

Tensor-splitting properties of n -inverse pairs of operators

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Abstract. We study n -inverse pairs of operators on the tensor product of Banach spaces. In particular we show that an n -inverse pair of elementary tensors of operators on the tensor product of two Banach spaces can arise *only* from l - and m -inverse pairs of operators on the individual spaces. This gives a converse to a result of Duggal and Müller (2013), and proves a conjecture of the second named author (2015). Our proof uses techniques from algebraic geometry, which generalize to other relations among operators in a tensor product. We apply this theory to obtain results for n -symmetries in a tensor product as well.

1. Introduction. Let $B(X)$ be the algebra of all bounded linear operators on a Banach space X . For $S, T \in B(X)$, we define

$$(1) \quad \beta_n(S, T) = \sum_{k=0}^n (-1)^{n-k} \binom{n}{k} S^k T^k.$$

As in Sid Ahmed [24] and Duggal and Müller [13], we say S is a *left n -inverse* of T (or T is a *right n -inverse* of S , or (S, T) is an *n -inverse pair*) if $\beta_n(S, T) = 0$. If $\beta_n(S, T) = 0$, but $\beta_{n-1}(S, T) \neq 0$, we say S is a *strict left n -inverse* of T . In fact, these definitions make sense for elements S and T in an arbitrary \mathbb{C} -algebra with identity.

This definition is of course a generalization of the definition of an ordinary left inverse—that S is a *left inverse* of T if and only if

$$ST - 1 = 0.$$

Loosely speaking, the expression (1) is obtained by substituting S for x and

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T for y in the expansion

$$(xy - 1)^n = \sum_{k=0}^n (-1)^{n-k} \binom{n}{k} x^k y^k,$$

always keeping powers of S to the left of powers of T .

The concept of n -inverse pairs of operators is motivated by n -isometries studied early in [2, 3, 4, 23] on Hilbert spaces and more recently in [9, 10, 11, 12, 15, 18, 25] on Hilbert spaces and in [6, 8, 16, 20] on Banach spaces. An operator T on a Hilbert space H is called an n -isometry if $\beta_n(T^*, T) = 0$, that is, if T^* is a left n -inverse of T .

If X and Y are Banach spaces, we let $X \overline{\otimes} Y$ denote the completion, endowed with a reasonable uniform cross norm, of the algebraic tensor product $X \otimes Y$ of X and Y . The initial objective of this paper is to prove the following theorem.

THEOREM 1.1. *Suppose $S_1, T_1 \in B(X)$ and $S_2, T_2 \in B(Y)$. Then the following are equivalent:*

- (a) $S_1 \otimes S_2$ is a (strict) left n -inverse of $T_1 \otimes T_2$ in $B(X \overline{\otimes} Y)$.
- (b) There exist positive integers l, m with $l + m = n + 1$ and $\lambda \in \mathbb{C}$ with $\lambda \neq 0$ such that S_1 is a (strict) left l -inverse of λT_1 in $B(X)$ and S_2 is a (strict) left m -inverse of $(1/\lambda)T_2$ in $B(Y)$.

That (b) implies (a) was proved by Duggal and Müller in [13, Theorem 2.3]. A corollary of their result for n -isometric tensor products is proved in [12, Theorem 2.10], which answers questions about m -isometric elementary operators acting on Hilbert–Schmidt operator ideals studied in [9] and [10].

The other implication was conjectured by the second named author in [17, Conjecture 20], and verified for small n and under some technical conditions. With some additional work, we can view [15, Theorem 7] for n -isometric elementary operators as a corollary of this result. An elementary operator (acting on Hilbert–Schmidt operator ideals) of length one is equivalent to the tensor product of two operators [14]. See [15] for more general elementary operators (such as generalized derivations) that are m -isometries.

In this paper, we will prove Theorem 1.1, and extend it to a more general set of relations among elements of a \mathbb{C} -algebra. Specifically, for any polynomial $p(x, y)$, we consider the relation obtained by substituting S for x and T for y into $p(x, y)^n$, always keeping powers of S to the left of powers of T .

Of particular interest are the cases where $p(x, y) = xy - 1$ as already discussed, and where $p(x, y) = x - y$. The latter yields

$$\gamma_n(S, T) = \sum_{k=0}^n (-1)^{n-k} \binom{n}{k} S^k T^{n-k}.$$

This relation is studied in [21] and [22] for bounded operators S and T on a Hilbert space. In this context, we say T is in the n th Helton class of S and write $T \in \text{Helton}_n(S)$ if $\gamma_n(S, T) = 0$. Furthermore, we say T is an n -symmetry if $\gamma_n(T^*, T) = 0$. The n -symmetric operators were introduced and studied in connection with Sturm–Liouville conjugate point theory by Helton [19] and further explored in [5]. They are inspirational in the study of m -isometries and more general hereditary roots in [3] and [25]. Interestingly, we prove in Section 4 that a direct analogue of Theorem 1.1 is possible essentially in exactly the two cases $xy - 1$ and $x - y$, and no others.

We consider several applications of this theory in Sections 5 and 6, and perhaps most interestingly we prove the following pair of theorems. If H and K are Hilbert spaces, we denote by $H \otimes K$ their Hilbert space tensor product.

THEOREM 1.2. *Suppose H and K are Hilbert spaces, $T_1 \in B(H)$ and $T_2 \in B(K)$, and both T_1 and T_2 are left invertible. Then the following are equivalent:*

- (a) $T_1 \otimes T_2$ is an n -symmetry in $B(H \otimes K)$.
- (b) There exist positive integers l, m with $l + m = n + 1$ and $\lambda \in \mathbb{C}$ with $|\lambda| = 1$ such that λT_1 is an l -symmetry in $B(H)$ and $\bar{\lambda} T_2$ is an m -symmetry in $B(K)$.

THEOREM 1.3. *Suppose H and K are Hilbert spaces, $T_1 \in B(H)$ and $T_2 \in B(K)$. Then the following are equivalent:*

- (a) $T_1 \otimes I_K + I_H \otimes T_2$ is an n -symmetry in $B(H \otimes K)$.
- (b) There exist positive integers l, m with $l + m = n + 1$ and $\lambda \in \mathbb{C}$ such that $T_1 + \lambda I_H$ is an l -symmetry in $B(H)$ and $T_2 - \lambda I_K$ is an m -symmetry in $B(K)$.

The outline of this paper is as follows. In Sections 2 and 3, we lay down the algebraic foundation for dealing with expressions such as (1) and show that our problem can be considered in a commutative algebra setting. In Section 4, we take advantage of the commutativity to prove our main theoretical results. In particular, we use Hilbert’s Nullstellensatz extensively, we use the notion of the height of an ideal, and we see that quasi-homogeneous polynomials play a special role. In Section 5, we need to briefly explain how the more general algebra results from previous sections imply Theorem 1.1, and show how the theory applies to n -symmetries and the Helton class of an operator. Finally, in Section 6, we study the nilpotent perturbation of a left n -inverse. In doing so, we see that the algebraic results apply in a much stronger way for n -symmetries, leading to the proof of Theorem 1.3.

2. Definitions and algebraic foundation. Let \mathbb{C} denote the field of complex numbers, and let \mathbb{C}^* denote the set of nonzero complex numbers. For us, a \mathbb{C} -algebra A is a complex vector space which is also an algebra with an identity. For elements S, T of a \mathbb{C} -algebra A , we define $\beta_n(S, T)$ and n -inverses as in the introduction.

