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Linear preservers of equivalence relations on infinite-dimensional spaces

by

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Abstract. Linear maps preserving equivalence, equivalence by unitaries or congruence, acting on an infinite-dimensional Banach/Hilbert space, are classified. In the first two cases a unified approach is used: we identify the orbit of the identity and show that it is invariant under the map under consideration. Known results on linear invertibility or unitary group preservers are then used.

1. Introduction. Linear preserver problems concern the characterization of linear maps on spaces/algebras of matrices or operators that leave certain properties, functions, subsets or relations invariant. One of the interesting topics is the study of similarity-preserving linear maps.

Throughout the paper, $\mathcal{H}$ denotes an infinite-dimensional complex Hilbert space and $\mathcal{X}$ stands for an infinite-dimensional complex Banach space.

The first result goes back to 1987 when Hiai [7] characterized linear maps ϕ defined on the algebra of all complex $n \times n$ matrices that preserve similarity, which means that if matrices A and B are *similar* ($B = SAS^{-1}$ for some invertible matrix S) then $\phi(A)$ and $\phi(B)$ are similar as well. Next, linear preservers of some other relations on matrix spaces were classified [9, 18]. Linear or merely additive preservers of similarity or unitary similarity on infinite-dimensional spaces have been considered by many authors [2, 3, 5, 6, 13–15, 19, 20, 24]; however, there are still some open questions. On infinite-dimensional Hilbert space, linear preservers of similarity (in one direction only) were classified by Šemrl [24], who proved that if ϕ is a bijective similarity-preserving linear map on $\mathcal{B}(\mathcal{H})$, then ϕ is either of the form $\phi(X) = cTXT^{-1}$ for every $X \in \mathcal{B}(\mathcal{H})$, or of the form $\phi(X) = cTX^tT^{-1}$

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for every $X \in \mathcal{B}(\mathcal{H})$, for some non-zero $c \in \mathbb{C}$ and an invertible operator $T \in \mathcal{B}(\mathcal{H})$. Here, X^t denotes the transpose of X with respect to an arbitrary but fixed orthonormal basis of $\mathcal{H}$. Later, Petek et al. [15] determined bijective linear maps which map every unitarily similar pair $A, B \in \mathcal{B}(\mathcal{H})$ to a unitarily similar pair (A and B are *unitarily similar* if $B = UAU^*$ for some unitary $U \in \mathcal{B}(\mathcal{H})$). Such a map ϕ is either of the form $\phi(X) = cUXU^*$ for every $X \in \mathcal{B}(\mathcal{H})$, or of the form $\phi(X) = cUX^tU^*$ for every $X \in \mathcal{B}(\mathcal{H})$, for some non-zero $c \in \mathbb{C}$ and a unitary operator $U \in \mathcal{B}(\mathcal{H})$.

It is our aim to find a complete description of bijective linear maps that preserve some other equivalence relations on an infinite-dimensional space. We restrict our attention to three relations: equivalence ($A, B \in \mathcal{B}(\mathcal{X})$ are *equivalent* if $B = TAS$ for some invertible $T, S \in \mathcal{B}(\mathcal{X})$); equivalence by unitaries ($A, B \in \mathcal{B}(\mathcal{H})$ are *equivalent by unitaries* if $B = UAW$ for some unitary $U, W \in \mathcal{B}(\mathcal{H})$); and congruence ($A, B \in \mathcal{B}(\mathcal{H})$ are *congruent* if $B = SAS^*$ for some invertible $S \in \mathcal{B}(\mathcal{H})$). For each of these relations the orbit of the identity operator is rather large. The common approach in classifying linear preservers of the (first and second) relations above is to reduce the problem to the classification of linear maps which preserve the orbit of the identity, which is the set of invertible operators and the unitary group of operators, respectively.

We will describe linear maps on Banach/Hilbert spaces preserving a given binary relation, which is a purely algebraic condition. As a result we find in particular that such maps are continuous. Therefore, our results are a contribution to automatic continuity results.

2. Preliminaries. Let $\mathcal{X}$ be an infinite-dimensional Banach space over $\mathbb{C}$, and $\mathcal{X}'$ its dual. We denote by $\mathcal{B}(\mathcal{X})$ the algebra of all bounded linear operators on $\mathcal{X}$, and by $\mathcal{F}(\mathcal{X})$ the ideal of finite-rank operators in $\mathcal{B}(\mathcal{X})$.

Every rank-one operator on $\mathcal{X}$ can be written as $x \otimes f$ for some non-zero vector $x \in \mathcal{X}$ and some non-zero functional $f \in \mathcal{X}'$. This operator is defined by $(x \otimes f)z = f(z)x$ for every $z \in \mathcal{X}$, and for every $A \in \mathcal{B}(\mathcal{X})$ we have $A(x \otimes f) = Ax \otimes f$ and $(x \otimes f)A = x \otimes A'f$, where A' denotes the adjoint operator of A. Such an operator is idempotent if $f(x) = 1$, and nilpotent if $f(x) = 0$. Observe that $x \otimes \lambda f = \lambda x \otimes f$ for every $\lambda \in \mathbb{C}$.

When $\mathcal{H}$ is a complex infinite-dimensional Hilbert space, we define rank-one operators as $(x \otimes y)z = \langle z, y \rangle x$ for every $z \in \mathcal{H}$, and $(x \otimes y)A = x \otimes A^*y$, for every $A \in \mathcal{B}(\mathcal{H})$, where A^* stands for the Hilbert space adjoint of A. Here, $\langle z, y \rangle$ denotes the inner product of $z, y \in \mathcal{H}$. Clearly, the operator $x \otimes y$ is idempotent if $\langle x, y \rangle = 1$, and nilpotent if $\langle x, y \rangle = 0$. Furthermore, $x \otimes \lambda y = \overline{\lambda} x \otimes y$ for every $\lambda \in \mathbb{C}$, and $(x \otimes y)^* = y \otimes x$ for all $x, y \in \mathcal{H}$.

Our first step will be to reduce the problem to the case of rank-one preserving maps. We will use the following result due to Kuzma regarding "rank-one-non-increasing" additive mappings.

THEOREM 2.1 ([16]). *Let $\phi : \mathcal{F}(\mathcal{X}) \to \mathcal{F}(\mathcal{X})$ be an additive map, which satisfies* $\operatorname{rank} \phi(X) \le 1$ *whenever* $\operatorname{rank} X = 1$. *Then one and only one of the following statements holds.*

(i) *There exist $g \in \mathcal{X}'$ and an additive map $\tau : \mathcal{F}(\mathcal{X}) \to \mathcal{X}$ such that*

$$\phi(X) = \tau(X) \otimes g \quad \text{for every } X \in \mathcal{F}(\mathcal{X}).$$

(ii) *There exist $a \in \mathcal{X}$ and an additive map $\varphi : \mathcal{F}(\mathcal{X}) \to \mathcal{X}'$ such that*

$$\phi(X) = a \otimes \varphi(X) \quad \text{for every } X \in \mathcal{F}(\mathcal{X}).$$

(iii) *There exist additive maps $T : \mathcal{X} \to \mathcal{X}$ and $S : \mathcal{X}' \to \mathcal{X}'$ such that*

$$\phi(x \otimes f) = Tx \otimes Sf \quad \text{for every } x \in \mathcal{X} \text{ and every } f \in \mathcal{X}'.$$

(iv) *There exist additive maps $T : \mathcal{X}' \to \mathcal{X}$ and $S : \mathcal{X} \to \mathcal{X}'$ such that*

$$\phi(x \otimes f) = Tf \otimes Sx \quad \text{for every } x \in \mathcal{X} \text{ and every } f \in \mathcal{X}'.$$

REMARK. If ϕ is linear, the maps τ and φ in (i) and (ii), respectively, are linear and cannot map a rank-one operator to zero.

