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On decompositions of quadrinomials and related Diophantine equations

by

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1. Introduction. In [8] Péter, Pintér and Schinzel give an inefficient criterion for the Diophantine equation of the form

$$ax^m + bx^n + c = dy^p + e^q,$$

where a, b, c, d, e rationals, $abde \neq 0$, $m > n > 0$, $p > q > 0$, $\gcd(m, n) = 1$, $\gcd(p, q) = 1$, and $m, p \geq 3$, to have infinitely many integer solutions.

In a later paper Schinzel [9] dropped the assumption $\gcd(m, n) = 1$, $\gcd(p, q) = 1$ and gave a necessary and sufficient condition for such an equation to have infinitely many integer solutions.

In a recent paper Kreso [5] proved the finiteness of integral solutions for the equation

$$a_1 x^{n_1} + a_2 x^{n_2} + \cdots + a_l x^{n_l} + a_{l+1} = b_1 y^{m_1} + b_2 y^{m_2},$$

where $l \geq 2$ and $m_1 > m_2$, $n_1 > \cdots > n_l$ are fixed positive integers satisfying $\gcd(m_1, m_2) = 1$, $\gcd(n_1, \ldots, n_l) = 1$, $a_1, \ldots, a_l, a_{l+1}, b_1, b_2$ are rationals, non-zero except possibly for a_{l+1}, with $n_1 \geq 3$, $m_1 \geq 2l(l-1)$ and $(n_1, n_2) \neq (m_1, m_2)$.

All the above mentioned results rely on the Bilu–Tichy theorem [1] and theorems concerning decompositions of trinomials [2]. No such results for equations with at least three non-zero coefficients of positive powers on both sides are known, mainly because we have no results concerning decompositions of lacunary polynomials with more than three non-zero coefficients [5]. Some partial results in this direction are given in [6].

In this note we describe all possible decompositions of quadrinomials. Then we use the Bilu–Tichy theorem to prove the following generalizations of Schinzel's and Kreso's results.

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Theorem (A). *Let* $f(x) = Ax^{n_1} + Bx^{n_2} + Cx^{n_3} + D$, $g(x) = Ex^{m_1} + Fx^{m_2} + Gx^{m_3} + H$ *with* $f, g \in \mathbb{Q}[x]$, $n_1 > n_2 > n_3 > 0$, $m_1 > m_2 > m_3 > 0$, *and* $\gcd(n_1, n_2, n_3) = 1$, $\gcd(m_1, m_2, m_3) = 1$, $(m_1, m_2, m_3) \neq (n_1, n_2, n_3)$, $ABC \neq 0$, $EFG \neq 0$ *and* $n_1, m_1 \geq 9$. *Then the equation*

$$f(x) = g(y)$$

has only finitely many integer solutions.

Theorem (B). *Let* $l \geq 4$ *and* $n_1 > \cdots > n_l > 0$, $m_1 > m_2 > m_3 > 0$ *be integers. Let*

$$f(x) = A_1x^{n_1} + A_2x^{n_2} + \cdots + A_lx^{n_l} + A_{l+1} \text{ and } g(x) = Ex^{m_1} + Fx^{m_2} + Gx^{m_3}$$

be polynomials with rational coefficients such that $\gcd(n_1, \ldots, n_l) = 1$, $\gcd(m_1, m_2, m_3) = 1$, $A_1 \ldots A_l \neq 0$, $EFG \neq 0$ *and* $m_1 \geq 2l(l-1)$, $n_1 > 2l$. *Then the equation*

$$f(x) = g(y)$$

has only finitely many integer solutions.

Our results are ineffective as we use the theorem of Bilu and Tichy which relies on the classical theorem of Siegel [10] on integral points.

2. Decompositions of quadrinomials.

In this section we describe decompositions of quadrinomials. We will use some classical lemmas. Let us recall the Mason–Stothers theorem [7, 11].

Theorem 2.1. *Let* K *be a field of characteristic zero. Let* $a, b, c \in K[x]$ *be relatively prime, not all constant, such that* $a + b = c$. *Then*

$$\max\{\deg a, \deg b, \deg c\} \leq \deg(\mathrm{rad}(abc)) - 1,$$

where $\mathrm{rad}(f)$ *is the product of the distinct irreducible factors of* f.

We will use the following lemma when the polynomials a, b, c in the previous theorem are not coprime.

Lemma 2.2. *Let* K *be a field of characteristic zero. Let* $a, b, c \in K[x]$ *be such that* $a + b = c$. *Moreover, assume that* a, b, c *are not proportional. Then*

$$\max\{\deg a, \deg b, \deg c\} \leq \deg a + \deg(\mathrm{rad}(bc)) - 1.$$

Proof. Define $d = \gcd(a, b, c)$ and write $a_1 = a/d$, $b_1 = b/d$, $c_1 = c/d$. Applying Theorem 2.1 to the equality

$$a_1 + b_1 = c_1,$$

we get

$$\max\{\deg a_1, \deg b_1, \deg c_1\} \leq \deg(\mathrm{rad}(a_1b_1c_1)) - 1.$$

Adding $\deg d$ to both sides yields

$$\begin{aligned}\max\{\deg a, \deg b, \deg c\} &\le \deg(\operatorname{rad}(a_1b_1c_1)) + \deg d - 1\\ &\le \deg(\operatorname{rad} a_1) + \deg d + \deg(\operatorname{rad}(b_1c_1)) - 1\\ &\le \deg a + \deg(\operatorname{rad}(b_1c_1)) - 1. \ \blacksquare\end{aligned}$$

Let us recall Hajós' lemma [4].

