1}^k\|u_i\|_{\gamma,p};$$

(3) *if $u_1,\ldots,u_k \in L^p_{\lambda,\gamma}(\mathbb{R}^n)$, then $u_1*\cdots*u_k \in L^p_{\lambda,k\gamma-(k-1)n/p'}(\mathbb{R}^n)$, and $B[u_1,\ldots,u_k] \in L^p_{\lambda,k\gamma+2-(k-1)n/p'}(\mathbb{R}^n)$, and*

$$\|B[u_1,\ldots,u_k]\|_{\lambda,k\gamma+2-(k-1)n/p',p} \le C_{\gamma,n}\prod_{i=1}^k\|u_i\|_{\lambda,\gamma,p}.$$

Proof. (1) For $k=1$, (3.3) is obvious. Suppose $k=k_0$, $u_1*\cdots*u_{k_0} \in L^p_{k_0\gamma-(k_0-1)n/p'}(\mathbb{R}^n)$, and

$$\|u_1*\cdots*u_{k_0}\|_{k_0\gamma-(k_0-1)n/p',p} \le C_{\gamma,n}\prod_{i=1}^{k_0}\|u_i\|_{\gamma,p}.$$

For $k = k_0 + 1$,

$$u_1 * \cdots * u_{k_0+1}(x) = \int_{\mathbb{R}^n} u_1 * \cdots * u_{k_0}(x-y) u_{k_0+1}(y)\,dy \tag{3.4}$$

$$\le \Big(\int_{\mathbb{R}^n} \omega^p_{k_0\gamma-(k_0-1)n/p'}(x-y)|u_1 * \cdots * u_{k_0}(x-y)|^p |\omega^p_\gamma(y)||u_{k_0+1}(y)|^p\,dy\Big)^{1/p}$$
$$\times \Big(\int_{\mathbb{R}^n} \omega_{(k_0-1)n-k_0\gamma p'}(x-y)\omega_{-\gamma p'}(y)\,dy\Big)^{1/p'}.$$

Using Lemma 3.2 and (3.4), we obtain

$$\omega^p_{(k_0+1)\gamma-k_0n/p'}(u_1 * \cdots * u_{k_0+1}(x))^p$$
$$\le C_{\gamma,n} \int_{\mathbb{R}^n} \omega^p_{k_0\gamma-(k_0-1)n/p'}(x-y)|u_1 * \cdots * u_{k_0}(x-y)|^p |\omega^p_\gamma(y)|\,|u_{k_0+1}(y)|^p\,dy.$$

Hence

$$\|u_1 * \cdots * u_{k_0+1}\|_{(k_0+1)\gamma-k_0n/p',p}$$
$$\le C_{\gamma,n}\Big(\int_{\mathbb{R}^n}\Big[\int_{\mathbb{R}^n} \omega^p_{k_0\gamma-(k_0-1)n/p'}(x-y)|u_1 * \cdots * u_{k_0}(x-y)|^p$$
$$\times |\omega^p_\gamma(y)|\,|u_{k_0+1}(y)|^p\,dy\Big]\,dx\Big)^{1/p}$$
$$= C_{\gamma,n}\|u_1 * \cdots * u_{k_0}\|_{k_0\gamma-(k_0-1)n/p',p}\|u_{k_0+1}\|_{\gamma,p} \le C_{\gamma,n}\prod_{i=1}^{k_0+1}\|u_i\|_{\gamma,p}.$$

So, by induction, we deduce (1).

(2) Using the above estimates, we obtain

$$\|B[u_1,\ldots,u_k]\|_{k\gamma+2-(k-1)n/p',p}$$
$$= \Big(\int_{\mathbb{R}^n}\Big(\frac{1}{1+|x|^2}\Big)^p \omega^p_{k\gamma+2-(k-1)n/p'}|u_1 * \cdots * u_k|^p\,dx\Big)^{1/p}$$
$$\le C\Big(\int_{\mathbb{R}^n} \omega^p_{k\gamma-(k-1)n/p'}|u_1 * \cdots * u_k|^p\,dx\Big)^{1/p} \le C_{\gamma,n}\prod_{i=1}^{k}\|u_i\|_{\gamma,p}.$$

(3) Similarly, we get

$$\|B[u_1,\ldots,u_k]\|_{\lambda,k\gamma+2-(k-1)n/p',p}$$
$$= \Big(\int_{\mathbb{R}^n} e^{\lambda p|x|}\Big(\frac{1}{1+|x|^2}\Big)^p \omega^p_{k\gamma+2-(k-1)n/p'}|u_1 * \cdots * u_k|^p\,dx\Big)^{1/p}$$
$$\le C\Big(\int_{\mathbb{R}^n} e^{\lambda p|x|}\omega^p_{k\gamma-(k-1)n/p'}|u_1 * \cdots * u_k|^p\,dx\Big)^{1/p} \le C_{\gamma,n}\prod_{i=1}^{k}\|u_i\|_{\lambda,\gamma,p}. \blacksquare$$

The following elementary inequality will be used repeatedly.