REMARK. When $\mathcal{X} = \mathcal{H}$ is a Hilbert space, under the stronger assumption that ϕ is linear, it is easy to see that T and S must be injective linear maps in case (iii) and injective conjugate-linear maps in case (iv).

We add a simple lemma which is likely well known.

LEMMA 2.2. *Let $A \in \mathcal{B}(\mathcal{X})$ be of rank one and $B \in \mathcal{B}(\mathcal{X})$ be non-zero. If* $\operatorname{rank}(A + \lambda B) = 1$ *for at least two non-zero $\lambda \in \mathbb{C}$, then* $\operatorname{rank} B = 1$.

Proof. By the assumption there exist distinct non-zero $\lambda_1, \lambda_2 \in \mathbb{C}$ such that

$$A + \lambda_1 B = x_1 \otimes f_1 \quad \text{and} \quad A + \lambda_2 B = x_2 \otimes f_2$$

for some non-zero $x_1, x_2 \in \mathcal{X}$ and non-zero $f_1, f_2 \in \mathcal{X}'$. Since $\operatorname{rank} A = 1$, it follows from $\lambda_2 x_1 \otimes f_1 - \lambda_1 x_2 \otimes f_2 = (\lambda_2 - \lambda_1)A$ that x_1 and x_2 are linearly dependent or f_1 and f_2 are linearly dependent. Thus $(\lambda_1 - \lambda_2)B = x_1 \otimes f_1 - x_2 \otimes f_2$ implies that B is of rank one. ∎

3. Equivalence preservers. Let $\mathcal{X}$ be an infinite-dimensional complex Banach space. Recall that operators $A, B \in \mathcal{B}(\mathcal{X})$ are *equivalent*, denoted by $A \sim B$, whenever there exist invertible operators $T, S \in \mathcal{B}(\mathcal{X})$ such that $A = TBS$. A linear map $\phi : \mathcal{B}(\mathcal{X}) \to \mathcal{B}(\mathcal{X})$ *preserves equivalence* if $A \sim B$ implies $\phi(A) \sim \phi(B)$; and ϕ *preserves equivalence in both directions* when $A \sim B$ if and only if $\phi(A) \sim \phi(B)$.

It is easy to see that all rank-one operators are mutually equivalent, so the *equivalence orbit* of a fixed rank-one operator (i.e. the set of all operators equivalent to that operator) consists of all rank-one operators.

Our first result is a characterization of linear maps on $\mathcal{B}(\mathcal{X})$ which preserve equivalence.

THEOREM 3.1. *Let $\mathcal{X}$ be an infinite-dimensional reflexive complex Banach space and $\phi : \mathcal{B}(\mathcal{X}) \to \mathcal{B}(\mathcal{X})$ a bijective linear map. Then the following statements are equivalent.*

(i) *ϕ preserves equivalence.*

(ii) *Either there exist invertible operators $T, S \in \mathcal{B}(\mathcal{X})$ such that*

$$\phi(X) = TXS \quad \text{for every } X \in \mathcal{B}(\mathcal{X}),$$

or there exist bounded bijective linear operators $T : \mathcal{X}' \to \mathcal{X}$ and $S : \mathcal{X} \to \mathcal{X}'$ such that

$$\phi(X) = TX'S \quad \text{for every } X \in \mathcal{B}(\mathcal{X}),$$

where X' stands for the adjoint of the operator X.

(iii) *ϕ preserves equivalence in both directions.*

Before starting the proof, let us recall a simple lemma, which will be needed in the proof.

LEMMA 3.2 ([25]). *Let $x \in \mathcal{X}$ and $f \in \mathcal{X}'$. Then $I - x \otimes f$ is invertible in $\mathcal{B}(\mathcal{X})$ if and only if $f(x) \neq 1$.*

The proof of Theorem 3.1 depends on the description of bijective linear maps which preserve invertibility, that is, map each invertible operator to an invertible operator; a characterization of such maps is the content of the next theorem.

THEOREM 3.3 ([25]). *Let $\phi : \mathcal{B}(\mathcal{X}) \to \mathcal{B}(\mathcal{X})$ be a bijective linear map preserving invertibility. Then either there exist invertible operators $T, S \in \mathcal{B}(\mathcal{X})$ such that $\phi(X) = TXS$ for every $X \in \mathcal{B}(\mathcal{X})$, or there exist bounded invertible operators $T : \mathcal{X}' \to \mathcal{X}$ and $S : \mathcal{X} \to \mathcal{X}'$ such that $\phi(X) = TX'S$ for every $X \in \mathcal{B}(\mathcal{X})$.*

Proof of Theorem 3.1. The implications (ii)⇒(iii) and (iii)⇒(i) are obvious. It remains to prove (i)⇒(ii), which will be done in several steps.

STEP 1. *ϕ is rank-one preserving.*

Choose any $P \in \mathcal{B}(\mathcal{X})$ of rank one. By the surjectivity of ϕ, there exists a non-zero $A \in \mathcal{B}(\mathcal{X})$ such that $\phi(A) = P$ and a normalized vector $e \in \mathcal{X}$ with $Ae \neq 0$. For a fixed non-zero $f_e \in \mathcal{X}'$, the operator $I - \lambda e \otimes f_e$ is invertible for infinitely many non-zero $\lambda \in \mathbb{C}$. For such λ, operating by ϕ on

$$A \sim A(I - \lambda e \otimes f_e) = A - \lambda Ae \otimes f_e \tag{3.1}$$

gives

$$P = \phi(A) \sim \phi(A) - \lambda\phi(Ae \otimes f_e) = P - \lambda\phi(Ae \otimes f_e). \tag{3.2}$$

Hence, $P - \lambda\phi(Ae \otimes f_e)$ is of rank one for infinitely many non-zero scalars λ and by Lemma 2.2, $\operatorname{rank}\phi(Ae \otimes f_e) = 1$. So we have found a rank-one operator which is mapped to a rank-one operator. Since all rank-one operators are equivalent, $\phi(R)$ is of rank one for every rank-one R.

STEP 2. *Either there exist injective linear maps $T : \mathcal{X} \to \mathcal{X}$ and $S : \mathcal{X}' \to \mathcal{X}'$ such that $\phi(x \otimes f) = Tx \otimes Sf$ for every rank-one operator $x \otimes f$, or there exist injective linear maps $T : \mathcal{X}' \to \mathcal{X}$ and $S : \mathcal{X} \to \mathcal{X}'$ such that $\phi(x \otimes f) = Tf \otimes Sx$ for every rank-one operator $x \otimes f$.*

We apply Theorem 2.1. In addition to the assumptions of Theorem 2.1, our map ϕ is linear and preserves rank-one operators. Therefore, it cannot map any rank-one operator to zero.

Assume firstly that $\phi(x \otimes f) = \tau(x \otimes f) \otimes g$ for every $x \otimes f \in \mathcal{B}(\mathcal{X})$, for some $0 \neq g \in \mathcal{X}'$ and a linear map $\tau : \mathcal{F}(\mathcal{X}) \to \mathcal{X}$. Choose any non-zero $x_0 \in \mathcal{X}$ and any non-zero $f_0 \in \mathcal{X}'$ linearly independent of g. By the surjectivity of ϕ, there exists $A \in \mathcal{B}(\mathcal{X})$ such that $\phi(A) = x_0 \otimes f_0$. Clearly, A is not of rank one, thus there exist normalized $e_1, e_2 \in \mathcal{X}$ such that Ae_1 and Ae_2 are linearly independent. For each $i = 1, 2$ choose $f_i \in \mathcal{X}'$ with $f_i(e_i) = 1$. Then it follows from $A \sim A(I - 2e_i \otimes f_i) = A - 2Ae_i \otimes f_i$ that

$$x_0 \otimes f_0 \sim x_0 \otimes f_0 - 2\phi(Ae_i \otimes f_i). \tag{3.3}$$

Hence, $x_0 \otimes f_0 - 2\tau(Ae_i \otimes f_i) \otimes g$ is of rank one. Since f_0 and g are linearly independent, $\tau(Ae_i \otimes f_i)$ and x_0 are necessarily linearly dependent, so the vectors $\tau(Ae_1 \otimes f_1)$ and $\tau(Ae_2 \otimes f_2)$ and consequently the operators $\phi(Ae_1 \otimes f_1)$ and $\phi(Ae_2 \otimes f_2)$ are linearly dependent too. As ϕ is an injective linear map, this contradicts the fact that Ae_1 and Ae_2 are linearly independent.