Lemma 2.3. *Let K be a field o characteristic 0. If $f \in K[x]$ has a root $z \ne 0$ of multiplicity n then f has at least $n+1$ terms.*

Proof. We use induction on n. When $n = 1$, the statement obviously holds. For $n > 1$ write $f(x) = x^k f_1(x)$ where $f_1(0) \ne 0$. Then z is a root of f_1' of multiplicity $n-1$ and f_1' has exactly one term less than f. The result follows. ∎

Lemma 2.4. *Let K be a field o characteristic 0. If $f \in K[x]$ satisfies the equation $f(x)^2 = x^{n_1} + Ax^{n_2} + B$ for some $A, B \in K \setminus \{0\}$ and $n_1 > n_2 > 0$ then f is a binomial.*

Proof. Suppose that f has at least three non-zero coefficients, say

$$f(x) = x^{k_1} + Ux^{k_2} + \cdots + Vx^{k_3} + W$$

for some $k_1 > k_2 \ge k_3 > 0$ and $UVW \ne 0$. Then

$$f(x)^2 = x^{2k_1} + 2Ux^{k_1+k_2} + \cdots + 2VWx^{k_3} + W^2,$$

so $f(x)^2$ has at least four non-zero terms. ∎

Lemma 2.5. *Let K be an algebraically closed field of characteristic 0. Let $f, g, h \in K[x]$ be such that $f(x) = g(h(x))$ and $\deg g > 1$. Then there exists $\gamma \in K$ such that*

$$\deg(\gcd(f(x) - \gamma, f'(x)) \ge \deg h.$$

Proof. Let β be a root of $g'(x)$. Define $\gamma = g(\beta)$. Then $h(x) - \beta$ divides both $f'(x)$ and $f(x) - \gamma$. ∎

Now we are ready to state the main theorem of this section.

Theorem 2.6. *Let K be an algebraically closed field of characteristic 0. Let $f(x) = Ax^{n_1} + Bx^{n_2} + Cx^{n_3} + D$ for some $A, B, C, D \in K$ such that $ABC \ne 0$ and $n_1 > n_2 > n_3 > 0$. Suppose that $f(x) = g(h(x))$ for some $g, h \in K[x]$. Then one of the following cases holds:*

1. *$g(x) = (Ax^{n_1/d} + Bx^{n_2/d} + Cx^{n_3/d} + D) \circ l$ and $h(x) = l^{-1} \circ x^d$ for some linear polynomial $l \in K[x]$, and positive integer $d \mid \gcd(n_1, n_2, n_3)$.*
2. *$g(x) = l(x)$ and $h(x) = l^{-1} \circ f(x)$ for some linear $l \in K[x]$.*
3. *$g(x) = (Ax^2 + D) \circ l$ and $h(x) = l^{-1} \circ (x^{n_1/2} + \frac{B}{2A}x^{n_3/2})$ where $l \in K[x]$ is some linear polynomial, $2n_2 = n_1 + n_3$ and $C = B^2/(4A)$.*

(4) *$g(x) = (Ax(x - c^2) + D) \circ l$ and $h(x) = l^{-1} \circ (x^{2n_3} + cx^{n_3})$ for some linear $l \in K[x]$ and non-zero c; moreover $n_1 = 4n_3$, $n_2 = 3n_3$.*

Proof. By replacing f and g by $(Ax + D)^{-1} \circ f$ and $(Ax + D)^{-1} \circ g$ we can assume that $A = 1$, $D = 0$. Moreover by replacing g, h by $g \circ l^{-1}$, $l \circ h$ for a suitable linear l, we can assume that g, h are monic and $g(0) = h(0) = 0$.

Write

$$g(x) = x^{a_0}(x - x_1)^{a_1} \cdots (x - x_k)^{a_k}$$

with $a_0, a_1, \ldots, a_k \in \mathbb{N}_+$, $x_i \neq 0$ for $i = 1, \ldots, k$ and $x_i \neq x_j$ for $i \neq j$, $i, j = 1, \ldots, n$. We have

$$h(x)^{a_0}(h(x) - x_1)^{a_1} \cdots (h(x) - x_k)^{a_k} = x^{n_1} + Bx^{n_2} + Cx^{n_3}.$$

We write $h(x) = x^d h_1(x)$, where $h_1(x)$ is some monic polynomial such that $h_1(0) \neq 0$. If $h_1 \equiv 1$ then $h(x) = x^d$ and $g(x) = x^{n_1/d} + Bx^{n_2/d} + Cx^{n_3/d}$. This corresponds to the first case on our list.

Suppose that $h_1 \not\equiv 1$, and thus h_1 has a non-zero root ξ. Then from Lemma 2.3 we see that the multiplicity of ξ as a root of $x^{n_1} + Bx^{n_2} + Cx^{n_3}$ is ≤ 2, and therefore $a_0 \in \{1, 2\}$. We consider these two cases separately.

