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INFINITE TENSOR PRODUCT ACTIONS OF COMPACT QUANTUM GROUPS

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Abstract. This article is a survey of infinite tensor product actions of compact quantum groups.

1. Introduction. Infinite tensor product (ITP) provides us with a useful tool to construct non-trivial actions of locally compact groups or quantum groups. For compact groups, any (non-trivial) ITP-action is always minimal, that is, the relative commutant of its fixed point algebra M^α inside the ambient von Neumann algebra M is trivial. (See [2, 3, 10] for this fact.) This phenomenon does not occur for general quantum groups. Indeed, for any compact quantum group $\mathbb{G}$ of non-Kac type, such actions are never minimal. This interesting fact is explained well as non-triviality of a Poisson boundary of a dual quantum group [3]. Namely, Izumi has shown that the relative commutant $(M^\alpha)' \cap M$ is naturally isomorphic to the Poisson boundary of the dual $\widehat{\mathbb{G}}$, and he also realized the boundary for the dual of $SU_q(2)$ as the standard quantum sphere $T \setminus SU_q(2)$ that had been studied by Podleś [9]. This result has been generalized by Izumi–Neshveyev–Tuset for $SU_q(N)$ [4] and by the author for general G_q, the q-deformation of a connected semisimple compact Lie group G [13].

In this note, we will further study ITP-actions for G_q. It can be shown that the Poisson boundary of its dual is a type I_∞ factor, and the ITP-factor has the tensor product decomposition $M = R \vee Q \cong R \otimes Q$, where Q denotes $(M^\alpha)' \cap M$ and R its relative commutant $Q' \cap M$. Then it turns out that the inclusion $M^\alpha \subset R$ is irreducible and of depth 2. Namely, this inclusion is obtained as a fixed point subfactor by a minimal action β of some compact quantum group $\mathbb{H}$. Our aim is to detect the mysterious quantum group $\mathbb{H}$ and to describe the minimal action β on R. The main result of [15] is the following.

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THEOREM 1.1. *The compact quantum group* $\mathbb{H}$ *is isomorphic to the maximal torus* T *of* G_q. *The minimal action* β_t, $t \in T$, *is obtained by* $\alpha_{w_0 t}$, *where* w_0 *denotes the longest element in the Weyl group.*

The action of the Weyl group W of G on T is introduced as follows. Regard P, the lattice of integrable weights, as the dual group of T. (See [15, Section 2.8].) Then for $w \in W$ and $t \in T$, we define $wt \in T$ by the formula $\langle wt, \mu\rangle := \langle t, w^{-1}\mu\rangle$ for $\mu \in P$.

This result enables us to classify ITP-actions of $SU_q(2)$ up to conjugacy.

2. Infinite tensor product actions. We freely use notation and terminology introduced in [8, 14, 16] and references therein.

2.1. Settings. Let v be a fixed unitary representation of a compact quantum group $\mathbb{G}$ on a finite dimensional Hilbert space H. We assume that v's irreducible components are generating all irreducibles of $\mathbb{G}$.

For $n \in \mathbb{N}$, let us denote by v_n the n-times tensor product representation, that is, $v_n := v_{1,n+1}v_{2,n+1}\cdots v_{n,n+1}$ which is a unitary belonging to $B(H)^{\otimes n} \otimes C(\mathbb{G})$. The v_n provides us with the right $\mathbb{G}$-action α_n on $B(H)^{\otimes n}$ defined by

$$\alpha_n(x) := v_n(x \otimes 1)v_n^*, \quad x \in B(H)^{\otimes n}.$$

It is easy to see that the actions α_n are compatible with the embeddings $B(H)^{\otimes n} \ni x \mapsto x \otimes 1 \in B(H)^{\otimes(n+1)}$. Hence α_n's extend to the right $\mathbb{G}$-action α on the UHF-algebra $B(H)^{\otimes\infty}$.