Note that if S is a left n -inverse of T , then S is a left m -inverse of T for all $m \geq n$. This follows from the recursive formula

$$(2) \quad \beta_n(S, T) = S\beta_{n-1}(S, T)T - \beta_{n-1}(S, T).$$

It is also true that if T has a left n -inverse for some n , then T is left-invertible, and T would therefore be left m -invertible for all m . This follows from (2) and

$$\beta_n(S, T) = \left(\sum_{k=1}^n (-1)^{n-k} \binom{n}{k} S^k T^{k-1} \right) T + (-1)^n.$$

Because of this fact, we avoid the term n -invertible even though it appears in the literature. We also define $\gamma_n(S, T)$ as in the introduction.

We begin by giving a general algebraic formalism for the construction of expressions such as $\beta_n(S, T)$ and $\gamma_n(S, T)$. In particular, we will make precise our loose explanation in the introduction that “powers of S are kept to the left of powers of T ”. This is important, for example, because if A is noncommutative, then of course $\beta_n(S, T)$ may not be equal to $(ST - 1)^n$. To deal with this discrepancy, we define a vector space homomorphism from the free commutative \mathbb{C} -algebra on x, y to the free \mathbb{C} -algebra on X, Y :

$$\Phi : \mathbb{C}[x, y] \rightarrow \mathbb{C}\langle X, Y \rangle$$

defined on the monomial basis of $\mathbb{C}[x, y]$ by

$$\Phi(x^i y^j) = X^i Y^j.$$

Here, $\mathbb{C}[x, y]$ is the commutative \mathbb{C} -algebra of formal polynomials in two *commuting* variables x, y , and $\mathbb{C}\langle X, Y \rangle$ is the \mathbb{C} -algebra of formal polynomials in two *noncommuting* variables X, Y (i.e. formal linear combinations of words in X, Y). In what follows, set

$$R = \mathbb{C}[x, y], \quad F = \mathbb{C}\langle X, Y \rangle.$$

Again, Φ is only a vector space homomorphism and not a \mathbb{C} -algebra homomorphism, so the multiplicative structure is not preserved. For example, $yx = xy$ in R , but

$$\Phi(yx) = XY \neq YX = \Phi(y)\Phi(x).$$

However, Φ is exactly the map we need to construct expressions like β_n and γ_n because

$$\Phi((xy - 1)^n) = \beta_n(X, Y), \quad \Phi((x - y)^n) = \gamma_n(X, Y).$$

Since F is a free object in the category of \mathbb{C} -algebras, for any \mathbb{C} -algebra A and $S, T \in A$, there is a unique \mathbb{C} -algebra homomorphism $\kappa : F \rightarrow A$ such that

$$\kappa(X) = S, \quad \kappa(Y) = T.$$

For a given element $\omega \in F$, we may write $\omega(S, T)$ for $\kappa(\omega)$. With this notation, S is a left n -inverse of T if and only if

$$\Phi((xy - 1)^n)(S, T) = 0.$$

Although Φ is not a \mathbb{C} -algebra homomorphism, it does behave like one in a crucial way shown in the following proposition.

PROPOSITION 2.1. *The inverse image of any two-sided ideal in $\mathbb{C}\langle X, Y \rangle$ under Φ is an ideal in $\mathbb{C}[x, y]$.*

Proof. Let \mathcal{I} be any two-sided ideal in $\mathbb{C}\langle X, Y \rangle$. Since Φ is a vector space homomorphism, it will suffice to check that $\Phi^{-1}(\mathcal{I})$ is closed under multiplication by a monomial.

Suppose $f(x, y) = \sum k_{ij}x^i y^j$ is such that $f \in \Phi^{-1}(\mathcal{I})$, where all but finitely many k_{ij} are zero. Then for any $a, b \geq 0$, we have

$$\begin{aligned} \Phi(x^a y^b \cdot f(x, y)) &= \Phi\left(\sum k_{ij}x^{i+a}y^{j+b}\right) \\ &= \sum k_{ij}X^{i+a}Y^{j+b} \\ &= X^a \cdot \left(\sum k_{ij}X^i Y^j\right) \cdot Y^b. \end{aligned}$$

The last line is clearly in \mathcal{I} , so $x^a y^b \cdot f(x, y)$ is in $\Phi^{-1}(\mathcal{I})$. ■

Notice that the proof of Proposition 2.1 would not go through with three or more variables, and indeed the conclusion would not hold. If $\Psi : \mathbb{C}[x, y, z] \rightarrow \mathbb{C}\langle X, Y, Z \rangle$ is the analogous vector space homomorphism for three variables, and $\mathcal{I} \subset \mathbb{C}\langle X, Y, Z \rangle$ is the two-sided ideal generated by XZ , then $xz \in \Psi^{-1}(\mathcal{I})$ but $xyz \notin \Psi^{-1}(\mathcal{I})$.

LEMMA 2.2. *Let I be an ideal in $\mathbb{C}[x, y]$. The inverse image under Φ of the ideal generated by $\Phi(I)$ is equal to I .*

Proof. This follows immediately from the observation that

$$\Phi(I) = \Phi(R) \cap \langle \Phi(I) \rangle. \quad \blacksquare$$

3. Tensor products. We investigate how relations in the tensor product of two possibly noncommutative \mathbb{C} -algebras descend to relations in the individual \mathbb{C} -algebras and vice versa. In this section, we will build a general framework which will allow us to use Proposition 2.1 to convert such questions to the commutative algebra setting. Our motivation lies in proving

Theorem 1.1, but in this section we maintain a much more general perspective that can be applied to other relations such as $\gamma_n(S, T)$, and also to nilpotent perturbations in Section 6.

In what follows, suppose A_1 and A_2 are \mathbb{C} -algebras, and $S_i, T_i \in A_i$. Also, define

$$\kappa_i : F \rightarrow A_i, \quad i = 1, 2, \quad X \mapsto S_i, \quad Y \mapsto T_i.$$

Now let

$$\delta : R \rightarrow R \otimes R$$

be any injective \mathbb{C} -algebra homomorphism. Letting $S = \mathbb{C}[x_1, y_1, x_2, y_2]$, we also recall the \mathbb{C} -algebra isomorphism

$$\mu : R \otimes R \rightarrow S, \quad x^\alpha y^\beta \otimes x^\gamma y^\delta \mapsto x_1^\alpha y_1^\beta x_2^\gamma y_2^\delta.$$

We then define ϵ and κ so that the following diagram commutes:

$$(3) \quad \begin{array}{ccccc} R & \xrightarrow{\Phi} & F & & \\ \downarrow \delta & & \downarrow \epsilon & \searrow \kappa & \\ S & \xleftarrow{\mu} & R \otimes R & \xrightarrow{\Phi \otimes \Phi} & F \otimes F & \xrightarrow{\kappa_1 \otimes \kappa_2} & A_1 \otimes A_2 \end{array}$$

In particular,

$$\epsilon(X) := (\Phi \otimes \Phi)\delta(x), \quad \epsilon(Y) := (\Phi \otimes \Phi)\delta(y), \quad \kappa := (\kappa_1 \otimes \kappa_2)\epsilon.$$

Recall the following linear algebra fact about tensor products.