CASE 1: $a_0 = 1$. If $g(x) = x$ then we get the trivial decomposition, i.e. (2). If $g(x) \neq x$, then $\deg g \geq 2$. We have

$$x^{n_1} + Bx^{n_2} + Cx^{n_3} = g(x^{n_3} h_1(x)).$$

Let us prove that $h_1(x) = h_2(x^{n_3})$ for some polynomial $h_2(x)$. Suppose otherwise; let $h_1(x)$ have non-zero coefficient c_ν of x^ν with $n_3 \nmid \nu$, and suppose ν is the smallest integer with this property. Let us prove that $g(h(x))$ has at least four non-zero coefficients. We have

$$g(h(x)) = g(x^{n_3} h_1(x)) = \sum_{i=1}^{\deg g} C_i (x^{n_3} h_1(x))^i,$$

where $g(x) = \sum_{i=1}^{\deg g} C_i x^i$. The polynomial $h_1(x)$ is not a monomial, and thus $C_{\deg g}(x^{n_3} h_1(x))^{\deg g}$ has at least two non-zero coefficients of powers which cannot cancel $C_j(x^{n_3} h_1(x))^j$ where $j < \deg g$. The coefficient of $x^{n_3+\nu}$ in $g(h(x))$ is equal to $C_1 c_v$; it does not vanish, being the coefficient of the lowest power which is not divisible by n_3. So in that case $g(h(x))$ has at least four non-zero coefficients, a contradiction. We have proved that $h_1(x) = h_2(x^{n_3})$.

As a consequence we get the equality

$$x^{n_1} + Bx^{n_2} + Cx^{n_3} = g(x^{n_3} h_2(x^{n_3})).$$

Therefore $n_3 \mid n_1$ and $n_3 \mid n_2$, say $m_1 n_3 = n_1$, $m_2 n_3 = n_2$, and

$$x^{m_1} + Bx^{m_2} + Cx = g(x h_2(x)). \tag{2.1}$$

Let us write $k = m_1 - m_2$.

We claim that $h_2(x) = h_3(x^k)$ for some $h_3 \in K[x]$. Equating the coefficients in (2.1) we get

$$h_2(x) = x^t + \frac{B}{\deg g} x^{t-k} + \text{l.o.t.}$$

for some t. Let us prove that if the coefficient D_s of x^s in $h_2(x)$ is non-zero then $s \equiv t \pmod{k}$. Suppose otherwise, and let ν be the highest power such that the coefficient D_ν of x^ν in $h_2(x)$ is non-zero and $\nu \not\equiv t \pmod{k}$. We have

$$x^{m_1} + Bx^{m_2} + Cx = x^{\deg g}\left(x^t + \frac{B}{\deg g} x^{t-k} + \text{l.o.t.}\right)^{\deg g} + \text{l.o.t.}$$

Observe that the coefficient of $x^{\deg g + (\deg g - 1)t + \nu}$ on the right hand side is equal to $D_\nu \deg g$. It cannot vanish because it is the coefficient of the highest power x^u which satisfies $u \neq \deg g + t \deg g \pmod{k}$. Of course $m_2 = \deg g + (\deg g - 1)t + t - k > \deg g + (\deg g - 1)t + \nu > 1$, so we arrive at a contradiction. We know that $h_2(0) \neq 0$, so we deduce $h_2(x) = h_3(x^k)$.

We thus proved that

$$g(xh_2(x)) = g(xh_3(x^k)) = x^{m_1} + Bx^{m_2} + C.$$

We will prove that $g(x) = xg_2(x^k)$ for some polynomial $g_2(x)$. Suppose that this is not the case and write $g(x) = \sum_{i=1}^{\deg g} C_i x^i$. Let ν be the smallest integer such that $\nu \not\equiv 1 \pmod{k}$ and $C_\nu \neq 0$. We have

$$\sum_{i=1}^{\deg g} C_i (xh_3(x^k))^i = x^{m_1} + Bx^{m_2} + C.$$

The coefficient of x^ν on the left hand side is equal to $C_\nu h_3(0) \neq 0$. Thus $\nu = m_1$ or $\nu = m_2$. On the other hand,

$$m_2 = m_1 - k = \deg(g(xh_3(x^k))) - k \geq \nu(k+1) - k = (\nu - 1)k + \nu > \nu,$$

so we arrive at a contradiction.

We have

$$xh_3(x^k) g_2(x^k h_3(x^k)^k) = x^{m_1} + Bx^{m_2} + Cx,$$

therefore $m_2 - 1 = kl$ and $m_1 - 1 = k(l+1)$ for some l. Consequently,

$$h_3(x) g_2(xh_3(x)^k) = x^{l+1} + Bx^l + C. \tag{2.2}$$

Let us prove that $k = 1$. We write $g_2(x) = \sum_{j=0}^{\deg g_2} W_j x^j$, and get

$$(W_0 h_3(x) - C) + (xh_3(x)^{k+1}) \sum_{j=1}^{\deg g_2} W_j (xh_3(x)^k)^{j-1} = x^{l+1} + Bx^l. \tag{2.3}$$

We have

$$\deg\operatorname{rad}\Big((x^{l+1}+Bx^l)xh_3(x)^{k+1}\sum_{j=1}^{\deg g_2}W_j(xh_3(x)^k)^{j-1}\Big) \\ \leq (k\deg h_3+1)(\deg g_2-1)+\deg h_3+2.$$

Moreover the appropriate polynomials in (2.3) are not proportional. Applying Lemma 2.2 we get

$$1+(k+1)\deg h_3+(k\deg h_3+1)(\deg g_2-1) \\ \leq \deg h_3+(k\deg h_3+1)(\deg g_2-1)+\deg h_3+2-1,$$

thus $(k-1)\deg h_3\leq 0$ and $k=1$. Therefore $h_2=h_3$.