Analogously we prove that the second statement of Theorem 2.1 cannot be true. Therefore ϕ satisfies either (iii) or (iv).

It is enough to consider one of these cases, since the proof of the other is almost the same. We will suppose that (iv) holds, i.e. there exist injective linear maps $T : \mathcal{X}' \to \mathcal{X}$ and $S : \mathcal{X} \to \mathcal{X}'$ such that

$$\phi(x \otimes f) = Tf \otimes Sx \quad \text{for every rank-one operator } x \otimes f.$$

STEP 3. *T and S are surjective maps.*

Assume that T is not surjective, so there exists a non-zero $x_0 \in \mathcal{X} \setminus \operatorname{Im} T$. Choose any non-zero $f_0 \in \mathcal{X}'$; since ϕ is bijective, there exists $A \in \mathcal{B}(\mathcal{X})$ such that $\phi(A) = x_0 \otimes f_0$. Because A is not of rank one, there exist normalized $e_1, e_2 \in \mathcal{X}$ such that Ae_1 and Ae_2 are linearly independent. As in (3.3), we infer $x_0 \otimes f_0 \sim x_0 \otimes f_0 - 2\phi(Ae_1 \otimes f_1)$ for some non-zero $f_1 \in \mathcal{X}'$ with $f_1(e_1) = 1$. Obviously, $x_0 \otimes f_0 - 2Tf_1 \otimes SAe_1$ is of rank one. Since Tf_1 and

x_0 are linearly independent, we deduce that SAe_1 and similarly SAe_2 are linearly dependent on f_0. Therefore, SAe_1 and SAe_2 are linearly dependent. By the injectivity of S it follows that Ae_1 and Ae_2 are linearly dependent, a contradiction. Thus T is surjective. In the same way we can prove the surjectivity of S.

STEP 4. *T and S are continuous.*

Since ϕ is bijective, there has to be a non-zero $A \in \mathcal{B}(\mathcal{X})$ with $\phi(A) = I$. Firstly, choose any non-zero $e_0 \in \mathcal{X}$ and non-zero $f_0 \in \mathcal{X}'$ such that $f_0(e_0) = 0$. By Lemma 3.2, the operator $I - \lambda_0 e_0 \otimes f_0$ is invertible for every $\lambda_0 \in \mathbb{C}$. From the relation $A \sim A(I - \lambda_0 e_0 \otimes f_0) = A - \lambda_0 Ae_0 \otimes f_0$ we get

$$I \sim I - \lambda_0 Tf_0 \otimes SAe_0 \quad \text{for every } \lambda_0 \in \mathbb{C}.$$

Thus $\lambda_0(SAe_0)(Tf_0) \neq 1$ for every scalar λ_0, by Lemma 3.2. Hence,

$$(SAe_0)(Tf_0) = 0 \quad \text{for every nilpotent } e_0 \otimes f_0 \in \mathcal{B}(\mathcal{X}). \tag{3.4}$$

Following steps similar to those used in [24, Proposition 3.1], we can prove that there exists $c \in \mathbb{C}$ such that

$$(SAe_1)(Tf_1) = c \quad \text{for every idempotent } e_1 \otimes f_1 \in \mathcal{B}(\mathcal{X}). \tag{3.5}$$

For completeness, let us sketch the proof. Take $e_1 \in \mathcal{X}$ and $f_1 \in \mathcal{X}'$ such that $f_1(e_1) = 1$. Then $I - \lambda_1 e_1 \otimes f_1$ is invertible for every scalar $\lambda_1 \neq 1$. As in (3.4), we have $\lambda_1(SAe_1)(Tf_1) \neq 1$ for every $\lambda_1 \neq 1$. Define $c = (SAe_1)(Tf_1) \in \mathbb{C}$; since $\lambda_1 c \neq 1$ for every scalar $\lambda_1 \neq 1$, it follows that

$$c = 0 \quad \text{or} \quad c = 1. \tag{3.6}$$

Next, consider $e_2 \in \mathcal{X}$ and $f_2 \in \mathcal{X}'$ such that $f_2(e_2) = 1$ and $f_1(e_2) = 0 = f_2(e_1)$. By applying (3.4) we get $(SAe_2)(Tf_1) = 0 = (SAe_1)(Tf_2)$. As the operator $(e_1 + e_2) \otimes (f_1 - f_2)$ is nilpotent, it must be that $(SAe_1)(Tf_1) = (SAe_2)(Tf_2)$. Finally, choose $e_3 \in \mathcal{X}$ and $f_3 \in \mathcal{X}'$ such that $f_3(e_3) = 1$ and then take $e_4 \in \mathcal{X}$ and $f_4 \in \mathcal{X}'$ linearly independent of e_3 and f_3, respectively, such that $f_4(e_4) = 1$, $f_4(e_1) = 0 = f_1(e_4)$ and $f_4(e_3) = 0 = f_3(e_4)$. By the same method as above, we get $c = (SAe_1)(Tf_1) = (SAe_4)(Tf_4) = (SAe_3)(Tf_3)$, which is the desired conclusion.

Take any $x \in \mathcal{X}$ and $f \in \mathcal{X}'$ with $f(x) \neq 0$. Replacing f_1 by $f(x)^{-1}f$ in (3.5) and using (3.4) we easily obtain

$$(SAx)(Tf) = cf(x) \quad \text{for every } x \in \mathcal{X} \text{ and every } f \in \mathcal{X}'. \tag{3.7}$$

Our next goal is to show that $c \neq 0$. Suppose that $c = 0$. By (3.7), we have

$$(SAx)(Tf) = 0 \quad \text{for every } x \in \mathcal{X} \text{ and every } f \in \mathcal{X}'.$$

Fix $x_0 \in \mathcal{X}$ and let $g_0 = SAx_0 \in \mathcal{X}'$. Then $g_0(Tf) = 0$ for every $f \in \mathcal{X}'$. The map T is surjective, so $g_0 = 0$. By the injectivity of S, we have $Ax_0 = 0$. The vector x_0 was arbitrary, thus $A = 0$, a contradiction.

Since $c \neq 0$, conclusions (3.6) and (3.7) imply

$$(SAx)(Tf) = f(x) \quad \text{for every } x \in \mathcal{X} \text{ and every } f \in \mathcal{X}'. \tag{3.8}$$

We now prove the continuity of the operator SA and in turn also of T. Let $x_n \to 0$ and $SAx_n \to g \in \mathcal{X}'$. Applying (3.8) gives

$$g(Tf) = 0 \quad \text{for every } f \in \mathcal{X}'.$$

The operator T is surjective, so $g = 0$. By the closed graph theorem, the operator SA is continuous. Furthermore, by the bijectivity of T, for every $y \in \mathcal{X}$ let $f_y = T^{-1}y$. Then by (3.8), we have

$$f_y(x) = (SAx)(Tf_y) = (SAx)(y) \quad \text{for every } x \in \mathcal{X}.$$

Since SAx is a bounded functional, we estimate

$$|f_y(x)| = |(SAx)(y)| \le \|SAx\| \cdot \|y\| \le \|SA\| \cdot \|x\| \cdot \|y\|$$

for every $x \in \mathcal{X}$. Hence, $\|T^{-1}y\| = \|f_y\| \le \|SA\| \cdot \|y\|$ for every $y \in \mathcal{X}$. It turns out that $\|T^{-1}\| \le \|SA\|$, so T^{-1} and consequently T is bounded.

