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A unified approach to approximation properties and ideals

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Abstract. We use a unified argument to obtain relationships between approximation properties and ideals in spaces of some operators. We prove that a Banach space X (respectively, the dual space X^* of X) has the metric approximation property if and only if for every Banach space Y and every operator T from Y to X (respectively, T from X to Y), there exists a

$$\varPhi \in \mathcal{HB}(\mathcal{F}(X)T,\ \mathrm{span}(\mathcal{F}(X)T \cup \{T\}))$$
$$(\text{respectively, } \varPhi \in \mathcal{HB}(T\mathcal{F}(X),\ \mathrm{span}(T\mathcal{F}(X) \cup \{T\})))$$

such that

$$\varPhi(x^* \otimes y)(R) = x^*(Ry) \text{ (respectively, } \varPhi(x^{**} \otimes y^*)(R) = x^{**}(R^{\mathrm{a}}y^*))$$

for every $x^* \in X^*$ and $y \in Y$ (respectively, $x^{**} \in X^{**}$ and $y^* \in Y^*$), and every $R \in \mathrm{span}(\mathcal{F}(X)T \cup \{T\})$ (respectively, $R \in \mathrm{span}(T\mathcal{F}(X) \cup \{T\})$).

1. Introduction and main results. A Banach space X is said to have the *approximation property* (*AP*) (resp. *compact AP* (*CAP*)) if for every compact subset K of X and every $\varepsilon > 0$, there exists an S in the space $\mathcal{F}(X) := \mathcal{F}(X, X)$ of all finite rank operators (resp. the space $\mathcal{K}(X)$ of all compact operators) from X into X such that $\sup_{x\in K} \|Sx - x\| \le \varepsilon$; moreover, if $\|S\| \le 1$, then we say that X has the *metric AP* (*MAP*) (resp. *MCAP*). Lima, Nygaard, and Oja [LNO] proved that a Banach space X has the AP if and only if $\mathcal{F}(Y, X)$ is an ideal in $\mathcal{W}(Y, X)$ for every Banach space Y, where $\mathcal{W}(Y, X)$ is the space of all weakly compact operators from Y to X (see below for the definition of an ideal). They also proved in [LNO] that if X has the CAP, then $\mathcal{K}(Y, X)$ is an ideal in $\mathcal{W}(Y, X)$ for every Banach space Y. But the converse does not hold in general (see [LNO, Example in Section 4]).

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V. Lima, Å. Lima, and Nygaard [LLN] obtained various characterizations of the CAP, and the main one is somewhat analogous to the above characterization of the AP. The main purpose of this paper is to obtain similar such characterizations of various approximation properties using an argument in [L1].

Let F be a linear subspace of a normed space E. A linear isometry $\phi : F^* \to E^*$ is called a *Hahn–Banach extension operator* if $(\phi f^*)(f) = f^*(f)$ for every $f \in F$ and $f^* \in F^*$. We say that F is an *ideal* in E if the set $\mathcal{HB}(F, E)$ is nonempty, where $\mathcal{HB}(F, E)$ is the collection of all Hahn–Banach extension operators from F^* to E^*. We see that F is an ideal in E if and only if $\overline{F}^{\|\cdot\|}$ (or F) is an ideal in $\overline{E}^{\|\cdot\|}$ (or E).

Let X be a Banach space and let $\mathcal{A}(X)$ be a convex subset of the space $\mathcal{L}(X)$ of all bounded operators from X to X. Then X is said to have the *$\mathcal{A}$-AP* if for every compact subset K of X and every $\varepsilon > 0$, there exists an $S \in \mathcal{A}(X)$ such that $\sup_{x\in K} \|Sx - x\| \le \varepsilon$; moreover, if $\|S\| \le 1$, then we say that X has the *$\mathcal{A}$-MAP*.

For a set A, we denote by $\langle A\rangle$ the linear span of A. Let X, Y, Z be Banach spaces. For a subset $\mathcal{A}$ of $\mathcal{L}(Y)$, $T \in \mathcal{L}(X, Y)$, and $R \in \mathcal{L}(Y, Z)$, let $R\mathcal{A}T := \{RST : S \in \mathcal{A}\}$ and $\mathcal{A}^{\mathrm{a}} := \{S^{\mathrm{a}} : S \in \mathcal{A}\}$, where S^{a} is the adjoint operator of S. We denote by $i_X : X \to X^{**}$ the canonical isometry. Recall that the projective tensor product $X \hat{\otimes}_\pi Y$ is a closed subspace of $X^{**} \hat{\otimes}_\pi Y^{**}$, and $(X \hat{\otimes}_\pi Y)^*$ is isometrically isomorphic to $\mathcal{L}(X, Y^*)$ (cf. [R, Section 2]). Moreover, $X^* \hat{\otimes}_\pi Y^{**}$ is embedded in $\mathcal{L}(Y, X)^*$. In this paper, all elementary tensors are elements of projective tensor products. For $T \in \mathcal{L}(Y, X)$, we have the dual actions $(x^* \otimes y)(T) = x^*(Ty)$ and $(x^* \otimes y^{**})(T) = y^{**}(T^{\mathrm{a}}x^*)$.

V. Lima and Å. Lima [LL1] proved that a Banach space X has the MAP (resp. MCAP) if and only if $\mathcal{F}(Y, X)$ (resp. $\mathcal{K}(Y, X)$) is an ideal in $\mathcal{L}(Y, X)$ for every Banach space Y.

In this paper, we prove:

Theorem 1.1. *Let X be a Banach space. Suppose that $\mathcal{A}(X)$ is a linear subspace of $\mathcal{K}(X)$ containing $\mathcal{F}(X)$ such that $i_X^{-1}S^{\mathrm{aa}}U \in \mathcal{A}(X)$ for every $S \in \mathcal{A}(X)$ and $U \in \mathcal{L}(X, X^{**})$. Then the following statements are equivalent:*

(a) *X has the $\mathcal{A}$-MAP.*

(b) *For every Banach space Y and every $T \in \mathcal{L}(Y, X)$, there exists a*

$$\Phi \in \mathcal{HB}(\mathcal{A}(X)T, \langle\mathcal{A}(X)T \cup \{T\}\rangle)$$

such that

$$\Phi(x^* \otimes y)(R) = x^*(Ry)$$

for every $x^* \in X^*$ and $y \in Y$, and every $R \in \langle\mathcal{A}(X)T \cup \{T\}\rangle$.

(c) *For every separable Banach space Z and every $T \in \mathcal{L}(Z, X)$, the space $\mathcal{A}(X)T$ is an ideal in $\langle\mathcal{A}(X)T \cup \{T\}\rangle$.*

(d) *For every equivalent renorming* $\hat{X}$ *of* X, *the space* $\mathcal{A}(X)I$ *is an ideal in* $\langle \mathcal{A}(X)I \cup \{I\}\rangle$, *where* $I : \hat{X} \to X$ *is the identity map.*

An unbounded version of Theorem 1.1 is the following, which is an extension of [LLN, Theorem 2.2]. Here τ_c is the topology of uniform convergence on each compact set.

THEOREM 1.2. *Let* X *be a Banach space. Suppose that* $\mathcal{A}(X)$ *is a linear subspace of* $\mathcal{K}(X)$ *such that* $S_1S_2 \in \mathcal{A}(X)$ *for all* $S_1, S_2 \in \mathcal{A}(X)$. *Then the following statements are equivalent:*

(a) X *has the* $\mathcal{A}$-*AP.*

(b) *For every Banach space* Y *and every* $T \in \mathcal{W}(Y,X)$,

$$T \in \overline{\{ST : S \in \mathcal{A}(X), \|ST\| \le \|T\|\}}^{\tau_c}.$$

(c) *For every Banach space* Y *and every* $T \in \mathcal{W}(Y,X)$, *there exists a*

$$\Phi \in \mathcal{HB}(\mathcal{A}(X)T, \langle \mathcal{A}(X)T \cup \{T\}\rangle)$$

such that

$$\Phi(x^* \otimes y)(R) = x^*(Ry)$$

for every $x^* \in X^*$ *and* $y \in Y$, *and every* $R \in \langle \mathcal{A}(X)T \cup \{T\}\rangle$.

(d) *For every separable reflexive Banach space* Z *and every* $T \in \mathcal{K}(Z,X)$, *there exists a*

$$\Theta \in \mathcal{L}((\mathcal{A}(X)T)^*, \langle \mathcal{A}(X)T \cup \{T\}\rangle^*)$$

such that

$$\Theta(x^* \otimes z)(T) = x^*(Tz)$$

for every $x^* \in X^*$ *and* $z \in Z$.

(e) *For every separable reflexive Banach space* Z *and every* $T \in \mathcal{K}(Z,X)$,

$$T \in \overline{\mathcal{A}(X)T}^{\tau_c}.$$

A Banach space X is said to have the *weak MAP* [LO1] if for every Banach space Y and every $R \in \mathcal{W}(X,Y)$, the identity map id_X is in $\overline{\{S \in \mathcal{F}(X) : \|RS\| \le \|R\|\}}^{\tau_c}$. When the space $\mathcal{F}(X)$ is replaced by a convex subset $\mathcal{A}(X)$ of $\mathcal{L}(X)$, we say that X has the *weak* $\mathcal{A}$-*MAP* [LisO]. V. Lima and Å. Lima [L2, LL2] studied the weak MAP and the weak $\mathcal{K}$-MAP from the point of view of ideals.

The following theorem is an extension of [LL2, Theorem 4.3].

