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An ordering of measures induced by plurisubharmonic functions

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Abstract. We study an ordering of measures induced by plurisubharmonic functions. This ordering arises naturally in connection with problems related to negative plurisubharmonic functions. We study maximality with respect to the ordering and a related notion of minimality for certain plurisubharmonic functions. The ordering is then applied to the problem of weak*-convergence of measures, in particular Monge–Ampère measures.

1. Introduction. A frequently studied question in pluripotential theory is whether a given sequence of Monge–Ampère measures is weak*-convergent and in that case to what measure. For example, if Ω is a bounded domain in $\mathbb{C}^n$ and $\{u_j\}_{j=1}^{\infty}$ is a sequence of locally bounded plurisubharmonic functions on Ω which is decreasing to a function $u \in \mathrm{PSH}(\Omega) \cap L^{\infty}_{\mathrm{loc}}(\Omega)$, then $(dd^c u_j)^n$ converges to $(dd^c u)^n$ in the weak* topology [3]. The same conclusion holds if $\mathrm{PSH}(\Omega) \cap L^{\infty}_{\mathrm{loc}}(\Omega)$ is replaced by the class $\mathcal{E}(\Omega)$, where Ω is a bounded hyperconvex domain (see Section 2 for definitions) [6]. A sufficient condition for weak*-convergence for the class $\mathcal{F}(\Omega)$ is given in [7]: if u_j tends to u in the sense of distributions and if there is a strictly plurisubharmonic function $v \in \mathcal{E}_0(\Omega)$ such that $\int_\Omega v(dd^c u_j)^n \to \int_\Omega v(dd^c u)^n$, then $(dd^c u_j)^n$ tends to $(dd^c u)^n$ in the weak* topology. (Note that just the convergence $u_j \to u$ as distributions is not enough [4].)

Looking at convergence in the sense of distributions instead of monotone convergence is one way to generalize this problem. Another way is to say that corresponding Monge–Ampère measures should be plurisubharmonically increasing in the sense described in Section 3. This is the background of the order relation between measures defined there.

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In Section 4 we study maximality with respect to the plurisubharmonic ordering and a related notion of minimality for functions in the class $\mathcal{F}(\Omega)$. For example we show that each finite measure on Ω with compact support is majorized by a maximal measure with the same total mass (Theorem 4.7). We also show that each function in $\mathcal{F}(\Omega)$ is minorized by a minimal function with the same total Monge–Ampère mass (Theorem 4.17).

In Section 5, the plurisubharmonic ordering is applied to the problem of weak*-convergence of measures, in particular Monge–Ampère measures. The two main results are the following (for definition of the class $\mathcal{F}_1(\Omega)$, see Section 2).

(Theorem 5.5) If $u, u_j \in \mathcal{F}_1(\Omega)$, $j \geq 0$, have the properties that u_j tends to u in the sense of distributions, $(dd^c u_j)^n$ is plurisubharmonically increasing and $u_j \geq u_0$ for each j, then $(dd^c u_j)^n$ tends to $(dd^c u)^n$ in the weak* topology and $\int_\Omega \varphi\, (dd^c u_j)$ decreases to $\int_\Omega \varphi\, (dd^c u)^n$ for each $\varphi \in \mathrm{PSH}^-(\Omega)$.

(Theorem 5.6) If $u, u_j \in \mathcal{F}(\Omega)$, $j \geq 1$, have the properties that u_j tends to u in the sense of distributions, $(dd^c u)^n$ is a maximal measure and $\int_\Omega (dd^c u_j)^n \to \int_\Omega (dd^c u)^n$, then $(dd^c u_j)^n$ tends to $(dd^c u)^n$ in the weak* topology.

2. Preliminaries. We first recall some definitions. Let Ω be a domain in $\mathbb{C}^n$, $n \geq 1$. Denote by $\mathrm{PSH}(\Omega)$ the plurisubharmonic functions on Ω and by $\mathrm{PSH}^-(\Omega)$ the subclass of non-positive functions. A domain $\Omega \subset \mathbb{C}^n$ is said to be *hyperconvex* if there exists a negative plurisubharmonic exhaustion function, i.e. a function $\varphi \in \mathrm{PSH}^-(\Omega)$ such that $\{z \in \Omega : \varphi(z) < -c\} \subset\subset \Omega$ for all $c > 0$. If Ω is a bounded hyperconvex domain, then it can be shown that the exhaustion function φ can be chosen in $C^\infty(\Omega) \cap C(\overline{\Omega})$ and such that $\int_\Omega (dd^c \varphi)^n < \infty$ (see [9]). This implies for example that the classes defined below are non-trivial.

Unless otherwise stated, throughout this paper, Ω will denote a bounded hyperconvex domain in $\mathbb{C}^n$. Also, by a *measure* we mean a positive regular Borel measure.

Let $\mathcal{E}_0(\Omega)$, $\mathcal{F}(\Omega)$, $\mathcal{F}_1(\Omega)$, $\mathcal{E}(\Omega)$, and $\mathcal{F}^a(\Omega)$ be the subclasses of $\mathrm{PSH}^-(\Omega)$ defined in [5] and [6]:

- $\mathcal{E}_0(\Omega)$: The set of functions $u \in \mathrm{PSH}(\Omega) \cap L^\infty(\Omega)$ such that $\int_\Omega (dd^c u)^n < \infty$ and $\lim_{z\to\xi} u(z) = 0$ for all $\xi \in \partial\Omega$.
- $\mathcal{F}(\Omega)$: The set of functions $u \in \mathrm{PSH}(\Omega)$ such that there is a sequence $\{u_j\}_{j=1}^\infty$ in $\mathcal{E}_0(\Omega)$ with $u_j \searrow u$ and $\sup_j \int_\Omega (dd^c u_j)^n < \infty$.
- $\mathcal{F}_1(\Omega)$: The set of functions $u \in \mathrm{PSH}(\Omega)$ such that there is a sequence $\{u_j\}_{j=1}^\infty$ in $\mathcal{E}_0(\Omega)$ with $u_j \searrow u$, $\sup_j \int_\Omega (dd^c u_j)^n < \infty$ and such that $\sup_j \int_\Omega (-u_j)\, (dd^c u_j)^n < \infty$.

$\mathcal{E}(\Omega)$: The set of functions $u \in \mathrm{PSH}(\Omega)$ such that for each $\omega \subset\subset \Omega$ there is function $u_\omega \in \mathcal{F}(\Omega)$ with $u_\omega \geq u$ on Ω and $u_\omega = u$ on ω.

$\mathcal{F}^a(\Omega)$: The set of functions $u \in \mathcal{F}(\Omega)$ such that $\int_E (dd^c u)^n = 0$ for each pluripolar set $E \subset \Omega$.

For each $f \in \mathrm{PSH}^-(\Omega)$ there is a decreasing sequence of functions in $\mathcal{E}_0(\Omega) \cap C(\overline{\Omega})$ converging to f [6, Theorem 2.1]. Using this fact and weak*-convergence, the complex Monge–Ampère operator is well defined on these classes. Moreover, functions in $\mathcal{F}(\Omega)$ have finite total Monge–Ampère mass, and integration by parts (with no boundary terms) is allowed in this class. We also have $\mathrm{PSH}^-(\Omega) \cap L^\infty_{\mathrm{loc}}(\Omega) \subset \mathcal{E}(\Omega)$ and $\mathcal{E}_0(\Omega) \subset \mathcal{F}_1(\Omega) \subset \mathcal{F}^a(\Omega) \subset \mathcal{F}(\Omega) \subset \mathcal{E}(\Omega)$. See [5] and [6] for details and further properties of these classes.

