

Normal regular Hermitian lattices over imaginary quadratic fields

by

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1. Introduction. A Hermitian lattice is said to be *regular* if it globally represents all the positive integers that are represented at every one of its local completions. Earnest and Khosravani [5] showed that every imaginary quadratic field allows finitely many classes of primitive regular binary Hermitian lattices. Their result was generalized by Chan and Rokicki [3] to CM extensions of number fields. The present author and his colleagues completed the list of binary normal regular Hermitian lattices over imaginary quadratic fields [12] and found infinitely many imaginary quadratic fields which allow binary subnormal regular Hermitian lattices [13]. Recently, Liu and Marino [14] showed the finiteness of strictly regular ternary Hermitian lattices.

In the present article, we show that every primitive normal regular Hermitian lattice over $\mathbb{Q}(\sqrt{-m})$ is indeed universal if and only if $m \neq 1, 2, 11, 23$ and $(m, 3 \cdot 5 \cdot 7) = 1$. The term *universal* means that the lattice represents all positive integers.

The classification of universal quadratic forms over \mathbb{Z} was completed thanks to the celebrated 15-Theorem proved by Conway, Schneeberger, and Bhargava [4, 1]. The author and his colleagues [11] have proved an analogous 15-Theorem for Hermitian lattices: if a positive definite integral Hermitian lattice represents 1, 2, 3, 5, 6, 7, 10, 13, 14, and 15, then it is universal.

2. Preliminaries. Let \mathcal{O}_E be the ring of integers of an imaginary quadratic field $E = \mathbb{Q}(\sqrt{-m})$, where m is a squarefree positive integer. Then $\mathcal{O}_E = \mathbb{Z}[\omega_m]$ with $\omega_m = \sqrt{-m}$ if $m \equiv 1, 2 \pmod{4}$ and $\omega_m = (1 + \sqrt{-m})/2$ if $m \equiv 3 \pmod{4}$.

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A *Hermitian space* (V, H) is a vector space over $E = \mathbb{Q}(\sqrt{-m})$ equipped with a bilinear map $H : V \times V \rightarrow E$ such that $H(\mathbf{v}, \mathbf{w}) = \overline{H(\mathbf{w}, \mathbf{v})}$ and $H(\alpha\mathbf{v}, \beta\mathbf{w}) = \alpha\beta H(\mathbf{v}, \mathbf{w})$ for any $\alpha, \beta \in E$ and $\mathbf{v}, \mathbf{w} \in V$.

Let L be a finitely generated \mathcal{O}_E -module on a Hermitian space (V, H) . If $H(\mathbf{v}) := H(\mathbf{v}, \mathbf{v}) > 0$ for all nonzero $\mathbf{v} \in L$, then we call L *positive definite*. Throughout this article we call a positive definite \mathcal{O}_E -module a *Hermitian lattice*, or briefly a *lattice*.

The *scale ideal* $\mathfrak{s}L$ is the fractional \mathcal{O}_E -ideal generated by $\{H(\mathbf{v}, \mathbf{w}) : \mathbf{v}, \mathbf{w} \in L\}$, and the *norm ideal* $\mathfrak{n}L$ is the \mathcal{O}_E -ideal generated by $\{H(\mathbf{v}) : \mathbf{v} \in L\}$. Clearly, $\mathfrak{n}L \subseteq \mathfrak{s}L$. The lattice L is said to be *normal* if equality holds, and *subnormal* otherwise. If $\mathfrak{n}L = \mathcal{O}_E$, then we call L *primitive*.

We define $\text{rank } L := \dim_E E \otimes L$. We can consider a Hermitian lattice of rank n as a quadratic \mathbb{Z} -module of rank $2n$ via a bilinear form $B(\mathbf{v}, \mathbf{w}) := \frac{1}{2} \text{Tr}_{E/\mathbb{Q}} H(\mathbf{v}, \mathbf{w})$. This \mathbb{Z} -module, called a *\mathbb{Z} -lattice associated with L* , will be distinguished by using subscript \mathbb{Z} .

If $n = \text{rank } L$, then we can write

$$L = \mathcal{A}_1 \mathbf{v}_1 + \cdots + \mathcal{A}_n \mathbf{v}_n$$

for some ideals $\mathcal{A}_1, \dots, \mathcal{A}_n \subseteq \mathcal{O}_E$ and vectors $\mathbf{v}_1, \dots, \mathbf{v}_n \in E \otimes L$. The *volume ideal* $\mathfrak{v}L$ of L is defined to be $\mathcal{A}_1 \overline{\mathcal{A}_1} \mathcal{A}_2 \overline{\mathcal{A}_2} \cdots \mathcal{A}_n \overline{\mathcal{A}_n} \det[H(\mathbf{v}_i, \mathbf{v}_j)]$.

EXAMPLE 2.1. Note that $L = \mathcal{O}_E \mathbf{v}_1 + (2, \omega_6) \mathbf{v}_2$ with

$$H(\mathbf{v}_1, \mathbf{v}_1) = 2, \quad H(\mathbf{v}_1, \mathbf{v}_2) = 0, \quad H(\mathbf{v}_2, \mathbf{v}_2) = 1/2$$

is a binary Hermitian lattice over $E = \mathbb{Q}(\sqrt{-6})$ and $\mathfrak{v}L = 2\mathcal{O}_E$. Then $\mathfrak{n}L$ and $\mathfrak{s}L$ are \mathcal{O}_E , and thus L is normal. The \mathbb{Z} -lattice associated with L is equipped with a quaternary quadratic form $Q(x, y, z, w) = 2x^2 + 12y^2 + 2z^2 + 3w^2$.

We can also write $L = \mathcal{O}_E \mathbf{v}_1 + \mathcal{O}_E(2\mathbf{v}_2) + \mathcal{O}_E(\omega_6 \mathbf{v}_2)$ and describe L by the 3×3 matrix

$$\mathcal{G}_L := \begin{bmatrix} H(\mathbf{v}_1, \mathbf{v}_1) & H(\mathbf{v}_1, 2\mathbf{v}_2) & H(\mathbf{v}_1, \omega_6 \mathbf{v}_2) \\ H(2\mathbf{v}_2, \mathbf{v}_1) & H(2\mathbf{v}_2, 2\mathbf{v}_2) & H(2\mathbf{v}_2, \omega_6 \mathbf{v}_2) \\ H(\omega_6 \mathbf{v}_2, \mathbf{v}_1) & H(\omega_6 \mathbf{v}_2, 2\mathbf{v}_2) & H(\omega_6 \mathbf{v}_2, \omega_6 \mathbf{v}_2) \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & -\omega_6 \\ 0 & \omega_6 & 3 \end{bmatrix}$$

formed using the generators $\mathbf{v}_1, 2\mathbf{v}_2, \omega_6 \mathbf{v}_2$. That is, if $x, y, z \in \mathcal{O}_E$, then

$$H(x\mathbf{v}_1 + (2y + \omega_6 z)\mathbf{v}_2) = [x \ y \ z] \mathcal{G}_L [x \ y \ z]^*.$$

We call such a matrix \mathcal{G}_L the *formal Gram matrix* of L . Henceforth, we simply call it the Gram matrix and often identify a lattice with its Gram matrix.

We define the *discriminant* dL of L as $\det \mathcal{G}_L$. Note that a nonfree positive definite lattice is of discriminant 0. If L is free, then $\mathfrak{v}L = (dL)\mathcal{O}_E$ and $N_{E/\mathbb{Q}}(\mathfrak{v}L) = |\mathcal{O}_E/\mathfrak{v}L| = (dL)^2$.

Assume that lattices L and K have Gram matrices \mathcal{G}_L and \mathcal{G}_K , respectively. If $\mathcal{G}_L = X\mathcal{G}_KX^*$ for some matrix X , then we write $L \rightarrow K$. If $L \rightarrow K$ and $K \rightarrow L$, then we say L is *isometric* to K and write $L \cong K$.

If a prime p does not split over $E = \mathbb{Q}(\sqrt{-m})$, then $E_p := \mathbb{Q}_p \otimes_{\mathbb{Q}} E = \mathbb{Q}_p(\sqrt{\theta})$ with $\theta \in \mathbb{Q}_p$ which has a natural involution and $\mathcal{O}_p = \mathbb{Z}_p \otimes_{\mathbb{Z}} \mathcal{O}_E$. If p splits, then $E_p = \mathbb{Q}_p \times \mathbb{Q}_p$, which has an involution $(a, b) = (b, a)$ and $\mathcal{O}_p = \mathbb{Z}_p \times \mathbb{Z}_p$. Thus, we can consider local lattices over E_p as \mathcal{O}_p -modules. We often write them as L_p to distinguish a local lattice from a global lattice L . The aforementioned terms can be defined for local lattices.

If a global lattice L represents all positive integers, we call it *universal*. If a local lattice L_p represents all elements of \mathbb{Z}_p , we call it *locally universal* or *\mathbb{Z}_p -universal* to specify p . It is clear that a regular lattice locally universal for every prime is globally universal.