THEOREM 1.3. *Let* X *be a Banach space. Suppose that* $\mathcal{A}(X)$ *is a linear subspace of* $\mathcal{K}(X)$ *such that* $S_1S_2 \in \mathcal{A}(X)$ *for all* $S_1, S_2 \in \mathcal{A}(X)$. *Then the following statements are equivalent:*

(a) X *has the weak* $\mathcal{A}$-*MAP.*

(b) *There exists a* $\phi \in \mathcal{HB}(X, X^{**})$ *such that* $\phi^{\mathrm{a}} i_{X^{**}} \in \overline{\mathcal{A}^{\mathrm{aa}}(X)}^{\mathrm{weak}^*}$ *in* $\mathcal{L}(X^{**})$.

(c) *There exists a* $\phi \in \mathcal{HB}(X, X^{**})$ *such that for every reflexive Banach space* Z *and every* $T \in \mathcal{W}(Z, X^{**})$, *there is a net* (S_α) *in* $\mathcal{A}(X)$ *with* $\sup_\alpha \|S_\alpha^{\mathrm{aa}} T\| \le \|T\|$ *such that* $(S_\alpha^{\mathrm{aa}} T)^{\mathrm{a}}(x^*) \to T^{\mathrm{a}}(\phi x^*)$ *for all* $x^* \in X^*$.

(d) *There exists a* $\phi \in \mathcal{HB}(X, X^{**})$ *such that for every Banach space* Y *and every* $T \in \mathcal{W}(Y, X^{**})$, *there exists a* $\varPhi \in \mathcal{HB}(\mathcal{A}^{\mathrm{aa}}(X)T, \langle \mathcal{A}^{\mathrm{aa}}(X)T \cup \{T\}\rangle)$ *such that*

$$\varPhi(x^* \otimes y^{**})(R) = y^{**}(R^a \phi x^*)$$

for every $x^* \in X^*$ *and* $y^{**} \in Y^{**}$, *and every* $R \in \langle \mathcal{A}^{\mathrm{aa}}(X)T \cup \{T\}\rangle$.

(e) *There exists a* $\phi \in \mathcal{HB}(X, X^{**})$ *such that for every separable reflexive Banach space* Z *and every* $T \in \mathcal{K}(Z, X^{**})$, *there exists a*

$$\varTheta \in \mathcal{L}((\mathcal{A}^{\mathrm{aa}}(X)T)^*, \langle \mathcal{A}^{\mathrm{aa}}(X)T \cup \{T\}\rangle^*)$$

such that

$$\varTheta(x^* \otimes z)(T) = \phi x^*(Tz)$$

for every $x^* \in X^*$ *and* $z \in Z$.

It was shown in [LLN, Lemma 3.5] that if X^* has the $\mathcal{K}^a$-AP, then for every Banach space Y and every $T \in \mathcal{W}(Y, X^{**})$, $\mathcal{K}^{\mathrm{aa}}(X)T$ is an ideal in $\langle \mathcal{K}^{\mathrm{aa}}(X)T \cup \{T\}\rangle$. But it was left open in [LLN] whether the converse holds. [LL2, Theorem 4.3] or Theorem 1.3 gives a negative answer to that question because it is well known that there exists a Banach space B having the MAP but with B^* not having the CAP.

The following theorem provides conditions equivalent to the $\mathcal{A}^{\mathrm{a}}$-AP in a similar form to Theorem 1.3.

THEOREM 1.4. *Let* X *be a Banach space. Suppose that* $\mathcal{A}(X)$ *is a linear subspace of* $\mathcal{K}(X)$ *such that* $S_1 S_2 \in \mathcal{A}(X)$ *for all* $S_1, S_2 \in \mathcal{A}(X)$. *Then the following statements are equivalent:*

(a) X^* *has the* $\mathcal{A}^{\mathrm{a}}$*-AP.*

(b) *For every reflexive Banach space* Z *and every* $T \in \mathcal{W}(Z, X^{**})$, *there exists a net* (S_α) *in* $\mathcal{A}(X)$ *with* $\sup_\alpha \|S_\alpha^{\mathrm{aa}} T\| \le \|T\|$ *such that* $(S_\alpha^{\mathrm{aa}} T)^{\mathrm{a}}(x^*) \to T^{\mathrm{a}}(i_{X^*} x^*)$ *for every* $x^* \in X^*$.

(c) *For every Banach space* Y *and every* $T \in \mathcal{W}(Y, X^{**})$, *there exists a*

$$\varPhi \in \mathcal{HB}(\mathcal{A}^{\mathrm{aa}}(X)T, \langle \mathcal{A}^{\mathrm{aa}}(X)T \cup \{T\}\rangle)$$

such that

$$\varPhi(x^* \otimes y^{**})(R) = y^{**}(R^{\mathrm{a}} i_{X^*} x^*)$$

for every $x^* \in X^*$ *and* $y^{**} \in Y^{**}$, *and every* $R \in \langle \mathcal{A}^{\mathrm{aa}}(X)T \cup \{T\}\rangle$.

(d) *For every separable reflexive Banach space Z and every $T \in \mathcal{K}(Z, X^{**})$, there exists a $\Theta \in \mathcal{L}((\mathcal{A}^{\mathrm{aa}}(X)T)^*, \langle \mathcal{A}^{\mathrm{aa}}(X)T \cup \{T\}\rangle^*)$ such that*

$$\Theta(x^* \otimes z)(T) = (Tz)(x^*)$$

for every $x^* \in X^*$ and $z \in Z$.

2. Proof of Theorem 1.1. The main argument in proofs of this paper is due to [L1]. The prototype of Lemma 2.1 is [L1, Lemma 3.3].

LEMMA 2.1. *Let X be a Banach space. Suppose that $\mathcal{A}(X)$ is a linear subspace of $\mathcal{K}(X)$ such that $S_1S_2 \in \mathcal{A}(X)$ for any $S_1, S_2 \in \mathcal{A}(X)$. If X has the $\mathcal{A}$-MAP, then for every Banach space Y and every $T \in \mathcal{L}(Y, X)$, for every finite-dimensional subspace G of $\langle \mathcal{A}(X)T \cup \{T\}\rangle$, every finite-dimensional subspace H of $\mathcal{L}(Y, X)^*$, and every $\varepsilon > 0$, there exists a linear contraction $U : G \to \mathcal{A}(X)T$ such that for every $R \in G \cap \mathcal{A}(X)T$,*

$$\|U(R) - R\| \le \varepsilon\|R\|,$$

and for every $x^* \otimes y \in H$ and every $R \in G$,

$$|x^*(U(R)y) - x^*(Ry)| \le \varepsilon\|x^* \otimes y\|\,\|R\|.$$

Proof. Let Y be a Banach space and let $T \in \mathcal{L}(Y, X)$. Let G be a finite-dimensional subspace of $\langle \mathcal{A}(X)T \cup \{T\}\rangle$ and let H be a finite-dimensional subspace of $\mathcal{L}(Y, X)^*$. Let $0 < \varepsilon < 1$. Since $\{R \in G \cap \mathcal{A}(X)T : \|R\| \le 1\}$ is a relatively compact subset of $\mathcal{K}(Y, X)$ and X has the $\mathcal{A}$-MAP, there exists a net (S_α) in $\mathcal{A}(X)$ with $\sup_\alpha \|S_\alpha\| \le 1$ such that

$$\sup\{\|S_\alpha R - R\| : R \in G \cap \mathcal{A}(X)T,\ \|R\| \le 1\} \to 0,$$

and for every $x^* \in X^*$, $y \in Y$ and $R \in G$, we have

$$\lim_\alpha x^*(S_\alpha Ry) = x^*(Ry).$$

Now, let $\delta > 0$ be such that $0 < (3 - 2\delta)\delta/(1 - \delta) \le \varepsilon$. Consider the finite-dimensional subspace $H_1 := \operatorname{span}\{x^* \otimes y : x^* \otimes y \in H\}$ of $X^* \hat{\otimes}_\pi Y$. We may assume that $H_1 \ne \{0\}$. Then there exist $u_1, \ldots, u_m$ in the unit sphere of H_1 such that the ball $(1 - \delta)B_{H_1}$ is a subset of the convex hull of $\{u_i\}_{i=1}^m$ (cf. [L2, Lemma 2.4]). For each $i = 1, \ldots, m$, choose a representation

$$u_i = \sum_{k=1}^{n_i} x_{i,k}^* \otimes y_{i,k},$$

where all $x_{i,k}^* \otimes y_{i,k}$ are in H. Let $\{R_j\}_{j=1}^l$ be a δ-net for B_G. Choose an α_0 such that

$$\sum_{k=1}^{n_i} |x_{i,k}^*(S_{\alpha_0}R_jy_{i,k}) - x_{i,k}^*(R_jy_{i,k})| \le \delta$$

for all $i = 1, \ldots, m$ and $j = 1, \ldots, l$, and

$$\sup\{\|S_{\alpha_0}R - R\| : R \in G \cap \mathcal{A}(X)T,\ \|R\| \le 1\} \le \delta.$$