LEMMA 3.1. *If $\phi_1 : V_1 \rightarrow W_1$ and $\phi_2 : V_2 \rightarrow W_2$ are vector space homomorphisms, then*

$$\ker(\phi_1 \otimes \phi_2) = (\ker(\phi_1) \otimes V_2) + (V_1 \otimes \ker(\phi_2)).$$

Proof. This is a standard result, but we include the proof for completeness. Let $V_i = V'_i \oplus \ker(\phi_i)$ be a splitting of the epimorphism $\phi_i : V_i \rightarrow \text{im}(\phi_i)$ for $i = 1, 2$. Then

$$V_1 \otimes V_2 = (V'_1 \otimes V'_2) \oplus (V'_1 \otimes \ker(\phi_2)) \oplus (\ker(\phi_1) \otimes V'_2) \oplus (\ker(\phi_1) \otimes \ker(\phi_2)).$$

However, the restriction $\phi_1 \otimes \phi_2 : V'_1 \otimes V'_2 \rightarrow \text{im}(\phi_1) \otimes \text{im}(\phi_2)$ is an isomorphism because of the splitting. Hence

$$\begin{aligned} \ker(\phi_1 \otimes \phi_2) &= (V'_1 \otimes \ker(\phi_2)) \oplus (\ker(\phi_1) \otimes V'_2) \oplus (\ker(\phi_1) \otimes \ker(\phi_2)) \\ &= (\ker(\phi_1) \otimes V_2) + (V_1 \otimes \ker(\phi_2)). \quad \blacksquare \end{aligned}$$

For any ideal $I \subseteq R$, we write

$$I' = \mu(I \otimes R), \quad I'' = \mu(R \otimes I)$$

for the corresponding ideals in S . Let

$$I := \Phi^{-1}(\ker \kappa_1), \quad J := \Phi^{-1}(\ker \kappa_2),$$

so that I and J are ideals in R by Proposition 2.1. We will also define for any $p \in R$,

$$\hat{p} := \mu\delta(p).$$

In what follows, we adopt the above notation under the assumptions that A_1, A_2 are \mathbb{C} -algebras, $S_i, T_i \in A_i$ for $i = 1, 2$, and that $\delta : R \rightarrow R \otimes R$ is any injective \mathbb{C} -algebra homomorphism.

The following theorem allows us to prove facts about relations in $A_1 \otimes A_2$ in the commutative algebra S .

THEOREM 3.2. *Let $p \in R$. Then $\hat{p} \in I' + J''$ if and only if $\Phi(p) \in \ker \kappa$.*

Proof. By Lemma 3.1, we have

$$\ker(\kappa_1\Phi \otimes \kappa_2\Phi) = I \otimes R + R \otimes J.$$

Hence by the commutativity of (3), $\Phi(p) \in \ker \kappa$ if and only if

$$(4) \quad \delta(p) \in I \otimes R + R \otimes J.$$

Thanks to the isomorphism μ , (4) is equivalent to $\hat{p} \in I' + J''$. ■

The following two results will be used in proving all of our major theorems.

LEMMA 3.3. *Suppose that $p, q_1, q_2 \in R$ and*

$$\hat{p} \in \langle q_1(x_1, y_1), q_2(x_2, y_2) \rangle.$$

If $\Phi(q_1^l) \in \ker \kappa_1$, $\Phi(q_2^m) \in \ker \kappa_2$, then $\Phi(p^n) \in \ker \kappa$, where $n = l + m - 1$.

Proof. By assumption, there exist $f, g \in S$ such that

$$(5) \quad \hat{p}^n = \sum_{k=0}^n \binom{n}{k} f^{n-k} q_1(x_1, y_1)^{n-k} g^k q_2(x_2, y_2)^k.$$

Notice that every summand of (5) is in either I' or J'' . Thus $\Phi(p^n) \in \ker \kappa$. ■

The following theorem is a variation on the above lemma, which operates under more technical conditions. However, it will allow us to obtain sharp “if and only if” statements like Theorems 1.1–1.3.

THEOREM 3.4. *Suppose that $p, q_1, q_2 \in R$ are such that there exist $f, g \in R$ satisfying:*

- (i) $\hat{p}(x_1, x_2, y_1, y_2) = f(x_2, y_2)q_1(x_1, y_1) + g(x_1, y_1)q_2(x_2, y_2)$.
- (ii) $\Phi(q_1^l) \in \ker \kappa_1$ and $\Phi(q_2^m) \in \ker \kappa_2$ for large enough l and m .
- (iii) For any i, j , $f^i q_2^j \in J$ if and only if $q_2^j \in J$, and $g^i q_1^j \in I$ if and only if $q_1^j \in I$.

Then $\Phi(p^n) \in \ker \kappa$ if and only if there exist l and m so that $l + m = n + 1$ and $\Phi(q_1^l) \in \ker \kappa_1$ and $\Phi(q_2^m) \in \ker \kappa_2$.

Proof. The “if” part follows directly from Lemma 3.3. In fact, if we assume that l, m are minimal such that $\Phi(q_1^l) \in \ker \kappa_1$ and $\Phi(q_2^m) \in \ker \kappa_2$, then applying (5) with $n = l + m - 1$, we obtain

$$\hat{p}^{l+m-2} = f(x_2, y_2)^{l-1} q_1(x_1, y_1)^{l-1} g(x_1, y_1)^{m-1} q_2(x_2, y_2)^{m-1} \text{ in } S/(I' + J'').$$

Applying μ^{-1} to both sides of the expression, we get

$$\delta(p^{l+m-2}) = g^{m-1} q_1^{l-1} \otimes f^{l-1} q_2^{m-1} \text{ in } R/I \otimes R/J.$$

By (iii), we see that $g^{m-1} q_1^{l-1} \notin I$ and $f^{l-1} q_2^{m-1} \notin J$, so by the commutativity of (3), $\Phi(p^{l+m-2}) \notin \ker \kappa$.

Thus if we assume $\Phi(p^n) \in \ker \kappa$, the minimal l, m chosen above must add to no more than $n + 1$, proving the “only if” part of the theorem. ■

REMARK 3.5. Condition (iii) of Theorem 3.4 is a rather technical one, and throughout the exposition we take care to include assumptions that ensure it will be satisfied. However, we note that (iii) is satisfied when f, q_2 generate R and g, q_1 generate R . This follows from Lemma 3.6 below.

LEMMA 3.6. *Suppose $g \in R$, and $I \subseteq R$ is an ideal such that $g^m \in I$, but $g^{m-1} \notin I$. If $f \in R$ is such that f, g generate R , then for any $k \geq 0$, $f^k g^{m-1} \notin I$.*

Proof. Since f, g generate R , there exist α, β such that $\alpha f + \beta g = 1$. By way of contradiction, let k be minimal such that $f^k g^{m-1} \in I$. Then

$$\alpha f^k g^{m-1} + \beta f^{k-1} g^m = (\alpha f + \beta g) f^{k-1} g^{m-1} = f^{k-1} g^{m-1}.$$

But since $g^m \in I$, both sides of the equation must be in I , violating the minimality of k . ■

The following lemma, which relies on Hilbert’s Nullstellensatz, will assist us in showing that condition (ii) of Theorem 3.4 is satisfied in all of its applications in later sections.