Every local lattice can be decomposed orthogonally into lines and planes, that is, lattices of the form $\mathcal{O}_p\mathbf{v}$ and $\mathcal{O}_p\mathbf{v}_1 + \mathcal{O}_p\mathbf{v}_2$ for some vectors $\mathbf{v}, \mathbf{v}_1, \mathbf{v}_2$ [6, 9, 10]. This decomposition is called the *Jordan splitting*. In particular, if p is unramified, then L_p has an orthogonal basis.

3. Watson transformation. Now we introduce the Watson transformation for Hermitian lattices, which was initiated in [3]. It plays a crucial role because it produces a simpler lattice from a given regular lattice.

DEFINITION 3.1 ([3, Definition 3.1]). For any positive integer n and any prime p , let

$$\begin{aligned} \Lambda_n(L) &= \{\mathbf{v} \in L : H(\mathbf{v} + \mathbf{w}) - H(\mathbf{w}) \in n\mathbb{Z} \text{ for all } \mathbf{w} \in L\}, \\ \Lambda_n(L_p) &= \{\mathbf{v} \in L_p : H(\mathbf{v} + \mathbf{w}) - H(\mathbf{w}) \in n\mathbb{Z}_p \text{ for all } \mathbf{w} \in L_p\}. \end{aligned}$$

We can rescale $\Lambda_n(L)$ to be primitive, which results in what is called the *Watson transformation* of L , denoted by $\lambda_n(L)$.

LEMMA 3.2 ([3, Theorem 3.2]). For a positive integer n and a prime p ,

- (1) $\Lambda_n(L)$ is a sublattice of L and $\Lambda_n(L_p)$ is a sublattice of L_p .
- (2) $\Lambda_n(L_p) = (\Lambda_n(L))_p$.
- (3) $\Lambda_n(L_p) = L_p$ whenever $p \nmid n$.
- (4) $n\Lambda_n(L) \subseteq n\mathcal{O}_E$ and $n\Lambda_n(L_p) \subseteq n\mathcal{O}_p$.

LEMMA 3.3. If L is regular, so is $\Lambda_p(L)$.

Proof. Suppose $a \rightarrow (\Lambda_p(L))_q$ for every prime q . Then $a \rightarrow L_q$ since $\Lambda_p(L) \subseteq L$. The regularity of L implies that $a \rightarrow L$. So there exists $\mathbf{v} \in L$ such that $H(\mathbf{v}) = a$.

From $\mathbf{v} \in \Lambda_p(L)_p = \Lambda_p(L_p)$, we have $H(\mathbf{v} + \mathbf{w}) - H(\mathbf{w}) \in p\mathbb{Z}_p$ for any $\mathbf{w} \in L \subset L_p$. Since $H(\mathbf{v} + \mathbf{w})$ and $H(\mathbf{w})$ are integers for any $\mathbf{w} \in L$, it

follows that $H(\mathbf{v} + \mathbf{w}) - H(\mathbf{w}) \in p\mathbb{Z}$. Hence we conclude that $\mathbf{v} \in \Lambda_p(L)$ and $a \rightarrow \Lambda_p(L)$. ■

Now we need some lemmas for the local representation of a Hermitian lattice. In these lemmas, L is a primitive normal Hermitian lattice.

LEMMA 3.4 (split case). *If p is a prime which splits in $E = \mathbb{Q}(\sqrt{-m})$, then L_p is \mathbb{Z}_p -universal.*

Proof. In this case, $E_p = \mathbb{Q}_p \times \mathbb{Q}_p$ and we can write L_p as a sum of lines [6]. Since L is primitive, L_p represents a unit, say a , in \mathbb{Z}_p . Then the unary sublattice $\langle a \rangle$ represents all elements of \mathbb{Z}_p [6, Lemma 1.1]. ■

LEMMA 3.5 (inert case). *If p is an inert prime in $E = \mathbb{Q}(\sqrt{-m})$ and L_p is not \mathbb{Z}_p -universal, then we obtain a \mathbb{Z}_p -universal normal lattice from L by applying $\lambda_p(\cdot)$ repeatedly.*

Proof. The Jordan splitting of L_p is orthogonally

$$L_p \cong \langle a_1 \rangle \perp \langle a_2 p^{i_2} \rangle \perp \cdots \perp \langle a_n p^{i_n} \rangle$$

with $a_i \in \mathbb{Z}_p^\times$ and $2 \leq i_2 \leq \cdots \leq i_n$ [9].

Let $\mathbf{v} = x_1 \mathbf{v}_1 + x_2 \mathbf{v}_2 + \cdots + x_n \mathbf{v}_n \in L_p$ with orthogonal vectors \mathbf{v}_i . By Lemma 3.2(4), if $\mathbf{v} \in \Lambda_p(L_p)$, then x_1 is a multiple of p . Conversely, if $\mathbf{v} = x_1 \mathbf{v}_1 + \cdots + x_n \mathbf{v}_n$ with $x_1 \in p\mathcal{O}_p$, then

$$H(\mathbf{v} + \mathbf{w}) - H(\mathbf{w}) = (x_1 \bar{x}_1 + x_1 \bar{y}_1 + \bar{x}_1 y_1) a_1 + p(\dots) \in p\mathbb{Z}_p$$

for arbitrary $\mathbf{w} = y_1 \mathbf{v}_1 + \cdots + y_n \mathbf{v}_n \in L_p$. Therefore,

$$\Lambda_p(L)_p \cong \langle a_1 p^2 \rangle \perp \langle a_2 p^{i_2} \rangle \perp \cdots \perp \langle a_n p^{i_n} \rangle.$$

Since $H(\Lambda_p(L)) \subseteq p\mathbb{Z}_p$,

$$\lambda_p(L) \cong \Lambda_p(L)^{1/p^2} = \langle a_1 \rangle \perp \langle a_2 p^{i_2-2} \rangle \perp \cdots \perp \langle a_n p^{i_n-2} \rangle.$$

By iterating this process, we see that $\lambda_p \circ \cdots \circ \lambda_p(L)$ contains a sublattice $\langle a, b \rangle$ or $\langle a, bp \rangle$ with $a, b \in \mathbb{Z}_p^\times$, which is \mathbb{Z}_p -universal. ■

LEMMA 3.6 (odd ramified case). *If p is an odd ramified prime in $E = \mathbb{Q}(\sqrt{-m})$ and L_p is not \mathbb{Z}_p -universal, then we obtain a \mathbb{Z}_p -universal lattice from L by applying $\lambda_p(\cdot)$ repeatedly.*

Proof. In this case, m is divisible by p and $E_p = \mathbb{Q}_p(\sqrt{\pi})$, where $\pi = p\epsilon$ for some unit $\epsilon \in \mathbb{Z}_p^\times$.

Note that L_p can be represented as an orthogonal sum of lines $\langle ap^i \rangle$ and hyperbolic planes

$$\mathbb{H}(i) = \begin{bmatrix} 0 & \sqrt{\pi}^i \\ \frac{1}{\sqrt{\pi}} i & 0 \end{bmatrix}$$

with i odd.

Since the non- \mathbb{Z}_p -universal L_p represents a unit, say a_1 , in \mathbb{Z}_p , the Jordan splitting of L_p with nonincreasing scales is

$$\langle a_1 \rangle \perp \langle a_2 p^{i_2} \rangle \perp K \quad \text{or} \quad \langle a_1 \rangle \perp \mathbb{H}(i_2) \perp K$$

with $a_i \in \mathbb{Z}_p^\times$, $i_2 \geq 1$, and $\mathfrak{s}K \subseteq p^{i_2} \mathcal{O}_p$ [8, Proposition 8.1].

Similarly to the proof of Lemma 3.5, $\Lambda_p(L_p) = \{\mathbf{v} \in L_p \mid H(\mathbf{v}) \in p\mathbb{Z}_p\}$. Consequently, $\Lambda_p(L_p)$ is one of the following:

$$\langle a_1 p \epsilon \rangle \perp \langle a_2 p^{i_2} \rangle \perp K \quad \text{or} \quad \langle a_1 p \epsilon \rangle \perp \mathbb{H}(i_2) \perp K.$$

Since $\lambda_p(L)_p = \Lambda_p(L)_p^{1/p}$, we can obtain a unimodular binary sublattice $\langle a, b \rangle$ or a hyperbolic plane $\mathbb{H}(-1)$, each of which is \mathbb{Z}_p -universal, by iterating the Watson transformation. ■

REMARK 3.7. Unlike unramified cases, $\lambda_p(L)$ can be subnormal for p ramified prime, though L is normal.

Before investigating the dyadic ramified primes, we give an easy lemma about representation of \mathbb{Z}_2 -lattices.