We now define the map $U : G \to \mathcal{A}(X)T$ by

$$U(R) = S_{\alpha_0}R.$$

Since for every $R := aST + bT \in G$, we have $S_{\alpha_0}R = (aS_{\alpha_0}S + bS_{\alpha_0})T \in \mathcal{A}(X)T$, U is well defined and linear, and $\|U(R)\| = \|S_{\alpha_0}R\| \le \|R\|$ for every $R \in G$. Also, for every $R \in G \cap \mathcal{A}(X)T$ with $\|R\| \le 1$, we have

$$\|U(R) - R\| = \|S_{\alpha_0}R - R\| \le \delta \le \varepsilon.$$

Now, let $x^* \otimes y \in H$ and $R \in B_G$. Choose a j so that $\|R - R_j\| \le \delta$. Then

$$(1-\delta)\frac{x^* \otimes y}{\|x^* \otimes y\|} = \sum_{i=1}^{m} \lambda_i u_i = \sum_{i=1}^{m} \lambda_i \sum_{k=1}^{n_i} x^*_{i,k} \otimes y_{i,k},$$

where $\lambda_i \ge 0$ for all i and $\sum_{i=1}^m \lambda_i = 1$. We now have

$$\begin{aligned}
&|x^*(U(R)y) - x^*(Ry)| \\
&\quad \le |x^*(U(R)y) - x^*(U(R_j)y)| + |x^*(U(R_j)y) - x^*(R_jy)| \\
&\qquad + |x^*(R_jy) - x^*(Ry)| \\
&\quad \le 2\delta\|x^* \otimes y\| + |x^*(U(R_j)y) - x^*(R_jy)| \\
&\quad = 2\delta\|x^* \otimes y\| \\
&\qquad + \frac{\|x^* \otimes y\|}{1-\delta}\Big|\Big(\sum_{i=1}^{m} \lambda_i \sum_{k=1}^{n_i} x^*_{i,k}(U(R_j)y_{i,k})\Big) - \Big(\sum_{i=1}^{m} \lambda_i \sum_{k=1}^{n_i} x^*_{i,k}(R_jy_{i,k})\Big)\Big| \\
&\quad \le 2\delta\|x^* \otimes y\| + \frac{\|x^* \otimes y\|}{1-\delta}\sum_{i=1}^{m} \lambda_i \sum_{k=1}^{n_i} |x^*_{i,k}(S_{\alpha_0}R_jy_{i,k}) - x^*_{i,k}(R_jy_{i,k})| \\
&\quad \le \frac{(3-2\delta)\delta}{1-\delta}\|x^* \otimes y\| \le \varepsilon\|x^* \otimes y\|. \ \blacksquare
\end{aligned}$$

One may use [L1, proof of Lemma 3.5] to obtain:

LEMMA 2.2. *Let X and Y be Banach spaces. Suppose that $\mathcal{C}$ and $\mathcal{B}$ are linear subspaces of $\mathcal{L}(Y, X)$ with $\mathcal{C} \subset \mathcal{B}$. If for every finite-dimensional subspaces $H \subset \mathcal{L}(Y, X)^*$ and $G \subset \mathcal{B}$, and every $\varepsilon > 0$, there exists a linear contraction $U : G \to \mathcal{C}$ such that for every $R \in G \cap \mathcal{C}$,*

$$\|U(R) - R\| \le \varepsilon\|R\|,$$

and for every $x^* \otimes y \in H$ and every $R \in G$,

$$|x^*(U(R)y) - x^*(Ry)| \le \varepsilon\|x^* \otimes y\|\,\|R\|,$$

then there exists a $\Phi \in \mathcal{HB}(\mathcal{C}, \mathcal{B})$ such that

$$\Phi(x^* \otimes y)(R) = x^*(Ry)$$

for every $x^* \in X^*$ and $y \in Y$, and every $R \in \mathcal{B}$.

Now, (a)⇒(b) of Theorem 1.1 is a consequence of Lemmas 2.1 and 2.2, and (b)⇒(c) and (b)⇒(d) are trivial.

For the proofs of the next Lemmas 2.3 and 2.4, we note that one can readily observe that [LL1, Theorems 3.1 and 3.3] are also true even if $\mathcal{A}(Y,X)$ and $\mathcal{B}(Y,X)$ are linear subspaces of $\mathcal{L}(Y,X)$ with $\mathcal{F}(Y,X) \subset \mathcal{A}(Y,X) \subset \mathcal{B}(Y,X) \subset \mathcal{L}(Y,X)$ for given Banach spaces X and Y.

Lemma 2.3. *Let X be a Banach space. Suppose that $\mathcal{A}(X)$ is a linear subspace of $\mathcal{W}(X)$ such that $\mathcal{F}(X) \subset \mathcal{A}(X)$, $i_X^{-1} S^{\mathrm{aa}} U \in \mathcal{A}(X)$ for every $S \in \mathcal{A}(X)$ and $U \in \mathcal{L}(X, X^{**})$. If for every separable ideal Y in X and every equivalent renorming $\hat{Y}$ of Y, $\mathcal{A}(X)\hat{I}$ is ideal in $\langle \mathcal{A}(X)\hat{I} \cup \{\hat{I}\}\rangle$, where $\hat{I} : \hat{Y} \to X$ is the inclusion map, then X has the $\mathcal{A}$-MAP.*

Proof. This proof is essentially due to [LL1, Theorem 1.1(c)⇒(a)]. Let K be a compact subset of X and let $\varepsilon > 0$. By a result of Sims and Yost [SY] there exists a separable ideal Y in X such that $K \subset Y \subset X$. Let $\varphi \in \mathcal{HB}(Y,X)$ and let $I : Y \to X$ be the inclusion map.

To use [LL1, Theorem 3.1], let $\hat{Y}$ be an equivalent renorming of Y. By assumption the space $\mathcal{A}(X)\hat{I}$ is an ideal in $\langle \mathcal{A}(X)\hat{I} \cup \{\hat{I}\}\rangle \subset \mathcal{L}(\hat{Y},X)$. To see that $\mathcal{F}(\hat{Y},X) \subset \mathcal{A}(X)\hat{I}$, we only observe that for every $y^* \otimes x \in \mathcal{F}(\hat{Y},X)$, we have $y^* \otimes x = (\varphi y^* \otimes x)\hat{I}$. Thus by [LL1, Theorem 3.1] there exists a $\Phi \in \mathcal{HB}(\mathcal{A}(X)I, \langle \mathcal{A}(X)I \cup \{I\}\rangle)$ such that

$$\Phi(x^* \otimes y)(R) = x^*(Ry)$$

for every $x^* \in X^*, y \in Y$ and $R \in \langle \mathcal{A}(X)I \cup \{I\}\rangle$. Since $\Phi^{\mathrm{a}}(I) \in (\mathcal{A}(X)I)^{**}$, by Goldstine's theorem there exists a net (T_α) in $\mathcal{A}(X)I$ with $\sup_\alpha \|T_\alpha\| \le 1$ and $T_\alpha \xrightarrow{\text{weak}^*} \Phi^{\mathrm{a}}(I)$ in $(\mathcal{A}(X)I)^{**}$. Thus for every $x^* \in X^*$ and $y \in Y$,

$$x^*(T_\alpha y) = (x^* \otimes y)(T_\alpha) \to \Phi^{\mathrm{a}}(I)(x^* \otimes y) = \Phi(x^* \otimes y)(I) = x^*(Iy).$$

By an argument of convex combinations we may assume that $T_\alpha \to I$ in the strong operator topology of $\mathcal{L}(Y,X)$.

Now, let $S_\alpha := i_X^{-1} T_\alpha^{\mathrm{aa}} \varphi^{\mathrm{a}} i_X \in \mathcal{A}(X)$ for every α. Then $\|S_\alpha\| \le 1$ and one may check that $S_\alpha y = T_\alpha y$ for every $y \in Y$. Thus by boundedness of the net (S_α),

$$\sup_{x \in K} \|S_\alpha x - x\| \to 0.$$

Hence we can conclude that there exists an $S \in \mathcal{A}(X)$ with $\|S\| \le 1$ such that $\sup_{x \in K} \|Sx - x\| \le \varepsilon$. ∎

(c)⇒(a) of Theorem 1.1 is a consequence of Lemma 2.3.

The prototype of the lemma below is [LL1, Corollary 3.4].

Lemma 2.4. *Suppose that $\mathcal{A}(X)$ is a linear subspace of $\mathcal{L}(X)$ such that $\mathcal{F}(X) \subset \mathcal{A}(X)$. If for every equivalent renorming $\hat{X}$ of X, the space $\mathcal{A}(X)\hat{I}$*

is ideal in $\langle \mathcal{A}(X)\hat{I} \cup \{\hat{I}\}\rangle$*, where* $\hat{I} : \hat{X} \to X$ *is the identity map, then* X *has the* $\mathcal{A}$*-MAP.*

Proof. By [LL1, Theorem 3.3] there exists a $\Phi \in \mathcal{HB}(\mathcal{A}(X), \langle \mathcal{A}(X) \cup \{\mathrm{id}_X\}\rangle)$ such that

$$\Phi(x^* \otimes x)(R) = x^*(Rx)$$

for every $x^* \in X^*$, $x \in X$ and $R \in \langle \mathcal{A}(X) \cup \{\mathrm{id}_X\}\rangle$. Since $\Phi^{\mathrm{a}}(\mathrm{id}_X) \in \mathcal{A}(X)^{**}$, by Goldstine's theorem there exists a net (S_α) in $\mathcal{A}(X)$ such that $\sup_\alpha \|S_\alpha\| \le 1$ and $S_\alpha \xrightarrow{\text{weak}^*} \Phi^{\mathrm{a}}(\mathrm{id}_X)$ in $\mathcal{A}(X)^{**}$. As in the proof of Lemma 2.3, we may assume that $\lim_\alpha \|S_\alpha x - x\| = 0$ for every $x \in X$. By boundedness of the net (S_α) we have $S_\alpha \xrightarrow{\tau_c} \mathrm{id}_X$. Hence X has the $\mathcal{A}$-MAP. ■

Theorem 1.1(d)⇒(a) is a consequence of Lemma 2.4.