For an ideal $I \subseteq \mathbb{C}[x_1, \dots, x_n]$, we write $V(I)$ for the variety along which the ideal vanishes. That is,

$$V(I) = \{\mathbf{a} \in \mathbb{C}^n \mid f(\mathbf{a}) = 0 \forall f \in I\}.$$

We also denote by \sqrt{I} the radical of the ideal I . That is,

$$\sqrt{I} = \{f \in I \mid f^L \in I \text{ for some } L > 0\}.$$

LEMMA 3.7. *Suppose $\hat{p} \in \sqrt{I' + J''}$ in S . Then for any $(a, b) \in V(I)$, $\hat{p}(a, b, x, y) \in \sqrt{J}$ in R , and for any $(c, d) \in V(J)$, $\hat{p}(x, y, c, d) \in \sqrt{I}$ in R .*

Proof. Notice that

$$V(I' + J'') = \{(a, b, c, d) \mid (a, b) \in V(I), (c, d) \in V(J)\}.$$

Since $\hat{p} \in \sqrt{I' + J''}$, \hat{p} vanishes at every point of $V(I' + J'')$. So if $(a, b) \in V(I)$, then $p(a, b, c, d) = 0$ for all $(c, d) \in V(J)$, and thus by Hilbert's Nullstellensatz, $p(a, b, x, y) \in \sqrt{J}$. Similarly, if $(c, d) \in V(J)$, $p(x, y, c, d) \in \sqrt{I}$. ■

4. Tensor-splitting properties. In this section, we will apply the results of Section 3 to the case where $\delta : R \rightarrow R \otimes R$ is defined by

$$(6) \quad \delta(x) = x \otimes x, \quad \delta(y) = y \otimes y.$$

Applying the definitions of Section 3, we see that $\epsilon(X) = X \otimes X$ and $\epsilon(Y) = Y \otimes Y$, and $\kappa(\omega) = \omega(S_1 \otimes S_2, T_1 \otimes T_2)$. In particular,

$$\kappa\Phi((xy - 1)^n) = \beta_n(S_1 \otimes S_2, T_1 \otimes T_2).$$

We introduce the notation

$$p_{\lambda, \mu}(x, y) := p(\lambda x, \mu y), \quad p_\lambda(x, y) := p(x, \lambda y)$$

for $p(x, y) \in R$, $\lambda, \mu \in \mathbb{C}$. Notice that

$$p_{a,b}(x, y) = \hat{p}(x, y, a, b) = \hat{p}(a, b, x, y).$$

Now proving Theorem 1.1 entails showing that, for $p(x, y) = xy - 1$,

$$(7) \quad \Phi(p^n) \in \ker \kappa \Leftrightarrow \begin{array}{l} \Phi(p_\lambda^l) \in \ker \kappa_1 \text{ and } \Phi(p_{1/\lambda}^m) \in \ker \kappa_2 \\ \text{for some } l, m > 0 \text{ with } l + m = n + 1, \lambda \in \mathbb{C}^*. \end{array}$$

It is natural to ask if (7), or some weaker version, is true for other polynomials $p \in R$. We therefore make the following definitions.

DEFINITION 4.1. Suppose $p \in R$. If for every pair of \mathbb{C} -algebras A_1, A_2 , and every $S_i, T_i \in A_i$ (not both zero), $i = 1, 2$,

$$(8) \quad \Phi(p^n)(S_1 \otimes S_2, T_1 \otimes T_2) = 0 \Rightarrow \begin{array}{l} \Phi(p_{a,b}^l)(S_1, T_1) = 0, \\ \Phi(p_{c,d}^m)(S_2, T_2) = 0 \\ \text{for some } a, b, c, d \in \mathbb{C} \\ \text{and large enough } l, m, \end{array}$$

then we say p has a *weak tensor-splitting property*.

If additionally we can replace the phrase “large enough l, m ” in (8) with “some positive l, m with $l + m = n + 1$ ” whenever S_i is left-invertible and T_i is right-invertible, then we say p has a *strong tensor-splitting property*.

DEFINITION 4.2. Suppose $p \in R$. If for every pair of \mathbb{C} -algebras A_1, A_2 , and every $S_i, T_i \in A_i$, $i = 1, 2$,

$$(9) \quad \begin{array}{l} \Phi(p^l)(S_1, T_1) = 0, \\ \Phi(p^m)(S_2, T_2) = 0 \end{array} \Rightarrow \begin{array}{l} \Phi(p^n)(S_1 \otimes S_2, T_1 \otimes T_2) = 0 \\ \text{for some large enough } n, \end{array}$$

then we say p has a *weak tensor-product property*.

If additionally we can replace the phrase “large enough n ” in (9) with “ $n = l + m - 1$ ”, then we say p has a *strong tensor-product property*.

In what follows, we use Theorem 3.2 to characterize polynomials with the above properties by working in the commutative algebra setting. Note that applying the notation of Section 3 to our situation where δ is defined in (6), we have

$$\hat{p}(x_1, y_1, x_2, y_2) = p(x_1x_2, y_1y_2).$$

Our first main result is the following.

THEOREM 4.3. *Every $p(x, y) \in \mathbb{C}[x, y]$ satisfies a weak tensor-splitting property.*

Proof. Suppose $\Phi(p^n) \in \ker \kappa$. By Theorem 3.2, this means $\hat{p}^n \in I' + J''$, where $I = \Phi^{-1}(\ker \kappa_1)$ and $J = \Phi^{-1}(\ker \kappa_2)$. Since S_i, T_i are assumed not to be both zero, I and J are proper ideals of R , and so $V(I)$ and $V(J)$ are nonempty by the Weak Nullstellensatz. Applying Lemma 3.7, we see that there exist a, b, c, d such that $p_{c,d} \in \sqrt{I}$ and $p_{a,b} \in \sqrt{J}$, and thus (again by Theorem 3.2) $\Phi(p_{c,d}^l) \in \ker \kappa_1$ and $\Phi(p_{a,b}^m) \in \ker \kappa_2$ for large enough l, m . ■

Theorem 4.3 may seem surprisingly strong, but as we shall see in Corollary 4.6, it turns out that the assumption $\hat{p} \in \sqrt{I' + J''}$ is quite restrictive for most polynomials p .

The *height* of a prime ideal \mathfrak{p} is the length n of a maximal ascending chain of prime ideals $\mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_n = \mathfrak{p}$. The height of any ideal I is the infimum of the heights of prime ideals containing I . If $I \subseteq \mathbb{C}[x, y]$, then the height of I is also equal to the complex codimension of $V(I)$ in \mathbb{C}^2 . The only height-0 ideal is all of R , and height-2 ideals in R are ones large enough that their vanishing set is a finite (or empty) set of points.

If A is any \mathbb{C} -algebra, we say $S \in A$ is *algebraic* if S satisfies a polynomial relation, that is, there exist scalars c_i such that $c_n S^n + \cdots + c_1 S + c_0 1 = 0$. In an operator algebra, the algebraic elements are called *algebraic operators*. In this sense, algebraic operators behave like finite matrices.

PROPOSITION 4.4. *Let A be any \mathbb{C} -algebra, and suppose $\kappa : F \rightarrow A$ is a \mathbb{C} -algebra homomorphism as before. Let $S = \kappa(X)$, $T = \kappa(Y)$. If $\Phi^{-1}(\ker \kappa)$ has height 2, then S and T are both algebraic.*

Proof. Suppose $I = \Phi^{-1}(\ker \kappa)$ has height 2 so that $V(I)$ is a finite collection of points $p_i = (a_i, b_i)$, $i = 1, \dots, r$. Hence \sqrt{I} is the product of the maximal ideals $\mathfrak{m}_{p_i} = \langle x - a_i, y - b_i \rangle$, so I contains $\alpha(x, y) = (x - a_1)^k \cdots (x - a_r)^k$ and $\beta(x, y) = (y - b_1)^k \cdots (y - b_r)^k$ for some k . But then $\Phi(\alpha), \Phi(\beta) \in \ker \kappa$, so $\Phi(\alpha)(S) = 0$ and $\Phi(\beta)(T) = 0$, and thus S and T are algebraic. ■

Because of Proposition 4.4, the only “interesting” (i.e. not both algebraic) pairs S, T in an algebra A are ones for which I has height 1. In what follows, we characterize polynomials $p \in R$ for which a height-1 ideal may arise in our context.