We conclude this section with a discussion on weak*-convergence of measures on Ω. Let $\{\mu_j\}_{j=1}^\infty$ and μ be measures on Ω. Consider the following properties:

$$\text{(i)}\ \lim_{j\to\infty} \int_\Omega \phi\, d\mu_j = \int_\Omega \phi\, d\mu, \quad \forall \phi \in C_0(\Omega),$$

$$\text{(ii)}\ \lim_{j\to\infty} \int_\Omega \phi\, d\mu_j = \int_\Omega \phi\, d\mu, \quad \forall\, \phi \in C_0^\infty(\Omega),$$

$$\text{(iii)}\ \lim_{j\to\infty} \int_\Omega \phi\, d\mu_j = \int_\Omega \phi\, d\mu, \quad \forall\, \phi \in \mathcal{E}_0(\Omega) \cap C(\overline{\Omega}).$$

Property (i) means, by definition, that μ_j tends weak* to μ on Ω. By standard measure theory, this is equivalent to (ii).

If we assume in addition that the measures involved are finite and that $\sup_j \int_\Omega d\mu_j < \infty$, then weak*-convergence is also equivalent to (iii). To see this, first assume that $\phi \in C_0^\infty(\Omega)$. Then $\phi = \phi_1 - \phi_2$ where $\phi_1, \phi_2 \in \mathcal{E}_0(\Omega) \cap C(\overline{\Omega})$ (see [6, Lemma 3.1]), so (iii) implies (ii). Next assume that $\phi \in \mathcal{E}_0(\Omega) \cap C(\overline{\Omega})$. For $\varepsilon > 0$, let $\phi_\varepsilon = \max\{\phi, -\varepsilon\}$. Then $\phi - \phi_\varepsilon \in C_0(\Omega)$ and we have

$$\int_\Omega \phi\, d\mu_j = \int_\Omega (\phi - \phi_\varepsilon)\, d\mu_j + \int_\Omega \phi_\varepsilon\, d\mu_j \geq \int_\Omega (\phi - \phi_\varepsilon)\, d\mu_j - \varepsilon M,$$

where $M = \sup_j \int_\Omega d\mu_j$. Therefore (i) implies that $\liminf_{j\to\infty} \int_\Omega \phi\, d\mu_j \geq \int_\Omega (\phi - \phi_\varepsilon)\, d\mu - \varepsilon M$, so $\liminf_{j\to\infty} \int_\Omega \phi\, d\mu_j \geq \int_\Omega \phi\, d\mu$ if we let $\varepsilon \searrow 0$. Moreover $\int_\Omega \phi\, d\mu \geq \limsup_{j\to\infty} \int_\Omega \phi\, d\mu_j$ (see Remark 1 below). Hence the claim is proved.

Remark 1. Assume that μ_j tends weak* to μ and that $f : \Omega \to [0, \infty]$ is lower semicontinuous. Choose a sequence $\{f_k\}_{k=1}^\infty$ in $C_0(\Omega)$ such that $f_k \geq 0$ and $f_k \nearrow f$ on Ω. Then $\int_\Omega f_k\, d\mu_j \leq \int_\Omega f\, d\mu_j$ for each j and k, which implies that $\int_\Omega f_k\, d\mu \leq \liminf_{j\to\infty} \int_\Omega f\, d\mu_j$. Hence $\int_\Omega f\, d\mu \leq \liminf_{j\to\infty} \int_\Omega f\, d\mu_j$ by the monotone convergence theorem. For example, with $f = 1$, we see that $\mu(\Omega) \le \liminf_{j\to\infty} \mu_j(\Omega)$.

Remark 2. If $\{\mu_j\}_{j=1}^{\infty}$ is a sequence of finite measures on Ω such that $\sup_j \int_\Omega d\mu_j < \infty$, then one can always find a weak*-convergent subsequence. This uses the fact that $C_0(\Omega)$ equipped with the sup-norm is separable, a diagonal argument and the Riesz representation theorem.

3. The plurisubharmonic ordering. In this section we define an ordering of measures induced by plurisubharmonic functions. We give some examples and basic properties of this order relation.

Definition 3.1. Let μ and ν be measures on Ω. Then we define

$$\mu \preceq \nu \iff \int_\Omega \varphi\, d\mu \ge \int_\Omega \varphi\, d\nu, \ \forall \varphi \in \mathcal{E}_0(\Omega) \cap C(\overline{\Omega}),$$

and we say that μ is *plurisubharmonically less than* ν. If $\{\mu_j\}_{j=1}^{\infty}$ is a sequence of measures such that $\mu_1 \preceq \mu_2 \preceq \cdots$, then we say that $\{\mu_j\}_{j=1}^{\infty}$ is *plurisubharmonically increasing*. Occasionally we write $\nu \succeq \mu$ instead of $\mu \preceq \nu$.

Proposition 3.2. *The plurisubharmonic order relation* $\preceq$ *satisfies the following:*

(a) *If* $\mu \le \nu$ *as measures, then* $\mu \preceq \nu$.
(b) *The relation* $\preceq$ *defines a partial order on the set of positive regular Borel measures on* Ω.
(c) *If* $\mu \preceq \nu$, *then* $\int_\Omega \varphi\, d\mu \ge \int_\Omega \varphi\, d\nu$ *for each* $\varphi \in \mathrm{PSH}^-(\Omega)$, *in particular* $\mu(\Omega) \le \nu(\Omega)$.

Proof. (a) This follows directly from the definition.

(b) If $\mu \preceq \nu$ and $\nu \preceq \eta$, then it is obvious that $\mu \preceq \eta$. Moreover, if $\mu \preceq \nu$ and $\nu \preceq \mu$, then $\int_\Omega \phi\, d\mu = \int_\Omega \phi\, d\nu$ for each $\phi \in C_0(\Omega)$ (cf. (iii)⇒(i) in Section 2), so $\mu = \nu$. Hence $\preceq$ defines a partial order.

(c) For the first part of the statement, choose $\{\varphi_j\}_{j=1}^{\infty} \subset \mathcal{E}_0(\Omega) \cap C(\overline{\Omega})$ such that $\varphi_j \searrow \varphi$. For the second part, choose $\varphi \equiv -1$. ∎

Remark. The plurisubharmonic ordering is not a total order. Consider for example the Dirac measures δ_z and δ_w, where $z, w \in \Omega$ and $z \ne w$. Choose $\varphi, \psi \in \mathrm{PSH}^-(\Omega)$ such that $\varphi(z) < \varphi(w)$ and $\psi(z) > \psi(w)$. Then $\int_\Omega \varphi\, d\delta_z < \int_\Omega \varphi\, d\delta_w$ and $\int_\Omega \psi\, d\delta_w > \int_\Omega \psi\, d\delta_w$, so neither $\delta_z \preceq \delta_w$ nor $\delta_z \succeq \delta_w$.

Example 3.3 shows that the converse of Proposition 3.2(a) is not true. We also see that $\mu \preceq \nu$ and $\int_\Omega d\mu = \int_\Omega d\nu$ can hold true even though $\mu \ne \nu$. (Compare Lemma 4.6 though.)

Example 3.3. Fix $z \in \Omega$ and choose $r > 0$ such that $\mathbb{B}(z, r) \subset\subset \Omega$. Let σ_r be the normalized area measure on the sphere $\partial\mathbb{B}(z, r)$ and δ_z the Dirac measure at z. Then $\sigma_r \preceq \delta_z$ by the mean-value inequality for subharmonic functions. Similarly, $\sigma_r \preceq \sigma_s$ if $r \geq s$.

The following example can be considered an inspiration for introducing the plurisubharmonic order relation.