We have interest in the von Neumann algebraic property of α. So, let us fix an invariant state ϕ on $B(H)$. Then the product state $\varphi = \otimes\phi$ is α-invariant, and α extends to the von Neumann closure M of $B(H)^{\otimes\infty}$ with respect to φ. This extended action, which we also denote by α, is called the *infinite tensor product action* or the *product type action.*

2.2. Poisson boundaries and depth 2-inclusions. Now let M^α be the fixed point algebra, that is, the collection of all x's in M such that $\alpha(x) = x \otimes 1$. Then $\mathbb{G}$ acts on the relative commutant $Q := (M^\alpha)' \cap M$ by restricting α. A very interesting phenomenon is the non-triviality of Q. When we think of a "compact group" $\mathbb{G}$, Q is always trivial. However, if $\mathbb{G}$ is a "compact quantum group", then Q is often non-trivial. This fact is explained by theory of Poisson boundary for quantum groups initiated by Izumi [3]. Indeed, Q is $\mathbb{G}$-isomorphic to the von Neumann algebra $H^\infty(\widehat{\mathbb{G}}, \mu)$, which we call the Poisson boundary, for some generating probability measure μ on $\mathrm{Irr}(\mathbb{G})$. For more details, see [3]. It is known that the triviality of the Poisson boundary implies the triviality of the scaling automorphism of $\mathbb{G}$ (which is equivalent to $\mathbb{G}$ being of Kac type). In particular, $Q \cong H^\infty(\widehat{\mathbb{G}}, \mu)$ is non-trivial if $\mathbb{G}$ is the q-deformation G_q of a connected semisimple compact Lie group G.

The description of $H^\infty(\widehat{G_q}, \mu)$ is given in [3] for $SU_q(2)$, [4] for $SU_q(N)$ and [13] for general G_q. Let us quickly explain the main result of these works. Consider first the faithful unital completely positive map Θ from $L^\infty(G_q)$ into $R(G_q)$ by

$$\Theta(x) := (\mathrm{id} \otimes h)(V^*(1 \otimes x)V), \quad x \in L^\infty(G_q),$$

where V denotes the right regular representation of G_q and $R(G_q)$ the right quantum group von Neumann algebra. Then we can show that Θ maps $L^\infty(G_q)$ into $H^\infty(\widehat{G_q}, \mu)$

that is an operator system in $R(G_q)$. In fact, when restricting Θ on $L^\infty(T \setminus G_q)$, we have a von Neumann isomorphism between $L^\infty(T \setminus G_q)$ and $H^\infty(\widehat{G_q}, \mu)$. Also note that Θ intertwines the following two symmetries: the right G_q-action

$$B(L^2(G_q)) \ni x \mapsto V(x \otimes 1)V^* \in B(L^2(G_q)) \otimes L^\infty(G_q),$$

and the left $\widehat{G_q}$-action

$$B(L^2(G_q)) \ni x \mapsto V^*(1 \otimes x)V \in R(G_q) \otimes B(L^2(G_q)).$$

Hence we have the G_q-isomorphisms:

$$(M^\alpha)' \cap M = Q \cong H^\infty(\widehat{G_q}, \mu) \xleftarrow{\Theta} L^\infty(T \setminus G_q).$$

Next we will use the following result:

THEOREM 2.1. *The von Neumann algebra $L^\infty(T \setminus G_q)$ is a type I_∞ factor.*

Before presenting a sketch of the proof, we shall continue our analysis. The result above implies that Q is also a type I_∞ factor, and therefore it follows from general theory of von Neumann algebras that M has the tensor product decomposition:

$$M = (Q' \cap M) \vee Q \cong (Q' \cap M) \otimes Q. \tag{2.1}$$

We denote by R the relative commutant $Q' \cap M$, that is, the relative bicommutant of M^α inside M. Note that R does not have the G_q-action by restriction of α. By definition of Q, we see that M^α is a von Neumann subalgebra of R. Thus it turns out from the decomposition above that the inclusion $M^\alpha \subset R$ is *irreducible*, that is, $(M^\alpha)' \cap R = \mathbb{C}$. Note that there exists a faithful normal conditional expectation from R onto M^α. Indeed this is obtained by restricting the averaging expectation $E_\alpha := (\mathrm{id} \otimes h) \circ \alpha$ on R, where h denotes the Haar state on G_q. The following simple lemma is our key for analysis of α.