A polynomial $q(x_1, \dots, x_n) \in \mathbb{C}[x_1, \dots, x_n]$ is called *quasi-homogeneous* if there exist coprime integers w_1, \dots, w_n called *weights* and d called the *quasi-degree* such that for any $\lambda \in \mathbb{C}^*$,

$$q(\lambda^{w_1}x_1, \dots, \lambda^{w_n}x_n) = \lambda^d q(x_1, \dots, x_n).$$

Equivalently, if we write

$$q(x_1, \dots, x_n) = \sum_{i_1, \dots, i_n} k_{i_1, \dots, i_n} x_1^{i_1} \cdots x_n^{i_n},$$

then $k_{i_1, \dots, i_n} \neq 0$ implies $w_1 i_1 + \cdots + w_n i_n = d$. Note that some authors restrict to the case where w_1, \dots, w_n, d are all positive, but we do not make that part of the definition.

PROPOSITION 4.5. *Let $p(x, y) \in R$ be irreducible. Then the following are equivalent:*

- (a) p is quasi-homogeneous.
- (b) $p(x, y) = A(x^\alpha y^\beta - B)$ or $p(x, y) = A(Bx^\alpha - y^\beta)$ for some coprime positive integers α and β and constants $A, B \in \mathbb{C}$.
- (c) $\hat{p} \in \langle p \rangle' + \langle p \rangle''$.
- (d) There exist height-1 ideals $I, J \subset R$ such that $\hat{p} \in \sqrt{I' + J''}$.
- (e) p has a strong tensor-product property.
- (f) p has a weak tensor-product property.

Proof. That (a) is equivalent to (b) is a standard result, and we omit the proof.

If $\alpha = \beta = 1$ and $A = B = 1$, then the implication (b) \Rightarrow (c) follows from

$$(10) \quad \begin{aligned} x_1 x_2 y_1 y_2 - 1 &= (x_1 y_1 - 1) x_2 y_2 + (x_2 y_2 - 1), \\ x_1 x_2 - y_1 y_2 &= (x_1 - y_1) x_2 + (x_2 - y_2) y_1. \end{aligned}$$

The general case follows from substituting x^α for x and y^β/B for y .

Condition (c) clearly implies (d), and (e) clearly implies (f). By Lemma 3.3, (c) implies (e).

If p has a weak tensor-product property, then let $A_1 = A_2 = F/\Phi(p)$, with canonical maps $\kappa_i : F \rightarrow A_i$. Then $I = J = \langle p \rangle$ are height-1 ideals, and since $\Phi(p) \in \ker \kappa_i$ for $i = 1, 2$, the weak tensor-product property shows that $\hat{p} \in \sqrt{I' + J''}$. Thus (f) implies (d).

Finally, we must prove (d) implies (a). Suppose I and J are height-1 ideals in R and $\hat{p}^n \in I' + J''$. First, note that if $p(x, y) = x$ or $p(x, y) = y$, then p is quasi-homogeneous. Otherwise, the vanishing set $V(J)$ contains some (c, d) with c, d both nonzero. Then by Lemma 3.7, we have $p_{c,d} \in \sqrt{I}$.

Since I has height 1, also \sqrt{I} has height 1, and $\langle p_{c,d} \rangle \subseteq \sqrt{I}$. But since $p_{c,d}$ is irreducible, $\langle p_{c,d} \rangle$ is prime of height 1, so $\sqrt{I} = \langle p_{c,d} \rangle$, and $V(I)$ is the codimension-1 set of (a, b) such that $p_{c,d}(a, b) = 0$.

Now suppose $(a_0, b_0) \in V(I)$ with a_0, b_0 nonzero, and let

$$p(x, y) = \sum_{i,j} k_{ij} x^i y^j$$

(where all but finitely many k_{ij} are zero). Let $(a, b) \in V(I)$ be such that the constants $\lambda = a/a_0$ and $\mu = b/b_0$ are nonzero and have modulus other than 1.

Then since $p_{a,b}$ and p_{a_0,b_0} are both irreducible and each generate \sqrt{J} , the polynomials must be constant multiples of one another. Hence, there is a constant M such that if $k_{i,j} \neq 0$, then $M a_0^i b_0^j = a^i b^j$, that is, $M = \lambda^i \mu^j$. Therefore, if $k_{i_1 j_1}$ and $k_{i_2 j_2}$ are both nonzero, then $\lambda^{i_1} \mu^{j_1} = \lambda^{i_2} \mu^{j_2}$, so $\lambda^{i_1 - i_2} = \mu^{j_2 - j_1}$. Let α, β be coprime integers such that $\alpha(i_1 - i_2) = \beta(j_2 - j_1)$. Then by the Chinese Remainder Theorem, there exists η such that $\eta^\alpha = \lambda$ and $\eta^\beta = \mu$. Letting $d = \alpha i_1 + \beta j_1$, we have $M = \eta^d$, and so for any i, j with $k_{i,j} \neq 0$, $\eta^{\alpha i + \beta j} = \eta^d$; since $|\eta| \neq 1$, we get $d = \alpha i + \beta j$. Therefore p_{a_0, b_0} and, as a corollary, p are quasi-homogeneous with weights α and β and quasi-degree d . ■

COROLLARY 4.6. *If p is irreducible and not quasi-homogeneous, and*

$$(11) \quad \Phi(p^n)(S_1 \otimes S_2, T_1 \otimes T_2) = 0,$$

then either S_1 and T_1 are both algebraic, or S_2 and T_2 are both algebraic.

The authors view Corollary 4.6 as an indication that unless p is quasi-homogeneous, condition (11) is in some sense too strong to be useful. In particular, if p is not quasi-homogeneous, then (11) implies that one of the two pairs of operators S_i, T_i was uninteresting in the sense that both operators were algebraic. This also sheds new light on Theorem 4.3; the weak tensor-splitting property of p may simply arise as a by-product of the implied algebraicness of the original operators. For this reason, we consider the tensor-splitting property to really only be meaningful in the case where p is quasi-homogeneous.

THEOREM 4.7. *Any irreducible quasi-homogeneous polynomial in two variables has a strong tensor-splitting property. Specifically, suppose that $p(x, y) = A(x^\alpha y^\beta - B)$ or $p(x, y) = A(Bx^\alpha - y^\beta)$ and let η be such that $\eta^\beta = B$; then if*

$$\Phi(p^n)(S_1 \otimes S_2, T_1 \otimes T_2) = 0$$

and S_i is left-invertible and T_i is right-invertible for $i = 1, 2$, then there exist $\lambda \in \mathbb{C}^$ and $l, m \geq 0$ with $l + m \leq n + 1$ such that*

$$\Phi(p_\lambda^l)(S_1, T_1) = 0, \quad \Phi(p_{\eta/\lambda}^m)(S_2, T_2) = 0.$$

Proof. We will restrict to the cases $p(x, y) = x - y$ and $p(x, y) = xy - 1$. To go to the general cases, one can substitute x^α and y^β/B for x and y respectively.