LEMMA 3.8. *Let $\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4$ be units in \mathbb{Z}_2 . Then the following quaternary quadratic forms are all \mathbb{Z}_2 -universal over \mathbb{Z} :*

$$\langle \epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4 \rangle_{\mathbb{Z}}, \quad \langle \epsilon_1, \epsilon_2, 2\epsilon_3, 2\epsilon_4 \rangle_{\mathbb{Z}}, \quad \langle \epsilon_1, 2\epsilon_2, 2\epsilon_3, 4\epsilon_4 \rangle_{\mathbb{Z}}.$$

LEMMA 3.9 (dyadic ramified case). *If $p = 2$ is ramified in $E = \mathbb{Q}(\sqrt{-m})$ and L_2 is not \mathbb{Z}_2 -universal, then we obtain a \mathbb{Z}_2 -universal lattice from L by applying $\lambda_2(\cdot)$ repeatedly.*

Proof. In this case, there are two types of extension: R-P (ramified-prime) and R-U (ramified-unit).

If E/\mathbb{Q} is R-P, then m is even and $E_2 = \mathbb{Q}_2(\sqrt{\pi})$, where $\pi = 2\epsilon$ for some $\epsilon \in \mathbb{Z}_2^\times$. Note that $\mathcal{O}_2 = \mathbb{Z}_2[\sqrt{\pi}]$. Since L is primitive, its Jordan splitting is

$$\langle a_1 \rangle \perp \langle 2^{i_2} a_2 \rangle \perp K \quad \text{or} \quad \langle a_1 \rangle \perp \begin{bmatrix} 2^s & \sqrt{\pi}^{i_2-1} \\ -\sqrt{\pi}^{i_2-1} & \eta 2^t \end{bmatrix} \perp K$$

with $a_1, a_2, \eta \in \mathbb{Z}_2^\times$, $s, t \in \mathbb{N}$ and some K such that $\mathfrak{s}K \subseteq 2^{i_2}$. Thus $i_2 \geq 2$ and $2 \leq s \leq t$. This is enough here, in contrast to Jacobowitz's elaborate theory [8, §9, §10]. If the second component represents an element in $2\mathbb{Z}_2^\times$, then L_2 is \mathbb{Z}_2 -universal by Lemma 3.8.

For this case, $\mathbf{v} = \sum_{i=1}^n \alpha_i \mathbf{v}_i \in \Lambda_2(L)_2$ if and only if $H(\mathbf{v}) \in 2\mathcal{O}_2$. Therefore, $\Lambda_2(L)_2$ is

$$\langle 2\epsilon a_1 \rangle \perp \langle 2^{i_2} a_2 \rangle \perp K \quad \text{or} \quad \langle 2\epsilon a_1 \rangle \perp \begin{bmatrix} 2^s & \sqrt{\pi}^{i_2-1} \\ -\sqrt{\pi}^{i_2-1} & \eta 2^t \end{bmatrix} \perp K.$$

Iterating the Watson transformation, we obtain a lattice containing $\langle a, b \rangle$ or $\langle a, 2b \rangle$ with $a, b \in \mathbb{Z}_2^\times$, which is \mathbb{Z}_2 -universal.

If E/\mathbb{Q} is R-U, $m \equiv 1 \pmod{4}$ and $E_2 = \mathbb{Q}_2(\sqrt{\pi})$ where $\pi = 1 + 2\epsilon$ for some $\epsilon \in \mathbb{Z}_2^\times$. Then $\mathcal{O}_2 = \mathbb{Z}_2[1 + \sqrt{\pi}]$. The Jordan splitting of L_2 is

$$\langle a_1 \rangle \perp \langle 2^{i_2} a_2 \rangle \perp K \quad \text{or} \quad \langle a_1 \rangle \perp \begin{bmatrix} 2^s & (1 + \sqrt{\pi})^{i_2} \\ (1 - \sqrt{\pi})^{i_2} & \eta 2^t \end{bmatrix} \perp K$$

with $a_1, a_2, \eta \in \mathbb{Z}_2^\times$, $s, t \in \mathbb{N}$ and some K such that $\mathfrak{s}K \subseteq 2^{i_2}\mathcal{O}_2$.

If $i_2 < 2$, then the second components represent an element in $2\mathbb{Z}_2^\times$ and L_2 is \mathbb{Z}_2 -universal. Thus we may assume that $i_2 \geq 2$, $2 \leq s \leq t$, and $\mathfrak{n}K \subseteq 2^2\mathcal{O}_2$.

For $\mathbf{v} = \sum_{i=1}^n \alpha_i \mathbf{v}_i$ with generators \mathbf{v}_i , we have $\mathbf{v} \in \Lambda_2(L)_2$ if and only if $H(\mathbf{v}) \in 2\mathcal{O}_2$. Therefore, $\Lambda_2(L)_2$ is

$$\langle 2\epsilon a_1 \rangle \perp \langle 2^{i_2} a_2 \rangle \perp K \quad \text{or} \quad \langle 2\epsilon a_1 \rangle \perp \begin{bmatrix} 2^s & (1 + \sqrt{\pi})^{i_2} \\ (1 - \sqrt{\pi})^{i_2} & \eta 2^t \end{bmatrix} \perp K.$$

We can construct $\lambda_2(L)_2$ by scaling $\Lambda_2(L)_2$ by $1/2$. Note that $(1 + \sqrt{\pi})^{i_2}/2 = ((1 + \pi)/2 + \sqrt{\pi})(1 + \sqrt{\pi})^{i_2-2} \in \mathcal{O}_2$. Iterating the Watson transformation, we obtain a regular lattice containing $\langle a, b \rangle$ or $\langle a, 2b \rangle$ with $a, b \in \mathbb{Z}_2^\times$, which is \mathbb{Z}_2 -universal. ■

REMARK 3.10. Note that $\lambda_2(L)$ can be subnormal with some entries of the form $(a + b\sqrt{-m})/2$ when 2 divides m . Otherwise, $\lambda_2(L)$ is still normal.

We denote by $\lambda_{(p)}(L)$ the \mathbb{Z}_p -universal lattice transformed from L in the previous lemmas. If L_p itself is \mathbb{Z}_p -universal, we define $\lambda_{(p)}(L)$ as L .

EXAMPLE 3.11. Let $L = \mathcal{O}_E \mathbf{v}_1 + \mathcal{O}_E \mathbf{v}_2 + \mathcal{O}_E \mathbf{v}_3$ be a lattice with Gram matrix

$$\langle 1 \rangle \perp \begin{bmatrix} 36 & 30 + \sqrt{-30} \\ 30 - \sqrt{-30} & 60 \end{bmatrix}$$

over $E = \mathbb{Q}(\sqrt{-30})$. Since $dL = 1230 = 2 \cdot 3 \cdot 5 \cdot 41$, L_p is locally universal for $p \neq 2, 3, 5, 41$. Jordan splittings for other primes are as follows:

$$\begin{aligned} L_2 &\cong \langle 1 \rangle \perp \begin{bmatrix} 2^2 & \sqrt{2} \\ -\sqrt{2} & 2^2(-3^2 \cdot 31^{-1}) \end{bmatrix}, & L_3 &\cong \langle 1 \rangle \perp \begin{bmatrix} 0 & \sqrt{-3} \\ -\sqrt{-3} & 0 \end{bmatrix}, \\ L_5 &\cong \langle 1 \rangle \perp \langle 1 \rangle \perp \langle 5 \rangle, & L_{41} &\cong \langle 1 \rangle \perp \langle 1 \rangle \perp \langle 41 \cdot 6 \rangle. \end{aligned}$$

Note that L_2 and L_3 are not locally universal.

If $\mathbf{v} = \alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 + \alpha_3 \mathbf{v}_3 \in \Lambda_2(L)$, then $N_{E/\mathbb{Q}}(\alpha_1)$ should be divisible by 2. Thus α_1 is in the ideal $(2, \sqrt{-30})$. Hence,

$$\lambda_2(L) = \begin{bmatrix} 4 & 2\sqrt{-30} \\ -2\sqrt{-30} & 30 \end{bmatrix} \perp \begin{bmatrix} 36 & 30 + \sqrt{-30} \\ 30 - \sqrt{-30} & 60 \end{bmatrix},$$

$$\lambda_2(L) = \begin{bmatrix} 2 & \sqrt{-30} \\ -\sqrt{-30} & 15 \end{bmatrix} \perp \begin{bmatrix} 18 & (30 + \sqrt{-30})/2 \\ (30 - \sqrt{-30})/2 & 30 \end{bmatrix}.$$

In a similar way,

$$\lambda_3(L) = \begin{bmatrix} 3 & \sqrt{-30} \\ -\sqrt{-30} & 10 \end{bmatrix} \perp \begin{bmatrix} 12 & (30 + \sqrt{-30})/3 \\ (30 - \sqrt{-30})/3 & 20 \end{bmatrix}.$$

Note that $\lambda_2(L)_2$ and $\lambda_3(L)_3$ are locally universal. Also,

$$\lambda_2 \circ \lambda_3(L) = \begin{bmatrix} 6 & \sqrt{-30} \\ -\sqrt{-30} & 5 \end{bmatrix} \perp \begin{bmatrix} 6 & (30 + \sqrt{-30})/6 \\ (30 - \sqrt{-30})/6 & 10 \end{bmatrix}.$$

4. Main theorem and its proof. Not every primitive normal regular Hermitian lattice is universal. From the list of binary normal regular lattices [12] we know that there exists a nonuniversal binary regular lattice over $\mathbb{Q}(\sqrt{-m})$ if $m = 1, 2, 3, 5, 6, 7, 10, 11$, or 15 . But some imaginary quadratic fields allow only universal lattices as regular lattices.