Example 3.4. Assume that u and v are functions in $\mathcal{F}(\Omega)$ such that $u \geq v$. Then $(dd^c u)^n \preceq (dd^c v)^n$, via integration by parts. Hence if $\{u_j\}_{j=1}^{\infty}$ is a decreasing sequence in $\mathcal{F}(\Omega)$, then $\{(dd^c u_j)^n\}_{j=1}^{\infty}$ is a plurisubharmonically increasing sequence of measures.

The next example shows that the converse of Example 3.4 is not true, i.e. $(dd^c u)^n \preceq (dd^c v)^n$ does not imply that $u \geq v$.

Example 3.5. Let Ω be the unit ball $\mathbb{B}$ in $\mathbb{C}^2$ and define $t(z) = |z|^2 - 1$ and $v(z) = |z|^3 - 1$. Then $t, v \in \mathcal{E}_0(\mathbb{B}) \cap C^2(\overline{\mathbb{B}})$ and $t \geq v$ on $\mathbb{B}$ so $(dd^c t)^2 \preceq (dd^c v)^2$ by Example 3.4. Using [14, Theorem 2 and Proposition 2.3], one can compute the solution u to the equation

$$(dd^c u)^2 = \frac{(dd^c v)^2 + (dd^c t)^2}{2}, \qquad u \in \mathcal{E}_0(\mathbb{B}) \cap C^2(\overline{\mathbb{B}}).$$

The solution is $u(z) = \frac{1}{27\sqrt{2}}((9|z|^2+4)^{3/2} - 13^{3/2})$. Then $(dd^c u)^2 \preceq (dd^c v)^2$, but $u(0) < -1$ so $u \ngeq v$.

We will use the following notation in connection with the plurisubharmonic ordering.

Definition 3.6. For a subset E of Ω we denote by $\widehat{E}$ the convex hull of E in Ω with respect to the family $\mathrm{PSH}(\Omega) \cap C(\overline{\Omega})$, i.e.

$$\widehat{E} = \Big\{ z \in \Omega : \varphi(z) \leq \sup_E \varphi,\ \forall \varphi \in \mathrm{PSH}(\Omega) \cap C(\overline{\Omega}) \Big\}.$$

Note that $\widehat{E}$ is closed in Ω. Moreover, if E is relatively compact in Ω then so is $\widehat{E}$. To see this, choose an exhaustion function $\varphi \in \mathrm{PSH}^-(\Omega) \cap C(\overline{\Omega})$. Then $\sup_E \varphi < 0$ and $\widehat{E} \subset \{z \in \Omega : \varphi(z) \leq \sup_E \varphi\} \subset\subset \Omega$.

Proposition 3.7. *Suppose that μ and ν are measures on Ω such that $\mu(\Omega) = \nu(\Omega) < \infty$. If $\mu \preceq \nu$ then* $\operatorname{supp} \nu \subset \widehat{\operatorname{supp} \mu}$.

Proof. For simplicity, assume that $\mu(\Omega) = 1$. Let $K = \operatorname{supp} \mu$. If $\widehat{K} = \Omega$ then there is nothing to prove. We therefore assume that $\Omega \setminus \widehat{K} \neq \emptyset$. Now suppose that $\nu(\Omega) = 1$ and $\operatorname{supp} \nu \not\subset \widehat{K}$. Since $\widehat{K}$ is closed in Ω, it follows that $\nu(\Omega \setminus \widehat{K}) > 0$. By the regularity of ν, we can find a compact set $L \subset \Omega \setminus \widehat{K}$ such that $\nu(L) > 0$. For each $z \in L$ there is a neighbourhood U of z and a function $\varphi \in \mathrm{PSH}(\Omega) \cap C(\overline{\Omega})$ such that $\varphi(\zeta) > \sup_K \varphi$ for each $\zeta \in U$. Choose $z_1, \ldots, z_N \in L$ such that the corresponding neighbourhoods

$U_1, \dots, U_N$ cover L. Let $\varphi_1, \dots, \varphi_N$ be the associated functions and define

$$M_j = \sup_K \varphi_j, \qquad M = M_1 + \dots + M_N.$$

Let $\psi \in \mathrm{PSH}(\Omega) \cap C(\overline{\Omega})$ be defined by

$$\psi = \max\{\varphi_1, M_1\} + \dots + \max\{\varphi_N, M_N\}.$$

Then $\psi \geq M$ on $\overline{\Omega}$, $\psi = M$ on K, and $\psi > M$ on L. Now let

$$C = \max_{\overline{\Omega}} \psi, \qquad \psi_0 = \psi - C, \qquad M_0 = M - C.$$

Then $\psi_0 \in \mathrm{PSH}^-(\Omega)$ and

$$\psi_0 \geq M_0 \quad \text{on } \Omega, \qquad \psi_0 = M_0 \quad \text{on } K, \qquad \psi_0 > M_0 \quad \text{on } L.$$

Since $\nu(L) > 0$, it follows that

$$\int_\Omega \psi_0 \, d\nu = \int_{\Omega \setminus L} \psi_0 \, d\nu + \int_L \psi_0 \, d\nu > M_0 = \int_\Omega \psi_0 \, d\mu.$$

Hence $\nu \not\preceq \mu$ and the proposition is proved. ∎

4. Maximal measures and minimal functions. Given any measure μ on Ω, we have $\frac{1}{2}\mu \preceq \mu \preceq 2\mu$. So if we want to study maximality with respect to the plurisubharmonic ordering, we have to use some kind of normalization. We therefore make the following definition.

Definition 4.1. A finite measure μ on Ω is said to be *maximal* if for any measure ν on Ω such that $\nu(\Omega) = \mu(\Omega)$, the relation $\nu \succeq \mu$ implies that $\nu = \mu$.

Example 4.2. Let Ω be the unit ball $\mathbb{B}$ in $\mathbb{C}^n$ and δ_0 the Dirac measure at 0. Assume that $\nu(\Omega) = 1$ and $\nu \succeq \delta_0$. Define

$$\varphi_j(z) = \max\{j^{-1} \log|z|, -1\}, \qquad j = 1, 2, \dots.$$

Then $\lim_{j\to\infty} \int_\Omega \varphi_j \, d\nu = -\nu(\{0\})$ and $-1 \leq \int_\Omega \varphi_j \, d\nu \leq \int_\Omega \varphi_j \, d\delta_0 = -1$ for every j, which implies that $\nu(\{0\}) = 1$. Hence $\nu = \delta_0$ and it follows that δ_0 is maximal. Similarly, the Dirac measure at an arbitrary point $z \in \Omega$ is maximal for any bounded hyperconvex domain Ω (use the pluricomplex Green's function for Ω with pole at z, see for example [12]).

If we write a maximal measure μ as the sum $\mu = \mu_1 + \mu_2$ of two finite measures, then these are maximal too. To see this, assume that μ_1 is not maximal. Then there is a measure $\nu \neq \mu_1$ such that $\nu(\Omega) = \mu_1(\Omega)$ and $\nu \succeq \mu_1$. Then $(\nu + \mu_2)(\Omega) = \mu(\Omega)$ and $\nu + \mu_2 \succeq \mu$ but $\nu + \mu_2 \neq \mu$ (by the finiteness of the measures), which is a contradiction. Moreover, the set of maximal measures on Ω is a cone, i.e. if μ is a maximal measure then so is $c\mu$ for any constant $c > 0$. As Example 4.16 shows, it is not a convex cone.

Using the following definition, we can present a sufficient condition for a measure to be maximal.

DEFINITION 4.3. We say that a set $K \subset\subset \Omega$ is an *interpolation set for* $\mathrm{PSH}^-(\Omega)$ if for each $f \in C(K)$ such that $f < 0$ there exists $\varphi \in \mathrm{PSH}^-(\Omega)$ such that $\varphi|_K = f$.