LEMMA 3.4. *If $a > 0$ and $b \in \mathbb{R}$, then*

$$\sup_{x\in\mathbb{R}^n} \omega_b(x) e^{-a|x|^2} \le C_b(1 + a^{-b/2}).$$

Proof. Similar to [6, Proposition 15]. ∎

LEMMA 3.5. *For $kn/((k+1)p') < \gamma < n/p'$, $\alpha, \eta, \lambda > 0$, if $u_1, \ldots, u_k \in L^p_{\lambda,\gamma}(\mathbb{R}^n)$, then*

$$\|e^{-\eta A} B[u_1, \cdots, u_k]\|_{\lambda,\alpha,p} \le C_{\gamma,n,\alpha,p}(1 + \eta^{\frac{k\gamma+2-\alpha-(k-1)n/p'}{2}}) \prod_{i=1}^{k} \|u_i\|_{\lambda,\gamma,p}.$$

Proof. First, from Lemma 3.4, we see that

$$\sup_{x\in\mathbb{R}^n} \omega_{\alpha-(k\gamma+2-(k-1)n/p')}(x) e^{-\eta|x|^2} \le C_{\gamma,n,\alpha,p}(1 + \eta^{\frac{k\gamma+2-\alpha-(k-1)n/p'}{2}}).$$

Combining this with Lemma 3.3, we obtain

$$\begin{aligned}
\|e^{-\eta A} B[u_1, \cdots, u_k]\|_{\lambda,\alpha,p} &\le \Big(\int_{\mathbb{R}^n} e^{p\lambda|x|} e^{-p\eta|x|^2} \omega^p_\alpha(x) |B[u_1, \ldots, u_k]|^p \, dx \Big)^{1/p} \\
&\le C_{\gamma,n,\alpha,p}(1 + \eta^{\frac{k\gamma+2-\alpha-(k-1)n/p'}{2}}) \|B[u_1, \ldots, u_k]\|_{\lambda,k\gamma+2-(k-1)n/p',p} \\
&\le C_{\gamma,n,\alpha,p}(1 + \eta^{\frac{k\gamma+2-\alpha-(k-1)n/p'}{2}}) \prod_{i=1}^{k} \|u_i\|_{\lambda,\gamma,p}. \ \blacksquare
\end{aligned}$$

PROPOSITION 3.6. *If α satisfies* (2.6), *$\lambda > 0$, $u_1, \ldots, u_k \in L^p_{\lambda,\alpha}(\mathbb{R}^n)$, then $b[u_1, \ldots, u_k] \in E$ and*

$$\|b[u_1, \cdots, u_k]\|_E \le C_{\alpha,n,p} e^{\lambda^2 T/2} (T + T^{\frac{(k-1)\alpha+4-(k-1)n/p'}{2}}) \prod_{i=1}^{k} \|u_i\|_E.$$

Proof. From Lemma 3.5, we obtain

$$\begin{aligned}
\|b[u_1, \ldots, u_k]\|_{\lambda t,\alpha,p} &= \int_0^t \|e^{-(t-s)A} B[u_1, \ldots, u_k](x, s)\|_{\lambda t,\alpha,p} \, ds \\
&\le e^{\lambda^2 t/2} \int_0^t \|e^{-\frac{(t-s)}{2} A} B[u_1, \ldots, u_k](x, s)\|_{\lambda s,\alpha,p} \, ds \\
&\le C_{\alpha,n,p} e^{\lambda^2 t/2} \int_0^t (1 + (t-s)^{\frac{(k-1)\alpha+2-(k-1)n/p'}{2}}) \prod_{i=1}^{k} \|u_i\|_{\lambda s,\alpha,p} \, ds \\
&\le C_{\alpha,n,p} e^{\lambda^2 t/2} \int_0^t (1 + (t-s)^{\frac{(k-1)\alpha+2-(k-1)n/p'}{2}}) \, ds \prod_{i=1}^{k} \|u_i\|_E \\
&\le C_{\alpha,n,p} e^{\lambda^2 T/2} (T + T^{\frac{(k-1)\alpha+4-(k-1)n/p'}{2}}) \prod_{i=1}^{k} \|u_i\|_E.
\end{aligned}$$

This means $b[u] \in E$, and

$$\|b[u_1, \cdots, u_k]\|_E \leq C_{\alpha,n,p} e^{\lambda^2 T/2} (T + T^{\frac{(k-1)\alpha + 4 - (k-1)n/p'}{2}}) \prod_{i=1}^{k} \|u_i\|_E. \ \blacksquare$$

4. Existence and analytic regularity of the mild solution. In this section, we use the fixed point theorem to prove existence and analytic regularity of the mild solution.

Proof of Theorem 2.3. Define

$$L : E \to E, \qquad Lu = e^{-tA} u_0 + b[\underbrace{u, \ldots, u}_{k}](x,t).$$

For convenience, we write $b[u]$ for $b[\underbrace{u, \ldots, u}_{k}]$.