3. Proof of Theorem 1.2. (a)⇒(b) is a consequence of [K3, Theorem 5.4].

Lemma 3.1. *Let* X *be a Banach space. Suppose that* $\mathcal{A}(X)$ *is a linear subspace of* $\mathcal{K}(X)$ *such that* $S_1S_2 \in \mathcal{A}(X)$ *for all* $S_1, S_2 \in \mathcal{A}(X)$*. If for every reflexive Banach space* Z *and every* $T \in \mathcal{W}(Z, X)$*,*

$$T \in \overline{\{ST : S \in \mathcal{A}(X), \|ST\| \le \|T\|\}}^{\tau_c},$$

then for every Banach space Y *and every* $T \in \mathcal{W}(Y, X)$*, for every finite-dimensional subspace* G *of* $\langle \mathcal{A}(X)T \cup \{T\}\rangle$*, every finite-dimensional subspace* H *of* $\mathcal{L}(Y, X)^*$*, and every* $\varepsilon > 0$*, there exists a linear contraction* $U : G \to \mathcal{A}(X)T$ *such that for every* $R \in G \cap \mathcal{A}(X)T$*,*

$$\|U(R) - R\| \le \varepsilon \|R\|,$$

and for every $x^* \otimes y \in H$ *and every* $R \in G$*,*

$$|x^*(U(R)y) - x^*(Ry)| \le \varepsilon \|x^* \otimes y\| \, \|R\|.$$

Proof. Let Y be a Banach space and let $T \in \mathcal{W}(Y, X)$. Let G be a finite-dimensional subspace of $\langle \mathcal{A}(X)T \cup \{T\}\rangle$ and let H be a finite-dimensional subspace of $\mathcal{L}(Y, X)^*$. Let $0 < \varepsilon < 1$. By [LNO, Theorem 2.3] there exist a reflexive Banach space Z, a norm one operator $J : Z \to X$ and a linear isometry $\Psi : G \to \mathcal{W}(Y, Z)$ such that $R = J\Psi(R)$ for all $R \in G$. Here the closed convex hull of $\bigcup_{R \in B_G} R(B_Y)$ is contained in $B_Z \subset B_X$.

Now, by our assumption there exists a net (S_α) in $\mathcal{A}(X)$ with $\sup_\alpha \|S_\alpha J\| \le \|J\| = 1$ such that

$$S_\alpha J \xrightarrow{\tau_c} J.$$

By [LNO, Theorem 2.3(ii)] we see that $K := \bigcup_{R \in B_{G \cap \mathcal{A}(X)T}} R(B_Y)$ is a relatively compact subset of Z. Thus

$$\sup_{z \in K} \|S_\alpha Jz - Jz\| \to 0$$

and for every $x^* \in X^*$, $y \in Y$ and $R \in G$, we have

$$\lim_\alpha x^*(S_\alpha Ry) = \lim_\alpha x^*(S_\alpha J\Psi(R)y) = x^*(J\Psi(R)y) = x^*(Ry).$$

Then we can use the proof of Lemma 2.1 to complete our proof. ∎

Now, (b)⇒(c) of Theorem 1.2 is a consequence of Lemmas 3.1 and 2.2, and (c)⇒(d) is trivial.

LEMMA 3.2. *Let X be a Banach space. Suppose that $\mathcal{A}(X)$ is a linear subspace of $\mathcal{L}(X)$. If for every separable reflexive Banach space Z and every $T \in \mathcal{K}(Z,X)$, there exists a*

$$\Theta \in \mathcal{L}((\mathcal{A}(X)T)^*, \langle \mathcal{A}(X)T \cup \{T\}\rangle^*)$$

such that

$$\Theta(x^* \otimes z)(T) = x^*(Tz)$$

for every $x^* \in X^*$ and $z \in Z$, then for every separable reflexive Banach space Z and every $T \in \mathcal{K}(Z,X)$,

$$T \in \overline{\mathcal{A}(X)T}^{\tau_c}.$$

Proof. Let Z be a separable reflexive Banach space and let $T \in \mathcal{K}(Z,X)$. Let Θ be the operator in the assumption. Then $\Theta^{\mathrm{a}}(T) \in (\mathcal{A}(X)T)^{**}$. By Goldstine's theorem there exists a net (S_α) in $\mathcal{A}(X)$ with $\sup_\alpha \|S_\alpha T\| \le \|\Theta^{\mathrm{a}}(T)\|$ such that

$$S_\alpha T \xrightarrow{\text{weak}^*} \Theta^{\mathrm{a}}(T).$$

Then for every $z \in Z$ and $x^* \in X^*$, we have

$$\lim_\alpha x^*(S_\alpha Tz) = \lim_\alpha (x^* \otimes z)(S_\alpha T) = \Theta^{\mathrm{a}}(T)(x^* \otimes z) = x^*(Tz).$$

By an argument of convex combinations and boundedness of $(S_\alpha T)$ we complete the proof. ∎

(d)⇒(e) of Theorem 1.2 is a consequence of Lemma 3.2, and (e)⇒(a) is a consequence of [K4, Theorem 1.4].

4. Proofs of Theorems 1.3 and 1.4.

The proof of the following lemma is the same as the one of [L2, Proposition 2.5].

LEMMA 4.1. *Let X be a Banach space. Suppose that $\mathcal{A}(X)$ is a convex subset of $\mathcal{L}(X)$. If X has the weak $\mathcal{A}$-MAP, then there exists a $\phi \in \mathcal{HB}(X, X^{**})$ such that $\phi^{\mathrm{a}} i_{X^{**}} \in \overline{\mathcal{A}^{\mathrm{aa}}(X)}^{\text{weak}^*}$ in $\mathcal{L}(X^{**})$.*

LEMMA 4.2. *Let X be a Banach space. Suppose that $\mathcal{A}(X)$ is a convex subset of $\mathcal{K}(X)$ containing 0. If there exists a $\phi \in \mathcal{HB}(X, X^{**})$ such that $\phi^{\mathrm{a}} i_{X^{**}} \in \overline{\mathcal{A}^{\mathrm{aa}}(X)}^{\text{weak}^*}$ in $\mathcal{L}(X^{**})$, then X has the weak $\mathcal{A}$-MAP.*

Proof. It was shown in [K1, Proposition 3.1] that X has the weak $\mathcal{A}$-MAP if and only if for every reflexive Banach space Z and every $R \in \mathcal{W}(X, Z)$,

$$R \in \overline{\{RS : S \in \mathcal{A}(X),\ \|RS\| \le \|R\|\}}^{\tau_c}.$$

We will use that characterization.

Let Z be a reflexive Banach space and let $R \in \mathcal{W}(X, Z)$. By assumption there exists a net (S_α) in $\mathcal{A}(X)$ such that for every $\sum_n z_n^* \otimes x_n^{**} \in Z^* \hat{\otimes}_\pi X^{**}$, we have

$$\begin{aligned}\sum_n (R^{\mathrm{aa}} S_\alpha^{\mathrm{aa}} x_n^{**})(z_n^*) &= \sum_n (S_\alpha^{\mathrm{aa}} x_n^{**})(R^{\mathrm{a}} z_n^*) \\ &\to \sum_n (\phi^{\mathrm{a}} i_{X^{**}} x_n^{**})(R^{\mathrm{a}} z_n^*) = \sum_n (R^{\mathrm{aa}} \phi^{\mathrm{a}} i_{X^{**}} x_n^{**})(z_n^*).\end{aligned}$$

Thus by Grothendieck's representation theorem of $(\mathcal{L}(X^{**}, Z^{**}), \tau_c)^*$ (cf. [LT, Proposition 1.e.3]), for every $f \in (\mathcal{L}(X^{**}, Z^{**}), \tau_c)^*$, we have $f(R^{\mathrm{aa}} S_\alpha^{\mathrm{aa}}) \to f(R^{\mathrm{aa}} \phi^{\mathrm{a}} i_{X^{**}})$, hence

$$R^{\mathrm{aa}} \phi^{\mathrm{a}} i_{X^{**}} \in \overline{\{R^{\mathrm{aa}} S^{\mathrm{aa}} : S \in \mathcal{A}(X)\}}^{\tau_c}.$$

By [K2, Corollary 1.4] there exists a net (S_α) in $\mathcal{A}(X)$ with

$$\sup_\alpha \|RS_\alpha\| \le \|R^{\mathrm{aa}} \phi^{\mathrm{a}} i_{X^{**}}\| \le \|R\|$$

such that $R^{\mathrm{aa}} S_\alpha^{\mathrm{aa}} \xrightarrow{\tau_c} R^{\mathrm{aa}} \phi^{\mathrm{a}} i_{X^{**}}$. In particular, for every $x \in X$ and $z^* \in Z^*$,

$$\begin{aligned} z^*(RS_\alpha x) &= (S_\alpha^{\mathrm{a}} R^a z^*)(x) = (R^{\mathrm{aa}} S_\alpha^{\mathrm{aa}} i_X x)(z^*) \\ &\to (R^{\mathrm{aa}} \phi^{\mathrm{a}} i_{X^{**}} i_X x)(z^*) \\ &= \phi^{\mathrm{a}} i_{X^{**}} i_X x(R^{\mathrm{a}} z^*) = (\phi R^a z^*)(i_X x) = z^*(Rx).\end{aligned}$$

By an argument using convex combinations and boundedness of (RS_α) we conclude that

$$R \in \overline{\{RS : S \in \mathcal{A}(X),\ \|RS\| \le \|R\|\}}^{\tau_c}. \ \blacksquare$$

Now, Theorem 1.3(a)⇔(b) is a consequence of Lemmas 4.1 and 4.2.