Suppose $\Phi(p^n) \in \ker \kappa$. Since we can assume that S_i, T_i both have one-sided inverses in A_i for $i = 1, 2$, we can conclude that $I = \Phi^{-1}(\ker \kappa_1)$ and $J = \Phi^{-1}(\ker \kappa_2)$ are both proper ideals, and there exist points $(a, b) \in V(I)$ and $(c, d) \in V(J)$ such that a, b, c, d are all nonzero.

We will now prove that $p, p_\lambda, p_{1/\lambda}$ satisfy the conditions of Theorem 3.4, whose conclusion is exactly what we are trying to prove. The following variation on (10) shows (i), i.e. $\hat{p} \in \langle p_\lambda, p_{1/\lambda} \rangle$:

$$\begin{aligned} x_1x_2y_1y_2 - 1 &= (x_2y_2/\lambda)(\lambda x_1y_1 - 1) + 1 \cdot (x_2y_2/\lambda - 1), \\ x_1x_2 - y_1y_2 &= x_2(x_1 - \lambda y_1) + \lambda y_1(x_2 - y_2/\lambda). \end{aligned}$$

We also just showed (ii), i.e. $p_\lambda^L \in I$ and $p_{1/\lambda}^M \in J$.

For (iii), we treat the two cases of $xy - 1$ and $x - y$ separately. If $p(x, y) = xy - 1$, then since $xy/\lambda, xy/\lambda - 1$ generate R , as do $1, \lambda xy - 1$, we may apply Remark 3.6.

If $p(x, y) = x - y$, we prove (iii) directly. Suppose $(\lambda y)^i(x - \lambda y)^j \in I$. Then

$$\begin{aligned} 0 &= \kappa_2 \Phi((\lambda y)^i(x - \lambda y)^j) = \lambda^i \kappa_2(\Phi((x - \lambda y)^j)Y^i) \\ &= \lambda^i \kappa_2(\Phi((x - \lambda y)^j))T_2^i. \end{aligned}$$

Since T_2 is right-invertible, this implies $\Phi((x - \lambda y)^j) \in \ker \kappa_2$, so $(x - \lambda y)^j \in I$. By a similar argument, if $x^i(x - y/\lambda)^j \in J$, then $(x - y/\lambda)^j \in J$. ■

REMARK 4.8. Note that as long as A_1, A_2 are nonzero, we do not need to include the invertibility of S_i, T_i as a separate condition for the case of $p(x, y) = xy - 1$. Indeed, the only time this condition is invoked in the proof is to show that a, b, c, d are nonzero, but this is implied by the fact that $(abxy - 1)^L \in I$ and $(cdxy - 1)^M \in J$ and that I, J are proper ideals.

5. Operator-theoretic results. We now briefly explain the operator-theoretic consequences of the results of the previous section.

LEMMA 5.1. *For Banach algebras B_1, B_2 , we let*

$$\iota : B_1 \otimes B_2 \rightarrow B_1 \overline{\otimes} B_2$$

be the inclusion of the algebraic tensor product into its completion with respect to some reasonable uniform cross norm. Then for any $\omega \in F$ and $\alpha, \beta \in B_1 \otimes B_2$, we have $\omega(\alpha, \beta) = 0$ if and only if $\iota(\omega(\alpha, \beta)) = 0$.

Proof. Follows immediately from the injectivity of ι . ■

The above lemma may be somewhat trivial, but for us it means that results from earlier sections regarding the algebraic tensor product of

\mathbb{C} -algebras apply equally well to the (completed) tensor product of Banach algebras.

Proof of Theorem 1.1. Note that for Banach spaces X and Y , $B(X)$ and $B(Y)$ are nonzero \mathbb{C} -algebras with identities. The Banach algebra $B(X \overline{\otimes} Y)$ is isomorphic to a completion of $B(X) \otimes B(Y)$ with respect to a reasonable uniform cross norm.

Thus, if we apply Theorem 4.7 with $A_1 = B(X)$ and $A_2 = B(Y)$, the algebraic assumption that

$$\Phi((xy - 1)^n)(S_1 \otimes S_2, T_1 \otimes T_2) = 0$$

in $B(X) \otimes B(Y)$ is equivalent to the same condition in $B(X \overline{\otimes} Y)$ by Lemma 5.1, and so the same conclusion holds.

We remark that the statement of Theorem 1.1 also holds when the parenthetical strictness condition is added to both (a) and (b). This follows from the original statement of Theorem 1.1 along with the recursive condition in (2). ■

We now move on to some applications of the theory to $p(x, y) = x - y$. Recall from the introduction that Helton classes were defined for operators on a Hilbert space, but the definition extends easily to arbitrary \mathbb{C} -algebras. In particular if S, T are elements of an arbitrary \mathbb{C} -algebra A , we say $T \in \text{Helton}_n(S)$ if

$$\Phi((x - y)^n)(S, T) = 0.$$

THEOREM 5.2. *Assume that $S_1, T_1 \in B(X)$ and $S_2, T_2 \in B(Y)$, and that S_1, S_2 are left-invertible, and T_1, T_2 are right-invertible. Then the following are equivalent:*

- (a) *The tensor product $T_1 \otimes T_2$ on $X \overline{\otimes} Y$ belongs to $\text{Helton}_n(S_1 \otimes S_2)$.*
- (b) *There exist m, l and $\lambda \in \mathbb{C}^*$ such that $m + l = n + 1$, $\lambda T_1 \in \text{Helton}_l(S_1)$ and $(1/\lambda)T_2 \in \text{Helton}_m(S_2)$.*

Proof. Using the same logic as in the proof of Theorem 1.1, we can apply Theorem 4.7 and Lemma 5.1 with $p(x, y) = x - y$. ■

In preparation for the proof of Theorem 1.2, we recall the definition of an n -symmetry from the introduction: if H is a Hilbert space, we say $T \in B(H)$ is an n -symmetry if

$$\gamma_n(T^*, T) = \Phi((x - y)^n)(T^*, T) = 0.$$

In what follows, we write $\sigma(T)$ and $\sigma_{\text{ap}}(T)$ for the spectrum and approximate point spectrum of T respectively.

LEMMA 5.3. *If T is not nilpotent and $\lambda T \in \text{Helton}_n(T^*)$, then $|\lambda| = 1$.*

Proof. Let r be the spectral radius of T , so that there exists $\rho \in \sigma_{\text{ap}}(T)$ with $|\rho| = r$. Note that by nonnilpotency, $\rho \neq 0$. Then there exists a sequence

of unit vectors $h_i \in H$ such that $\|(T - \rho I_H)h_i\| \rightarrow 0$ as $i \rightarrow 0$. We have

$$\begin{aligned} \langle \gamma_n(T^*, \lambda T)h_i, h_i \rangle &= \sum_{k=0}^n (-1)^{n-k} \binom{n}{k} \langle (T^*)^k (\lambda T)^{n-k} h_i, h_i \rangle \\ &= \sum_{k=0}^n (-1)^{n-k} \binom{n}{k} \langle (\lambda T)^{n-k} h_i, T^k h_i \rangle. \end{aligned}$$