By applying ϕ on the relations $A \sim (I - \lambda_0 e_0 \otimes f_0)A$, $\lambda_0 \in \mathbb{C}$, and $A \sim (I - \lambda_1 e_1 \otimes f_1)A$, $\lambda_1 \neq 1$, where $e_0 \otimes f_0$ is any nilpotent and $e_1 \otimes f_1$ any idempotent, and by a similar reasoning to that used above, we obtain

$$(Sx)(TA'f) = f(x) \quad \text{for every } x \in \mathcal{X} \text{ and every } f \in \mathcal{X}'. \tag{3.9}$$

As $\mathcal{X}$ is reflexive by assumption, let $i : \mathcal{X} \to \mathcal{X}''$ denote the canonical isometric embedding. By setting $h = Sx \in \mathcal{X}'$ in (3.9), it follows that

$$|i(S^{-1}h)f| = |f(S^{-1}h)| = |h(TA'f)| \le \|h\| \cdot \|TA'\| \cdot \|f\|, \tag{3.10}$$

and we infer that S^{-1} is continuous, so S is continuous as well.

Step 5. *$\phi^{-1}(I)$ is invertible.*

Denote $A = \phi^{-1}(I)$. In order to show that A is invertible, note that $T : \mathcal{X}' \to \mathcal{X}$ is a bounded linear operator, so its adjoint $T' : \mathcal{X}' \to \mathcal{X}''$ exists and $T'h = h \circ T : \mathcal{X}' \to \mathbb{C}$. Now, choose any $x \in \mathcal{X}$ and apply (3.8) to get

$$i(x)(f) = f(x) = (SAx)(Tf) = (SAx \circ T)(f) = (T'SAx)(f)$$

for every $f \in \mathcal{X}'$. Thus $i(x) = T'SAx$ for every $x \in \mathcal{X}$. Therefore $i = T'SA$ is bijective, and consequently the operator $A = (T'S)^{-1}i$ is invertible.

As there exists an invertible operator which is mapped to I, the map ϕ preserves invertibility, and by Theorem 3.3 we complete the proof. ∎

Let us close this section with a characterization of bijective equivalence-preserving linear maps on $\mathcal{B}(\mathcal{H})$. As the proof is essentially the same as the proof of Theorem 3.1, it is omitted.

Theorem 3.4. *Let $\phi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ be a bijective linear map preserving equivalence. Then there exist invertible operators $T, S \in \mathcal{B}(\mathcal{H})$ such that* *either*

$$\phi(X) = TXS \quad \textit{for every } X \in \mathcal{B}(\mathcal{H}),$$

or

$$\phi(X) = TX^tS \quad \textit{for every } X \in \mathcal{B}(\mathcal{H}),$$

where X^t denotes the transpose of X with respect to an arbitrary but fixed orthonormal basis in $\mathcal{H}$.

4. Preservers of equivalence by unitaries. Recall that $A, B \in \mathcal{B}(\mathcal{H})$ are *equivalent by unitaries*, denoted by $A \simeq B$, whenever there exist unitary operators $U, W \in \mathcal{B}(\mathcal{H})$ such that $A = UBW$. A map ϕ *preserves equivalence by unitaries* if $A \simeq B$ implies $\phi(A) \simeq \phi(B)$.

It is easy to verify that $x \otimes y \simeq e \otimes f$ if and only if $\|x\| \cdot \|y\| = \|e\| \cdot \|f\|$, and $I \simeq U$ if and only if $U \in \mathcal{B}(\mathcal{H})$ is unitary.

Let us give a simple technical lemma.

LEMMA 4.1. *Let $A \in \mathcal{B}(\mathcal{H})$. If $I \simeq I + (\mu - 1)A$ for every $\mu \in \mathbb{C}$ with $|\mu| = 1$, then A is a projection (i.e. $A^* = A = A^2$).*

Proof. Since $I + (\mu - 1)A$ is a unitary operator for $|\mu| = 1$ we have

$$(I + (\mu - 1)A)^*(I + (\mu - 1)A) = I,$$

and hence

$$(\mu - 1)A + \overline{\mu - 1}\, A^* + |\mu - 1|^2 A^*A = 0$$

By taking $\mu = -1$ and then $\mu = i$, we obtain

$$A + A^* - 2A^*A = 0 \quad \text{and} \quad (i - 1)A - (i + 1)A^* + 2A^*A = 0.$$

Summing up these equations, we get $A = A^*$, and by the first equality $A^2 = A$, as desired. ∎

In our next result we determine linear maps on $\mathcal{B}(\mathcal{H})$ which preserve equivalence by unitaries.

THEOREM 4.2. *Let $\phi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ be a bijective linear map. Then the following statements are equivalent.*

(i) *ϕ preserves equivalence by unitaries.*
(ii) *There exist a non-zero $c \in \mathbb{C}$ and unitary operators $U, W \in \mathcal{B}(\mathcal{H})$ such that either*

$$\phi(X) = cUXW \quad \textit{for every } X \in \mathcal{B}(\mathcal{H}),$$

or

$$\phi(X) = cUX^tW \quad \textit{for every } X \in \mathcal{B}(\mathcal{H}),$$

where X^t denotes the transpose of X with respect to an arbitrary but fixed orthonormal basis in $\mathcal{H}$.

(iii) *ϕ preserves equivalence by unitaries in both directions (i.e. $A \simeq B$ if and only if $\phi(A) \simeq \phi(B)$).*

The proof of Theorem 4.2 applies a well-known result on the unitary group preserving maps, which we recall below.

THEOREM 4.3 ([21]). *Let $\phi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ be a linear bijective map which preserves the unitary group. Then there exist unitary operators $U, W \in \mathcal{B}(\mathcal{H})$ such that either $\phi(X) = UXW$ for every $X \in \mathcal{B}(\mathcal{H})$, or $\phi(X) = UX^tW$ for every $X \in \mathcal{B}(\mathcal{H})$, where X^t denotes the transpose of X with respect to an arbitrary but fixed orthonormal basis in $\mathcal{H}$.*

REMARK. In [21], the continuity of ϕ was assumed. However, this assumption is superfluous. By [22] every self-adjoint operator $H \in \mathcal{B}(\mathcal{H})$ can be written as a linear combination of two unitary operators $W_1, W_2 \in \mathcal{B}(\mathcal{H})$ as $H = \frac{\|H\|}{2}(W_1 + W_2)$. Then it follows easily that for every $A \in \mathcal{B}(\mathcal{H})$ there exist unitary operators $U_i \in \mathcal{B}(\mathcal{H})$, $i = 1, 2, 3, 4$, such that $A = \sum_{i=1}^4 \alpha_i U_i$, where $|\alpha_i| \leq \|A\|/2$ for $i = 1, 2, 3, 4$. Since ϕ preserves the unitary group, it is bounded and therefore continuous.

Proof of Theorem 4.2. We only have to prove (i)$\Rightarrow$(ii), since (ii)$\Rightarrow$(iii) and (iii)$\Rightarrow$(i) are obvious. The structure of the proof will be similar to that of the proof of Theorem 3.1.

STEP 1. *ϕ is rank-one preserving.*

The proof is similar to that of Step 1 in the proof of Theorem 3.1, so it is omitted.