MAIN THEOREM 4.1. *Every primitive normal regular Hermitian lattice is universal over $\mathbb{Q}(\sqrt{-m})$ if and only if $m \neq 1, 2, 11, 23$ and m is prime to $3 \cdot 5 \cdot 7$.*

We will show that a primitive normal regular lattice L defined over $\mathbb{Q}(\sqrt{-m})$ with m as in the Main Theorem is \mathbb{Z}_p -universal for every prime p . To do this, we construct a binary \mathbb{Z}_p -universal lattice from L via the Watson transformation. But there is an obstacle: the binary sublattice could be subnormal. So we proceed in five steps:

- STEP 1. Find odd ramified primes p for which L_p may not be \mathbb{Z}_p -universal.
- STEP 2. Show L_p is \mathbb{Z}_p -universal for p found in Step 1.
- STEP 3. Show L_2 is \mathbb{Z}_2 -universal.
- STEP 4. Show L_p is \mathbb{Z}_p -universal for other primes p .
- STEP 5. Give some nonuniversal primitive normal regular lattices for the exceptional m 's.

Now we start the main proof. Let S be the set of primes that divide $2m \cdot \mathbf{N}_{E/\mathbb{Q}}(\mathbf{v}L)$. If $p \notin S$, then L_p is \mathbb{Z}_p -universal. Thus $\lambda_{(p)}(L) = L$ for almost all p .

Applying $\lambda_{(q)}(\cdot)$ to L for all $q \neq p$, we eventually obtain a lattice which is \mathbb{Z}_q -universal for every $q \neq p$ by the lemmas in the previous section. Denote the lattice obtained by applying the Watson transform at all but the target prime p by $\text{WAT}_p(L)$. Note that $\text{WAT}_p(L)_p = L_p$ by Lemma 3.2(3). If

q_1, \dots, q_n are primes other than p such that $\lambda_{(q_i)}(L)$ is subnormal, then $Q := q_1 \cdots q_n$ divides m . If no such q_i 's exist, we set $Q = 1$. Then we have a useful lemma.

LEMMA 4.2. *Let p be an odd ramified prime. Suppose $\text{WAT}_p(L)$ contains a sublattice*

$$\ell = \begin{bmatrix} s & (a + b\omega_m)/Q \\ (a + b\bar{\omega}_m)/Q & t \end{bmatrix}.$$

If ℓ_p is not \mathbb{Z}_p -universal, then

$$Q \mid \text{Tr}_{E/\mathbb{Q}}(a + b\omega_m) \quad \text{and} \quad p \mid 4st - (\text{Tr}_{E/\mathbb{Q}}(a + b\omega_m)/Q)^2.$$

Proof. The sublattice ℓ is produced by scaling a normal lattice

$$\ell^Q = \begin{bmatrix} sQ & a + b\omega_m \\ a + b\bar{\omega}_m & tQ \end{bmatrix}.$$

Since $\mathfrak{n}(\ell^Q)$ is supposed to be included in $q\mathcal{O}_E$ for every prime $q \mid Q$, we see that q divides $T := \text{Tr}_{E/\mathbb{Q}}(a + b\omega_m)$. Thus, $Q \mid T$.

We find that ℓ is not \mathbb{Z}_p -universal only when

$$d\ell = st - \frac{\text{N}_{E/\mathbb{Q}}(a + b\omega_m)}{Q^2} = \frac{1}{4Q} \left[\left(4st - \left(\frac{T}{Q} \right)^2 \right) Q - 4b^2 \left(\frac{m}{Q} \right) \right]$$

is divisible by p . So p should divide $4st - (T/Q)^2$, since $p \mid (m/Q)$. ■

4.1. Step 1. Assume that $m > 11$ and m is prime to $3 \cdot 5 \cdot 7$. Given a primitive normal regular lattice L over $\mathbb{Q}(\sqrt{-m})$, suppose L_p is not \mathbb{Z}_p -universal for an odd ramified prime p . That is, p is a divisor of m and is not 3, 5, 7.

Let us consider $\text{WAT}_p(L)$. If we could construct a binary unimodular \mathbb{Z}_p -sublattice, then $\text{WAT}_p(L)_p = L_p$ would be \mathbb{Z}_p -universal. Thus, $\text{WAT}_p(L)_p$ cannot contain any binary unimodular sublattice. Since L is primitive, $\text{WAT}_p(L)_p$ should represent a unit in \mathbb{Z}_p^\times . We consider two cases according to whether the unit is a square unit (Step 1-1) or a nonsquare unit (Step 1-2).

4.1.1. STEP 1-1. Suppose that $1_p \rightarrow \text{WAT}_p(L)_p$. Then $1 \rightarrow \text{WAT}_p(L)$ globally by regularity of L .

STEP 1-1-1. If $\left(\frac{2}{p}\right) = 1$, then $\text{WAT}_p(L)$ contains a binary sublattice

$$\ell \cong \begin{bmatrix} 1 & (a + b\omega_m)/Q \\ (a + b\bar{\omega}_m)/Q & 2 \end{bmatrix} \cong \left\langle 1, 2 - \frac{\text{N}_{E/\mathbb{Q}}(a + b\omega_m)}{Q^2} \right\rangle.$$

By Lemma 4.2, ℓ is not \mathbb{Z}_p -universal when $4 \cdot 1 \cdot 2 - A^2$ is divisible by p for some integer A . But there is no such A . So $L_p = \text{WAT}_p(L)_p$ is \mathbb{Z}_p -universal.

STEP 1-1-2. Now, assume that $\left(\frac{2}{p}\right) = -1$ and $\left(\frac{3}{p}\right) = 1$. In this case, the sublattice $\langle 1, 3 - N_{E/\mathbb{Q}}(a + b\omega_m)/Q^2 \rangle$ is not \mathbb{Z}_p -universal only when $4 \cdot 1 \cdot 3 - 1^2 = 11$ is divisible by p . Thus, if a nonuniversal normal regular lattice could exist, then it might be non- \mathbb{Z}_{11} -universal.

STEP 1-1-3. If $\left(\frac{2}{p}\right) = \left(\frac{3}{p}\right) = -1$, then $\left(\frac{6}{p}\right) = 1$ and $\text{WAT}_p(L)$ thus contains $\langle 1, 6 - N_{E/\mathbb{Q}}(a + b\omega_m)/Q^2 \rangle$. The candidate is $p = 23$.

4.1.2. STEP 1-2. Now, suppose that $1_p \not\rightarrow \text{WAT}_p(L)$ and $\Delta_p \rightarrow \text{WAT}_p(L)$.

STEP 1-2-1. If $\left(\frac{2}{p}\right) = \left(\frac{3}{p}\right) = -1$, then $\text{WAT}_p(L)_p$ can contain a non- \mathbb{Z}_p -universal sublattice $\left[\begin{smallmatrix} 2 & (a+b\omega_m)/Q \\ (a+b\bar{\omega}_m)/Q & 3 \end{smallmatrix} \right]$ only when $p = 23$. But $\left(\frac{2}{23}\right) = \left(\frac{3}{23}\right) = 1$, contrary to the condition $\left(\frac{2}{p}\right) = \left(\frac{3}{p}\right) = -1$.

STEP 1-2-2. If $\left(\frac{2}{p}\right) = -1$ and $\left(\frac{3}{p}\right) = 1$, then $\left(\frac{6}{p}\right) = -1$ and $\text{WAT}_p(L)$ contains a sublattice $\left[\begin{smallmatrix} 2 & (a+b\omega_m)/Q \\ (a+b\bar{\omega}_m)/Q & 6 \end{smallmatrix} \right]$. For this sublattice not to be \mathbb{Z}_p -universal, p should be 11, 13, 23, or 47 by Lemma 4.2. But 23 and 47 are excluded since $\left(\frac{2}{23}\right) = \left(\frac{2}{47}\right) = 1$.

STEP 1-2-3. Finally, consider the case $\left(\frac{2}{p}\right) = 1$. Note that $p \equiv \pm 1 \pmod{8}$. In this case, quadratic nonresidues could be large. So we need some tools to bound them.

STEP 1-2-3-1. Assume that $p \equiv 1 \pmod{8}$. Granville, Mollin, and Williams [7] proved that if $D > 3705$ is a fundamental discriminant of a real quadratic field, then the Kronecker symbol $\left(\frac{D}{q}\right)$ is -1 for some prime $q < \sqrt{D}/2$ (see also Sun [15]). Thus, if $p > 3705$ is a prime with $p \equiv 1 \pmod{4}$, then there exists a quadratic nonresidue $q < \sqrt{p}/2$ by the quadratic reciprocity law.

Since we assume $\left(\frac{2}{p}\right) = 1$, q and $2q$ are two quadratic nonresidues and $\text{WAT}_p(L)$ contains a sublattice $\left[\begin{smallmatrix} q & (a+b\omega_m)/Q \\ (a+b\bar{\omega}_m)/Q & 2q \end{smallmatrix} \right]$. If this lattice is not \mathbb{Z}_p -universal, then p divides $8q^2 - A^2$ for some integer A by Lemma 4.2.