We set $k=1$ and $h_2=h_3$ in (2.2) to obtain

$$g(xh_2(x))=xh_2(x)g_2(xh_2(x))=x^{l+2}+Bx^{l+1}+Cx.$$

Set $F(x)=g(xh_2(x))$. By Lemma 2.5 there exists $\lambda\in K$ such that

$$\deg(\gcd(F(x)-\lambda,F'(x))\geq\deg xh_2(x),$$

and thus

$$\deg\operatorname{rad}((F(x)-\lambda)(x^{l+2}+Bx^{l+1}))\leq\deg F-\deg(\gcd(F(x)-\lambda,F'(x))+2 \\ \leq\deg F-\deg(xh_2(x))+2.$$

Since $Cx-\lambda$ and $x^{l+2}+Bx^{l+1}$ are not proportional, applying Lemma 2.2 to the equation

$$F(x)-\lambda=(x^{l+2}+Bx^{l+1})+(Cx-\lambda)$$

yields

$$\deg F\leq 1+\big(\deg F-\deg(xh_2(x))+2\big)-1.$$

Consequently, $\deg h_2\leq 1$. Let $h_2(x)=x+c$ for some non-zero c. We have

$$g(x(x+c))=x^{l+2}+Bx^{l+1}+Cx.$$

The left hand side is symmetric with respect to the line $x=-c/2$, and thus so is the right hand side. We get

$$x^{l+2}+Bx^{l+1}+Cx=(-x-c)^{l+2}+B(-x-c)^{l+1}+C(-x-c).$$

Taking the second derivative of both sides gives

$$(l+2)(l+1)x^l+B(l+1)lx^{l-1}=(l+2)(l+1)(-x-c)^l+B(l+1)l(-x-c)^{l-1},$$

which is equivalent to

$$((l+2)x+Bl)x^{l-1}=(x+c)^{l-1}((l+2)(x+c)-Bl).$$

As $c\neq 0$, we have $l=2$. As a consequence, $\deg g=2$ and $g(x)=x(x+b)$ for some non-zero b. Finally, the coefficient of x^2 in $g(x(x+c))$ is zero, which implies $b = -c^2$. Summing up: for $a_0 = 1$ we get the solution of $g(h(x)) = f(x)$ of the form

$$g(x) = x^2 - c^2x, \qquad h(x) = x^{n_3}(x^{n_3} + c),$$

which corresponds to (4).

CASE 2: $a_0 = 2$. We know that $a_0d = n_3$, so $d = n_3/2$. If $g(x) = x^2$ then $h(x)^2 = x^{n_1} + Bx^{n_2} + Cx^{n_3}$ and $h_1(x)^2 = x^{n_1-n_3} + Bx^{n_2-n_3} + C$. Lemma 2.4 implies that $h(x)$ is a binomial, which corresponds to (3).

Now assume that $g(x) \neq x^2$, so g has at least one non-zero root, and $\deg g(x) \geq 3$.

Let us prove that $h_1(x) = h_2(x^{n_3/2})$ for some monic polynomial h_2. Suppose not, and assume that $h_1(x)$ has a non-zero coefficient c_ν of x^ν with $d = n_3/2 \nmid \nu$. Choose ν to be the smallest integer with this property. We will prove that $g(h(x))$ has at least four non-zero coefficients. We have

$$g(h(x)) = x^{n_3}h_1(x)^2(x^dh_1(x)-x_1)^{a_1}\cdots(x^dh_1(x)-x_k)^{a_k} = \sum_{i=2}^{\deg g} C_i(x^dh_1(x))^i,$$

where $g(x) = \sum_{i=2}^{\deg g} C_ix^i$. As $h_1(x)$ is not a monomial, $C_{\deg g}(x^dh_1(x))^{\deg g}$ has at least two non-zero coefficients, of powers which cannot cancel $C_j(x^dh_1(x))^j$ where $j < \deg g$. Moreover, the coefficient of $x^{n_3+\nu}$ in $g(h(x))$ is $2C_2c_vh_1(0)$. It cannot vanish, being the coefficient of the lowest power which is not divisible by d. So in that case $g(h(x))$ has at least four non-zero coefficients, a contradiction.

We have proved that $h_1(x) = h_2(x^{n_3/2})$, and thus

$$x^{n_1} + Bx^{n_2} + Cx^{n_3} = g(x^dh_2(x^d)),$$

where $d = n_3/2$. Therefore $d \mid n_1$, $d \mid n_2$, say $m_1d = n_1$, $m_2d = n_2$, and

$$x^{m_1} + Bx^{m_2} + Cx^2 = g(xh_2(x)). \tag{2.4}$$

Let us write $k = m_1 - m_2$. We claim that $h_2(x) = h_3(x^k)$. Equating the coefficients in (2.4) we get

$$h_2(x) = x^t + \frac{B}{\deg g}x^{t-k} + \text{l.o.t.}$$

for some t. Let us prove that if the coefficient D_s of x^s in $h_2(x)$ is non-zero then $s \equiv t \pmod{k}$. Suppose otherwise, and let ν be the highest power such that $h_2(x)$ has coefficient $D_\nu \neq 0$ and $\nu \not\equiv t \pmod{k}$. We have

$$x^{m_1} + Bx^{m_2} + Cx^2 = x^{\deg g}\left(x^t + \frac{B}{\deg g}x^{t-k} + \text{l.o.t.}\right)^{\deg g} + \text{l.o.t.}$$

Observe that the coefficient of $x^{\deg g + (\deg g - 1)t + \nu}$ on the right hand side is $D_\nu \deg g$. It cannot vanish because it is the coefficient of the highest power x^u which satisfies $u \not\equiv \deg g + t\deg g \pmod{k}$. Of course $m_2 = \deg g +$

$(\deg g-1)t+t-k>\deg g+(\deg g-1)t+\nu>2$, so we arrive at a contradiction. We know that $h_2(0)\neq 0$, and thus $h_2(x)=h_3(x^k)$.