LEMMA 4.3. *Let X be a Banach space. Suppose that $\mathcal{A}(X)$ is a convex subset of $\mathcal{K}(X)$ containing* 0. *Let $\phi \in \mathcal{L}(X^*, X^{***})$ with $\|\phi\| \le 1$. If $\phi^{\mathrm{a}} i_{X^{**}} \in \overline{\mathcal{A}^{\mathrm{aa}}(X)}^{\mathrm{weak}^*}$ in $\mathcal{L}(X^{**})$, then for every reflexive Banach space Z and every $T \in \mathcal{W}(Z, X^{**})$, there exists a net (S_α) in $\mathcal{A}(X)$ with* $\sup_\alpha \|S_\alpha^{\mathrm{aa}} T\| \le \|T\|$ *such that $(S_\alpha^{\mathrm{aa}} T)^{\mathrm{a}}(x^*) \to T^{\mathrm{a}}(\phi x^*)$ for every $x^* \in X^*$.*

Proof. Let Z be a reflexive Banach space and let $T \in \mathcal{W}(Z, X^{**})$. Let $R \in \mathcal{W}(X^*, Z^*)$ be such that $T = R^a$. By assumption there exists a net (S_α)

in $\mathcal{A}(X)$ such that for every $\sum_n x_n^* \otimes z_n \in X^* \mathbin{\hat{\otimes}_\pi} Z$, we have

$$\begin{aligned}\sum_n z_n(RS_\alpha^{\mathrm{a}} x_n^*) &= \sum_n (S_\alpha^{\mathrm{aa}} T z_n)(x_n^*)\\ &\to \sum_n \phi^{\mathrm{a}} i_{X^{**}}(Tz_n)(x_n^*)\\ &= \sum_n \phi(x_n^*)(Tz_n) = \sum_n z_n(T^a \phi x_n^*).\end{aligned}$$

It follows that for every $f \in (\mathcal{L}(X^*, Z^*), \tau_c)^*$, we have $f(RS_\alpha^{\mathrm{a}}) \to f(T^a\phi)$. Thus

$$T^a\phi \in \overline{\{RS^{\mathrm{a}} : S \in \mathcal{A}(X)\}}^{\tau_c}.$$

By [K2, Corollary 1.4] there exists a net (S_α) in $\mathcal{A}(X, X)$ with

$$\sup_\alpha \|S_\alpha^{\mathrm{aa}} T\| = \sup_\alpha \|RS_\alpha^{\mathrm{a}}\| \le \|T\|$$

such that $RS_\alpha^{\mathrm{a}} \xrightarrow{\tau_c} T^a\phi$. Now, for every $x^* \in X^*$ and $z \in Z$, we have

$$z((S_\alpha^{\mathrm{aa}} T)^{\mathrm{a}} x^*) = (S_\alpha^{\mathrm{aa}} Tz)(x^*) = z(RS_\alpha^{\mathrm{a}} x^*) \to z(T^{\mathrm{a}}\phi x^*).$$

By an argument of convex combinations we complete the proof. ∎

Theorem 1.3(b)⇒(c) is a consequence of Lemma 4.3.

Since X^* has the $\mathcal{A}^{\mathrm{a}}$-AP if and only if $i_{X^*}^{\mathrm{a}} i_{X^{**}} \in \overline{\mathcal{A}^{\mathrm{aa}}(X)}^{\mathrm{weak}^*}$, by Lemma 4.3 we have:

Corollary 4.4. *Let X be a Banach space. Suppose that $\mathcal{A}(X)$ is a convex subset of $\mathcal{K}(X)$ containing* 0. *If X^* has the $\mathcal{A}^{\mathrm{a}}$-AP, then for every reflexive Banach space Z and every $T \in \mathcal{W}(Z, X^{**})$, there exists a net (S_α) in $\mathcal{A}(X)$ with* $\sup_\alpha \|S_\alpha^{\mathrm{aa}} T\| \le \|T\|$ *such that $(S_\alpha^{\mathrm{aa}} T)^{\mathrm{a}}(x^*) \to T^{\mathrm{a}}(i_{X^*} x^*)$ for every $x^* \in X^*$.*

Theorem 1.4(a)⇒(b) is a consequence of Corollary 4.4.

The following lemma stems from [L1, Lemma 3.3].

Lemma 4.5. *Let X be a Banach space. Suppose that $\mathcal{A}(X)$ is a linear subspace of $\mathcal{K}(X)$ such that $S_1 S_2 \in \mathcal{A}(X)$ for all $S_1, S_2 \in \mathcal{A}(X)$. Let $\phi \in \mathcal{HB}(X, X^{**})$. If for every reflexive Banach space Z and every $T \in \mathcal{W}(Z, X^{**})$, there exists a net (S_α) in $\mathcal{A}(X)$ with* $\sup_\alpha \|S_\alpha^{\mathrm{aa}} T\| \le \|T\|$ *such that*

$$(S_\alpha^{\mathrm{aa}} T)^{\mathrm{a}}(x^*) \to T^{\mathrm{a}}(\phi x^*)$$

for every $x^* \in X^*$, then for every Banach space Y and every $T \in \mathcal{W}(Y, X^{**})$, for every finite-dimensional subspace G of $\langle \mathcal{A}^{\mathrm{aa}}(X)T \cup \{T\}\rangle$ and every finite-dimensional subspace H of $\mathcal{L}(Y, X)^*$, and every $\varepsilon > 0$, there exists a linear contraction $U : G \to \mathcal{A}^{\mathrm{aa}}(X)T$ such that for every $R \in G \cap \mathcal{A}^{\mathrm{aa}}(X)T$,

$$\|U(R) - R\| \le \varepsilon \|R\|,$$

and for every $x^* \otimes y^{**} \in H$ *and every* $R \in G$,

$$|y^{**}(U(R)^{\mathrm{a}}x^*) - y^{**}(R^{\mathrm{a}}\phi x^*)| \le \varepsilon \|x^* \otimes y^{**}\| \, \|R\|.$$

Proof. Let Y be a Banach space and let $T \in \mathcal{W}(Y, X^{**})$. Let G be a finite-dimensional subspace of $\langle \mathcal{A}^{\mathrm{aa}}(X)T \cup \{T\}\rangle$ and let H be a finite-dimensional subspace of $\mathcal{L}(Y,X)^*$. Let $0 < \varepsilon < 1$. By [LNO, Theorem 2.3] there exist a reflexive Banach space Z, a norm one operator $J : Z \to X^{**}$ and a linear isometry $\Psi : G \to \mathcal{W}(Y, Z)$ such that $R = J\Psi(R)$ for all $R \in G$ and the closed convex hull of $\bigcup_{R\in B_G} R(B_Y)$ is contained in $B_Z \subset B_{X^{**}}$.

Now, by our assumption there exists a net (S_α) in $\mathcal{A}(X)$ with $\sup_\alpha \|S_\alpha^{\mathrm{aa}} J\| \le \|J\|$ such that

$$(S_\alpha^{\mathrm{aa}} J)^{\mathrm{a}}(x^*) \to J^{\mathrm{a}}(\phi x^*)$$

for every $x^* \in X^*$. Consider the operators $S_\alpha^{\mathrm{aa}} J$ and J from $Z \cap i_X(X)$ to X. Then for every $z \in Z \cap i_X(X)$ and every $x^* \in X^*$, we have

$$x^*(S_\alpha^{\mathrm{aa}} Jz) = ((S_\alpha^{\mathrm{aa}} J)^{\mathrm{a}} x^*)(z) \to (J^{\mathrm{a}}\phi x^*)(z) = \phi x^*(Jz) = x^*(Jz).$$

By an argument of convex combinations we see that there exists a net (T_β) in the convex hull of $\{S_\alpha\}$ such that for every $z \in Z \cap i_X(X)$ and every $x^* \in X^*$,

$$T_\beta^{\mathrm{aa}} Jz \to Jz \quad \text{and} \quad (T_\beta^{\mathrm{aa}} J)^{\mathrm{a}}(x^*) \to J^{\mathrm{a}}(\phi x^*).$$

By boundedness of the net $(T_\beta^{\mathrm{aa}} J)$ the first convergence implies

$$(T_\beta^{\mathrm{aa}} J)|_{Z\cap i_X(X)} \xrightarrow{\tau_c} J|_{Z\cap i_X(X)}.$$

Also, from the second convergence, for every $x^* \in X^*$, we have

$$\begin{aligned}
&\sup_{y^{**}\in B_{Y^{**}},\, R\in B_G} |y^{**}((T_\beta^{\mathrm{aa}} R)^{\mathrm{a}} x^*) - y^{**}(R^{\mathrm{a}}\phi x^*)| \\
&\quad = \sup_{y^{**}\in B_{Y^{**}},\, R\in B_G} |y^{**}\Psi(R)^a((T_\beta^{\mathrm{aa}} J)^{\mathrm{a}} x^*) - y^{**}(R^a\phi x^*)| \\
&\quad = \sup_{y^{**}\in B_{Y^{**}},\, R\in B_G} |y^{**}\Psi(R)^a((T_\beta^{\mathrm{aa}} J)^{\mathrm{a}} x^*) - y^{**}\Psi(R)^a(J^{\mathrm{a}}\phi x^*)| \\
&\quad \le \|(T_\beta^{\mathrm{aa}} J)^{\mathrm{a}}(x^*) - J^{\mathrm{a}}(\phi x^*)\| \to 0.
\end{aligned}$$

Since by [LNO, Theorem 2.3(ii)], $K := \bigcup_{R\in B_{G\cap \mathcal{A}^{\mathrm{aa}}(X)T}} R(B_Y)$ is a relatively compact subset of $Z \cap i_X(X)$,

$$\sup_{z\in K} \|T_\beta^{\mathrm{aa}} Jz - Jz\| \to 0.$$

Now, let $\delta > 0$ be such that $0 < \delta/(1-\delta) \le \varepsilon$. Consider the finite-dimensional subspace $H_1 := \operatorname{span}\{x^* \otimes y^{**} : x^* \otimes y^{**} \in H\}$ of $X^* \hat{\otimes}_\pi Y^{**}$. Then there exist $u_1, \ldots, u_m \in S_{H_1}$ such that $(1-\delta)B_{H_1}$ is a subset of the convex hull of $\{u_i\}_{i=1}^m$. For each $i = 1, \ldots, m$, choose a representation

$$u_i = \sum_{k=1}^{n_i} x_{i,k}^* \otimes y_{i,k}^{**},$$

where all $x_{i,k}^* \otimes y_{i,k}^{**}$ are in H. Choose a β_0 such that

$$\sum_{k=1}^{n_i} |y_{i,k}^{**}((T_{\beta_0}^{\mathrm{aa}} R)^{\mathrm{a}} x_{i,k}^*) - y_{i,k}^{**}(R^{\mathrm{a}} \phi x_{i,k}^*)| \le \delta$$

for all $i = 1, \ldots, m$ and all $R \in B_G$, and

$$\sup_{z \in K} \|T_{\beta_0}^{\mathrm{aa}} Jz - Jz\| \le \varepsilon.$$

We now define the map $U : G \to \mathcal{A}^{\mathrm{aa}}(X)T$ by

$$U(R) = T_{\beta_0}^{\mathrm{aa}} R.$$

Then U is well defined and linear, and for every $R \in G$, we have

$$\|U(R)\| = \|T_{\beta_0}^{\mathrm{aa}} J\Psi(R)\| \le \|\Psi(R)\| = \|R\|.$$

Also, for every $R \in B_{G \cap \mathcal{A}^{\mathrm{aa}}(X)T}$,

$$\begin{aligned}\|U(R) - R\| &= \sup_{y \in B_Y} \|T_{\beta_0}^{\mathrm{aa}} Ry - Ry\| = \sup_{y \in B_Y} \|T_{\beta_0}^{\mathrm{aa}} JRy - JRy\| \\ &\le \sup_{z \in K} \|T_{\beta_0}^{\mathrm{aa}} Jz - Jz\| \le \varepsilon.\end{aligned}$$