Assume that $p > 3705$. If A is even, say $2r$, then p should divide $2q^2 - r^2$. Since $2q^2 < 4q^2 < p$, we obtain $2q^2 - r^2 = 0$, which is absurd.

If A is odd, $8q^2 - A^2 \equiv 7 \pmod{8}$ and thus $(8q^2 - A^2)/p \equiv 7 \pmod{8}$. Then $(8q^2 - A^2)/p \geq 7$. But $7p \leq 8q^2 - A^2 \leq 8q^2 < 2p$, a contradiction.

Assume now $p < 3705$. Let q_1 and q_2 be the two least quadratic nonresidues. Since $\left(\frac{2q_1}{p}\right) = \left(\frac{2}{p}\right)\left(\frac{q_1}{p}\right) = -1$, we have $q_1 < q_2 \leq 2q_1$ and $\text{WAT}_p(L)$ contains a sublattice $\left[\begin{smallmatrix} q_1 & (a+b\omega_m)/Q \\ (a+b\bar{\omega}_m)/Q & q_2 \end{smallmatrix} \right]$. We should find when $4q_1q_2 - A^2$ is divisible by p for some integer A , by Lemma 4.2.

From Table 1, we have $q_1q_2 \leq 13 \cdot 17 = 221$. Thus, $4q_1q_2 - A^2 < p$ for $4 \cdot 221 < p < 3705$. This means $4q_1q_2 - A^2$ vanishes, but this is impossible

for two least quadratic nonresidues q_1 and q_2 . So the above sublattice is \mathbb{Z}_p -universal.

Table 1. Two least quadratic nonresidues for $p \equiv 1 \pmod{8}$

p	q_1, q_2	p	q_1, q_2	p	q_1, q_2
17	3, 5	433 – 593	≤ 10	1489	7, 13
41	3, 6	601	7, 11	1553, 1601	≤ 6
73	5, 7	617 – 761	≤ 10	1609	7, 14
89	3, 6	769	7, 11	1657 – 2153	≤ 13
97	5, 7	809 – 977	≤ 7	2161	7, 14
113, 137	3, 5	1009	11, 13	2273 – 2473	≤ 13
193	5, 10	1033 – 1097	≤ 10	2521	11, 17
233	3, 5	1129	11, 13	2593 – 2657	≤ 7
241	7, 11	1153, 1193	≤ 7	2689	13, 17
257	3, 5	1201	11, 13	2713 – 3457	≤ 14
281	3, 6	1217 – 1297	≤ 11	3529	13, 17
313 – 401	≤ 10	1321	7, 13	3593 – 3761	≤ 10
409	7, 11	1361 – 1481	≤ 6		

Similarly, $q_1q_2 \leq 7 \cdot 11 = 77$ for $p < 4 \cdot 221$. That is, if $4 \cdot 77 < p$, then $4q_1q_2 - A^2$ vanishes. This is impossible. So the sublattice is \mathbb{Z}_p -universal.

The sublattice is also \mathbb{Z}_p -universal for $41 \leq p \leq 4 \cdot 77 = 308$. For example, $p = 41$ cannot divide $4 \cdot 3 \cdot 6 - A^2$.

The only exception occurs when $p = 17$. That is, $4q_1q_2 - A^2 = 4 \cdot 3 \cdot 5 - 3^2 = 51$ is divisible by 17.

STEP 1-2-3-2. Now, finally, consider $p \equiv -1 \pmod{8}$. We have

$$q < (2p)^{2/5} + 3(2p)^{1/5} + 1,$$

where an odd prime q is the least quadratic nonresidue. This was proved for $p \not\equiv 1 \pmod{8}$ by Brauer [2]. If $p \geq 2034743$, the above inequality yields $2q^2 < p/4$. If $\text{WAT}_p(L)$ contains a sublattice $\left[\begin{smallmatrix} q & (a+b\omega_m)/Q \\ (a+b\bar{\omega}_m)/Q & 2q \end{smallmatrix} \right]$, then by Lemma 4.2 we should check when $8q^2 - A^2$ vanishes for some integer A since $8q^2 < p$. But this value cannot vanish.

For $p < 2034743$, let q_1 and q_2 be the two least quadratic nonresidues. Then $\text{WAT}_p(L)$ contains a sublattice $\left[\begin{smallmatrix} q_1 & (a+b\omega_m)/Q \\ (a+b\bar{\omega}_m)/Q & q_2 \end{smallmatrix} \right]$ of discriminant divisible by p .

From Table 2, $q_1q_2 \leq 47 \cdot 53 = 2497$. If $4 \cdot 2497 < p < 2034743$, then $p/4 > q_1q_2$, and thus $4q_1q_2 - A^2 = 0$. But this does not hold for the two least quadratic nonresidues q_1 and q_2 .

The same reasoning shows

$$q_1 q_2 \leq 19 \cdot 29 = 551 \quad \text{for } p < 4 \cdot 2497 = 9988,$$

$$q_1 q_2 \leq 17 \cdot 19 = 323 \quad \text{for } p < 4 \cdot 551 = 2204,$$

$$q_1 q_2 \leq 13 \cdot 17 = 221 \quad \text{for } p < 4 \cdot 323 = 1292.$$

Hence $4q_1 q_2 - A^2$ cannot be divisible by p for $4 \cdot 221 = 884 < p < 2034743$.

Table 2. Two least quadratic nonresidues for $p \equiv -1 \pmod{8}$

p	q_1, q_2	p	q_1, q_2	p	q_1, q_2
7	3, 5	599	7, 11	15791	23, 29
23	5, 7	607	3, 5	15823 – 18143	≤ 23
31	3, 6	631	3, 6	18191	29, 31
47	5, 10	647	5, 10	18199 – 31327	≤ 29
71	7, 11	719	11, 17	31391	31, 37
79	3, 6	727	3, 5	31511 – 38567	≤ 31
103	3, 5	743	5, 7	38639	29, 41
127	3, 5	751	3, 6	38671 – 95383	≤ 37
151	3, 6	823	3, 5	95471	29, 43
167	5, 10	839	11, 13	95479 – 366727	≤ 41
191	7, 11	863	5, 7	366791	43, 47
199	3, 6	887 – 1103	≤ 14	366967 – 514711	≤ 43
223	3, 5	1151	13, 17	514751	43, 47
239	7, 13	1223 – 1303	≤ 10	514783 – 572687	≤ 41
263	5, 7	1319	13, 17	572711	29, 47
271	3, 6	1327 – 1487	≤ 11	572791 – 701383	≤ 41
311	11, 17	1511	11, 22	701399	41, 47
359	7, 13	1543 – 4007	≤ 19	701447 – 1062263	≤ 43
367	3, 5	4079	11, 22	1062311	29, 47
383	5, 10	4111 – 5647	≤ 19	1062343 – 1139503	≤ 43
431	7, 13	5711	19, 29	1139519	41, 47
439	3, 6	5743 – 6367	≤ 17	1139623 – 1412711	≤ 43
463	3, 5	6551	17, 29	1412759	41, 59
479	13, 17	6599 – 10487	≤ 23	1412767 – 2449871	≤ 53
487	3, 5	10559	23, 29		
503	5, 10	10567 – 15767	≤ 23		

When $31 \leq p < 884$, we can verify that there exists no integer A such that $4q_1 q_2 - A^2$ is divisible by p . The only exception is $4 \cdot 5 \cdot 7 - 5^2$ for $p = 23$.

From the above results, we can summarize that if L_p is not locally universal for an odd ramified prime p , then p is 11 (Steps 1-1-2, 1-2-2), 13 (Step 1-2-2), 17 (Step 1-2-3-1), or 23 (Steps 1-1-3, 1-2-3-2).

4.2. Step 2. We know that if p is an odd divisor of m and is not 11, 13, 17, 23, then L_p is locally universal. Recall that L_q is also locally universal for $q \notin S = \{p : p \mid 2m \cdot N_{E/\mathbb{Q}}(\mathfrak{v}L)\}$.

Now, let $\widehat{L} = \text{WAT}_{S \cap \{11, 13, 17, 23\}}(L)$ be the lattice obtained from L by applying $\lambda_{(q)}(\cdot)$ for $q \notin S \cap \{11, 13, 17, 23\}$. By the lemmas in the previous section, \widehat{L}_q is normal for every $q \neq 2$.

If \widehat{L} contains a locally nonuniversal sublattice $\left[\begin{smallmatrix} s & (a+b\omega_m)/Q \\ (a-b\omega_m)/Q & t \end{smallmatrix} \right]$, then $Q = 1$ or $Q = 2$ and $d\ell = st - N_{E/\mathbb{Q}}(a + b\omega)/Q^2 \in p\mathbb{Z}_p$ for $p \in S \cap \{11, 13, 17, 23\}$. Note that $Q = 2$ is possible when m is even.