We can write

$$x^2h_3(x^k)^2 \mid g(xh_2(x))=x^{m_1}+Bx^{m_2}+Cx^2,$$

therefore

$$\begin{aligned} h_3(x^k)\mid \frac{d}{dx}(x^{m_1-2}+Bx^{m_2-2}+C) &= (m_1-2)x^{m_1-3}+(m_2-2)Bx^{m_2-3}\\ &= x^{m_2-3}\big((m_1-2)x^k+(m_2-2)B\big). \end{aligned}$$

We know that $h_3(0)\neq 0$, so $h_3(x^k)\mid (m_1-2)x^k+(m_2-2)B$, in particular $\deg h_3\le 1$. If $h_3(x)$ is constant then so is $h(x)$, and we arrive at a contradiction.

Suppose that $h_3(x)$ is linear. Then $h_2(x)=x^k+c$ for some non-zero c. Write $g(x)=x^2g_2(x)$. Then

$$x^{m_1}+Bx^{m_2}+Cx^2=x^2(x^k+c)^2g_2(x(x^k+c)),$$

and thus

$$x^{m_1-2}+Bx^{m_2-2}+C=(x^k+c)^2g_2(x(x^k+c)),$$

Let us prove that $g_2(x)=g_3(x^k)$ for some $g_3(x)$. Suppose not, and let x^u be the lowest power such that g_2 has non-zero coefficient of x^u and $u\not\equiv 0 \pmod k$. Then the coefficient of x^u in $(x^k+c)^2g_2(x(x^k+c))$ is non-zero, so $u=m_2-2$ or $u=m_1-2$. In both cases we have

$$\begin{aligned} m_2-2=(m_1-2)-k &= \big(2k+\deg g_2\cdot(k+1)\big)-k\\ &\ge k+u(k+1)\ge k+u\ge k+(m_2-2)>m_2-2, \end{aligned}$$

a contradiction. Therefore $g_2(x)=g_3(x^k)$.

We have

$$x^{m_1-2}+Bx^{m_2-2}+C=(x^k+c)^2g_3(x^k(x^k+c)^k),$$

in particular $k\mid m_1-2$ and $k\mid m_2-2$, say $k(s+1)=m_1-2$. We have

$$x^{s+1}+Bx^s+C=(x+c)^2g_3(x(x+c)^k). \tag{2.5}$$

Let $g_3(x)=\sum_{j=0}^{\deg g_3}W_jx^j$, and write

$$x^{s+1}+Bx^s=(W_0(x+c)^2-C)+x(x+c)^{k+2}\sum_{j=1}^{\deg g_3}W_j(x(x+c)^k)^{j-1}. \tag{2.6}$$

Then

$$\begin{aligned} \deg\operatorname{rad}\Big((x^{s+1}+Bx^s)x(x+c)^{k+2}\sum_{j=1}^{\deg g_3}W_j(x(x+c)^k)^{j-1}\Big)&\\ \le (k+1)(\deg g_3-1)+3.& \end{aligned}$$

The polynomials in (2.6) are not proportional, because at least two of them have different degrees. Applying Lemma 2.2 to (2.6) we get

$$k+3+(k+1)(\deg g_3-1)\le 2+(k+1)(\deg g_3-1)+3-1,$$

so $k=1$. We plug this into (2.5) to obtain

$$x^{s+1}+Bx^s+C=(x+c)^2g_2(x(x+c)). \tag{2.7}$$

Consequently,

$$x^{s+3}+Bx^{s+2}+Cx^2=(x+c)^2x^2g_2(x(x+c));$$

the right hand side is symmetric with respect to the line $x=-c/2$, and thus so is the left hand side. We get

$$x^{s+3}+Bx^{s+2}+Cx^2=(-x-c)^{s+3}+B(-x-c)^{s+2}+C(-x-c)^2.$$

Taking the second derivative of both sides we obtain

$$\begin{aligned}(s+3)(s+2)x^{s+1}&+(s+2)(s+1)Bx^s\\&=(s+3)(s+2)(-x-c)^{s+1}+(s+2)(s+1)B(-x-c)^s.\end{aligned}$$

If $s>1$ then the only multiple root of the left hand side is $x=0$, and the only multiple root of the right hand side is $x=-c$, thus $s=1$. By comparing degrees in (2.7) we find that $\deg g_2=0$, a contradiction as $g_2(x)$ has a non-zero root. ■

REMARK 2.7. In each of cases (1)–(4) of Theorem 2.6 the polynomial $g\circ h$ is indeed a quadrinomial.

3. A Diophantine equation. In this section we give a sufficient condition for two quadrinomials to have only finitely many common integral points. We recall the Bilu–Tichy result. To state it we will need the notion of standard pairs over $\mathbb{Q}$. We list them in the table below.