LEMMA 2.2. *The inclusion $M^\alpha \subset R$ is of depth* 2.

The "depth 2" means the following. Let an irreducible inclusion of factors $A \subset B$ be given. Let $A \subset B \subset B_1 \subset B_2$ be the basic extension due to Jones. Then we say that $A \subset B$ is of depth 2 when $A' \cap B_2$ is a type I factor. So, to show the previous lemma, we need to compute the basic extension of $M^\alpha \subset R$. Noting the decomposition (2.1) and the type I-factoriality of Q, we see that it suffices to show that $(M^\alpha)' \cap M_2$ is a type I factor, where $M^\alpha \subset M \subset M_1 \subset M_2$ is the basic extension, and this analysis is not so involved if we compare the extension with the crossed product extension: $M^\alpha \subset M \subset M \rtimes_\alpha G_q \subset M \otimes B(L^2(G_q))$. Indeed, we have $(M^\alpha)' \cap M_2 \cong Q \otimes B(L^2(G_q))$ that is a factor of type I.

Sketch of proof of Theorem 2.1. By [14, Theorem 4.7], we know that it suffices to show that the maximal torus T of G_q is faithfully acting on the center $Z(L^\infty(G_q))$ from the left. Let us use the notation introduced in [5]. For a dominant integral weight $\Lambda \in P_+$, we let $a_\Lambda := C^\Lambda_{\Lambda, w_0\Lambda}$ be the matrix element of the irreducible representation associated with Λ. Here, w_0 denotes the longest element in the Weyl group. Let $a_\Lambda = v_\Lambda |a_\Lambda|$ be the polar decomposition in $L^\infty(G_q)$. Then we can show that v_Λ belongs to the center of $L^\infty(G_q)$ by using the formulae in [5, Chapter 3]. Since v_Λ has Λ as the spectrum of the left action $T \curvearrowright Z(L^\infty(G_q))$ and Λ is arbitrary, we are done. ∎

2.3. The mysterious quantum group. Let us focus on the inclusion $M^\alpha \subset R$ that is irreducible and of depth 2. By a basic result of subfactor theory, this inclusion is coming from a compact quantum group action [1, 6, 12]. Namely, there exist a compact quantum group $\mathbb{H}$ and a minimal action β of $\mathbb{H}$ on R such that $M^\alpha = R^\beta$.

So, next our task is to detect the mysterious quantum group $\mathbb{H}$. Our strategy is to give an irreducible decomposition of M^α-M^α-bimodule $L^2(R)$. Then we can actually prove the following result.

THEOREM 2.3. *The compact quantum group* $\mathbb{H}$ *is isomorphic to the maximal torus* T *of* G_q.

We can predict a candidate for $\mathbb{H}$ as follows. The inclusion $M^\alpha \subset R$ is essentially same as $M^\alpha \subset M$ since Q is a type I factor. Then we compute $(M^\alpha)' \cap (M \rtimes_\alpha G_q) = Q \rtimes_\alpha G_q$ that is isomorphic to $L^\infty(T \setminus G_q) \rtimes_\alpha G_q = B(L^2(G_q))^T$, the fixed point algebra by the left T-action on $B(L^2(G_q))$. Thus the irreducible components of the M^α-M^α-bimodule $L^2(M)$ are labelled by the dual group $\widehat{T}$. To show $\mathbb{H}$ exactly coincides with T, we must prove that each irreducible bimodule comes from an automorphism on M^α. Let us briefly explain how to construct such automorphisms.

Consider an irreducible and depth 2 inclusion $A \subset B$ coming from an outer action θ of a discrete group Γ, that is, $A \subset B$ is isomorphic to $A \subset A \rtimes_\theta \Gamma$. Then θ is implemented by unitaries in B. So, we need to find a unitary in R which normalizes M^α.