PROPOSITION 4.4. *If μ is a finite measure on Ω such that $\widehat{\operatorname{supp}\mu}$ is contained in some interpolation set K for* $\mathrm{PSH}^-(\Omega)$, *then μ is maximal.*

Proof. Assume that ν is a measure on Ω such that $\nu(\Omega) = \mu(\Omega)$ and $\mu \preceq \nu$. Then by Proposition 3.7, $\operatorname{supp}\nu \subset \widehat{\operatorname{supp}\mu} \subset K$. Let $f \in C(K)$ be such that $f < 0$. Then there is $\varphi \in \mathrm{PSH}^-(\Omega)$ such that $\varphi|_K = f$. Therefore

$$\int_\Omega f\,d\mu = \int_\Omega \varphi\,d\mu \ge \int_\Omega \varphi\,d\nu = \int_\Omega f\,d\nu.$$

Since $\mu(\Omega) = \nu(\Omega)$, we conclude that $\int_\Omega f\,d\mu \ge \int_\Omega f\,d\nu$ for each $f \in C_0(\Omega)$ such that $f \le 0$. Hence, $\mu \le \nu$ as measures, so $\mu = \nu$. ∎

EXAMPLE 4.5. Let $K = \{z_1, \dots, z_N\}$ be a finite set of points in Ω and assume that numbers $c_j < 0$, $j = 1, \dots, N$ are given. Let G_j be the multipole pluricomplex Green's function for Ω with poles at $\{z_1, \dots, z_N\} \setminus \{z_j\}$ (see [13]). Then $G_j(z_k) = -\infty$ if $k \ne j$ and $G_j > -\infty$ elsewhere. Multiplying by a suitable positive constant, we may assume that $G_j(z_j) = c_j$. Therefore $\varphi = \max\{G_1, \dots, G_N\}$ defines a negative plurisubharmonic function such that $\varphi(z_j) = c_j$, $j = 1, \dots, N$. Hence, K is an interpolation set for $\mathrm{PSH}^-(\Omega)$. Moreover, $\widehat{K} = K$ (consider for example the multipole pluricomplex Green's function for Ω with poles at $\{z_1, \dots, z_N\}$). Therefore, $\mu = \sum_{j=1}^N a_j \delta_{z_j}$, where δ_{z_j} is the Dirac measure at $z_j \in \Omega$ and $a_j > 0$, is a maximal measure.

Next we show that each finite measure with compact support is majorized (in the plurisubharmonic ordering) by a maximal measure with the same total mass. We begin with a useful lemma.

LEMMA 4.6. *Assume that μ and ν are measures on Ω such that $\mu \preceq \nu$. If $\int_\Omega \psi\,d\mu = \int_\Omega \psi\,d\nu > -\infty$ for some negative strictly plurisubharmonic function ψ on Ω, then $\mu = \nu$.*

Proof. Let $f \in C_0^\infty(\Omega)$. Choose a constant $c \ge 0$ large enough so that $f + c\psi \in \mathrm{PSH}^-(\Omega)$. Then

$$\int_\Omega f\,d\mu + c\alpha = \int_\Omega (f + c\psi)\,d\mu \ge \int_\Omega (f + c\psi)\,d\nu = \int_\Omega f\,d\nu + c\alpha,$$

where $\alpha = \int_\Omega \psi\,d\mu = \int_\Omega \psi\,d\nu$, so $\int_\Omega f\,d\mu \ge \int_\Omega f\,d\nu$. Since this argument can also be applied to the function $-f$, it follows that $\mu = \nu$. ∎

THEOREM 4.7. *Let μ be a finite measure on Ω with compact support. Then there is a maximal measure μ_0 such that $\mu_0 \succeq \mu$ and $\mu_0(\Omega) = \mu(\Omega)$.*

Proof. We may assume that μ is a probability measure, i.e. $\mu(\Omega) = 1$. Let $K_0 = \widehat{\operatorname{supp}\mu}$, which is a compact subset of Ω, and let $\mathcal{M}$ be the set of probability measures on Ω. Define

$$\mathcal{M}_\mu = \{\nu \in \mathcal{M} : \nu \succeq \mu\}.$$

Note that by Proposition 3.7, $\operatorname{supp}\nu \subset K_0$ for each $\nu \in \mathcal{M}_\mu$. Choose a negative continuous strictly plurisubharmonic function ψ and define

$$\alpha = \sup\Big\{\int_\Omega (-\psi)\, d\nu : \nu \in \mathcal{M}_\mu\Big\}$$

(which is finite since ψ is bounded on K_0). Let $\{\nu_j\}_{j=1}^\infty$ be a sequence in $\mathcal{M}_\mu$ such that $\int_\Omega (-\psi)\, d\nu_j \to \alpha$. Taking a subsequence if necessary, we may assume that ν_j tends weak* to some measure μ_0, where $\mu_0(\Omega) \le 1$ (see Section 2, Remarks 1 and 2). If $\varphi \in \mathcal{E}_0(\Omega) \cap C(\overline{\Omega})$, then $\int_\Omega \varphi\, d\mu_0 = \lim_{j\to\infty} \int_\Omega \varphi\, d\nu_j \le \int_\Omega \varphi\, d\mu$, which implies that $\mu_0 \succeq \mu$. Therefore $\mu_0(\Omega) = 1$, so $\mu_0 \in \mathcal{M}_\mu$.

Next, choose $f \in C_0(\Omega)$ such that $f = 1$ on K_0. It follows that

$$\int_\Omega (-\psi)\, d\mu_0 = \int_\Omega f(-\psi)\, d\mu_0 = \lim_{j\to\infty} \int_\Omega f(-\psi)\, d\nu_j = \lim_{j\to\infty} \int_\Omega (-\psi)\, d\nu_j = \alpha.$$

Now, suppose that $\nu \in \mathcal{M}$ satisfies $\nu \succeq \mu_0$. Then $\nu \in \mathcal{M}_\mu$, which implies that $\alpha \ge \int_\Omega (-\psi)\, d\nu \ge \int_\Omega (-\psi)\, d\mu_0 = \alpha$, so $\int_\Omega (-\psi)\, d\nu = \alpha$. Hence $\nu = \mu_0$ by Lemma 4.6, so μ_0 is maximal. ∎

We now turn to a notion of minimality for certain plurisubharmonic functions, which is related to the plurisubharmonic ordering and its maximal measures.

Definition 4.8. A function $u \in \mathcal{F}(\Omega)$ is said to be *minimal* if for any function $v \in \mathcal{F}(\Omega)$ such that $\int_\Omega (dd^c v)^n = \int_\Omega (dd^c u)^n$, the relation $v \le u$ implies that $v = u$.

For functions in $\mathcal{F}(\Omega)$ we have the following identity principle (see [8, Theorem 3.15]):

$$u, v \in \mathcal{F}(\Omega),\ u \ge v,\ (dd^c u)^n = (dd^c v)^n \ \Rightarrow\ u = v. \tag{4.1}$$

Combining this with Example 3.4, we get the next proposition.

Proposition 4.9. *If $u \in \mathcal{F}(\Omega)$ is such that $(dd^c u)^n$ is a maximal measure, then u is a minimal function.*

Example 4.10. Let Ω be the unit ball $\mathbb{B}(0,1)$ in $\mathbb{C}^n$. Then

$$(dd^c \log|z|)^n = (2\pi)^n \delta_0$$

is a maximal measure by Example 4.2, so $\log|z|$ is a minimal function.

One might ask if there is a converse of Proposition 4.9. The answer is affirmative if $n = 1$ (see Proposition 4.11 below). In higher dimension, the answer is unknown to the author.