Kind	Standard pair (or switched)	Parameter restrictions
first	$(x^m, ax^rp(x)^m)$	$r<m$, $\gcd(r,m)=1$, $r+\deg p>0$
second	$(x^2,(ax^2+b)p(x)^2)$	
third	$(D_m(x,a^n),D_n(x,a^m))$	$\gcd(m,n)=1$
fourth	$(a^{-m/2}D_m(x,a),-b^{-n/2}D_n(x,b))$	$\gcd(m,n)=2$
fifth	$((ax^2-1)^3,3x^4-4x^3)$	

Here a,b are non-zero rationals, m,n are positive integers, r is a non-negative integer, $p\in\mathbb{Q}[x]$ (maybe constant) and $D_n(x,a)$ is the nth Dickson polynomial with parameter a, given by

$$D_n(x,a)=\sum_{i=0}^{[n/2]}\frac{n}{n-i}\binom{n-i}{i}(-a)^i x^{n-2i}.$$

Now we are ready to recall the theorem of Bilu and Tichy [1].

THEOREM 3.1. *Let $f,g\in\mathbb{Q}[x]$ be non-constant. Then the following assertions are equivalent:*

- *The equation $f(x)=g(y)$ has infinitely many rational solutions with a bounded denominator.*
- *We have*
$$f(x)=\varphi(f_1(\lambda(x))),\quad g(x)=\varphi(g_1(\mu(x))),$$
where $\varphi\in\mathbb{Q}[x]$, $\lambda,\mu\in\mathbb{Q}[x]$ are linear polynomials, and (f_1,g_1) is a standard pair over $\mathbb{Q}$ such that the equation $f_1(x)=g_1(y)$ has infinitely many rational solutions with a bounded denominator.

Before stating the main theorem we prove some lemmas concerning decompositions of certain polynomials. As the main ingredient in the proofs we will use the classical theorem of Gessel and Viennot [3] concerning determinants with binomial coefficients.

THEOREM 3.2. *Let $0\le a_1<\cdots<a_n$ and $0\le b_1<\cdots<b_n$ be integers. Then the determinant*

$$\det\left(\left[\binom{a_i}{b_j}\right]_{i,j=1,\dots,n}\right)$$

is non-negative, and positive iff $b_i\le a_i$ for all i.

LEMMA 3.3. *Let $f,g\in\mathbb{Q}[x]$ and $u,v\in\mathbb{Q}$ be non-zero such that*

$$f(x)=g(ux+v).$$

Suppose that $g(x)$ has exactly l non-zero coefficients and $f(x)$ has exactly k non-zero coefficients, and $n=\deg f=\deg g$. Then

$$n+2\le k+l.$$

Proof. We write $g(x)=\sum_{i=1}^{l}C_{n_i}x^{n_i}$, where $(n_i)_{i=1}^{l}$ is a decreasing sequence of non-zero integers, and the C_{n_i} are non-zero rationals. The coefficient of x^j in the polynomial $g(ux+v)$ is equal to

$$u^j v^{-j}\sum_{i=1}^{l}C_{n_i}\binom{n_i}{j}v^{n_i}.$$

Suppose this coefficient vanishes for $j=m_1,\dots,m_s$, where $s=n+1-k\ge l$ and $(m_i)_{i=1}^{s}$ is a decreasing sequence of non-zero integers. We observe that the vector $(C_{n_1}v^{n_1},\dots,C_{n_l}v^{n_l})$ is perpendicular to every row of the matrix

$$\left[\binom{n_i}{m_j}\right]_{i=1,\dots,l,\,j=1,\dots,s}.$$

We set $t=\max\{i \mid n_i \ge m_i,\ 1\le i\le l\}$; of course $n=n_1>m_1$, and so t is well-defined. By Lemma 3.2 the determinant of the matrix

$$M=\left[\binom{n_i}{m_j}\right]_{i,j=1,\dots,t}$$

is non-zero. Moreover the vector $(C_{n_1}v^{n_1},\dots,C_{n_t}v^{n_t})$ is in the kernel of M, therefore $v=0$. This contradicts the assumption $v\ne 0$. We have proved that $s=n+1-k<l$, therefore $n+2\le k+l$. ∎

LEMMA 3.4. *Let $n_1>\dots>n_s>0$ be integers and $A_1,\dots,A_{s+1}$ be non-zero rational numbers, and set $f(x)=A_1x^{n_1}+\dots+A_sx^{n_s}+A_{s+1}$. If*

$$D_{n_1}(x,\gamma)=f(ux+v)$$

for some $u,v,\gamma\in\mathbb{Q}$ such that $u\gamma\ne 0$, then $n_1\le 2s$.

Proof. If $v=0$ then

$$D_{n_1}(x,\gamma)=\sum_{i=0}^{[n_1/2]}\frac{n_1}{n_1-i}\binom{n_1-i}{i}(-\gamma)^i x^{n_1-2i}=f(ux).$$

The polynomial on the left hand side has exactly $[n_1/2]+1$ non-zero coefficients, while $f(ux)$ has s or $s+1$ non-zero coefficients. Hence $[n_1/2]+1\le s+1$, so $n_1\le 2s+1$. The equality $n_1=2s+1$ cannot hold, because it implies that $A_{s+1}=0$ and $f(ux)$ has s non-zero coefficients, so $[n_1/2]+1\le s$ and thus $n_1\le 2s-1$.