We know that $L^\infty(T \setminus G_q)$ is a type I factor by Theorem 2.1, and $L^\infty(G_q)$ has the tensor product decomposition

$$\begin{aligned} L^\infty(G_q) &= (L^\infty(T \setminus G_q)' \cap L^\infty(G_q)) \vee L^\infty(T \setminus G_q) \\ &\cong (L^\infty(T \setminus G_q)' \cap L^\infty(G_q)) \otimes L^\infty(T \setminus G_q). \end{aligned}$$

The relative commutant $L^\infty(T \setminus G_q)' \cap L^\infty(G_q)$ is actually equal to the center $Z(L^\infty(G_q))$. Hence we have

$$L^\infty(G_q) = Z(L^\infty(G_q)) \vee L^\infty(T \setminus G_q) \cong Z(L^\infty(G_q)) \otimes L^\infty(T \setminus G_q). \tag{2.2}$$

Moreover we see that $Z(L^\infty(G_q))$ is T-isomorphic to $L^\infty(T)$ that is generated by unitary group v_μ for $\mu \in \widehat{T}$. It is now important to compare the decompositions (2.1) and (2.2). We want to examine whether the inclusion $L^\infty(T \setminus G_q) \subset L^\infty(G_q)$ is G_q-embeddable into $Q \subset M$. Then we could find a unitary element which normalizes M^α since each v_μ would be mapped into $Q' \cap M = R$.

Let us start from a G_q-isomorphism π from $L^\infty(T \setminus G_q)$ onto Q that is already given via theory of Poisson boundary. Consider the unitary

$$w_\mu := (v_\mu^* \otimes 1)\delta(v_\mu). \tag{2.3}$$

Since the coproduct δ is left-T-equivariant, w_μ belongs to $L^\infty(T \setminus G_q) \otimes L^\infty(G_q)$. Hence we can consider $w_\mu^o := (\pi \otimes \mathrm{id})(w_\mu)$ in $Q \otimes L^\infty(G_q)$. It is easy to check that w_μ^o is a unitary α-cocycle, that is, we have $(\mathrm{id} \otimes \delta)(w_\mu^o) = (w_\mu^o \otimes 1)(\alpha \otimes \mathrm{id})(w_\mu^o)$. Then we prove the following result.

THEOREM 2.4. *The* α*-cocycles* w_μ^o *for* $\mu \in \widehat{T}$ *are coboundaries in* M *when* M^α *is infinite. Namely, there exists a unitary* u *in* M *such that* $w_\mu^o = (u^* \otimes 1)\alpha(u)$.

Sketch of proof. This result is proved by using 2-by-2 matrix trick. We denote w_μ^o simply by w. Let $N := M_2(\mathbb{C})\otimes M$ and $\beta := \mathrm{id}\otimes\alpha$. Then the unitary $v := e_{11}\otimes 1_M\otimes 1+e_{22}\otimes w$, which belongs to $N\otimes L^\infty(G_q)$, is a β-cocycle. Consider the perturbed G_q-action $\gamma(\cdot) := v\beta(\cdot)v^*$. If we show the projections e_{11} and e_{22} are Murray–von Neumann equivalent inside N^γ, then we are done.

Since a crossed product is not affected by 1-cocycle perturbations, we see that $N\rtimes_\gamma G_q$ is isomorphic to $N\rtimes_\beta G_q = M_2(\mathbb{C})\otimes(M\rtimes_\alpha G_q)$ that is also isomorphic to the factor $M_2(\mathbb{C})\otimes M_1$. Thus $N\rtimes_\gamma G_q$ is a factor and so is N^γ because N^γ is a corner of $N\rtimes_\gamma G_q$. Hence if we show that e_{11} and e_{22} are infinite projections, then $e_{11}\sim e_{22}$.