STEP 2. *Either there exist injective linear maps $T, S : \mathcal{H} \to \mathcal{H}$ such that $\phi(x \otimes y) = Tx \otimes Sy$ for every rank-one operator $x \otimes y$, or there exist conjugate-linear maps $T, S : \mathcal{H} \to \mathcal{H}$ such that $\phi(x \otimes y) = Ty \otimes Sx$ for every rank-one operator $x \otimes y$.*

The map ϕ takes one of the forms from Theorem 2.1 by Step 1. Firstly, we assume that there exist a non-zero $b \in \mathcal{H}$ and a linear map $\tau : \mathcal{F}(\mathcal{H}) \to \mathcal{H}$, where $\operatorname{rank} X = 1$ forces $\tau(X) \neq 0$, such that $\phi(X) = \tau(X) \otimes b$ for every $X \in \mathcal{F}(\mathcal{H})$. As ϕ is surjective, there exists $A \in \mathcal{B}(\mathcal{H})$ such that $\phi(A) = I$. Clearly, A is not of rank one, so there exist $e_1, e_2 \in \mathcal{H}$ such that Ae_1 and Ae_2 are linearly independent. By the assumption, we obtain

$$\phi(Ae_i \otimes e_i) = \tau(Ae_i \otimes e_i) \otimes b \quad \text{for } i = 1, 2.$$

As $I + (\mu - 1)e_i \otimes e_i$ is unitary, for every unimodular $\mu \in \mathbb{C}$, we have

$$A \simeq A(I + (\mu - 1)e_i \otimes e_i) = A + (\mu - 1)Ae_i \otimes e_i,$$

which further implies

$$I \simeq I + (\mu - 1)\phi(Ae_i \otimes e_i) \quad \text{for every } \mu \text{ with } |\mu| = 1. \tag{4.1}$$

By Lemma 4.1 the operators $\phi(Ae_1 \otimes e_1)$ and $\phi(Ae_2 \otimes e_2)$ are projections, thus

$$\tau(Ae_1 \otimes e_1) = \frac{1}{\|b\|^2} b = \tau(Ae_2 \otimes e_2),$$

and consequently $\phi(Ae_1 \otimes e_1) = \phi(Ae_2 \otimes e_2)$. Since ϕ is injective, we deduce that $Ae_1 \otimes e_1 = Ae_2 \otimes e_2$, which contradicts linear independence of Ae_1 and Ae_2. In the same way, we can see that the second statement of Theorem 2.1 cannot be true. Therefore ϕ satisfies either (iii) or (iv).

Again, we will only consider case (iv), i.e. we suppose there exist injective conjugate-linear maps $T, S : \mathcal{H} \to \mathcal{H}$ such that

$$\phi(x \otimes y) = Ty \otimes Sx \quad \text{for every rank-one operator } x \otimes y.$$

Step 3. *There exist positive constants α and β such that $\|Sx\| = \alpha\|x\|$ and $\|Tx\| = \beta\|x\|$ for all $x \in \mathcal{H}$. Consequently, both T and S are continuous.*

Fix orthonormal vectors $e, f \in \mathcal{H}$. Then it can be easily checked that for any $x, y \in \mathcal{H}$,

$$x \otimes y \simeq \|x\| e \otimes y \quad \text{and} \quad x \otimes y \simeq \|y\| x \otimes f.$$

By operating with ϕ on these relations, we derive that

$$Ty \otimes Sx \simeq \|x\| Ty \otimes Se \quad \text{and} \quad Ty \otimes Sx \simeq \|y\| Tf \otimes Sx,$$

which immediately implies

$$\|Ty\| \cdot \|Sx\| = \|x\| \cdot \|Ty\| \cdot \|Se\| \quad \text{and} \quad \|Ty\| \cdot \|Sx\| = \|y\| \cdot \|Tf\| \cdot \|Sx\|.$$

Setting $\alpha = \|Se\| > 0$ and $\beta = \|Tf\| > 0$, we arrive at

$$\|Sx\| = \alpha\|x\| \quad \text{and} \quad \|Ty\| = \beta\|y\|, \tag{4.2}$$

for all $x, y \in \mathcal{H}$.

Step 4. *$\phi^{-1}(I)$ is a unitary operator multiplied by a non-zero scalar.*

Let $\phi(A) = I$. If A is a scalar operator, we are done. So, assume that A is not scalar. As $A \neq 0$, there exists a normalized vector $e \in \mathcal{H}$ with $Ae \neq 0$. As in (4.1), we infer

$$I \simeq I + (\mu - 1)\phi(Ae \otimes e) \quad \text{for every } \mu \text{ with } |\mu| = 1.$$

By Lemma 4.1, the operator $\phi(Ae \otimes e) = Te \otimes SAe$ is a projection of rank one. Hence, $\lambda_e Te = SAe$ for some non-zero $\lambda_e \in \mathbb{C}$. By (4.2) and from

$$1 = \langle Te, SAe \rangle = \langle Te, \lambda_e Te \rangle = \overline{\lambda_e}\|Te\|^2 = \overline{\lambda_e}\beta^2$$

it follows that

$$Te = \beta^2 SAe \quad \text{for every } e \in \mathcal{H} \text{ with } Ae \neq 0. \tag{4.3}$$

As A is bounded, thus continuous, and T and S are also continuous, (4.3) can be extended to

$$Te = \beta^2 SAe \quad \text{for every } e \in \mathcal{H}.$$

This yields $T = \beta^2 SA$, and by applying Step 3 we get

$$I = \frac{1}{\beta^2} T^*T = \frac{1}{\beta^2}(\beta^2 SA)^*(\beta^2 SA) = \beta^2 A^*(S^*S)A = \alpha^2\beta^2 A^*A. \tag{4.4}$$

By the same method, $\phi(f \otimes A^*f) = TA^*f \otimes Sf$ is a projection of rank one for every non-zero $f \in \mathcal{H}$ with $A^*f \neq 0$. From continuity of A, T and S it follows that $S = \alpha^2 TA^*$. Thus,

$$I = \frac{1}{\alpha^2} S^*S = \alpha^2 (TA^*)^*(TA^*) = \alpha^2 A(T^*T)A^* = \alpha^2\beta^2 AA^*. \tag{4.5}$$

Because of (4.4) and (4.5), the operator $\alpha\beta A$ is unitary and $\phi(\frac{1}{\alpha\beta}A) = I$, as desired.

Replacing ϕ by $(\alpha\beta)^{-1}\phi$, we see that ϕ preserves unitary operators, and by Theorem 4.3, the proof is completed. ■

5. Congruence preservers. Recall that $A, B \in \mathcal{B}(\mathcal{H})$ are *congruent*, denoted by $A \equiv B$, whenever there exists a bijective operator $S \in \mathcal{B}(\mathcal{H})$ such that $A = SBS^*$. A map ϕ *preserves congruence* when $A \equiv B$ implies $\phi(A) \equiv \phi(B)$. We say that $A \in \mathcal{B}(\mathcal{H})$ is *positive* if $\langle Ax, x\rangle \geq 0$ for every $x \in \mathcal{H}$.

It is clear that for any $A, B \in \mathcal{B}(\mathcal{H})$, $A \equiv B$ implies $A \sim B$, and if $A \equiv B$ then either both A and B are self-adjoint, or neither is. Moreover, if $A \equiv B$ and A is positive, then B is positive as well.

In the next proposition we give a classification of congruence classes for 2×2 complex matrices, which is actually a special case of the Claim in [11, Theorem 1.1(b)].

PROPOSITION 5.1 ([11]). *Every 2×2 complex matrix is congruent to a matrix of exactly one of the following types:*

$$\begin{bmatrix} 0 & 1 \\ \lambda & 0 \end{bmatrix}, \quad \mu\begin{bmatrix} 0 & 1 \\ 1 & i \end{bmatrix}, \quad \begin{bmatrix} \mu_1 & 0 \\ 0 & \mu_2 \end{bmatrix}, \quad \begin{bmatrix} \mu & 0 \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$

where μ, μ_1, μ_2 are unimodular scalars and $|\lambda| > 1$.

We continue with two elementary observations; the first lemma is derived from the previous proposition, and the second one will be needed in the last step of the proof of the main result of this section.