Finally, let $x^* \otimes y^{**} \in H$ and let $R \in B_G$. Then

$$(1-\delta)\frac{x^* \otimes y^{**}}{\|x^* \otimes y^{**}\|} = \sum_{i=1}^m \lambda_i u_i = \sum_{i=1}^m \lambda_i \sum_{k=1}^{n_i} x_{i,k}^* \otimes y_{i,k}^{**},$$

where $\lambda_i \ge 0$ for all i and $\sum_{i=1}^m \lambda_i = 1$. We now have

$$\begin{aligned}&|y^{**}(U(R)^{\mathrm{a}} x^*) - y^{**}(R^{\mathrm{a}} \phi x^*)| \\ &= |(x^* \otimes y^{**})(U(R)) - (\phi \otimes \mathrm{id}_{Y^{**}})(x^* \otimes y^{**})(R)| \\ &= \frac{\|x^* \otimes y^{**}\|}{1-\delta}\Big|\Big(\sum_{i=1}^m \lambda_i \sum_{k=1}^{n_i} x_{i,k}^* \otimes y_{i,k}^{**}\Big)(U(R)) \\ &\qquad - (\phi \otimes \mathrm{id}_{Y^{**}})\Big(\sum_{i=1}^m \lambda_i \sum_{k=1}^{n_i} x_{i,k}^* \otimes y_{i,k}^{**}\Big)(R)\Big| \\ &\le \frac{\|x^* \otimes y^{**}\|}{1-\delta} \sum_{k=1}^{n_i} |y_{i,k}^{**}((T_{\beta_0}^{\mathrm{aa}} R)^{\mathrm{a}} x_{i,k}^*) - y_{i,k}^{**}(R^{\mathrm{a}} \phi x_{i,k}^*)| \\ &\le \frac{\delta}{1-\delta}\|x^* \otimes y^{**}\| \le \varepsilon \|x^* \otimes y^{**}\|. \ \blacksquare\end{aligned}$$

Since $i_{X^*} \in \mathcal{HB}(X, X^{**})$, by Lemma 4.5 we have:

Corollary 4.6. *Let X be a Banach space. Suppose that $\mathcal{A}(X)$ is a linear subspace of $\mathcal{K}(X)$ such that $S_1 S_2 \in \mathcal{A}(X)$ for all $S_1, S_2 \in \mathcal{A}(X)$. If for every reflexive Banach space Z and every $T \in \mathcal{W}(Z, X^{**})$, there exists a net (S_α) in $\mathcal{A}(X)$ with $\sup_\alpha \|S_\alpha^{\mathrm{aa}} T\| \le \|T\|$ such that*

$$(S_\alpha^{\mathrm{aa}} T)^{\mathrm{a}}(x^*) \to T^{\mathrm{a}}(i_{X^*} x^*)$$

for every $x^* \in X^*$, *then for every Banach space* Y *and every* $T \in \mathcal{W}(Y, X^{**})$, *for every finite-dimensional subspace* G *of* $\langle \mathcal{A}^{\mathrm{aa}}(X)T \cup \{T\}\rangle$ *and every finite-dimensional subspace* H *of* $\mathcal{L}(Y,X)^*$, *and every* $\varepsilon > 0$, *there exists a linear contraction* $U : G \to \mathcal{A}^{\mathrm{aa}}(X)T$ *such that for every* $R \in G \cap \mathcal{A}^{\mathrm{aa}}(X)T$,

$$\|U(R) - R\| \le \varepsilon \|R\|,$$

and for every $x^* \otimes y^{**} \in H$ *and every* $R \in G$,

$$|y^{**}(U(R)^{\mathrm{a}} x^*) - y^{**}(R^{\mathrm{a}} i_{X^*} x^*)| \le \varepsilon \|x^* \otimes y^{**}\| \, \|R\|.$$

As in the proof of [L1, Lemma 3.5], we have the following lemma. Here we note that the map ϕ need not be a Hahn–Banach extension operator.

Lemma 4.7. *Let* X *and* Y *be Banach spaces and let* $\phi \in \mathcal{L}(X^*, X^{***})$. *Suppose that* $\mathcal{C} \subset \mathcal{L}(Y, X)$ *and* $\mathcal{B} \subset \mathcal{L}(Y, X^{**})$ *are linear subspaces such that* $\mathcal{C} \subset \mathcal{B}$. *If for any finite-dimensional subspaces* $H \subset \mathcal{L}(Y,X)^*$ *and* $G \subset \mathcal{B}$, *and every* $\varepsilon > 0$, *there exists a linear contraction* $U : G \to \mathcal{C}$ *such that for every* $R \in G \cap \mathcal{C}$,

$$\|U(R) - R\| \le \varepsilon \|R\|,$$

and for every $x^* \otimes y^{**} \in H$ *and every* $R \in G$,

$$|y^{**}(U(R)^{\mathrm{a}} x^*) - y^{**}(R^{\mathrm{a}} \phi x^*)| \le \varepsilon \|x^* \otimes y^{**}\| \, \|R\|,$$

then there exists a $\Phi \in \mathcal{HB}(\mathcal{C}, \mathcal{B})$ *such that*

$$\Phi(x^* \otimes y^{**})(R) = y^{**}(R^{\mathrm{a}} \phi x^*)$$

for every $x^* \in X^*$ *and* $y^{**} \in Y^{**}$, *and every* $R \in \mathcal{B}$.

Theorem 1.3(c)⇒(d) is a consequence of Lemmas 4.5 and 4.7, and Theorem 1.4(b)⇒(c) is a consequence of Corollary 4.6 and Lemma 4.7.

Theorem 1.3(d)⇒(e) and Theorem 1.4(c)⇒(d) are trivial.

Lemma 4.8. *Let* X *be a Banach space and let* $\phi \in \mathcal{L}(X^*, X^{***})$. *Suppose that* $\mathcal{A}(X)$ *is a linear subspace of* $\mathcal{L}(X)$. *If for every separable reflexive Banach space* Z *and every* $T \in \mathcal{K}(Z, X^{**})$, *there exists a* $\Theta \in \mathcal{L}((\mathcal{A}^{\mathrm{aa}}(X)T)^*, \langle \mathcal{A}^{\mathrm{aa}}(X)T \cup \{T\}\rangle^*)$ *such that*

$$\Theta(x^* \otimes z)(T) = \phi x^*(Tz)$$

for every $x^* \in X^*$ *and* $z \in Z$, *then* $\phi^{\mathrm{a}} i_{X^{**}} \in \overline{\mathcal{A}^{\mathrm{aa}}(X)}^{\mathrm{weak}^*}$ *in* $\mathcal{L}(X^{**})$.

Proof. Let Z be a separable reflexive Banach space Z and let $T \in \mathcal{K}(Z, X^{**})$. Since $\Theta^a(T) \in (\mathcal{A}^{\mathrm{aa}}(X)T)^{**}$, by Goldstine's theorem there exists a net (S_α) in $\mathcal{A}(X)$ with $\sup_\alpha \|S_\alpha^{\mathrm{aa}} T\| \le \|\Theta^a(T)\|$ such that $S_\alpha^{\mathrm{aa}} T \xrightarrow{\mathrm{weak}^*} \Theta^a(T)$ in $(\mathcal{A}^{\mathrm{aa}}(X)T)^{**}$. Thus for every $x^* \in X^*$ and $z \in Z$, we have

$$\begin{aligned}\lim_\alpha (S_\alpha^{\mathrm{aa}} T z)(x^*) &= \lim_\alpha (x^* \otimes z)(S_\alpha^{\mathrm{aa}} T) \\ &= \Theta^a(T)(x^* \otimes z) = (\phi x^*)(Tz).\end{aligned}$$

We have shown that for every separable reflexive Banach space Z and every $T \in \mathcal{K}(Z, X^{**})$, there exists a net (S_α) in $\mathcal{A}(X)$ with $\sup_\alpha \|S_\alpha^{\mathrm{aa}} T\| \le \|\Theta\|\,\|T\|$ such that

$$\lim_\alpha (S_\alpha^{\mathrm{aa}} Tz)(x^*) = (\phi x^*)(Tz) = (\phi^{\mathrm{a}} i_{X^{**}} Tz)(x^*)$$

for every $x^* \in X^*$ and $z \in Z$. By boundedness of the net $(S_\alpha^{\mathrm{aa}} T)$ we see that

$$\phi^{\mathrm{a}} i_{X^{**}} T \in \overline{\{S^{\mathrm{aa}} T : S \in \mathcal{A}(X),\ \|S^{\mathrm{aa}} T\| \le \|\Theta\|\,\|T\|\}}^{\mathrm{weak}^*}$$

in $\mathcal{L}(Z, X^{**})$.