For e_{11}, we have $e_{11}N^\gamma e_{11}\cong M^\alpha$ that is infinite by our assumption. For e_{22}, we have $e_{22}N^\gamma e_{22}\cong M^{\alpha^w}$, where α^w is the perturbed action of α by w. Since w is evaluated in $Q=(M^\alpha)'\cap M$, M^{α^w} contains M^α. Hence e_{22} is also infinite. ∎

We assume the infiniteness of M^α from now on. From the previous result, we can obtain a unitary u_μ in M such that $w_\mu^o=(u_\mu^*\otimes 1)\alpha(u_\mu)$. The u_μ is considered as a copy of v_μ in M.

Lemma 2.5. *For any $\mu\in\widehat{T}$, u_μ belongs to $R=Q'\cap M$.*

Sketch of proof. Let us first prove $u_\mu M^\alpha u_\mu^*\subset M^\alpha$. For $x\in M^\alpha$, we have

$$\alpha(u_\mu x u_\mu^*) = (u_\mu\otimes 1)w_\mu^o\cdot(x\otimes 1)\cdot(w_\mu^o)^*(u_\mu^*\otimes 1) = u_\mu x u_\mu^*\otimes 1$$

since $x\otimes 1\in M^\alpha\otimes\mathbb{C}$ is commuting with $w_\mu^o\in Q\otimes L^\infty(G_q)$.

Hence $\theta_\mu := \mathrm{Ad}\,u_\mu$ gives an endomorphism on M^α, and $\rho_\mu := \mathrm{Ad}\,u_\mu^*$ gives an endomorphism on Q. Let $x\in Q$. Then

$$\begin{aligned}\alpha(\rho_\mu(x)) &= \alpha(u_\mu^*)\alpha(x)\alpha(u_\mu) = (w_\mu^o)^*(u_\mu^*\otimes 1)\alpha(x)(u_\mu\otimes 1)w_\mu^o\\ &= (w_\mu^o)^*(\rho_\mu\otimes\mathrm{id})(\alpha(x))w_\mu^o.\end{aligned}\tag{2.4}$$

Now we observe that w_μ^o is commuting with $\alpha(Q)$. This fact is equivalent to the commutativity of w_μ with $\delta(L^\infty(T\setminus G_q))$. This indeed holds because, in (2.3), v_μ is a central element in $L^\infty(G_q)$.

Multiplying w_μ^o and $(w_\mu^o)^*$ in (2.4) from the left and right, respectively, we obtain the equality $\alpha\circ\rho_\mu=(\rho_\mu\otimes\mathrm{id})\circ\alpha$ on Q. This means that ρ_μ is G_q-equivariant. By the following general result on $L^\infty(T\setminus G_q)$, we see that $\rho_\mu=\mathrm{id}$, and u_μ is contained in $Q'\cap M=R$. ∎

Theorem 2.6. *Let ρ be a G_q-equivariant endomorphism on $L^\infty(T\setminus G_q)$. Then ρ equals the identity map on $L^\infty(T\setminus G_q)$.*

To show the previous theorem, we need the following result due to Stokman–Dijkhuizen [11].

Theorem 2.7 (Stokman–Dijkhuizen). *Let $\chi\colon C(T\setminus G_q)\to\mathbb{C}$ be a unital $*$-homomorphism. Then χ equals the restriction of the counit ε.*

Proof of Theorem 2.6. Let ρ be a G_q-equivariant endomorphism on $L^\infty(T\setminus G_q)$. By Stokman–Dijkhuizen's result, we have $\varepsilon\circ\rho=\varepsilon$ on $C(T\setminus G_q)$. Then we have, for $x\in C(T\setminus G_q)$,

$$\rho(x) = (\varepsilon\otimes\mathrm{id})(\delta(\rho(x))) = (\varepsilon\circ\rho\otimes\mathrm{id})(\delta(x)) = (\varepsilon\otimes\mathrm{id})(\delta(x)) = x. \blacksquare$$