Thus, as $i \rightarrow \infty$,

$$\begin{aligned} \langle \gamma_n(T^*, \lambda T)h_i, h_i \rangle &\rightarrow \sum_{k=0}^n (-1)^{n-k} \binom{n}{k} \langle (\lambda \rho)^{n-k} h_i, \rho^k h_i \rangle \\ &= \sum_{k=0}^n (-1)^{n-k} \binom{n}{k} (\lambda \rho)^{n-k} \bar{\rho}^k \langle h_i, h_i \rangle \\ &= (\lambda \rho - \bar{\rho})^n. \end{aligned}$$

However, by assumption, $\gamma_n(T^*, \lambda T) = 0$ and $\rho \neq 0$, so $|\lambda| = 1$. ■

Proof of Theorem 1.2. We can apply Theorem 5.2 directly to show that (b) implies (a) in Theorem 1.2. To prove the converse, assume $T_1 \otimes T_2$ is an n -symmetry, so that Theorem 5.2 tells us

$$\Phi((x - y)^l)(T_1^*, \lambda T_1) = 0, \quad \Phi((x - y)^m)(T_2^*, (1/\lambda)T_2) = 0.$$

Thus it only remains to be shown that $|\lambda| = 1$, but this is the content of Lemma 5.3. ■

6. Nilpotent perturbation of a left n -inverse. Inspired by the construction of n -isometries using the sum of isometries and nilpotent operators in [1] and [7], the second named author proved in [17, Theorem 2] that if S is a left m -inverse of T and if Q is a nilpotent operator of order l such that $QS = SQ$, then $S + Q$ is an n -inverse of T , where $n = l + m - 1$. In this section we show that the theorem can be extended in a certain context, and prove the converse using the algebraic geometry approach developed in previous sections. The same result actually applies in a much stronger way to Helton classes, where we use it to prove Theorem 1.3.

We achieve these theorems using the same framework built in Section 3, but the map $\delta : R \rightarrow R \otimes R$ is different than in Section 4.

PROPOSITION 6.1. *Let A_1, A_2 be \mathbb{C} -algebras and $S, T \in A_1$ and $Q \in A_2$ with S, T not nilpotent, and Q nonzero. Also let $p \in \mathbb{C}[x, y]$. If $\Phi(p^l)(S + \lambda 1, T) = 0$ in A_1 and $(Q - \lambda 1)^m = 0$ in A_2 , then*

$$\Phi(p^n)(S \otimes 1 + 1 \otimes Q, T \otimes 1) = 0$$

where $n = l + m - 1$.

Proof. We will apply Lemma 3.3 using $\delta : R \rightarrow R \otimes R$ defined by $\delta(x) = x \otimes 1 + 1 \otimes x$ and $\delta(y) = y \otimes 1$. Say $\kappa_1 : F \rightarrow A_1$ is defined by $\kappa_1(X) = S$ and $\kappa_1(Y) = T$, and $\kappa_2 : F \rightarrow A_2$ is defined by $\kappa_2(X) = Q$ and $\kappa_2(Y) = 0$.

First notice that if $p \in R$, then $\mu\delta(p) = p(x_1 + x_2, y_1)$. Supposing $p(x, y) = \sum k_{ij}x^i y^j$, we can write

$$\begin{aligned} p(x_1 + x_2, y_1) &= \sum k_{ij}(x_1 + x_2)^i y_1^j \\ &= \sum k_{ij}(x_1 + \lambda)^i y_1^j + (x_2 - \lambda)g_{ij} \quad \text{for some } g_{ij} \\ &= \left(\sum k_{ij}(x_1 + \lambda)^i y_1^j \right) + (x_2 - \lambda)g \quad \text{for some } g \\ &= p(x_1 + \lambda, y_1) + (x_2 - \lambda)g. \end{aligned}$$

Let $q_1(x, y) = p(x + \lambda, y)$ and $q_2(x, y) = x - \lambda$. Then we can see that $\hat{p} \in \langle q_1(x_1, y_1), q_2(x_2, y_2) \rangle$. Also, notice that $\Phi(q_1^l) \in \ker \kappa_1$ and $\Phi(q_2^m) \in \ker \kappa_2$ by assumption. Thus by Lemma 3.3, $\Phi(p^n) \in \ker \kappa$. ■

We say $p(x, y)$ is *linear in x* if $p(x, y)$ can be written as $\alpha(y)x + \beta(y)$ for some α, β .

THEOREM 6.2. *Let A_1, A_2 be \mathbb{C} -algebras and $S, T \in A_1$ and $Q \in A_2$ with S, T not both zero and Q nonzero. Also let $p \in \mathbb{C}[x, y]$ be any irreducible polynomial that is linear in x . Then the following are equivalent:*

- (a) $\Phi(p^n)(S \otimes 1 + 1 \otimes Q, T \otimes 1) = 0$.
- (b) *There exist positive integers l, m with $l + m = n + 1$ and $\lambda \in \mathbb{C}$ such that $\Phi(p^l)(S + \lambda 1, T) = 0$ and $(Q - \lambda 1)^m = 0$.*

Proof. We will use the same definitions of δ and κ_i as in the proof of Proposition 6.1.

Because of Proposition 6.1, we need only prove that (a) implies (b). Since Q is assumed to be nonzero, $\ker \kappa_2$ is a proper ideal, so $V(J)$ is nonempty, where $J = \Phi^{-1}(\ker \kappa_2)$. Therefore, let (λ, d) be any point in $V(J)$ and set

$$q_1(x, y) := \hat{p}(x, y, \lambda, d) = p(x + \lambda, y).$$

Then by Lemma 3.7, $q_1 \in \sqrt{I}$.

By the linearity assumption we can write $p(x, y) = \alpha(y)x + \beta(y)$. We claim that there exists $(a, b) \in V(I)$ with $\alpha(b) \neq 0$. Indeed, if no such (a, b) exists, then $\alpha(y) \in \sqrt{I}$, but then since $q_1(x, y) = \alpha(y)(x + \lambda) + \beta(y) \in \sqrt{I}$, we also have $\beta(y) \in \sqrt{I}$, and since p was assumed to be irreducible, this means $I = R$, contrary to the assumption that S and T are not both zero.

Therefore, pick $(a, b) \in V(I)$ so that $\alpha(b) \neq 0$. Then by Lemma 3.7,

$$\begin{aligned} q_2(x, y) &:= \hat{p}(a, b, x, y) = p(x + a, b) = p(a + \lambda, b) + \alpha(b)(x - \lambda) \\ &= \alpha(b)(x - \lambda) \in \sqrt{I}. \end{aligned}$$

And since $\alpha(b) \neq 0$, this means $q_2 \in \sqrt{I}$.

We now show that \hat{p}, q_1, q_2 satisfy the conditions of Theorem 3.4. The proof of Proposition 6.1 shows that (i) is satisfied, and we have just shown that (ii) holds. We use Remark 3.5 to prove (iii). In particular, q_2 and 1 clearly generate R , and since α and β share no roots, $q_1 = \alpha(y)(x + \lambda) + \beta(y)$ and $\alpha(y)$ also generate R . ■

COROLLARY 6.3. *Let X, Y be Banach spaces, and assume $S, T \in B(X)$ and $Q \in B(Y)$. Then the following are equivalent:*

- (a) *The tensor sum $S \otimes I_Y + I_X \otimes Q$ on $X \overline{\otimes} Y$ is a strict left n -inverse of $T \otimes I_Y$.*
- (b) *There exist positive l and m such that $l + m = n - 1$ and some constant $\lambda \in \mathbb{C}$ such that $S + \lambda I_X$ is a strict left l -inverse of T , and $Q - \lambda I_Y$ is a nilpotent operator of order m .*

Proof. This follows directly from Theorem 6.2 using $p(x, y) = xy - 1$ and Lemma 5.1. ■

Corollary 6.3 was proven in [17, Theorem 22] using operator-theoretic techniques, but here we see that it follows purely from algebraic considerations. We could make a similar statement about Helton classes, but we will see in Proposition 6.5 that a much stronger result is possible when $p(x, y) = x - y$.