PROPOSITION 4.11. *If $n = 1$ and $u \in \mathcal{F}(\Omega)$ is a minimal function, then $dd^c u$ is a maximal measure.*

Proof. Note that if $n = 1$ then $(dd^c u)^1 = \Delta u$, the Laplacian of u. Moreover, the class $\mathcal{F}(\Omega)$ can be characterized as follows. Given a finite measure μ on $\Omega \subset \mathbb{C}$, define the potential p_μ of μ by $p_\mu(z) = \int_\Omega G(z,w)\,d\mu$, where $G(z,w)$ is the (normalized) Green's function for Ω. Then $p_\mu \in \mathcal{F}(\Omega)$ and $\Delta p_\mu = \mu$. Conversely, if $u \in \mathcal{F}(\Omega)$ then $u = p_{\Delta u}$.

Now assume that $u \in \mathcal{F}(\Omega)$ is minimal. Suppose that $\mu \succeq \Delta u$ and $\int_\Omega d\mu = \int_\Omega \Delta u$. Then $p_\mu(z) = \int_\Omega G(z,w)\,d\mu \le \int_\Omega G(z,w)\Delta u = u(z)$, since $G(z,w)$ is subharmonic in w. The minimality of u then implies that $p_\mu = u$, so $\mu = \Delta u$. Hence Δu is maximal and the proposition is proved. ∎

The next propositon gives another condition for $u \in \mathcal{F}(\Omega)$ to be minimal.

PROPOSITION 4.12. *Assume that $u \in \mathcal{F}(\Omega)$ and that $(dd^c u)^n$ is carried by a pluripolar set. Then u is a minimal function.*

Proof. We will make use of the following facts:

(i) If $u, v \in \mathcal{E}(\Omega)$ and $u \ge v$, then
$$\chi_{\{u=-\infty\}}(dd^c u)^n \le \chi_{\{v=-\infty\}}(dd^c v)^n$$
(see [1, Lemma 4.1]).

(ii) If $(dd^c u)^n$ is carried by a pluripolar set, then
$$(dd^c u)^n = \chi_{\{u=-\infty\}}(dd^c u)^n$$
(see [6, Theorem 5.11]).

Assume that $v \in \mathcal{F}(\Omega)$, $v \le u$ and $\int_\Omega (dd^c v)^n = \int_\Omega (dd^c u)^n$. Then from (i) and (ii) it follows that
$$\int_\Omega (dd^c v)^n \ge \int_\Omega \chi_{\{v=-\infty\}}(dd^c v)^n \ge \int_\Omega \chi_{\{u=-\infty\}}(dd^c u)^n = \int_\Omega (dd^c u)^n,$$
and so the inequalities above are in fact equalities. Hence $(dd^c v)^n = \chi_{\{v=-\infty\}}(dd^c v)^n$, so $(dd^c u)^n \le (dd^c v)^n$ by (i) and (ii). Therefore $(dd^c u)^n = (dd^c v)^n$ (since the measures have the same total mass), so (4.1) implies that $v = u$, and the proof is complete. ∎

There are functions in $\mathcal{F}(\Omega)$ whose Monge–Ampère measure is maximal and lives on a non-pluripolar set. This will be shown in Example 4.15, where we also see that there are bounded minimal functions. We will need the following proposition and lemma.

PROPOSITION 4.13. *Assume that μ is a finite measure on Ω such that $\widehat{\operatorname{supp}\mu}$ is contained in a level set $\{z \in \Omega : \psi(z) = s\}$, $s > -\infty$, where $\psi < 0$ is a strictly plurisubharmonic function on Ω. Then μ is maximal.*

Proof. Suppose that $\nu \succeq \mu$ and $\nu(\Omega) = \mu(\Omega)$. Then $\operatorname{supp}\nu \subset \{z \in \Omega : \psi(z) = s\}$ by Proposition 3.7, hence

$$\int_\Omega \psi\, d\nu = \int_\Omega s\, d\nu = \int_\Omega s\, d\mu = \int_\Omega \psi\, d\mu > -\infty.$$

Therefore Lemma 4.6 implies that $\nu = \mu$, and the proof is complete. ∎

LEMMA 4.14. *Let Ω_1 and Ω_2 be bounded hyperconvex domains in $\mathbb{C}$. Assume that $u \in \mathcal{E}(\Omega_1) \cap C(\Omega_1)$ and $v \in \mathcal{E}(\Omega_2) \cap C(\Omega_2)$ are such that $u, v \geq -1$ and*

$$\int_{\{u>-1\}} dd^c u = \int_{\{v>-1\}} dd^c v = 0. \tag{4.2}$$

Let $\psi(z_1, z_2) = \max\{u(z_1), v(z_2)\}$. Then

$$(dd^c\psi)^2 = (dd^c u) \wedge (dd^c v) \qquad \text{on } \Omega_1 \times \Omega_2.$$

Proof. Let $\varepsilon > 0$. If $z \in \Omega_1 \times \Omega_2$ with $u(z) < v(z)$, then by continuity $u < v$ in a neighbourhood of z, which implies that $dd^c \max\{u, v\} - dd^c v = 0$ in a neighbourhood of z. If $u(z) \geq v(z)$ then $u(z) > v(z) - \varepsilon$, and it follows that $dd^c \max\{u, v - \varepsilon\} - dd^c u = 0$ in a neighbourhood of z. Therefore

$$(dd^c \max\{u, v - \varepsilon\} - dd^c u) \wedge (dd^c \max\{u, v\} - dd^c v) = 0.$$

Moreover, since the functions involved are in $\mathcal{E}(\Omega_1 \times \Omega_2)$ (see for example [2, Corollary 2.1]), we may let $\varepsilon \searrow 0$ to conclude that

$$(dd^c \max\{u, v\} - dd^c u) \wedge (dd^c \max\{u, v\} - dd^c v) = 0.$$

Again, let $\varepsilon > 0$. If $u(z) < v(z)$, then by continuity $v > -1$ in a neighbourhood of z, so $dd^c v = 0$ there by assumption (4.2). If $u(z) \geq v(z)$, then by the same argument as above, $dd^c \max\{u, v - \varepsilon\} - dd^c u = 0$ in a neighbourhood of z. Hence $(dd^c \max\{u, v - \varepsilon\} - dd^c u) \wedge (dd^c v) = 0$, and again we may let $\varepsilon \searrow 0$. Therefore

$$(dd^c \max\{u, v\} - dd^c u) \wedge (dd^c v) = 0,$$

and by the same argument (interchanging the roles of u and v)

$$(dd^c \max\{u, v\} - dd^c v) \wedge (dd^c u) = 0.$$

Expanding and combining the expressions above it follows that

$$\begin{aligned}
&(dd^c \max\{u,v\})^2 \\
&\quad = (dd^c \max\{u,v\}) \wedge (dd^c v) + (dd^c \max\{u,v\}) \wedge (dd^c u) - (dd^c u) \wedge (dd^c v) \\
&\quad = (dd^c u) \wedge (dd^c v),
\end{aligned}$$

and the proof is complete. ∎

EXAMPLE 4.15. We first look at the unit disc $\mathbb{D}$ in $\mathbb{C}$. Let $K = \{z = \frac{1}{2}e^{i\theta} : \varepsilon \le \theta \le 2\pi\}$, for some (small) $\varepsilon > 0$, and let $\psi(z) = |z|^2 - 1$. Then ψ is a negative strictly plurisubharmonic function on $\mathbb{D}$ and $K \subset \{\psi = -3/4\}$. Let h_K be the relative extremal function for K in $\mathbb{D}$, i.e. $h_K = \sup\{u \in \mathrm{SH}^-(\mathbb{D}) : u|_K \le -1\}$. Then $h_K \in \mathcal{E}_0(\mathbb{D}) \cap C(\bar{\mathbb{D}})$ and $h_K = -1$ on K. Moreover, h_K is harmonic on the connected set $\mathbb{D} \setminus K$, which implies that $h_K > -1$ on $\mathbb{D} \setminus K$. Hence $\widehat{K} = K$ and we can use Proposition 4.13 to deduce that

$$\mu(\mathbb{D}) < \infty,\ \operatorname{supp}\mu \subset K \ \Rightarrow\ \mu \text{ is maximal.}$$

For example, $\mu = \Delta h_K$ is a maximal measure with $\operatorname{supp}\mu = K$, which is a nonpolar set, and h_K is a bounded minimal function.