Recall the endomorphism θ_μ on M^α introduced in the proof of Lemma 2.5. We will show the surjectivity of θ_μ. Using the equality $w_\mu w_\nu = w_{\mu+\nu}$ for $\mu, \nu \in \widehat{T}$, we can show that $c_{\lambda,\mu} := u_\lambda u_\mu u_{\lambda+\mu}^*$ is fixed by α. This means (θ, c) is a *cocycle action* of $\widehat{T}$ on M^α, that is, for all $\lambda, \mu, \nu \in \widehat{T}$,

$$\theta_\lambda \circ \theta_\mu = \operatorname{Ad} c_{\lambda,\mu} \circ \theta_{\lambda+\mu}, \quad c_{\lambda,\mu} c_{\lambda+\mu,\nu} = \theta_\lambda(c_{\mu,\nu}) c_{\lambda,\mu+\nu}.$$

Then $\theta_\mu \circ \theta_{-\mu} = \operatorname{Ad} c_{\mu,-\mu}$. This, in particular, implies the surjectivity of θ_μ.

So, u_μ is a normalizer of M^α. Moreover, by standard discussion on discrete group actions, we may and do assume that $c = 1$. Namely, we can take u_μ's so that they form a group representation of $\widehat{T}$. Then the automorphisms $\theta_\mu := \operatorname{Ad} u_\mu$ on M^α give a $\widehat{T}$-action.

With a little more effort, we can describe the following irreducible decomposition of the M^α-M^α-bimodule, which proves Theorem 2.3:

$$L^2(R) \cong \bigoplus_{\lambda \in \widehat{T}} L^2(M^\alpha)_{\theta_\lambda}.$$

Here, $L^2(M^\alpha)_{\theta_\lambda}$ denotes the M^α-M^α-bimodule defined by $x \cdot \xi \cdot y := x \xi \theta_\lambda(y)$ for $x, y \in M^\alpha$ and $\xi \in L^2(M^\alpha)$.

Note it turns out from Theorem 2.3 that R is the crossed product generated by M^α and u_μ's. The desired minimal action $T \curvearrowright R$ is nothing but the dual action $\hat{\theta}$.

We have specified the mysterious quantum subgroup $\mathbb{H}$. It would be, however, insufficient since Theorem 2.3 only states that the inclusion $M^\alpha \subset R$ is obtained by *some* minimal action of the maximal torus T. Next aim is to clarify this minimal action. Recall the decomposition (2.2). It is now not so difficult to show the following result.

Lemma 2.8. *There exists a G_q-embedding π of $L^\infty(G_q)$ into M such that*

- $\pi(v_\mu) = u_\mu$, $\mu \in \widehat{T}$;
- $\pi(L^\infty(T \setminus G_q)) = Q$.

Let γ and γ^R be the left and right action of T on $L^\infty(G_q)$, respectively. Then we have the following for $\mu \in P_+$:

$$\begin{aligned}\alpha_{w_0 t}(u_\mu) &= \alpha_{w_0 t}(\pi(v_\mu)) = \pi(\gamma^R_{w_0 t}(v_\mu)) \\ &= \langle w_0 t, w_0 \mu \rangle u_\mu = \langle t, \mu \rangle u_\mu \\ &= \hat{\theta}_t(u_\mu).\end{aligned}$$

Hence we have the following result.

Theorem 2.9. *The minimal action $\hat{\theta}_t$, $t \in T$, is given by the restriction of $\alpha_{w_0 t}$ on R.*

2.4. Induced actions. Using the previous theorem, we can prove the following result.

Theorem 2.10. *The ITP-action of G_q is induced from a minimal action of T on a factor.*