Suppose that $v\ne 0$ and $n_1\ge 2s+1$. From Lemma 3.3 we get

$$\deg D_{n_1}(x,\gamma)=n_1\le [n_1/2]+1+(s+1)-2=[n_1/2]+s,$$

therefore $n_1\le 2s$. ∎

Now we are ready to state the main theorems of this section.

THEOREM (A). *Let $f(x)=Ax^{n_1}+Bx^{n_2}+Cx^{n_3}+D$ and $g(x)=Ex^{m_1}+Fx^{m_2}+Gx^{m_3}+H$ with $f,g\in\mathbb{Q}[x]$, $n_1>n_2>n_3>0$, $m_1>m_2>m_3>0$, and $\gcd(n_1,n_2,n_3)=1$, $\gcd(m_1,m_2,m_3)=1$, $(m_1,m_2,m_3)\ne(n_1,n_2,n_3)$, $ABC\ne 0$, $EFG\ne 0$ and $n_1,m_1\ge 9$. Then the equation*

$$f(x)=g(y)$$

has only finitely many integer solutions.

Proof. If the equation has infinitely many integer solutions, then

$$f=\varphi\circ f_1\circ\lambda,\quad g=\varphi\circ g_1\circ\mu,$$

where $\varphi, \mu, \lambda, f_1, g_1 \in \mathbb{Q}[x]$, (f_1, g_1) is a standard pair, and μ, λ are linear polynomials.

Let us consider standard pairs of the first kind. By symmetry we can assume that

$$f_1(x) = x^m, \quad g_1(x) = ax^r p(x)^m$$

for $a \in \mathbb{Q} \setminus \{0\}$, $0 \leq r < m$, $\gcd(r, m) = 1$, $p(x) \in \mathbb{Q}[x]$ and $r + \deg p > 0$. From Theorem 2.6 we get $\deg \varphi \leq 2$ or $\deg f_1 = \deg g_1 = 1$. If $\deg \varphi = 1$ then

$$\varphi^{-1} \circ f(x) = x^m \circ \lambda = (ux + v)^m$$

for some $u, v \in \mathbb{Q}$. The polynomial $\varphi^{-1} \circ f$ has at least three non-zero coefficients, so $v \neq 0$. Therefore $\varphi^{-1} \circ f$ has exactly four non-zero coefficients and $(ux+v)^m$ has exactly $m+1$ non-zero coefficients. Therefore $m = \deg f = 3$, which contradicts $\deg f \geq 9$.

If $\deg \varphi = 2$ then from Theorem 2.6 we deduce that $x^m \circ \lambda$ has two or three non-zero coefficients. As in the previous case we get $m = 1$ or $m = 2$, so $\deg f \leq 4$, a contradiction.

If $\deg f_1 = \deg g_1 = 1$ then $m = 1$, $r = 0$, $\deg p = 1$. So $f = g \circ l$ for some linear l, and from Lemma 3.3 we find that either $(m_1, m_2, m_3) = (n_1, n_2, n_3)$ or $\deg f \leq 4 + 4 - 2 = 6$, a contradiction.

Let us consider pairs of the second kind. By symmetry we can assume that $f_1(x) = x^2$. Then from Theorem 2.6 we get $\deg \varphi \leq 2$, and therefore $\deg f \leq 4$, a contradiction.

Let us consider pairs of the fifth kind. By symmetry we can assume that $f_1(x) = 3x^4 - 4x^3$ and $g_1 = (ax^2 - 1)^3$. Then from Theorem 2.6 we get $\deg \varphi \leq 2$. Therefore $\deg f = \deg(\varphi \circ f_1 \circ \lambda) \leq 8$, a contradiction.

Let us consider pairs of the third and fourth kind. If $\deg f_1 = \deg g_1 = 1$ then we proceed as in the case of pairs of the first kind. We can assume that $\deg f_1 \geq 2$. From the definition of pairs of the third and fourth kind we have $f_1 = aD_m(x, \gamma)$ for some $a, \gamma \in \mathbb{Q} \setminus \{0\}$. From Theorem 2.6 we get $\deg \varphi \leq 2$. If $\deg \varphi = 1$ then $D_m(x, \gamma) = \varphi^{-1} \circ f \circ \lambda^{-1}$, and Lemma 3.4 yields $\deg f = m \leq 6$, a contradiction. If $\deg \varphi = 2$ then $f_1 \circ \lambda$ has two or three non-zero coefficients. Therefore Lemma 3.4 implies that $\deg f_1 \leq 4$, so $\deg f \leq 8$, a contradiction. ∎

To prove a theorem concerning equations with lacunary polynomials we will use the following result of Zannier [12].

Theorem 3.5 (Zannier). *Suppose that* $g, h \in \mathbb{C}[x]$ *are non-constant,* $h(x)$ *is not of the shape* $ax^n + b$, *and* $g(h(x))$ *has at most* $l > 0$ *non-zero coefficients of positive powers. Then* $\deg g \leq 2l(l-1)$.

Now we are ready to prove

Theorem (B). *Let $l \geq 4$ and $n_1 > \cdots > n_l > 0$, $m_1 > m_2 > m_3 > 0$ be integers. Let*

$$f(x) = A_1 x^{n_1} + \cdots + A_l x^{n_l} + A_{l+1} \quad \text{and} \quad g(x) = Ex^{m_1} + Fx^{m_2} + Gx^{m_3}$$

be polynomials with rational coefficients such that $\gcd(n_1, \ldots, n_l) = 1$, $\gcd(m_1, m_2, m_3) = 1$, $A_1 \cdots A_l \neq 0$, $EFG \neq 0$ *and* $m_1 \geq 2l(l-1)$, $n_1 > 2l$. *Then the equation*

$$f(x) = g(y)$$

has only finitely many integer solutions.