Define

$$B = \{u \in E : \|u - e^{-tA} u_0\|_E \leq \|e^{-tA} u_0\|_E\}.$$

For $u \in B$, combining Propositions 3.1 and 3.6, we get

$$\begin{aligned}
\|Lu - e^{-tA} u_0\|_E = \|b[u]\|_E &\leq C_{\alpha,n,p} e^{\lambda^2 T/2} (T + T^{\frac{(k-1)\alpha + 4 - (k-1)n/p'}{2}}) \|u\|_E^k \\
&\leq C_{\alpha,n,p} e^{\lambda^2 T/2} (T + T^{\frac{(k-1)\alpha + 4 - (k-1)n/p'}{2}}) 2^k \|e^{-tA} u_0\|_E^k \\
&\leq C_{\alpha,n,p} e^{(k+1)\lambda^2 T/4} (T + T^{\frac{(k-1)\alpha + 4 - (k-1)n/p'}{2}}) 2^k \|u_0\|_{\alpha,p}^{k-1} \|e^{-tA} u_0\|_E.
\end{aligned}$$

For $u_1, u_2 \in B$, from Propositions 3.1 and 3.6, we obtain

$$\begin{aligned}
\|Lu_1 - Lu_2\|_E &= \|b[u_1] - b[u_2]\|_E \\
&= \|b[u_1, \ldots, u_1] - b[u_2, u_1, \ldots, u_1] + b[u_2, u_1, \ldots, u_1] - \cdots \\
&\quad + b[u_2, \ldots, u_2]\|_E \\
&\leq k C_{\alpha,n,p} e^{\lambda^2 T/2} (T + T^{\frac{(k-1)\alpha + 4 - (k-1)n/p'}{2}}) \max\{\|u_1\|_E^{k-1}, \|u_2\|_E^{k-1}\} \\
&\quad \times \|u_1 - u_2\|_E \\
&\leq k 2^{k-1} C_{\alpha,n,p} e^{(k+1)\lambda^2 T/4} (T + T^{\frac{(k-1)\alpha + 4 - (k-1)n/p'}{2}}) \|u_0\|_{\alpha,p}^{k-1} \|u_1 - u_2\|_E.
\end{aligned}$$

For any $\lambda > 0$, we take $T > 0$ such that

$$a := k 2^{k-1} C_{\alpha,n,p} e^{\frac{(k+1)\lambda^2 T}{4}} (T + T^{\frac{(k-1)\alpha + 4 - (k-1)n/p'}{2}}) \|u_0\|_{\alpha,p}^{k-1} < 1. \tag{4.1}$$

Then, for $u_1, u_2 \in B$, we have $Lu_1, Lu_2 \in B$ and $\|Lu_1 - Lu_2\|_E \leq a\|u_1 - u_2\|_E$. Hence $L : E \to E$ is a contraction provided that $T, \lambda > 0$ satisfy (4.1). From the Banach fixed point theorem, there exists a $u \in B$ such that

$$u(x,t) = Lu = e^{-At} u_0 + b[u](x,t).$$

This implies the existence of a mild solution. Furthermore, since $u \in E$, for $\lambda > 0$ we can deduce the following analytic regularity of u:

$$\sup_{0\leq t\leq T} \|u(x,t)\|_{\lambda t,\alpha,p} < \infty.$$

Moreover, if $\alpha < n/p' - 2/(k-1)$, then $((k-1)\alpha + 4 - (k-1)n/p')/2 < 1$. Taking a sufficiently large constant $C'_{\lambda,k,\alpha,n,p} > 0$ we obtain

$$a \leq C'_{\lambda,k,\alpha,n,p} T^{\frac{(k-1)\alpha+4-(k-1)n/p'}{2}} \|u_0\|^{k-1}_{\alpha,p} \quad \text{for } T \leq \frac{4}{(k+1)\lambda^2}.$$

Then defining

$$C_{\lambda,k,\alpha,n,p} = \min\left\{\frac{C'_{\lambda,k,\alpha,n,p}}{2}, \frac{2}{(k+1)\lambda^2}\right\},$$

and with T as in (2.7), we calculate that $a < 1$ in (4.1), which implies the existence of a mild solution in the case $\alpha < n/p' - 2/(k-1)$.

If $\alpha \geq n/p' - 2/(k-1)$, then for a suitable constant $\bar{C}'_{\lambda,k,\alpha,n,p} > 0$ we have

$$a \leq \bar{C}'_{\lambda,k,\alpha,n,p} T \|u_0\|^{k-1}_{\alpha,p} \quad \text{for } T \leq \frac{4}{(k+1)\lambda^2}.$$

Then defining

$$\bar{C}_{\lambda,k,\alpha,n,p} = \min\left\{\frac{\bar{C}'_{\lambda,k,\alpha,n,p}}{2}, \frac{2}{(k+1)\lambda^2}\right\},$$

and with T as in (2.8), we obtain the existence of a mild solution in the case $\alpha \geq n/p' - 2/(k-1)$. ■

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