The induction is defined as follows. Let $\beta \colon T \curvearrowright N$ be an action. Put $N \otimes L^\infty(G_q)$ and $\Gamma := \operatorname{id} \otimes \delta$. The von Neumann algebra $N \otimes_T L^\infty(G_q)$ is the collection of all x in $N \otimes L^\infty(G_q)$ such that $(\beta_t \otimes \operatorname{id})(x) = (\operatorname{id} \otimes \gamma_t)(x)$, where $\gamma \colon T \curvearrowright L^\infty(G_q)$ denotes the left action. Then the action $\Gamma \colon G_q \curvearrowright N \otimes_T L^\infty(G_q)$ is called the induced action. The previous theorem means that any ITP-action of G_q is induced by some action of T. Note that any such induced action $\Gamma = \mathrm{Ind}_T^{G_q} N$ on $S := N \otimes_T L^\infty(G_q)$ satisfies $S^\Gamma = N^T \otimes \mathbb{C}$ and $(S^\Gamma)' \cap S = \mathbb{C} \otimes L^\infty(T \setminus G_q)$. So, Theorem 2.10 seems natural. However, not all induced actions are of ITP-type. (See Theorem 2.14 below.)

Using the previous result, we can classify ITP-actions of $SU_q(2)$. First of all, recall that the minimal action $\beta_t = \hat{\theta}_t \colon T \curvearrowright R$ is nothing but the restriction of $\alpha_{w_0 t}$. Since $Q = (M^\alpha)' \cap M$ is a type I factor, β is in fact cocycle conjugate to $\alpha_{w_0 t}$. Here, the cocycle conjugacy means that we can find an isomorphism $\Psi \colon R \to M$ and one cocycle v in M such that $\Psi \circ \beta_t \circ \Psi^{-1} = \mathrm{Ad}\, v_t \circ \alpha_{w_0 t}$, $t \in T$.

Next we note that $\alpha_{w_0 t}$ is an ITP-action of T on M. In particular, it is *invariantly approximately inner* [7]. Such actions are classified by using the classification result of Rohlin flows on injective factors [7].

Our classification results are stated as follows. The key fact is that the fixed point factor M^{α_T} is not of type III_0.

THEOREM 2.11. *A product type action α is unique up to conjugacy if M^α is of type III_1. More precisely, such α is conjugate to $\mathrm{Ind}_T^{G_q}(\mathrm{id}_{R_\infty} \otimes \gamma)$, where R_∞ denotes the injective type III_1 factor and γ the unique minimal action of T on the type II_1 injective factor R_0.*

When $G_q = SU_q(2)$, we can further prove the following results. In Theorem 2.12 and 2.14, the maximal torus of $SU_q(2)$ is identified with $\mathbb{R}/2\pi\mathbb{Z} = [0, 2\pi)$.

THEOREM 2.12. *If $G_q = SU_q(2)$, and M^α is of type II, then M^α and M must be of type II_1 and III_q, respectively. Moreover, α is conjugate to the induction of the torus action $\sigma^{\varphi_q}_{t/\log q}$, where φ_q denotes the Powers state on the Powers factor R_q of type III_q.*

THEOREM 2.13. *If $G_q = SU_q(2)$, and M^α is of type III_λ with $0 < \lambda < 1$, then $\mathrm{mod}(\theta) = q$ or $\lambda^{1/2} q$ in $\mathbb{R}_{>0}/\lambda^{\mathbb{Z}}$. In each case, α is unique up to conjugacy.*

In the last theorem, $\mathrm{mod}(\theta)$ denotes the Connes–Takesaki module of θ. We can consider $\mathrm{mod}(\theta)$ as an obstruction for α to be ITP-type. Indeed, we have the following.

THEOREM 2.14. *Let $G_q = SU_q(2)$ and $0 < \lambda < 1$. Suppose that μ satisfies $0 < \mu < 1$ and $\mu/q \notin (\lambda^{1/2})^{\mathbb{Z}_+}$. Then the induced action $\mathrm{Ind}_T^{G_q}(\mathrm{id}_{R_\lambda} \otimes \sigma^{\varphi_\mu}_{t/\log \mu})$ is not of ITP-type. In particular, for any $0 < \lambda < 1$, there exist uncountably many, non-product type, mutually non-cocycle conjugate actions of $SU_q(2)$ on the injective type III_1 factor with fixed point factor of type III_λ.*

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