Table 3 shows numbers represented by L depending on $S \cap \{11, 13, 17, 23\}$. If $S \cap \{11, 13, 17, 23\} = \emptyset$, go to Step 3. By Lemma 4.2, $p \mid (4st - A^2)$ for some A . Among the 80 cases in Table 3, this is possible only for five cases: Nos. 1, 2, 4, 6, and 8.

4.2.1. No. 1: $11 \mid (4 \cdot 1 \cdot 3 - 1^2)$. In this case \widehat{L} contains a lattice $\ell = \left[\begin{smallmatrix} 1 & (a+b\omega_m)/Q \\ (a-b\omega_m)/Q & 3 \end{smallmatrix} \right]$ with $11 \mid (3Q^2 - N_{E/\mathbb{Q}}(a + b\omega_m))$. This is possible only for $Q = 2$ and $a + b\omega_m = 1$, since $m > 11$. Hence, we can say that m is even and $\ell = \left[\begin{smallmatrix} 1 & 1/2 \\ 1/2 & 3 \end{smallmatrix} \right]$ must be produced by scaling $\left[\begin{smallmatrix} 2 & 1 \\ 1 & 6 \end{smallmatrix} \right]$ by $1/2$ in calculating $\lambda_2(L)$. But $\langle a_1 \rangle \perp \left[\begin{smallmatrix} 2 & 1 \\ 1 & 6 \end{smallmatrix} \right]$ with $a_1 \in \mathbb{Z}_2$ before being transformed is locally universal, and thus $\left[\begin{smallmatrix} 1 & 1/2 \\ 1/2 & 3 \end{smallmatrix} \right]$ cannot be produced.

4.2.2. No. 2: $11 \mid (4 \cdot 2 \cdot 7 - 1^2)$. In this case, if \widehat{L} contains a non- \mathbb{Z}_{11} -universal sublattice, it is $\left[\begin{smallmatrix} 2 & 1/2 \\ 1/2 & 7 \end{smallmatrix} \right]$ or $\left[\begin{smallmatrix} 2 & (1+\omega_{22})/2 \\ (1-\omega_{22})/2 & 7 \end{smallmatrix} \right]$. But, by a reasoning similar to the previous case, these lattices cannot be produced in calculating $\lambda_2(L)$ since $7 \in \mathbb{Z}_2$.

4.2.3. No. 4: $13 \mid (4 \cdot 2 \cdot 5 - 1^2)$. Similarly to the previous case, since $5 \in \mathbb{Z}_2^\times$, \widehat{L} cannot contain a non- \mathbb{Z}_{13} -universal sublattice produced by $\lambda_{(2)}(\cdot)$.

4.2.4. No. 6: $17 \mid (4 \cdot 3 \cdot 5 - 3^2)$. Since $3, 5 \in \mathbb{Z}_2$, \widehat{L} cannot contain a non- \mathbb{Z}_{17} -universal sublattice produced by $\lambda_{(2)}(\cdot)$.

4.2.5. No. 8: $23 \mid (4 \cdot 5 \cdot 7 - 5^2)$. Since $5, 7 \in \mathbb{Z}_2$, \widehat{L} cannot contain non- \mathbb{Z}_{23} -universal sublattice produced by $\lambda_{(2)}(\cdot)$ unless $m = 23$.

Let us scrutinize regular sublattices for $m = 23$. Note that $\lambda_2(L)$ is normal, since $2 \nmid 23$. From the above results, we can assume that $\text{WAT}_{23}(L)$ contains a normal sublattice $\ell = \left[\begin{smallmatrix} 5 & 2+\omega_{23} \\ 2+\bar{\omega}_{23} & 7 \end{smallmatrix} \right]$.

Table 3. Square and nonsquare units for $p = 11, 13, 17, 23$

No.	L_{11}	L_{13}	L_{17}	L_{23}	L	No.	L_{11}	L_{13}	L_{17}	L_{23}	L
1	1_{11}	\cdot	\cdot	\cdot	1, 3	41	1_{11}	\cdot	1_{17}	1_{23}	1, 26
2	Δ_{11}	\cdot	\cdot	\cdot	2, 7	42	1_{11}	\cdot	1_{17}	Δ_{23}	15, 38
3	\cdot	1_{13}	\cdot	\cdot	1, 3	43	1_{11}	\cdot	Δ_{17}	1_{23}	3, 31
4	\cdot	Δ_{13}	\cdot	\cdot	2, 5	44	1_{11}	\cdot	Δ_{17}	Δ_{23}	5, 14
5	\cdot	\cdot	1_{17}	\cdot	1, 2	45	Δ_{11}	1_{13}	1_{17}	\cdot	30, 35
6	\cdot	\cdot	Δ_{17}	\cdot	3, 5	46	Δ_{11}	1_{13}	Δ_{17}	\cdot	10, 29
7	\cdot	\cdot	\cdot	1_{23}	1, 2	47	Δ_{11}	1_{13}	\cdot	1_{23}	29, 35
8	\cdot	\cdot	\cdot	Δ_{23}	5, 7	48	Δ_{11}	1_{13}	\cdot	Δ_{23}	10, 17
9	1_{11}	1_{13}	\cdot	\cdot	1, 3	49	Δ_{11}	Δ_{13}	1_{17}	\cdot	2, 19
10	1_{11}	Δ_{13}	\cdot	\cdot	5, 15	50	Δ_{11}	Δ_{13}	Δ_{17}	\cdot	6, 7
11	1_{11}	\cdot	1_{17}	\cdot	1, 15	51	Δ_{11}	Δ_{13}	\cdot	1_{23}	2, 6
12	1_{11}	\cdot	Δ_{17}	\cdot	3, 5	52	Δ_{11}	Δ_{13}	\cdot	Δ_{23}	7, 19
13	1_{11}	\cdot	\cdot	1_{23}	1, 3	53	Δ_{11}	\cdot	1_{17}	1_{23}	2, 13
14	1_{11}	\cdot	\cdot	Δ_{23}	5, 14	54	Δ_{11}	\cdot	1_{17}	Δ_{23}	19, 21
15	Δ_{11}	1_{13}	\cdot	\cdot	10, 17	55	Δ_{11}	\cdot	Δ_{17}	1_{23}	6, 29
16	Δ_{11}	Δ_{13}	\cdot	\cdot	2, 6	56	Δ_{11}	\cdot	Δ_{17}	Δ_{23}	7, 10
17	Δ_{11}	\cdot	1_{17}	\cdot	2, 13	57	\cdot	1_{13}	1_{17}	1_{23}	1, 35
18	Δ_{11}	\cdot	Δ_{17}	\cdot	6, 7	58	\cdot	1_{13}	1_{17}	Δ_{23}	30, 38
19	Δ_{11}	\cdot	\cdot	1_{23}	2, 6	59	\cdot	1_{13}	Δ_{17}	1_{23}	3, 29
20	Δ_{11}	\cdot	\cdot	Δ_{23}	7, 10	60	\cdot	1_{13}	Δ_{17}	Δ_{23}	10, 14
21	\cdot	1_{13}	1_{17}	\cdot	1, 30	61	\cdot	Δ_{13}	1_{17}	1_{23}	2, 47
22	\cdot	1_{13}	Δ_{17}	\cdot	3, 10	62	\cdot	Δ_{13}	1_{17}	Δ_{23}	15, 19
23	\cdot	1_{13}	\cdot	1_{23}	1, 3	63	\cdot	Δ_{13}	Δ_{17}	1_{23}	6, 31
24	\cdot	1_{13}	\cdot	Δ_{23}	10, 14	64	\cdot	Δ_{13}	Δ_{17}	Δ_{23}	5, 7
25	\cdot	Δ_{13}	1_{17}	\cdot	2, 15	65	1_{11}	1_{13}	1_{17}	1_{23}	1, 179
26	\cdot	Δ_{13}	Δ_{17}	\cdot	5, 6	66	1_{11}	1_{13}	1_{17}	Δ_{23}	38, 42
27	\cdot	Δ_{13}	\cdot	1_{23}	2, 6	67	1_{11}	1_{13}	Δ_{17}	1_{23}	3, 82
28	\cdot	Δ_{13}	\cdot	Δ_{23}	5, 7	68	1_{11}	1_{13}	Δ_{17}	Δ_{23}	14, 113
29	\cdot	\cdot	1_{17}	1_{23}	1, 2	69	1_{11}	Δ_{13}	1_{17}	1_{23}	47, 59
30	\cdot	\cdot	1_{17}	Δ_{23}	15, 19	70	1_{11}	Δ_{13}	1_{17}	Δ_{23}	15, 67
31	\cdot	\cdot	Δ_{17}	1_{23}	3, 6	71	1_{11}	Δ_{13}	Δ_{17}	1_{23}	31, 58
32	\cdot	\cdot	Δ_{17}	Δ_{23}	5, 7	72	1_{11}	Δ_{13}	Δ_{17}	Δ_{23}	5, 37
33	1_{11}	1_{13}	1_{17}	\cdot	1, 38	73	Δ_{11}	1_{13}	1_{17}	1_{23}	35, 87
34	1_{11}	1_{13}	Δ_{17}	\cdot	3, 14	74	Δ_{11}	1_{13}	1_{17}	Δ_{23}	30, 43
35	1_{11}	1_{13}	\cdot	1_{23}	1, 3	75	Δ_{11}	1_{13}	Δ_{17}	1_{23}	29, 62
36	1_{11}	1_{13}	\cdot	Δ_{23}	14, 38	76	Δ_{11}	1_{13}	Δ_{17}	Δ_{23}	10, 61
37	1_{11}	Δ_{13}	1_{17}	\cdot	15, 47	77	Δ_{11}	Δ_{13}	1_{17}	1_{23}	2, 123
38	1_{11}	Δ_{13}	Δ_{17}	\cdot	5, 31	78	Δ_{11}	Δ_{13}	1_{17}	Δ_{23}	19, 21
39	1_{11}	Δ_{13}	\cdot	1_{23}	31, 47	79	Δ_{11}	Δ_{13}	Δ_{17}	1_{23}	6, 41
40	1_{11}	Δ_{13}	\cdot	Δ_{23}	5, 15	80	Δ_{11}	Δ_{13}	Δ_{17}	Δ_{23}	7, 57