Proof. If the equation has infinitely many integer solutions, then

$$f = \varphi \circ f_1 \circ \lambda, \quad g = \varphi \circ g_1 \circ \mu,$$

where $\varphi, \mu, \lambda, f_1, g_1 \in \mathbb{Q}[x]$, (f_1, g_1) is a standard pair, and μ, λ are linear polynomials.

Suppose that $\deg \varphi > 2$. From Theorem 2.6 we know that either $g_1 \circ \mu = \rho \circ x^d$ for some linear $\rho \in \mathbb{Q}[x]$ and positive integer $d \geq 2$, or $\deg(g_1 \circ \mu) = 1$. The first case is impossible in view of the assumption $\gcd(m_1, m_2, m_3) = 1$. In the second case we get

$$f = \varphi \circ f_1 \circ \lambda = g \circ (g_1 \circ \mu)^{-1} \circ f_1 \circ \lambda.$$

From Zannier's theorem we get $(g_1 \circ \mu)^{-1} \circ f_1 \circ \lambda = ux^k + v$ for some rationals u, v and positive integer k. We claim that $k = 1$. Suppose not; then all non-zero coefficients of f are of powers divisible by k, which contradicts the assumption $\gcd(n_1, \ldots, n_l) = 1$. In the case of $k = 1$ we have the equation $f = g \circ \rho$ for some linear $\rho \in \mathbb{Q}[x]$. From Lemma 3.3 we get

$$2l(l-1) \leq \deg g \leq (l+1) + 3 - 2 = l + 2,$$

or $f(x) = g(ux)$ for some non-zero rational u. The former is impossible since $l \geq 4$. The latter is impossible since g has three non-zero coefficients and f has at least four non-zero coefficients.We have proved that $\deg \varphi \leq 2$.

Let us consider standard pairs of the first kind. Suppose that $g_1(x) = x^m$ for some m. If $\deg \varphi = 1$ then

$$\varphi^{-1} \circ g = x^m \circ \mu = (ux + v)^m$$

for some rationals u, v. Suppose that $v = 0$. Then $g(x) = \varphi((ux)^m)$ has at most two non-zero coefficients, a contradiction. If $v \neq 0$, we find that $(ux+v)^m = \varphi^{-1} \circ g$ has $m + 1 = \deg g + 1 > 4$ non-zero coefficients, whereas g has three non-zero coefficients, a contradiction.

If $\deg \varphi = 2$ then Theorem 2.6 implies that $g_1 \circ \mu = x^m \circ \mu$ has two or three non-zero coefficients, as in the previous case we get $m = 1$ or $m = 2$. This contradicts $2m = \deg g > 4$.

Suppose that (f_1, g_1) is a switched pair of the first kind, namely $g_1(x) = ax^r p(x)^m$, $f_1(x) = x^m$. If $\deg \varphi = 1$ then

$$\varphi \circ (ux + v)^m = f$$

for some rationals u, v. If $v = 0$ then the polynomial $\varphi \circ (ux + v)^m$ has at most two non-zero coefficients, a contradiction because $l \geq 4$. If $v \neq 0$ then $\varphi \circ (ux + v)^m$ has non-zero coefficient of every non-zero power, which contradicts $\deg f > 2l$.

If $\deg \varphi = 2$, write $\varphi(x) = e_1 x^2 + e_2 x + e_3$ for some $e_1, e_2, e_3 \in \mathbb{Q}$. From the equality

$$f(x) = \varphi \circ (ux + v)^m = e_1(ux + v)^{2m} + e_2(ux + v)^m + e_3$$

we see that $2l < \deg f = 2m$. If $v = 0$ then f has at most three non-zero coefficients, contrary to $l \geq 4$. If $v \neq 0$ then f has non-zero coefficients of $x^{m+1}, x^{m+2}, \ldots, x^{2m}$. On the other hand, f has at most $l < m$ non-zero coefficients of non-zero powers, a contradiction.

Observe that (f_1, g_1) cannot be a standard pair of the second kind, since $\deg(\varphi \circ x^2) \leq 4$ and $n_1, m_1 > 4$.

Let us consider pairs of the third and fourth kind. In both cases we have $g_1 = aD_m(x, \gamma)$ for some $a, \gamma \in \mathbb{Q} \setminus \{0\}$. If $\deg \varphi = 1$ then we have $aD_m(x, \gamma) = \varphi^{-1} \circ g \circ \lambda^{-1}$, and from Lemma 3.4 we get $\deg g = m \leq 6$, a contradiction. If $\deg \varphi = 2$ then $g_1 \circ \lambda$ has two or three non-zero coefficients. Therefore Lemma 3.4 yields $\deg g_1 \leq 4$, so $\deg g \leq 8$, a contradiction.

Finally, (f_1, g_1) cannot be a standard pair of the fifth kind. Since $\deg g_1 \leq 6$ and $\deg \varphi \leq 2$, we deduce that $24 \leq 2l(l-1) \leq \deg g = \deg(\varphi \circ g_1 \circ \mu) \leq 12$, a contradiction. ∎

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