LEMMA 5.2. *Let A be a non-zero 2×2 complex matrix. If $A \equiv \alpha A$ for every unimodular $\alpha \in \mathbb{C}$, then $A \equiv \left[\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right]$.*

Proof. Firstly, if $A \equiv \left[\begin{smallmatrix} 0 & 1 \\ \lambda & 0 \end{smallmatrix}\right]$ for some $|\lambda| > 1$, then the relation $A \equiv \alpha A$, for every unimodular scalar α, implies

$$\begin{bmatrix} 0 & 1 \\ \lambda & 0 \end{bmatrix} \equiv \alpha \begin{bmatrix} 0 & 1 \\ \lambda & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & \overline{\alpha} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ \alpha^2\lambda & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \alpha \end{bmatrix} \equiv \begin{bmatrix} 0 & 1 \\ \alpha^2\lambda & 0 \end{bmatrix}.$$

The first and the last matrix are congruent and both are in canonical form. Hence, $\lambda = \lambda\alpha^2$ for every unimodular $\alpha \in \mathbb{C}$, a contradiction.

Next, suppose that $A \equiv \mu\left[\begin{smallmatrix} 0 & 1 \\ 1 & i \end{smallmatrix}\right]$ for some unimodular $\mu \in \mathbb{C}$. Thus $\mu\left[\begin{smallmatrix} 0 & 1 \\ 1 & i \end{smallmatrix}\right] \equiv \alpha\mu\left[\begin{smallmatrix} 0 & 1 \\ 1 & i \end{smallmatrix}\right]$ and by Proposition 5.1, since $|\alpha\mu| = 1$, it follows that $\mu = \alpha\mu$, for every unimodular $\alpha \in \mathbb{C}$, a contradiction.

By the same argument, the matrix A is neither congruent to $\left[\begin{smallmatrix} \mu_1 & 0 \\ 0 & \mu_2 \end{smallmatrix}\right]$ for any unimodular $\mu_1, \mu_2 \in \mathbb{C}$, nor to $\left[\begin{smallmatrix} \mu & 0 \\ 0 & 0 \end{smallmatrix}\right]$ for any unimodular $\mu \in \mathbb{C}$. Therefore $A \equiv \left[\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right]$, as claimed. ∎

LEMMA 5.3. *Let $A, B \in \mathcal{B}(\mathcal{H})$ be positive. If*

$$A - \lambda e \otimes Be - \lambda Be \otimes e + \lambda^2 \langle Be, e\rangle e \otimes e$$

is positive for every normalized $e \in \mathcal{H}$ and every real $\lambda \neq 1$, then $A - B$ is positive as well.

Proof. Choose any normalized $e \in \mathcal{H}$. Since the operator $A - \lambda e \otimes Be - \lambda Be \otimes e + \lambda^2\langle Be, e\rangle e \otimes e$, $\lambda \in \mathbb{R} \setminus \{1\}$, is positive, we actually have

$$\langle Ax, x\rangle - \lambda\langle x, Be\rangle\langle e, x\rangle - \lambda\langle x, e\rangle\langle Be, x\rangle + \lambda^2\langle Be, e\rangle\langle x, e\rangle\langle e, x\rangle \geq 0$$

for every $x \in \mathcal{H}$. By inserting $x := e$ it follows easily that

$$\langle Be, e\rangle(\lambda - 1)^2 + \langle (A - B)e, e\rangle \geq 0$$

for every real $\lambda \neq 1$. As $\langle Be, e\rangle \geq 0$, it must be that $\langle (A - B)e, e\rangle \geq 0$ for every normalized $e \in \mathcal{H}$ and in fact for every $e \in \mathcal{H}$, as claimed. ∎

Our last result is a representation of linear maps on $\mathcal{B}(\mathcal{H})$ which preserve congruence.

THEOREM 5.4. *Let $\phi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ be a bijective linear map. Then the following statements are equivalent.*

(i) *ϕ preserves congruence.*
(ii) *There exist a unimodular $\mu \in \mathbb{C}$ and an invertible operator $S \in \mathcal{B}(\mathcal{H})$ such that either*

$$\phi(X) = \mu SXS^* \quad \text{for every } X \in \mathcal{B}(\mathcal{H}),$$

or

$$\phi(X) = \mu SX^tS^* \quad \text{for every } X \in \mathcal{B}(\mathcal{H}),$$

where X^t denotes the transpose of X with respect to an arbitrary but fixed orthonormal basis of $\mathcal{H}$.

(iii) *ϕ preserves congruence in both directions (i.e. $A \equiv B$ if and only if $\phi(A) \equiv \phi(B)$).*

Proof. As in the previous sections, we only have to prove (i)⇒(ii), and the proof is in several steps.

STEP 1. *There exists a unimodular $\mu \in \mathbb{C}$ such that for every non-zero $x \in \mathcal{H}$ there exists a non-zero $p \in \mathcal{H}$ with $\phi(x \otimes x) = \mu p \otimes p$.*

Let $P \in \mathcal{B}(\mathcal{H})$ be any projection of rank one. As ϕ is surjective, there exists $A \in \mathcal{B}(\mathcal{H})$ such that $\phi(A) = P$. Because $A \neq 0$, there exists a normalized $e \in \mathcal{H}$ with $\langle Ae, e\rangle \neq 0$. By Lemma 3.2, the operator $I - \lambda e \otimes e$ is invertible for every $\lambda \in \mathbb{R} \setminus \{1\}$, so acting by ϕ on both sides of

$$A \equiv (I - \lambda e \otimes e)A(I - \lambda e \otimes e)^* = A - \lambda(e \otimes A^*e + Ae \otimes e) + \lambda^2\langle Ae, e\rangle e \otimes e$$

gives

$$P \equiv P - \lambda\phi(e \otimes A^*e + Ae \otimes e) + \lambda^2\langle Ae, e\rangle\phi(e \otimes e). \tag{5.1}$$

Hence,

$$P - \lambda\phi(e \otimes A^*e + Ae \otimes e) + \lambda^2\langle Ae, e\rangle\phi(e \otimes e) \tag{5.2}$$

is a self-adjoint operator of rank one. By dividing (5.2) by λ^2 and sending λ to infinity, we arrive at $\operatorname{rank}\langle Ae, e\rangle\phi(e \otimes e) \leq 1$. Moreover, $\langle Ae, e\rangle \neq 0$ and by the injectivity of ϕ, $\operatorname{rank}\phi(e \otimes e) = 1$. Thus,

$$\phi(e \otimes e) = a \otimes b \tag{5.3}$$

for some non-zero $a, b \in \mathcal{H}$. As the operator (5.2) is self-adjoint, we have

$$\lambda\phi(e\otimes A^*e+Ae\otimes e)-\lambda^2\langle Ae, e\rangle a\otimes b = \lambda\phi(e\otimes A^*e+Ae\otimes e)^*-\lambda^2\overline{\langle Ae, e\rangle}\, b\otimes a$$

for every $\lambda \in \mathbb{R} \setminus \{1\}$. Therefore

$$\langle Ae, e\rangle a \otimes b = \overline{\langle Ae, e\rangle}\, b \otimes a,$$

which implies that a and b are linearly dependent. Then there exists a non-zero $\alpha \in \mathbb{C}$ such that $a = \alpha b = |\alpha|\mu b$ for some unimodular $\mu \in \mathbb{C}$. By (5.3),

$$\phi(e \otimes e) = |\alpha|\mu b \otimes b = \mu p \otimes p,$$

where $p = \sqrt{|\alpha|}\, b \in \mathcal{H}$. Since $x \otimes x \equiv e \otimes e$ for every non-zero $x \in \mathcal{H}$, we deduce that $\phi(x \otimes x) \equiv \mu p \otimes p$ for every non-zero $x \in \mathcal{H}$.

Note that the same unimodular $\mu \in \mathbb{C}$ applies for every $x \otimes x$, so by replacing ϕ with $\overline{\mu}\phi$, we may further assume that for every non-zero $x \in \mathcal{H}$ we have $\phi(x \otimes x) = p \otimes p$ for some non-zero $p \in \mathcal{H}$, depending on x.