Now, let $\sum_{n=1}^\infty x_n^* \otimes x_n^{**} \in X^* \hat{\otimes}_\pi X^{**}$. We may assume that $\sum_{n=1}^\infty \|x_n^*\| < \infty$ and $1 \ge \|x_n^{**}\| \to 0$. Consider the closed balanced convex hull $\overline{\mathrm{bco}}\{x_n^{**}\} \subset B_{X^{**}}$. Then by [LNO, Lemmas 1.1 and 1.2] there exist a separable reflexive Banach space $Z \subset X^{**}$ with $\overline{\mathrm{bco}}\{x_n^{**}\} \subset B_Z$ such that the inclusion map $J : Z \to X^{**}$ is a norm one compact operator. By the above result,

$$\phi^{\mathrm{a}} i_{X^{**}} J \in \overline{\{S^{\mathrm{aa}} J : S \in \mathcal{A}(X),\ \|S^{\mathrm{aa}} J\| \le \|\Theta\|\}}^{\mathrm{weak}^*}.$$

We now have

$$\begin{aligned}
\mathrm{Re} \sum_{n=1}^\infty (\phi^{\mathrm{a}} i_{X^{**}} x_n^{**})(x_n^*) &= \mathrm{Re} \sum_{n=1}^\infty (\phi^{\mathrm{a}} i_{X^{**}} J x_n^{**})(x_n^*) \\
&\le \sup \Big\{ \mathrm{Re} \sum_{n=1}^\infty (S^{\mathrm{aa}} J x_n^{**})(x_n^*) : S \in \mathcal{A}(X),\ \|S^{\mathrm{aa}} J\| \le \|\Theta\| \Big\} \\
&= \sup \Big\{ \mathrm{Re} \sum_{n=1}^\infty (S^{\mathrm{aa}} x_n^{**})(x_n^*) : S \in \mathcal{A}(X),\ \|S^{\mathrm{aa}} J\| \le \|\Theta\| \Big\} \\
&\le \sup \Big\{ \mathrm{Re} \sum_{n=1}^\infty (S^{\mathrm{aa}} x_n^{**})(x_n^*) : S \in \mathcal{A}(X) \Big\}.
\end{aligned}$$

Hence $\phi^{\mathrm{a}} i_{X^{**}} \in \overline{\mathcal{A}^{\mathrm{aa}}(X)}^{\mathrm{weak}^*}$ by an application of the separation theorem. ∎

COROLLARY 4.9. *Let X be a Banach space. Suppose that $\mathcal{A}(X)$ is a linear subspace of $\mathcal{L}(X)$. If for every separable reflexive Banach space Z and every $T \in \mathcal{K}(Z, X^{**})$, there exists a $\Theta \in \mathcal{L}((\mathcal{A}^{\mathrm{aa}}(X)T)^*, \langle \mathcal{A}^{\mathrm{aa}}(X)T \cup \{T\}\rangle^*)$ such that*

$$\Theta(x^* \otimes z)(T) = (Tz)(x^*)$$

for every $x^* \in X^*$ and $z \in Z$, then X^* has the $\mathcal{A}^{\mathrm{a}}$-AP.

Theorem 1.3(e)⇒(b) is a consequence of Lemma 4.8, and Theorem 1.4 (d)⇒(a) is a consequence of Corollary 4.9.

5. Approximation properties for dual spaces. In this section, we obtain some symmetric forms of statements in Theorems 1.1 and 1.2 to characterize some approximation properties for dual spaces. It is well known that X^* has the MAP if and only if X^* has the $\mathcal{F}^{\mathrm{a}}$-MAP (cf. [J, Lemma 2]).

Theorem 5.1. *Let X be a Banach space. Suppose that $\mathcal{A}(X)$ is a linear subspace of $\mathcal{K}(X)$ containing $\mathcal{F}(X)$ such that $S_1S_2 \in \mathcal{A}(X)$ for all $S_1, S_2 \in \mathcal{A}(X)$. Then the following statements are equivalent:*

(a) *X^* has the $\mathcal{A}^{\mathrm{a}}$-MAP.*

(b) *For every Banach space Y and every $T \in \mathcal{L}(X,Y)$, there exists a*

$$\Psi \in \mathcal{HB}(T\mathcal{A}(X), \langle T\mathcal{A}(X) \cup \{T\}\rangle)$$

such that

$$\Psi(x^{**} \otimes y^*)(R) = x^{**}(R^{\mathrm{a}}y^*)$$

for every $x^{**} \in X^{**}$ and $y^* \in Y^*$, and every $R \in \langle T\mathcal{A}(X) \cup \{T\}\rangle$.

(c) *For every equivalent renorming $\hat{X}$ of X, the space $I\mathcal{A}(X)$ is ideal in $\langle I\mathcal{A}(X) \cup \{I\}\rangle$, where $I : X \to \hat{X}$ is the identity map.*

Proof. (a)⇒(b). Let Y be a Banach space and let $T \in \mathcal{L}(X,Y)$. Consider the subspaces $\mathcal{A}^{\mathrm{a}}(X)T^a$ and $\langle \mathcal{A}^{\mathrm{a}}(X)T^a \cup \{T^a\}\rangle$ of $\mathcal{L}(Y^*, X^*)$. By (a) and Lemmas 2.1 and 2.2 there exists a

$$\Phi \in \mathcal{HB}(\mathcal{A}^{\mathrm{a}}(X)T^a, \langle \mathcal{A}^{\mathrm{a}}(X)T^a \cup \{T^a\}\rangle)$$

such that

$$\Phi(x^{**} \otimes y^*)(R^a) = x^{**}(R^{\mathrm{a}}y^*)$$

for every $x^{**} \in X^{**}$ and $y^* \in Y^*$, and every $R \in \langle T\mathcal{A}(X) \cup \{T\}\rangle$.

Now, let $i_1 : \mathcal{A}^{\mathrm{a}}(X)T^a \to T\mathcal{A}(X)$ and $i_2 : \langle T\mathcal{A}(X) \cup \{T\}\rangle \to \langle \mathcal{A}^{\mathrm{a}}(X)T^a \cup \{T^a\}\rangle$ be the isometries defined by $i_1(U^a) = U$ and $i_2(V) = V^a$. Then a simple verification shows that the map $i_2^{\mathrm{a}}\Phi i_1^{\mathrm{a}} : (T\mathcal{A}(X))^* \to \langle T\mathcal{A}(X) \cup \{T\}\rangle^*$ is a Hahn–Banach extension operator. Moreover, for every $x^{**} \in X^{**}$ and $y^* \in Y^*$, and every $R \in \langle T\mathcal{A}(X) \cup \{T\}\rangle$, we have

$$i_2^{\mathrm{a}}\Phi i_1^{\mathrm{a}}(x^{**} \otimes y^*)(R) = \Phi i_1^{\mathrm{a}}(x^{**} \otimes y^*)(R^a) = x^{**}(R^{\mathrm{a}}y^*).$$

(b)⇒(c) is trivial.

(c)⇒(a). By an application of [LL1, Theorem 4.3] there exists a

$$\Phi \in \mathcal{HB}(\mathcal{A}(X), \langle \mathcal{A}(X) \cup \{\mathrm{id}_X\}\rangle)$$

such that

$$\Phi(x^{**} \otimes x^*)(R) = x^{**}(R^{\mathrm{a}}x^*)$$

for every $x^{**} \in X^*$ and $x^* \in X^*$, and every $R \in \langle \mathcal{A}(X) \cup \{\mathrm{id}_X\}\rangle$. Since $\Phi^{\mathrm{a}}(\mathrm{id}_X) \in \mathcal{A}(X)^{**}$, by Goldstine's theorem there exists a net (S_α) in $\mathcal{A}(X)$ such that $\sup_\alpha \|S_\alpha\| \le 1$ and $S_\alpha \xrightarrow{\text{weak}^*} \Phi^{\mathrm{a}}(\mathrm{id}_X)$ in $\mathcal{A}(X)^{**}$. As in the proof of Lemma 2.3, we may assume that $\lim_\alpha \|S_\alpha^{\mathrm{a}}x^* - x^*\| = 0$ for every $x^* \in X^*$. By boundedness of the net (S_α^{a}) we see that $S_\alpha^{\mathrm{a}} \xrightarrow{\tau_c} \mathrm{id}_{X^*}$. Hence X^* has the $\mathcal{A}^{\mathrm{a}}$-MAP. ■

We remark that X^* has the $\mathcal{A}^{\mathrm{a}}$-AP if and only if X^* has the weak $\mathcal{A}^{\mathrm{a}}$-MAP whenever $\mathcal{A}(X)$ is a convex subset of $\mathcal{K}(X)$ containing 0 (see [K1, Theorem 1.2]).

THEOREM 5.2. *Let X be a Banach space. Suppose that $\mathcal{A}(X)$ is a linear subspace of $\mathcal{K}(X)$ such that $S_1S_2 \in \mathcal{A}(X)$ for all $S_1, S_2 \in \mathcal{A}(X)$. Then the following statements are equivalent:*

(a) *X^* has the $\mathcal{A}^{\mathrm{a}}$-AP.*

(b) *For every Banach space Y and every $T \in \mathcal{W}(X,Y)$, there exists a*
$$\Psi \in \mathcal{HB}(T\mathcal{A}(X), \langle T\mathcal{A}(X) \cup \{T\}\rangle)$$
such that
$$\Psi(x^{**} \otimes y^*)(R) = x^{**}(R^{\mathrm{a}}y^*)$$
for every $x^{**} \in X^{**}$ and $y^* \in Y^*$, and every $R \in \langle T\mathcal{A}(X) \cup \{T\}\rangle$.