Since 10 is a nonsquare unit in \mathbb{Z}_{23} and $10 \nrightarrow \ell$, we can escalate ℓ to

$$\begin{bmatrix} 5 & 2 + \omega_{23} & \alpha \\ 2 + \bar{\omega}_{23} & 7 & \beta \\ \bar{\alpha} & \bar{\beta} & 10 \end{bmatrix}$$

with some $\alpha, \beta \in \mathbb{Z}[\omega_{23}]$. Since every binary or ternary sublattice is of discriminant divisible by 23, we obtain seven candidates:

$$\begin{aligned} \text{vol} = 23\mathcal{O}_E : & \begin{bmatrix} 5 & 2 + \omega_{23} & 1 + 2\omega_{23} \\ 2 + \bar{\omega}_{23} & 7 & 2\omega_{23} \\ 1 + 2\bar{\omega}_{23} & 2\bar{\omega}_{23} & 10 \end{bmatrix}, \begin{bmatrix} 5 & 2 + \omega_{23} & -3 + 2\omega_{23} \\ 2 + \bar{\omega}_{23} & 7 & -1 \\ -3 + 2\bar{\omega}_{23} & -1 & 10 \end{bmatrix}, \\ \text{vol} = 46\mathcal{O}_E : & \begin{bmatrix} 5 & 2 + \omega_{23} & 2 \\ 2 + \bar{\omega}_{23} & 7 & 2\omega_{23} \\ 2 & 2\bar{\omega}_{23} & 10 \end{bmatrix}, \\ \text{vol} = 69\mathcal{O}_E : & \begin{bmatrix} 5 & 2 + \omega_{23} & 1 + 2\omega_{23} \\ 2 + \bar{\omega}_{23} & 7 & 1 \\ 1 + 2\bar{\omega}_{23} & 1 & 10 \end{bmatrix}, \begin{bmatrix} 5 & 2 + \omega_{23} & -3 + 2\omega_{23} \\ 2 + \bar{\omega}_{23} & 7 & -2 + 2\omega_{23} \\ -3 + 2\bar{\omega}_{23} & -2 + 2\bar{\omega}_{23} & 10 \end{bmatrix}, \\ \text{vol} = 138\mathcal{O}_E : & \begin{bmatrix} 5 & 2 + \omega_{23} & -2 \\ 2 + \bar{\omega}_{23} & 7 & -2 + 2\omega_{23} \\ -2 & -2 + 2\bar{\omega}_{23} & 10 \end{bmatrix} \leftarrow 14, \\ \text{vol} = 207\mathcal{O}_E : & \begin{bmatrix} 5 & 2 + \omega_{23} & 2 \\ 2 + \bar{\omega}_{23} & 7 & 1 \\ 2 & 1 & 10 \end{bmatrix} \leftarrow 14. \end{aligned}$$

We can check that the first five lattices are regular. The last two are almost regular with the only exception 14.

Now we can conclude that L_p itself is \mathbb{Z}_p -universal for each odd ramified prime p if $m \neq 1, 2, 11, 23$ and $(m, 3 \cdot 5 \cdot 7) = 1$. But $\mathbb{Q}(\sqrt{-23})$ allows primitive normal regular lattices which are not \mathbb{Z}_{23} -universal.

4.3. Step 3. Consider $p = 2$. Assume that $m \neq 1, 2, 11, 23$ and m is prime to $3 \cdot 5 \cdot 7$. Let L be a non- \mathbb{Z}_2 -universal lattice. Note that $\text{WAT}_2(L)$ is normal and \mathbb{Z}_p -universal for odd primes p from the previous steps.

We consider eight cases depending on m modulo 16. We denote the four types of units in \mathbb{Z}_2 by $1_2, 3_2, 5_2, 7_2$.

4.3.1. $m \equiv 1 \pmod{8}$. Note that $\langle 1 \rangle_2$ is associated with $\langle 1, m \rangle_{\mathbb{Z}_2} \cong \langle 1, 1 \rangle_{\mathbb{Z}_2}$ and it thus represents $1_2, 5_2, 2 \cdot 1_2, 2 \cdot 5_2$.

If $1_2 \rightarrow \text{WAT}_2(L)_2$, then $1, 2 \rightarrow \text{WAT}_2(L)$. Thus $\text{WAT}_2(L)$ contains $\langle 1, 1 \rangle$ or $\langle 1, 2 \rangle$, each of which is \mathbb{Z}_2 -universal.

Now assume that $3_2 \rightarrow \text{WAT}_2(L)_2$. Then $3, 6 \rightarrow \text{WAT}_2(L)$ and $\begin{bmatrix} 3 & a+b\omega_m \\ a+b\bar{\omega}_m & 6 \end{bmatrix}$ is contained in $\text{WAT}_2(L)$. If the discriminant is not divisible by 4, the binary sublattice is \mathbb{Z}_2 -universal by Lemma 3.8. The discriminant

is divisible by 4 only when $a + b\omega_m = \pm 1 \pm \omega_{17}$. The sublattice $\begin{bmatrix} 3 & 1+\omega_{17} \\ 1+\bar{\omega}_{17} & 6 \end{bmatrix}$ represents 7_2 locally, but does not represent 15 globally. So, it is escalated by attaching a vector \mathbf{v} with $H(\mathbf{v}) = 15$. But this escalation process results in \mathbb{Z}_2 -universal lattices.

On the other hand, since $\langle 5 \rangle_2$ represents 1_2 locally and $\langle 7 \rangle_2$ represents 3_2 locally, we return to the previous cases.

4.3.2. $m \equiv 3 \pmod{8}$. Note that $\langle 1 \rangle_2$ represents all units in \mathbb{Z}_2 . If $\text{WAT}_2(L)_2$ represents a unit, it represents 1_2 and 3_2 . Thus $\text{WAT}_2(L)$ contains $\langle 1, 2 \rangle$ or $\langle 1, 3 \rangle$. These two lattices are \mathbb{Z}_2 -universal.

4.3.3. $m \equiv 5 \pmod{8}$. Note that $\langle 1 \rangle_2$ represents $1_2, 5_2, 2 \cdot 3_2, 2 \cdot 7_2$. If $1_2 \rightarrow \text{WAT}_2(L)_2$ or $5_2 \rightarrow \text{WAT}_2(L)_2$, then $\text{WAT}_2(L)$ represents both 1 and 5. Then $\text{WAT}_2(L)$ contains $\langle 1, 1 \rangle$, $\langle 1, 4 \rangle$, or $\langle 1, 5 \rangle$. From Lemma 3.8, $\langle 1, 1 \rangle_2$ and $\langle 1, 5 \rangle_2$ represent all elements of \mathbb{Z}_2 . Since $\langle 1, 4 \rangle_2$ represents 1_2 and $2 \cdot 1_2$, $\text{WAT}_2(L)$ contains $\langle 1, 2 \rangle$ or $\langle 1, 1 \rangle$, each of which is \mathbb{Z}_2 -universal.

If $3_2 \rightarrow \text{WAT}_2(L)_2$ or $7_2 \rightarrow \text{WAT}_2(L)_2$, then $\text{WAT}_2(L)$ represents 2 and 3. Thus $\text{WAT}_2(L)$ contains $\begin{bmatrix} 2 & a \\ a & 3 \end{bmatrix}$ with $a = 0, 1$, or 2. These three sublattices represent all elements of \mathbb{Z}_2 locally.

4.3.4. $m \equiv 7 \pmod{8}$. For any unit $\epsilon \in \mathbb{Z}_2$, the unary local sublattice $\langle \epsilon \rangle_2$ is associated with the isotropic \mathbb{Z}_2 -lattice $\langle \epsilon, -\epsilon \rangle_{\mathbb{Z}_2}$. So $\text{WAT}_2(L)_2$ is \mathbb{Z}_2 -universal.