STEP 2. *ϕ is rank-one preserving.*

It is easy to see that every rank-one operator is either congruent to $\mu e \otimes e$ for some $|\mu| = 1$ and $e \neq 0$, or to $e \otimes f$ where e and f are linearly independent vectors. By Step 1 it is enough to show that for arbitrary linearly independent $e, f \in \mathcal{H}$, the operator $\phi(e \otimes f)$ is of rank one.

To do so, fix linearly independent $e, f \in \mathcal{H}$. Without loss of generality, we may assume that $\phi(e\otimes e) = e\otimes e$ and $\phi(f\otimes f) = p\otimes p$, for some non-zero $p \in \mathcal{H}$. It is clear that there exists an invertible $S \in \mathcal{B}(\mathcal{H})$ such that $Se = e$ and $Sf = p$. By replacing ϕ with $S^{-1}\phi(\cdot)(S^{-1})^*$ we may further assume that $\phi(e \otimes e) = e \otimes e$ and $\phi(f \otimes f) = f \otimes f$.

By applying Step 1 we deduce $\phi((e + f) \otimes (e + f)) = a \otimes a$ for some non-zero $a \in \mathcal{H}$. On the other hand, by the linearity of ϕ, we have $\phi(e \otimes e)$ $+\, \phi(e \otimes f + f \otimes e) + \phi(f \otimes f) = a \otimes a$, and thus

$$\phi(e \otimes f + f \otimes e) = a \otimes a - e \otimes e - f \otimes f$$

is a self-adjoint operator of rank at most three. We can easily verify that $e \otimes f + f \otimes e \equiv -(e \otimes f + f \otimes e)$, and therefore

$$\phi(e \otimes f + f \otimes e) \equiv -\phi(e \otimes f + f \otimes e). \tag{5.4}$$

Observe that self-adjoint operators of finite rank are congruent if and only if they have the same number of positive eigenvalues and the same number of negative eigenvalues [8]. Now, as $\phi(e\otimes f + f\otimes e)$ is a self-adjoint operator of rank at most three and because (5.4) is true, it follows that $\phi(e\otimes f + f\otimes e)$ is of rank two with one positive and one negative eigenvalue. Since e and f are linearly independent, we have $a \in \operatorname{span}\{e, f\}$. With respect to the decomposition $\mathcal{H} = \operatorname{span}\{e, f\} \oplus \{e, f\}^\perp$ the operator $\phi(e \otimes f + f \otimes e)$ has a matrix representation

$$\phi(e \otimes f + f \otimes e) = \begin{bmatrix} a_{11} & a_{12} \\ \overline{a}_{12} & a_{22} \end{bmatrix} \oplus 0,$$

for some $a_{11}, a_{12}, a_{22} \in \mathbb{C}$. Similarly, repeating the same procedure with ie instead of e shows that the operator

$$\phi(ie \otimes f - if \otimes e) = b \otimes b - e \otimes e - f \otimes f$$

is of rank two for some $b \in \operatorname{span}\{e, f\}$. Then

$$\phi(ie \otimes f - if \otimes e) = \begin{bmatrix} b_{11} & b_{12} \\ \overline{b}_{12} & b_{22} \end{bmatrix} \oplus 0$$

for some $b_{11}, b_{12}, b_{22} \in \mathbb{C}$.

Since $2e \otimes f = (e \otimes f + f \otimes e) - i(ie \otimes f - if \otimes e)$, we get

$$\phi(2e \otimes f) = \begin{bmatrix} a_1 - ib_1 & a_2 - ib_2 \\ \overline{a}_2 - i\overline{b}_2 & a_4 - ib_4 \end{bmatrix} \oplus 0.$$

By [20, Proposition 3] every nilpotent N of rank one is unitarily similar and thus congruent to αN, for every unimodular $\alpha \in \mathbb{C}$. Thus $2e\otimes f \equiv \alpha(2e\otimes f)$

for every $\alpha \in \mathbb{C}$ with $|\alpha| = 1$. This yields

$$\phi(2e \otimes f) \equiv \alpha\phi(2e \otimes f) \quad \text{for every scalar } \alpha \text{ with } |\alpha| = 1.$$

Since $\phi(2e \otimes f)$ can be viewed as a 2×2 square matrix, by Lemma 5.2 the operator $\phi(2e \otimes f)$ must be of rank one, as desired.

STEP 3. *Either there exists an injective linear map $T : \mathcal{H} \to \mathcal{H}$ such that $\phi(x \otimes y) = Tx \otimes Ty$ for every rank-one operator $x \otimes y$, or there exists an injective conjugate-linear map $T : \mathcal{H} \to \mathcal{H}$ such that $\phi(x \otimes y) = Ty \otimes Tx$ for every rank-one operator $x \otimes y$.*

Since ϕ preserves rank-one operators, one and only one of the statements from Theorem 2.1 is true. Suppose that it is the first one, i.e. there exist a non-zero $c \in \mathcal{H}$ and a linear map $\tau : \mathcal{B}(\mathcal{H}) \to \mathcal{H}$, which cannot map any rank-one operator to zero, such that $\phi(X) = \tau(X) \otimes c$ for every $X \in \mathcal{F}(\mathcal{H})$. By taking any linearly independent $x, y \in \mathcal{H}$ we get

$$\phi(x \otimes x) = \tau(x \otimes x) \otimes c = p \otimes p \quad \text{and} \quad \phi(y \otimes y) = \tau(y \otimes y) \otimes c = q \otimes q$$

for some non-zero $p, q \in \mathcal{H}$ by Step 1. Hence p and q are each linearly dependent on c. Thus p and q are linearly dependent, which contradicts the injectivity of ϕ.

By the same argument, (ii) of Theorem 2.1 cannot be true, so ϕ satisfies either (iii) or (iv). Assume that (iii) holds and that there exist injective linear maps $T, S : \mathcal{H} \to \mathcal{H}$ such that $\phi(x \otimes y) = Tx \otimes Sy$ for every rank-one operator $x \otimes y$. Since we already know that for every $x \in \mathcal{H}$ we have $\phi(x \otimes x) = Tx \otimes Sx = p \otimes p$ for some $p \in \mathcal{H}$, there actually exists $\lambda > 0$ such that $Sx = \lambda Tx$ for every $x \in \mathcal{H}$. This immediately implies that $\phi(x \otimes x) = \lambda Tx \otimes Tx = \sqrt{\lambda}Tx \otimes \sqrt{\lambda}Tx$. Therefore, replacing T by $(1/\sqrt{\lambda})T$ yields $\phi(x \otimes y) = Tx \otimes Ty$ for every rank-one operator $x \otimes y$.