It is natural to ask how Corollary 6.3 can be applied to n -isometries. The following result (stated here in the equivalent tensor product of operators instead of elementary operators) is proved in [15, Theorem 12].

THEOREM 6.4. *Suppose that H and K are Hilbert spaces. Assume that $S \otimes I_K + I_H \otimes Q$ is a strict n -isometry and*

$$(12) \quad \sigma(S \otimes I_K + I_H \otimes Q) \neq \{\pm e^{\pm i\alpha} e^{i\theta} \text{ for some } \alpha, \theta \in [0, 2\pi)\}.$$

Then there exist m and l such that $m + 2l = n + 2$ and $S + \lambda I_H$ (or $Q - \lambda I_K$) is a strict m -isometry, and $Q + \lambda I_K$ (or $S - \lambda I_H$) is a nilpotent operator of order l for some constant $\lambda \in \mathbb{C}$.

The spectral condition (12) is necessary by [15, Proposition 14].

Somewhat surprisingly, Theorem 6.2 can be strengthened significantly when applied to $p(x, y) = x - y$, and so we treat this case separately.

PROPOSITION 6.5. *Suppose that A_1, A_2 are \mathbb{C} -algebras and $S_i, T_i \in A_i$ are nonzero for $i = 1, 2$. Then the following are equivalent:*

- (a) $\gamma_n(S_1 \otimes 1 + 1 \otimes S_2, T_1 \otimes 1 + 1 \otimes T_2) = 0$.
- (b) *There exist $l, m \geq 0$ and $\lambda \in \mathbb{C}$ such that $l + m = n + 1$ and that $\gamma_l(S_1 + \lambda 1, T_1) = 0$ and $\gamma_m(S_2 - \lambda 1, T_2) = 0$.*

Proof. Define $\delta : R \rightarrow R \otimes R$ by $\delta(x) = x \otimes 1 + 1 \otimes x$ and $\delta(y) = y \otimes 1 + 1 \otimes y$ and let $p(x, y) = x - y$. Thus $\hat{p}(x_1, y_1, x_2, y_2) = x_1 + x_2 - y_1 - y_2$. Define $\kappa_i : F \rightarrow A_i$ via $\kappa_i(X) = S_i$ and $\kappa_i(Y) = T_i$.

Since S_i, T_i are nonzero, I and J are proper ideals. Thus there exists $(c, d) \in V(J)$, and so by Theorem 4.3, if we let $\lambda = c - d$,

$$q_1(x, y) := p(x + \lambda, y) = x - y + \lambda \in \sqrt{I}.$$

Then let $(a, b) \in V(I)$, so that $p(a + c, b + d) = a - b + \lambda$, and thus $a - b = -\lambda$. Therefore, by Theorem 4.3,

$$q_2(x, y) := p(x - \lambda, y) = x - y - \lambda \in \sqrt{J}.$$

We have now shown that p, q_1, q_2 satisfy the conditions of Theorem 3.4. In particular, $\hat{p}(x_1, y_1, x_2, y_2) = 1 \cdot q_1(x_1, y_1) + 1 \cdot q_2(x_2, y_2)$, so (i) is satisfied. We have just shown that (ii) holds, and (iii) is satisfied trivially. ■

REMARK 6.6. Condition (b) of Proposition 6.5 could have been equivalently stated as $\gamma_l(S_1 + \alpha 1, T_1 - \beta 1) = 0$ and $\gamma_m(S_2 - \alpha 1, T_2 + \beta 1) = 0$ by making the substitution $\alpha + \beta = \lambda$.

Before proving Theorem 1.3, which interestingly requires no spectral condition like Theorem 6.4, we prove the following lemma.

LEMMA 6.7. *If $\gamma_n(T^* + \lambda I, T - \lambda I) = 0$, then λ is pure imaginary, so that $T - \lambda I$ is an n -symmetry.*

Proof. Let $\alpha \in \sigma_{\text{ap}}(T)$, so that there exists a sequence of unit vectors $h_i \in H$ such that $\|(T - \alpha I)h_i\| \rightarrow 0$ as $i \rightarrow \infty$. Then

$$\begin{aligned} \langle \gamma_n(T^* + \lambda I, T - \lambda I)h_i, h_i \rangle &= \sum_{k=0}^n (-1)^{n-k} \binom{n}{k} \langle (T^* + \lambda)^k (T - \lambda I)^{n-k} h_i, h_i \rangle \\ &= \sum_{k=0}^n (-1)^{n-k} \binom{n}{k} \langle (T - \lambda I)^{n-k} h_i, (T + \bar{\lambda})^k h_i \rangle. \end{aligned}$$

Thus, as $i \rightarrow \infty$, we have

$$\begin{aligned} \langle \gamma_n(T^* + \lambda I, T - \lambda I)h_i, h_i \rangle &\rightarrow \sum_{k=0}^n (-1)^{n-k} \binom{n}{k} (\alpha - \lambda)^{n-k} (\bar{\alpha} + \lambda)^k \langle h_i, h_i \rangle \\ &= (\alpha - \bar{\alpha} - 2\lambda)^n. \end{aligned}$$

Hence since $\gamma_n(T^* + \lambda I, T - \lambda I) = 0$, we must have $\lambda = (\alpha - \bar{\alpha})/2$, and so λ is pure imaginary.

Thus, $T^* + \lambda I = (T - \lambda I)^*$, so the original assumption shows that $T^* - \lambda I$ is an n -symmetry. ■

Proof of Theorem 1.3. We start by proving that (b) implies (a), so we begin by supposing that

$$\gamma_l(T_1^* + \bar{\lambda}I_H, T_1 + \lambda I_H) = 0, \quad \gamma_m(T_2^* - \bar{\lambda}I_K, T_2 - \lambda I_K) = 0.$$

Then by Proposition 6.5, if $n = l + m - 1$, we have

$$\begin{aligned} \gamma_n((T_1^* + \bar{\lambda}I_H) \otimes I_K + I_H \otimes (T_2^* - \bar{\lambda}I_K), (T_1 + \lambda I_H) \otimes I_K + I_H \otimes (T_2 - \lambda I_K)) \\ = \gamma_n(T_1^* \otimes I_K + I_H \otimes T_2^*, T_1 \otimes I_K + I_H \otimes T_2) = 0, \end{aligned}$$

which proves the claim.

To prove that (a) implies (b), we assume

$$\gamma_n(T_1^* \otimes I_K + I_H \otimes T_2^*, T_1 \otimes I_K + I_H \otimes T_2) = 0.$$

Using Remark 6.6 and Proposition 6.5 we can conclude that for some λ and some positive l, m with $l + m = n + 1$, we have

$$\gamma_l(T_1^* - \lambda I_H, T_1 + \lambda I_H) = 0, \quad \gamma_m(T_2^* + \lambda I_K, T_2 - \lambda I_H) = 0.$$

Finally, applying Lemma 6.7, we see that λ is pure imaginary, hence $\bar{\lambda} = -\lambda$, and the claim is proved. ■

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