Next, look at the unit bidisc $\mathbb{D} \times \mathbb{D}$ in $\mathbb{C}^2$ and the subset $K \times K$, with K as above. Then $K \times K \subset \{\psi = -3/2\}$, where $\psi(z,w) = |z|^2 + |w|^2 - 2$ is a negative strictly plurisubharmonic function. Define the function

$$g(z,w) = \max\{h_K(z), h_K(w)\} \in \mathcal{E}_0(\mathbb{D} \times \mathbb{D}) \cap C(\overline{\mathbb{D} \times \mathbb{D}}).$$

Using the properties of h_K it follows that $g = -1$ on $K \times K$ and $g > -1$ on $\mathbb{D} \times \mathbb{D} \setminus K \times K$. Hence $\widehat{K \times K} = K \times K$, so by Proposition 4.13,

$$\mu(\mathbb{D} \times \mathbb{D}) < \infty,\ \operatorname{supp}\mu \subset K \times K \ \Rightarrow\ \mu \text{ is maximal.}$$

For example $(dd^c g)^2 = dd^c h_K \wedge dd^c h_K = \Delta h_K \times \Delta h_K$ (see Lemma 4.14), so $(dd^c g)^2$ is a maximal measure with support $K \times K$, which is a non-pluripolar set, and g is a bounded minimal function.

The function g is actually the relative extremal function for $K \times K$ in $\mathbb{D} \times \mathbb{D}$, i.e.

$$g = h_{K \times K} = \sup\{u \in \mathrm{PSH}^-(\mathbb{D} \times \mathbb{D}) : u|_{K \times K} \le -1\}.$$

This can be seen for example as follows. On $K \times K$ we have $g = h_{K \times K} = -1$, and outside of $K \times K$ we have $(dd^c g)^2 = (dd^c h_{K \times K})^2 = 0$. Therefore Lemma 4.9 in [10] implies that $g = h_{K \times K}$ on $\mathbb{D} \times \mathbb{D}$, since $g, h_{K \times K} \in \mathcal{F}^a(\mathbb{D} \times \mathbb{D})$.

EXAMPLE 4.16. Let Ω be the unit disc $\mathbb{D}$ in $\mathbb{C}$. Define

$$S_1 = \{z = \tfrac{1}{2}e^{i\theta} : 0 \le \theta \le \pi\}, \qquad S_2 = \{z = \tfrac{1}{2}e^{i\theta} : \pi < \theta < 2\pi\}.$$

Let σ be the area measure on the circle $\partial\mathbb{D}(0,1/2)$ and define $\mu_j = \sigma|_{S_j}$ for $j = 1, 2$. Then μ_1 and μ_2 are maximal measures by Example 4.15, but the sum $\mu_1 + \mu_2 = \sigma$ is not maximal (see Example 3.3). Hence, the set of maximal measures on Ω is not a convex cone.

We conclude this section by showing that each function in $\mathcal{F}(\Omega)$ is minorized by a minimal function with the same total Monge–Ampère mass (cf. Theorem 4.7).

THEOREM 4.17. *For each $u \in \mathcal{F}(\Omega)$, there exists a minimal function $u_0 \in \mathcal{F}(\Omega)$ such that $u_0 \le u$ and $\int_\Omega (dd^c u_0)^n = \int_\Omega (dd^c u)^n$.*

Proof. Define

$$S = \Big\{ v \in \mathcal{F}(\Omega) : v \le u,\ \int_\Omega (dd^c v)^n = \int_\Omega (dd^c u)^n \Big\}.$$

Let T be a totally ordered subset of S, and t be the function defined by $t(z) = \inf\{v(z) : v \in T\}$. We shall prove that $t \in S$, which means that t is a lower bound for T in S. Once this is done, we can argue as follows: Since T was arbitrary, we can use Zorn's lemma to deduce that there is a minimal element u_0 of S. But then u_0 have the required properties, and the proof would be complete.

We need to prove that $t \in S$. It is obvious that $t \le u$, so it remains to prove that $t \in \mathcal{F}(\Omega)$ and $\int_\Omega (dd^c t)^n = \int_\Omega (dd^c u)^n$. Let $\{K_k\}_{k=1}^\infty$ be a compact exhaustion of Ω and $\{g_j\}_{j=1}^\infty$ a sequence of functions in $C(\Omega)$ such that $g_j > t$ and $g_j \searrow t$ as $j \to \infty$ (note that t by construction is upper semicontinuous). For j and k fixed, construct a function $v_k^j \in T$ as follows. For each $z \in K_k$, choose $v_z \in T$ such that $v_z(z) < g_j(z)$ and consider the open set

$$U_z = \{w \in \Omega : v_z(w) < g_j(w)\}.$$

Take $z_1, \dots, z_N \in K_k$ such that the corresponding sets $U_{z_1}, \dots, U_{z_N}$ cover K_k. Since T is totally ordered, we may choose v_k^j to be the smallest of the functions $v_{z_1}, \dots, v_{z_N}$, which implies that $v_k^j < g_j$ on K_k. Now let $u_1 = v_1^1$ and u_j be the smallest of the functions $\{u_1, \dots, u_{j-1}, v_j^j\}$ if $j \ge 2$, which is possible since T is totally ordered. Then $\{u_j\}_{j=1}^\infty$ is a decreasing sequence of functions in T such that $u_j \le v_j^j < g_j$ on K_j. Therefore $u_j \in \mathcal{F}(\Omega)$, $\int_\Omega (dd^c u_j)^n = \int_\Omega (dd^c u)^n$ and $u_j \searrow t$ when $j \to \infty$. Hence $t \in \mathcal{F}(\Omega)$ and $\int_\Omega (dd^c t)^n = \int_\Omega (dd^c u)^n$ (by [8, proof of Lemma 3.2]), so the proof is complete. ∎

5. Weak*-convergence. In this section we use the plurisubharmonic ordering to obtain some results on weak*-convergence of measures.

PROPOSITION 5.1. *Suppose that $\{\mu_j\}_{j=1}^\infty$ is a plurisubharmonically increasing sequence of measures on Ω such that* $\sup_j \int_\Omega d\mu_j < \infty$. *Then μ_j*

tends weak to a measure ν on Ω. Moreover, $\int_\Omega \varphi\, d\mu_j \searrow \int_\Omega \varphi\, d\nu$ for each $\varphi \in \mathrm{PSH}^-(\Omega)$.*

Proof. Let $\varphi \in \mathrm{PSH}^-(\Omega) \cap L^\infty(\Omega)$. Then

$$0 \geq \int_\Omega \varphi\, d\mu_1 \geq \int_\Omega \varphi\, d\mu_2 \geq \cdots \geq \Big(\inf_\Omega \varphi\Big) \cdot \Big(\sup_j \int_\Omega d\mu_j\Big) > -\infty,$$

so $\lim_{j\to\infty} \int_\Omega \varphi\, d\mu_j > -\infty$. By the same reasoning as in Section 2, it follows that the limit exists for each $\varphi \in C_0(\Omega)$. Hence, this defines a measure ν on Ω, to which μ_j tends in weak* topology. Moreover, we know that $\lim_{j\to\infty} \int_\Omega \varphi\, d\mu_j = \int_\Omega \varphi\, d\nu$ for each $\varphi \in \mathcal{E}_0(\Omega) \cap C(\overline{\Omega})$.