Similarly, if (iv) of Theorem 2.1 is true, there exists an injective conjugate-linear map T such that $\phi(x \otimes y) = Ty \otimes Tx$ for every rank-one operator $x \otimes y$. In this case, $J : \mathcal{H} \to \mathcal{H}$ defined by $Jx = \sum_{i \in I} \overline{\langle x, e_i \rangle} e_i$ is a bijective conjugate-linear map, where $\{e_i \mid i \in I\}$ is a fixed orthonormal basis of $\mathcal{H}$. Then $X^t = JXJ$ denotes the transpose of X with respect to that basis. Now, replace ϕ by $X \mapsto \phi(X^t)$, $X \in \mathcal{B}(\mathcal{H})$. This gives $\phi(x \otimes y) = TJx \otimes TJy$, where $T \circ J$ is an injective linear map on $\mathcal{H}$. So, without loss of generality we can and will assume that there exists an injective linear map $T : \mathcal{H} \to \mathcal{H}$ such that

$$\phi(x \otimes y) = Tx \otimes Ty \quad \text{for every rank-one operator } x \otimes y.$$

STEP 4. *Let $A \in \mathcal{B}(\mathcal{H})$. If $\phi(A)$ is positive, then A is positive.*

Denote $\phi(A) = B$ and assume that B is positive. We want to show that $\langle Ax, x \rangle \geq 0$ for every $x \in \mathcal{H}$. If $B = 0$, this is obviously true, so suppose $B \neq 0$. Choose a normalized $e \in \mathcal{H}$. By Lemma 3.2, the operator $I - \lambda e \otimes e$

is invertible for every real $\lambda \neq 1$. Thus,

$$A \equiv A - \lambda e \otimes A^* e - \lambda Ae \otimes e + \lambda^2 \langle Ae, e\rangle e \otimes e$$

for every $\lambda \in \mathbb{R} \setminus \{1\}$. Since ϕ preserves congruence, we obtain

$$B \equiv B - \lambda Te \otimes TA^* e - \lambda TAe \otimes Te + \lambda^2 \langle Ae, e\rangle Te \otimes Te,$$

which immediately implies that the operator

$$B - \lambda Te \otimes TA^* e - \lambda TAe \otimes Te + \lambda^2 \langle Ae, e\rangle Te \otimes Te$$

is positive. Therefore,

$$\langle Bx, x\rangle - \lambda(\langle x, TA^* e\rangle \langle Te, x\rangle + \langle x, Te\rangle \langle TAe, x\rangle) + \lambda^2 \langle Ae, e\rangle |\langle x, Te\rangle|^2 \geq 0$$

for every $x \in \mathcal{H}$ and every real $\lambda \neq 1$, and hence $\langle Ae, e\rangle \geq 0$ for every $e \in \mathcal{H}$.

Step 5. *The injective linear map $T : \mathcal{H} \to \mathcal{H}$ such that $\phi(x \otimes y) = Tx \otimes Ty$ for all $x, y \in \mathcal{H}$ is surjective and bounded.*

Suppose T is not surjective, so there exists $u \in \mathcal{H} \setminus \operatorname{Im} T$. By the surjectivity of ϕ and Step 4 there exists a positive $A \in \mathcal{B}(\mathcal{H})$ with $\phi(A) = u \otimes u$.

If A is a scalar operator, then $\phi(I) = \alpha u \otimes u$ for some non-zero scalar α. From the relation $I \equiv I + e_1 \otimes e_1$ for any normalized $e_1 \in \mathcal{H}$ it follows that $\alpha u \otimes u \equiv \alpha u \otimes u + Te_1 \otimes Te_1$. Hence, $\alpha u \otimes u + Te_1 \otimes Te_1$ is of rank one, which contradicts the assumption that u and Te_1 are linearly independent.

Thus A is not scalar, so there exists a normalized $e_2 \in \mathcal{H}$ such that e_2 and Ae_2 are linearly independent. The operator $I + e_2 \otimes e_2$ is obviously invertible, and since ϕ preserves congruence, the relation

$$A \equiv (I + e_2 \otimes e_2) A (I + e_2 \otimes e_2)$$

implies

$$u \otimes u \equiv u \otimes u + Te_2 \otimes TAe_2 + TAe_2 \otimes Te_2 + \langle Ae_2, e_2\rangle Te_2 \otimes Te_2,$$

and consequently the operator

$$u \otimes u + Te_2 \otimes \left(TAe_2 + \frac{\langle Ae_2, e_2\rangle}{2} Te_2 \right) + \left(TAe_2 + \frac{\langle Ae_2, e_2\rangle}{2} Te_2 \right) \otimes Te_2$$

is of rank one. On the other hand, since T is injective and $u \notin \operatorname{Im} T$, we see that $\{u, Te_2, TAe_2\}$ is a set of linearly independent vectors, so the set $\left\{u, Te_2, TAe_2 + \frac{1}{2}\langle Ae_2, e_2\rangle Te_2\right\}$ is linearly independent too, a contradiction.

To prove that T is bounded, observe that the operator $I - \lambda e \otimes e$ is positive for every normalized $e \in \mathcal{H}$ and every $\lambda < 1$. By Step 4, let $\phi(B) = I$ for some positive $B \in \mathcal{B}(\mathcal{H})$. Apply Step 4 once again to see that the operator $\phi^{-1}(I - \lambda e \otimes e) = B - \lambda T^{-1} e \otimes T^{-1} e$ is positive as well. Thus, $\langle Bx, x\rangle - \lambda |\langle x, T^{-1} e\rangle|^2 \geq 0$ for every $x \in \mathcal{H}$ and every $\lambda < 1$. By inserting

$x := T^{-1}e$ we get

$$\langle BT^{-1}e, T^{-1}e\rangle - \lambda \|T^{-1}e\|^4 \ge 0.$$

Obviously $T^{-1}e \neq 0$, therefore

$$\lambda \le \frac{\langle BT^{-1}e, T^{-1}e\rangle}{\|T^{-1}e\|^4}$$

for every $\lambda < 1$. Hence,

$$1 \le \frac{\langle BT^{-1}e, T^{-1}e\rangle}{\|T^{-1}e\|^4}.$$

By the Cauchy–Schwarz inequality and the boundedness of B,

$$\|T^{-1}e\|^4 \le \langle BT^{-1}e, T^{-1}e\rangle \le \|BT^{-1}e\| \cdot \|T^{-1}e\| \le \|B\| \cdot \|T^{-1}e\|^2$$

for every normalized $e \in \mathcal{H}$. We arrive at $\|T^{-1}e\| \le \sqrt{\|B\|}$ for every normalized $e \in \mathcal{H}$. Thus T^{-1} is bounded, and hence so is T.

With this at hand, we can replace ϕ with $T^{-1}\phi(\cdot)(T^{-1})^*$ and without loss of generality we further assume that

$$\phi(x \otimes y) = x \otimes y \quad \text{for every rank-one operator } x \otimes y.$$

Step 6. *$\phi(A) = A$ for every positive $A \in \mathcal{B}(\mathcal{H})$.*

Choose any positive $B \in \mathcal{B}(\mathcal{H})$. By the bijectivity of ϕ and Step 4 there exists exactly one positive $A \in \mathcal{B}(\mathcal{H})$ such that $\phi(A) = B$. Our aim is to show that $A = B$. In order to do this, take any normalized $e \in \mathcal{H}$. By Lemma 3.2, the operator $I - \lambda e \otimes e$ is invertible for every real $\lambda \neq 1$.

Firstly, apply $A \equiv (I - \lambda e \otimes e)A(I - \lambda e \otimes e)$ to get positivity of the operator $B - \lambda e \otimes Ae - \lambda Ae \otimes e + \lambda^2 \langle Ae, e\rangle e \otimes e$. By Lemma 5.3, we obtain

$$\langle (B - A)e, e\rangle \ge 0 \quad \text{for every } e \in \mathcal{H}.$$

Then from the relation $B \equiv (I - \lambda e \otimes e)B(I - \lambda e \otimes e)$ it follows that the operator $B - \lambda e \otimes Be - \lambda Be \otimes e + \lambda^2 \langle Be, e\rangle e \otimes e$ is positive. By Step 4 the operator $A - \lambda e \otimes Be - \lambda Be \otimes e + \lambda^2 \langle Be, e\rangle e \otimes e$ is positive as well. Hence, Lemma 5.3 gives

$$\langle (A - B)e, e\rangle \ge 0 \quad \text{for every } e \in \mathcal{H}.$$

Therefore,

$$\langle (B - A)e, e\rangle = 0 \quad \text{for every } e \in \mathcal{H},$$

and thus $A = B$.

We finally complete the proof of Theorem 5.4 by invoking the well known fact that every $A \in \mathcal{B}(\mathcal{H})$ can be written as a linear combination of two self-adjoint operators, and every self-adjoint operator can be written as a difference of two positive operators in $\mathcal{B}(\mathcal{H})$. ∎

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