4.3.5. $m \equiv 2 \cdot 1 \pmod{16}$. In this case, $\langle 1 \rangle_2$ represents $1_2, 3_2, 2 \cdot 1_2, 2 \cdot 3_2$. If $1_2 \rightarrow \text{WAT}_2(L)_2$, then $2 \cdot 1_2 \rightarrow \text{WAT}_2(L)_2$. Thus $\text{WAT}_2(L)$ should contain a binary sublattice $\langle 1, 1 \rangle$ or $\langle 1, 2 \rangle$. These sublattices are all \mathbb{Z}_2 -universal from Lemma 3.8. If $3_2 \rightarrow \text{WAT}_2(L)_2$, then $1_2 \rightarrow \text{WAT}_2(L)_2$ and we return to the previous case.

If $5_2 \rightarrow \text{WAT}_2(L)_2$ or $7_2 \rightarrow \text{WAT}_2(L)_2$, then $\text{WAT}_2(L)$ represents 5 and 7 globally. Thus $\text{WAT}_2(L)$ should contain a sublattice $\begin{bmatrix} 5 & a+b\omega_m \\ a+b\bar{\omega}_m & 7 \end{bmatrix}$. Every candidate is \mathbb{Z}_2 -universal except for $a + b\omega_m = 1 + \omega_{34}$. But all regular lattices escalated from $\begin{bmatrix} 5 & 1+\omega_{34} \\ 1+\bar{\omega}_{34} & 7 \end{bmatrix}$ are \mathbb{Z}_2 -universal.

4.3.6. $m \equiv 2 \cdot 3 \pmod{16}$. Note that $\langle 1 \rangle_2$ represents $1_2, 7_2, 2 \cdot 3_2, 2 \cdot 5_2$. If $1_2 \rightarrow \text{WAT}_2(L)_2$ or $7_2 \rightarrow \text{WAT}_2(L)_2$, then $\text{WAT}_2(L)$ represents both 1 and 6. Thus $\text{WAT}_2(L)$ contains $\begin{bmatrix} 1 & a \\ a & 6 \end{bmatrix}$. These sublattices are \mathbb{Z}_2 -universal.

If $3_2 \rightarrow \text{WAT}_2(L)_2$ or $5_2 \rightarrow \text{WAT}_2(L)_2$, then $\text{WAT}_2(L)$ represents 2 and 3. Thus $\text{WAT}_2(L)$ contains $\begin{bmatrix} 2 & a \\ a & 3 \end{bmatrix}$. These sublattices are also \mathbb{Z}_2 -universal.

4.3.7. $m \equiv 2 \cdot 5 \pmod{16}$. Note that $\langle 1 \rangle_2$ represents $1_2, 3_2, 2 \cdot 5_2, 2 \cdot 7_2$. If $\text{WAT}_2(L)_2$ represents 1_2 or 3_2 , then it should represent both 1 and 3. If $\text{WAT}_2(L)_2$ represents 5_2 or 7_2 , then it should represent 2 and 5. Thus $\text{WAT}_2(L)$ contains $\begin{bmatrix} 1 & a \\ a & 3 \end{bmatrix}$ or $\begin{bmatrix} 2 & a \\ a & 5 \end{bmatrix}$. These sublattices are all \mathbb{Z}_2 -universal.

4.3.8. $m \equiv 2 \cdot 7 \pmod{16}$. Note that $\langle 1 \rangle_2$ represents $1_2, 7_2, 2 \cdot 1_2, 2 \cdot 7_2$. If $1_2 \rightarrow \text{WAT}_2(L)_2$ or $7_2 \rightarrow \text{WAT}_2(L)_2$, then $\text{WAT}_2(L)$ represents both 1 and 2. If $\text{WAT}_2(L)_2$ represents 3_2 or 5_2 , then it represents both 3 and 5. Thus $\text{WAT}_2(L)$ contains $\begin{bmatrix} 1 & a \\ a & 2 \end{bmatrix}$ or $\begin{bmatrix} 3 & a \\ a & 5 \end{bmatrix}$. These lattices are \mathbb{Z}_2 -universal.

From the above results, we conclude that L_2 is \mathbb{Z}_2 -universal if $m \neq 1, 2, 11, 23$ and $(m, 3 \cdot 5 \cdot 7) = 1$.

4.4. Step 4. Now, consider an odd unramified prime p . Then $\text{WAT}_p(L)$ is normal and \mathbb{Z}_q -universal for all $q \neq p$. Since L is primitive, $L_p = \text{WAT}_p(L)_p$ represents a unit, say ϵ . Then $\langle \epsilon \rangle_p$ is associated with $\langle 1, m \rangle_{\mathbb{Z}_p}$, and thus it represents all unit elements in \mathbb{Z}_p . Therefore, $\text{WAT}_p(L)$ contains a binary sublattice $\begin{bmatrix} 1 & a+b\omega_m \\ a+b\omega_m & 2 \end{bmatrix} \cong \langle 1, 2 - N_{E/\mathbb{Q}}(a + b\omega_m) \rangle$, which is isometric to either $\langle 1, 1 \rangle$ or $\langle 1, 2 \rangle$. But both are \mathbb{Z}_p -universal.

We conclude that if $m \neq 1, 2, 11, 23$ and m is divisible by neither 3, 5, nor 7, then every primitive normal regular lattice is universal over $\mathbb{Q}(\sqrt{-m})$.

4.5. Step 5. To complete the proof, we find some nonuniversal regular lattices L over $\mathbb{Q}(\sqrt{-m})$.

We can find $\langle 1, 4 \rangle$ over $\mathbb{Q}(\sqrt{-1})$, $\langle 1, 8 \rangle$ over $\mathbb{Q}(\sqrt{-2})$, and $\langle 1, 4 \rangle$ over $\mathbb{Q}(\sqrt{-11})$ [12]. We already found such a lattice over $\mathbb{Q}(\sqrt{-23})$ in 4.2.5.

For $\mathbb{Q}(\sqrt{-m})$ with $(m, 3 \cdot 5 \cdot 7) \neq 1$, it is easy to check that the following lattices are regular over the stated imaginary quadratic fields:

$$\begin{aligned} \langle 1, 3, 3, 3 \rangle & \text{ over } \mathbb{Q}(\sqrt{-m}) \text{ with } 3 \mid m, \\ \langle 1, 5, 5, 5 \rangle & \text{ over } \mathbb{Q}(\sqrt{-m}) \text{ with } 5 \mid m, \\ \begin{bmatrix} 3 & 1 \\ 1 & 5 \end{bmatrix} \perp \langle 7, 7 \rangle & \text{ over } \mathbb{Q}(\sqrt{-m}) \text{ with } 7 \mid m. \end{aligned}$$

The above results entail that every primitive normal regular Hermitian lattice is universal over $\mathbb{Q}(\sqrt{-m})$ if and only if $m \neq 1, 2, 11, 23$ and $(m, 3 \cdot 5 \cdot 7) = 1$.

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References

- [1] M. Bhargava, *On the Conway–Schneeberger fifteen theorem*, in: *Contemp. Math.* 272, Amer. Math. Soc., 2000, 27–37.
- [2] A. Brauer, *Über den kleinsten quadratischen Nichtrest*, *Math. Z.* 33 (1931), 161–176.

- [3] W. K. Chan and A. Rokicki, *Positive definite binary hermitian forms with finitely many exceptions*, J. Number Theory 124 (2007), 167–180.
- [4] J. H. Conway, *Universal quadratic forms and the fifteen theorem*, in: Contemp. Math. 272, Amer. Math. Soc., 2000, 23–26.
- [5] A. G. Earnest and A. Khosravani, *Representation of integers by positive definite binary Hermitian lattices over imaginary quadratic fields*, J. Number Theory 62 (1997), 368–374.
- [6] L. Gerstein, *Classes of definite Hermitian forms*, Amer. J. Math. 100 (1978), 81–97.
- [7] A. Granville, R. A. Mollin, and H. C. Williams, *An upper bound on the least inert prime in a real quadratic field*, Canad. J. Math. 52 (2000), 369–380.
- [8] R. Jacobowitz, *Hermitian forms over local fields*, Amer. J. Math. 84 (1962), 441–465.
- [9] N. Jacobson, *A note on hermitian forms*, Bull. Amer. Math. Soc. 46 (1940), 264–268.
- [10] A. A. Johnson, *Integral representation of hermitian forms over local fields*, J. Reine Angew. Math. 229 (1968), 57–80.
- [11] B. M. Kim, J. Y. Kim, and P.-S. Park, *The fifteen theorem for universal Hermitian lattices over imaginary quadratic fields*, Math. Comp. 79 (2010), 1123–1144.
- [12] B. M. Kim, J. Y. Kim, and P.-S. Park, *Complete classification of binary normal regular Hermitian lattices*, J. Math. Soc. Japan 63 (2011), 1001–1025.
- [13] B. M. Kim, J. Y. Kim, and P.-S. Park, *Infinitely many regular subnormal binary Hermitian lattices over imaginary quadratic fields*, East Asian Math. J. 28 (2012), 333–337.
- [14] J. Liu and A. Marino, *Strictly regular ternary Hermitian forms*, J. Number Theory 168 (2016), 374–385.
- [15] Z.-W. Sun, *Simple arguments on consecutive power residues*, J. Number Theory 124 (2007), 57–61.

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