Now, let $\varphi \in \mathrm{PSH}^-(\Omega)$. Then again $\int_\Omega \varphi\, d\mu_j$ is decreasing in j, but the limit might be $-\infty$. In any case, we have

$$\lim_{j\to\infty} \int_\Omega \varphi\, d\mu_j \leq \int_\Omega \varphi\, d\nu$$

(see Section 2; note that $f = -\varphi$ is a non-negative lower semicontinuous function). Assume that $\lim_{j\to\infty} \int_\Omega \varphi\, d\mu_j < \int_\Omega \varphi\, d\nu$. Choose j_0 such that $\int_\Omega \varphi\, d\mu_{j_0} < \int_\Omega \varphi\, d\nu$, and a sequence $\{\varphi_k\}_{k=1}^\infty$ in $\mathcal{E}_0(\Omega) \cap C(\overline{\Omega})$ such that $\varphi_k \searrow \varphi$. Then k_0 may be chosen such that $\int_\Omega \varphi_{k_0}\, d\mu_{j_0} < \int_\Omega \varphi\, d\nu$. It follows that

$$\int_\Omega \varphi_{k_0}\, d\nu = \lim_{j\to\infty} \int_\Omega \varphi_{k_0}\, d\mu_j \leq \int_\Omega \varphi_{k_0}\, d\mu_{j_0} < \int_\Omega \varphi\, d\nu \leq \int_\Omega \varphi_{k_0}\, d\nu,$$

using, in turn, the facts that $\varphi_{k_0} \in \mathcal{E}_0(\Omega) \cap C(\overline{\Omega})$, that $\int_\Omega \varphi_{k_0}\, d\mu_j$ is decreasing in j, the choice of k_0, and finally the inequality $\varphi \leq \varphi_{k_0}$. This is a contradiction, hence $\lim_{j\to\infty} \int_\Omega \varphi\, d\mu_j = \int_\Omega \varphi\, d\nu$, and the proposition is proved. ∎

If $\mu_j = (dd^c u_j)^n$, where $u_j \in \mathcal{F}(\Omega)$ tends to $u \in \mathcal{F}(\Omega)$ as distributions, then we can relate the limit measure in the above proposition to $(dd^c u)^n$ as follows.

Proposition 5.2. *Assume that $\{u_j\}_{j=1}^\infty$ is a sequence in $\mathcal{F}(\Omega)$ such that*

(a) *u_j tends to $u \in \mathcal{F}(\Omega)$ in the sense of distributions,*
(b) *$\{(dd^c u_j)^n\}_{j=1}^\infty$ is plurisubharmonically increasing,*
(c) *$\sup_j \int_\Omega (dd^c u_j)^n < \infty$.*

Then $(dd^c u_j)^n$ is weak-convergent and $\nu = \lim_{j\to\infty} (dd^c u_j)^n \succeq (dd^c u)^n$. Moreover $\int_\Omega \varphi\, (dd^c u_j) \searrow \int_\Omega \varphi\, d\nu$ for each $\varphi \in \mathrm{PSH}^-(\Omega)$.*

Proof. By Proposition 5.1, we only need to prove that $(dd^c u)^n \preceq \nu$. This follows from [7, Lemma 2.1], since it is shown there that if $\{u_j\}_{j=1}^\infty \subset \mathcal{F}(\Omega)$ satisfies condition (a), then $\int_\Omega \varphi\, (dd^c u)^n \geq \limsup_{j\to\infty} \int_\Omega \varphi\, (dd^c u_j)^n$ for each $\varphi \in \mathrm{PSH}^-(\Omega)$. ∎

Unfortunately, the conditions in Proposition 5.2 are not sufficient for $(dd^c u)^n = \lim_{j\to\infty}(dd^c u_j)^n$ to hold. This can be seen in the following example.

EXAMPLE 5.3. Let Ω be the unit bidisc $\mathbb{D}\times\mathbb{D}$ in $\mathbb{C}^2$. Choose a sequence $\{\varepsilon_j\}_{j=1}^{\infty}$ such that $0<\varepsilon_j<1$ and $\varepsilon_j \searrow 0$. Define

$$u_j(z,w) = (1/\varepsilon_j)\cdot \max\left\{\log|z|,\, \varepsilon_j^2\log|w|,\, -1\right\}.$$

Then $u_j \in \mathrm{PSH}^-(\Omega)\cap C(\overline{\Omega})$, $u_j=0$ on the boundary $\partial\Omega$, and $\lim_{j\to\infty} u_j(z,w) = 0$ whenever $w\neq 0$. Moreover, we can write

$$u_j = (1/\varepsilon_j)\cdot \max\bigl\{\max\{\log|z|,-1\},\max\{\varepsilon_j^2\log|w|,\,-1\}\bigr\}$$

and use Lemma 4.14 to compute

$$\begin{aligned}(dd^c u_j)^2 &= 1/\varepsilon_j^2\cdot(dd^c\max\{\log|z|,-1\})\wedge(dd^c\max\{\varepsilon_j^2\log|w|,\,-1\})\\ &= (dd^c\max\{\log|z|,-1\})\wedge(dd^c\max\{\log|w|,\,-1/\varepsilon_j^2\})\\ &= (2\pi)^2\cdot\sigma\times\sigma_j,\end{aligned}$$

where σ and σ_j denote the normalized area measures on the circles $\partial\mathbb{D}(0,e^{-1})$ and $\partial\mathbb{D}(0,e^{(-1/\varepsilon_j^2)})$ respectively. Hence, using the mean-value inequality for subharmonic functions (in the w-variable), it follows that $\{(dd^c u_j)^2\}_{j=1}^{\infty}$ is plurisubharmonically increasing. We also see that

$$\lim_{j\to\infty}(dd^c u_j)^2 = (2\pi)^2\cdot\sigma\times\delta_0.$$

Thus, we have constructed a sequence $\{u_j\}_{j=1}^{\infty}$ in $\mathcal{E}_0(\Omega)\cap C(\overline{\Omega})$ which tends to 0 a.e. (dV) on Ω and has the properties that $\{(dd^c u_j)^2\}_{j=1}^{\infty}$ is plurisubharmonically increasing and $\int_\Omega (dd^c u_j)^2 = (2\pi)^2$ for each j. But $\lim_{j\to\infty}(dd^c u_j)^2 \neq 0 = (dd^c 0)^2$.

The next theorem gives sufficient conditions for weak*-convergence. We begin with a lemma.

LEMMA 5.4. *If $\mu \preceq \nu$ and ν vanishes on pluripolar sets, then μ vanishes on pluripolar sets.*

Proof. Let $E\subset\Omega$ be pluripolar. Then there is $h\in\mathrm{PSH}^-(\Omega)$ such that $E\subset\{z\in\Omega : h(z)=-\infty\} = E_h$ (see e.g. [6, Theorem 5.8]). For $\varepsilon>0$, define $h_\varepsilon = \max\{\varepsilon h,-1\}$. Then $h_\varepsilon=-1$ on E_h and $h_\varepsilon<0$ on Ω. Moreover, if $\varepsilon\searrow 0$ then $h_\varepsilon \nearrow 0$ on $\Omega\setminus E_h$. Hence,

$$\mu(E)\le\mu(E_h)\le\int_\Omega(-h_\varepsilon)\,d\mu\le\int_\Omega(-h_\varepsilon)\,d\nu=\int_{\Omega\setminus E_h}(-h_\varepsilon)\,d\nu.$$

Letting $\varepsilon\searrow 0$, we find that $\mu(E)=0$. ∎

THEOREM 5.5. *Assume that $\{u_j\}_{j=0}^{\infty}$ is a sequence in $\mathcal{F}_1(\Omega)$ such that*

(a) *u_j tends to $u\in\mathcal{F}_1(\Omega)$ in the sense of distributions,*

(b) $\{(dd^c u_j)^n\}_{j=1}^{\infty}$ *is plurisubharmonically increasing,*
(c) $u_j \geq u_0$ *for each* j.

Then $(dd^c u_j)^n$ *tends weak* to* $(dd^c u)^n$. *Moreover* $\int_\Omega \varphi\,(dd^c u_j) \searrow \int_\Omega \varphi\,(dd^c u)^n$ *for each* $\varphi \in \mathrm{PSH}^-(\Omega)$.

Proof. We will make use of the following facts:

(i) If $v, w \in \mathcal{F}_1(\Omega)$, then

$$\int_\Omega -v\,(dd^c w)^n \leq \Big(\int_\Omega -v\,(dd^c v)^n\Big)^{\frac{1}{n+1}} \cdot \Big(\int_\Omega -w\,(dd^c w)^n\Big)^{\frac{n}{n+1}}.$$

(ii) Assume that μ is a finite measure on Ω which vanishes on pluripolar sets. If $v_0, v_j \in \mathcal{E}(\Omega)$, $v_j \geq v_0$ for each j, $\int_\Omega -v_0\,d\mu < \infty$, and $v_j \to v$ as distributions, then $\lim_{j\to\infty} \int_\Omega v_j\,d\mu = \int_\Omega v\,d\mu$.

(iii) If $v, v_j \in \mathcal{F}_1(\Omega)$, $v_j \to v$ as distributions, and $\lim_{j\to\infty} \int_\Omega v_j\,(dd^c v_j)^n = \int_\Omega v\,(dd^c v)^n$, then $(dd^c v_j)^n$ tends weak* to $(dd^c v)^n$.

Here, (i) follows from [5, Theorems 3.2 and 3.8], (ii) is [11, Lemma 1.4], and (iii) is (part of) [11, Theorem 2.3].

Now, note that (c) implies that $(dd^c u_j)^n \preceq dd^c u_0$ (see Example 3.4). In particular Proposition 5.2 applies, so $(dd^c u)^n \preceq \nu \preceq (dd^c u_0)^n$, where $\nu = \lim_{j\to\infty}(dd^c u_j)^n$. Therefore $\int_\Omega -u_0\,d\nu \leq \int_\Omega -u_0\,(dd^c u_0)$, which is finite since $u_0 \in \mathcal{F}_1(\Omega)$ (see [5]), and ν vanishes on pluripolar sets (by Lemma 5.4, since $\mathcal{F}_1(\Omega) \subset \mathcal{F}^a(\Omega)$). The same holds with ν replaced by $(dd^c u_j)^n$. Hence, (ii) implies that

$$\text{(5.1)}\qquad \lim_{j\to\infty} \int_\Omega u_j\,d\nu = \int_\Omega u\,d\nu \quad \text{and} \quad \lim_{k\to\infty} \int_\Omega u_k\,(dd^c u_j)^n = \int_\Omega u\,(dd^c u_j)^n.$$

By (i),

$$\int_\Omega -u\,(dd^c u_j)^n \leq \Big(\int_\Omega -u\,(dd^c u)^n\Big)^{\frac{1}{n+1}} \cdot \Big(\int_\Omega -u_j\,(dd^c u_j)^n\Big)^{\frac{n}{n+1}},$$

and moreover $\lim_{j\to\infty} \int_\Omega -u\,(dd^c u_j)^n = \int_\Omega -u\,d\nu$ and $\int_\Omega -u_j\,(dd^c u_j)^n \leq \int_\Omega -u_j\,d\nu \to \int_\Omega -u\,d\nu$, by Proposition 5.2 and (5.1). This implies that

$$\int_\Omega -u\,d\nu \leq \Big(\int_\Omega -u\,(dd^c u)^n\Big)^{\frac{1}{n+1}} \cdot \Big(\int_\Omega -u\,d\nu\Big)^{\frac{n}{n+1}},$$

so $\int_\Omega -u\,d\nu \leq \int_\Omega -u\,(dd^c u)^n$. The opposite inequality follows from Proposition 5.2, so we conclude that $\int_\Omega -u\,d\nu = \int_\Omega -u\,(dd^c u)^n$. Furthermore,

$$\begin{aligned}\int_\Omega u\,d\nu &= \lim_{j\to\infty} \int_\Omega u\,(dd^c u_j)^n = \lim_{j\to\infty} \lim_{k\to\infty} \int_\Omega u_k\,(dd^c u_j)^n \\ &\geq \lim_{k\to\infty} \int_\Omega u_k\,(dd^c u_k)^n \geq \lim_{k\to\infty} \int_\Omega u_k\,d\nu = \int_\Omega u\,d\nu,\end{aligned}$$

where we have used (5.1) in the second and the last equality. The first inequality holds since we may assume that $j \leq k$ and we then know that $(dd^c u_j)^n \preceq (dd^c u_k)^n$, and the second one since $\int_\Omega \varphi\,(dd^c u_k)^n \geq \int_\Omega \varphi\,d\nu$ for each $\varphi \in \mathrm{PSH}^-(\Omega)$ (Proposition 5.2). Hence

$$\int_\Omega u\,(dd^c u)^n = \int_\Omega u\,d\nu = \lim_{k\to\infty} \int_\Omega u_k\,(dd^c u_k)^n,$$

so from (iii) it follows that $\nu = \lim_{j\to\infty}(dd^c u_j)^n = (dd^c u)^n$. Therefore $\int_\Omega \varphi\,(dd^c u_j)$ decreases to $\int_\Omega \varphi\,(dd^c u)^n$ for each $\varphi \in \mathrm{PSH}^-(\Omega)$, by Proposition 5.2, and the proof is complete. ∎

We conclude this paper with a theorem which relates weak*-convergence to the concept of maximal measures defined in Section 4.

THEOREM 5.6. *Suppose that* $\{u_j\}_{j=1}^\infty$ *is a sequence in* $\mathcal{F}(\Omega)$ *such that*

(a) u_j *tends to* $u \in \mathcal{F}(\Omega)$ *in the sense of distributions,*
(b) $(dd^c u)^n$ *is a maximal measure,*
(c) $\lim_{j\to\infty} \int_\Omega (dd^c u_j)^n = \int_\Omega (dd^c u)^n$.

Then $(dd^c u_j)^n$ *tends weak* to* $(dd^c u)^n$.

Proof. Let $\mu_j = (dd^c u_j)^n$ and let $\{\mu_{j_k}\}_{k=1}^\infty$ be any weak*-convergent subsequence. Such a sequence exists because $\sup_j \int_\Omega (dd^c u_j)^n < \infty$ by (c). Denote the limit measure by ν and let $\varphi \in \mathcal{E}_0(\Omega) \cap C(\overline{\Omega})$. Then property (a) implies that

$$\int_\Omega \varphi\,d\nu = \lim_{k\to\infty} \int_\Omega \varphi\,d\mu_{j_k} \leq \int_\Omega \varphi (dd^c u)^n$$

(see proof of Proposition 5.2). Therefore $\nu \succeq (dd^c u)^n$. Moreover, $\int_\Omega d\nu \leq \liminf_{k\to\infty} \int_\Omega d\mu_{j_k} = \int_\Omega (dd^c u)^n$ by (c). Hence $\int_\Omega d\nu = \int_\Omega (dd^c u)^n$, so using (b) we conclude that $\nu = (dd^c u)^n$. Now since $\{\mu_{j_k}\}$ was arbitrary, it follows that the sequence $\{\mu_j\}_{j=1}^\infty$ itself is weak*-convergent to $(dd^c u)^n$. ∎

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