(c) *For every separable reflexive Banach space Z and every $T \in \mathcal{K}(X,Z)$, there exists a*
$$\Theta \in \mathcal{L}((T\mathcal{A}(X))^*, \langle T\mathcal{A}(X) \cup \{T\}\rangle^*)$$
such that
$$\Theta(x^{**} \otimes z^*)(T) = x^{**}(T^{\mathrm{a}}z^*)$$
for every $x^{**} \in X^{**}$ and $z^* \in Z^*$.

Proof. (a)⇒(b). Let Y be a Banach space and let $T \in \mathcal{W}(X,Y)$. Consider the subspaces $\mathcal{A}^{\mathrm{a}}(X)T^{\mathrm{a}}$ and $\langle \mathcal{A}^{\mathrm{a}}(X)T^{\mathrm{a}} \cup \{T^{\mathrm{a}}\}\rangle$ of $\mathcal{W}(Y^*, X^*)$. By (a) and Theorem 1.2 there exists a
$$\Phi \in \mathcal{HB}(\mathcal{A}^{\mathrm{a}}(X)T^{\mathrm{a}}, \langle \mathcal{A}^{\mathrm{a}}(X)T^{\mathrm{a}} \cup \{T^{\mathrm{a}}\}\rangle)$$
such that
$$\Phi(x^{**} \otimes y^*)(R) = x^{**}(R^{\mathrm{a}}y^*)$$
for every $x^{**} \in X^{**}$ and $y^* \in Y^*$, and every $R \in \langle T\mathcal{A}(X) \cup \{T\}\rangle$. Then (b) follows from the proof of Theorem 5.1(a)⇒(b).

(b)⇒(c) is trivial.

(c)⇒(a). We use Theorem 1.2(e) to show (a). Let Z be a separable reflexive Banach space and let $T \in \mathcal{K}(Z, X^*)$. Consider $T^{\mathrm{a}}i_X \in \mathcal{K}(X, Z^*)$. Then let
$$\Theta \in \mathcal{L}((T^{\mathrm{a}}i_X\mathcal{A}(X))^*, \langle T^{\mathrm{a}}i_X\mathcal{A}(X) \cup \{T^{\mathrm{a}}i_X\}\rangle^*)$$
be the operator in (c). Then $\Theta^{\mathrm{a}}(T^{\mathrm{a}}i_X) \in (T^{\mathrm{a}}i_X\mathcal{A}(X))^{**}$. By Goldstine's theorem there exists a net (S_α) in $\mathcal{A}(X)$ with $\sup_\alpha \|S_\alpha^{\mathrm{a}}T\| = \sup_\alpha \|T^{\mathrm{a}}i_XS_\alpha\| \le \|\Theta^{\mathrm{a}}(T^{\mathrm{a}}i_X)\|$ such that
$$T^{\mathrm{a}}i_XS_\alpha \xrightarrow{\text{weak}^*} \Theta^{\mathrm{a}}(T^{\mathrm{a}}i_X).$$

Then for every $z \in Z$ and $x^{**} \in X^{**}$, we have

$$\begin{aligned}\lim_\alpha x^{**}(S_\alpha^{\mathrm{a}} Tz) &= \lim_\alpha x^{**}(T^{\mathrm{a}} i_X S_\alpha)^{\mathrm{a}} z = \lim_\alpha (x^{**} \otimes z)(T^{\mathrm{a}} i_X S_\alpha)\\ &= \Theta^{\mathrm{a}}(T^{\mathrm{a}} i_X)(x^{**} \otimes z) = \Theta(x^{**} \otimes z)(T^{\mathrm{a}} i_X) = x^{**}((T^{\mathrm{a}} i_X)^{\mathrm{a}} z) = x^{**}(Tz).\end{aligned}$$

By an argument of convex combinations and boundedness of $(S_\alpha^{\mathrm{a}} T)$ there exists a net (T_α) in $\mathcal{A}(X)$ such that $T_\alpha^{\mathrm{a}} T \xrightarrow{\tau_c} T$. Hence $T \in \overline{\mathcal{A}^{\mathrm{a}}(X)T}^{\tau_c}$. ∎

6. Extensions of some results. We need some approximation properties of finite rank operators to extend some results in this paper.

Lemma 6.1. *Let Z and X be Banach spaces. If $T : Z \to X^*$ is an injective operator, then $Z^* = \overline{T^{\mathrm{a}} i_X(X)}^{\text{weak}^*}$.*

Proof. By injectivity of T and weak* to weak* continuity of T^{a} we have

$$Z^* = \overline{T^{\mathrm{a}}(X^{**})}^{\text{weak}^*} = \overline{T^{\mathrm{a}} i_X(X)}^{\text{weak}^*}. \ \blacksquare$$

Corollary 6.2. *Let Z and X be Banach spaces. If Z is reflexive and $T : Z \to X^*$ is an injective operator, then $Z^* = \overline{T^{\mathrm{a}} i_X(X)}^{\|\cdot\|}$.*

Proposition 6.3. *Let Z and X be Banach spaces. Suppose that Z is reflexive.*

(a) *If $T : Z \to X$ is an injective operator, then*

$$\mathcal{F}(Z, X) \subset \overline{\{ST : S \in \mathcal{F}(X)\}}^{\|\cdot\|}.$$

(b) *If $T : Z \to X^*$ is an injective operator, then*

$$\mathcal{F}(Z, X^*) \subset \overline{\{S^{\mathrm{a}} T : S \in \mathcal{F}(X)\}}^{\|\cdot\|}.$$

Proof. (a) Since the operator $i_X T : Z \to X^{**}$ is injective, by Corollary 6.2 we have

$$Z^* = \overline{(i_X T)^{\mathrm{a}} i_{X^*}(X^*)}^{\|\cdot\|}.$$

Let $R = \sum_{k=1}^n z_k^* \otimes x_k \in \mathcal{F}(Z, X)$. We may assume that $\sum_{k=1}^n \|x_k\| = 1$. Let $\varepsilon > 0$. For every $k = 1, \ldots, n$, there exists an $x_k^* \in X^*$ such that

$$\|z_k^* - (i_X T)^{\mathrm{a}} i_{X^*}(x_k^*)\| \le \varepsilon.$$

Let $S := \sum_{k=1}^n i_{X^*}(x_k^*) \otimes x_k \in \mathcal{F}(X^{**}, X)$. Then

$$S i_X T = \sum_{k=1}^n (i_X T)^{\mathrm{a}} i_{X^*}(x_k^*) \otimes x_k.$$

We see that $\|R - S i_X T\| \le \varepsilon$. Hence

$$R \in \overline{\{S i_X T : S \in \mathcal{F}(X^{**}, X)\}}^{\|\cdot\|} \subset \overline{\{ST : S \in \mathcal{F}(X)\}}^{\|\cdot\|}.$$

(b) By Corollary 6.2, $Z^* = \overline{T^{\mathrm{a}} i_X(X)}^{\|\cdot\|}$. Let $R = \sum_{k=1}^n z_k^* \otimes x_k^* \in \mathcal{F}(Z, X^*)$. We may assume that $\sum_{k=1}^n \|x_k^*\| = 1$. Let $\varepsilon > 0$. For every

$k = 1, \dots, n$, there exists an $x_k \in X$ such that

$$\|z_k^* - T^{\mathrm{a}} i_X(x_k)\| \le \varepsilon.$$

Let $S := \sum_{k=1}^n x_k^* \otimes x_k \in \mathcal{F}(X, X)$. Then

$$S^{\mathrm{a}} T = \sum_{k=1}^n T^{\mathrm{a}} i_X(x_k) \otimes x_k^* \quad \text{and} \quad \|R - S^{\mathrm{a}} T\| \le \varepsilon.$$

Hence $R \in \overline{\{S^{\mathrm{a}} T : S \in \mathcal{F}(X)\}}^{\|\cdot\|}$. ■

We now extend Theorem 1.2.

Corollary 6.4. *Let X be a Banach space. Suppose that $\mathcal{A}(X)$ is a linear subspace of $\mathcal{K}(X)$ containing $\mathcal{F}(X)$ such that $S_1 S_2 \in \mathcal{A}(X)$ for all $S_1, S_2 \in \mathcal{A}(X)$. Then X has the $\mathcal{A}$-AP if and only if for every separable reflexive Banach space Z and every $T \in \mathcal{K}(Z, X)$, $\mathcal{A}(X)T$ is an ideal in $\langle \mathcal{A}(X)T \cup \{T\} \rangle$.*

Proof. By Theorem 1.2 we only need to show the "if" part. Let K be a compact subset of the unit ball of X and let $\varepsilon > 0$. Then there exist a separable reflexive Banach space Z, which is a linear subspace of X, such that the inclusion map $J : Z \to X$ is a norm one compact operator and K is a compact subset of B_Z. By Proposition 6.3(a) we have

$$\mathcal{F}(Z, X) \subset \overline{\mathcal{F}(X)J}^{\|\cdot\|} \subset \overline{\mathcal{A}(X)J}^{\|\cdot\|}.$$

Now, by assumption $\overline{\mathcal{A}(X)J}^{\|\cdot\|}$ is an ideal in

$$\overline{\langle \mathcal{A}(X)J \cup \{J\} \rangle}^{\|\cdot\|} = \overline{\langle \overline{\mathcal{A}(X)J}^{\|\cdot\|} \cup \{J\} \rangle}^{\|\cdot\|}.$$

Thus $\overline{\mathcal{A}(X)J}^{\|\cdot\|}$ is an ideal in $\langle \overline{\mathcal{A}(X)J}^{\|\cdot\|} \cup \{J\} \rangle$. By [LNO, Lemma 1.4],

$$J \in \overline{\{U \in \overline{\mathcal{A}(X)J}^{\|\cdot\|} : \|U\| \le 1\}}^{\tau_c}.$$

Choose a $U \in \overline{\mathcal{A}(X)J}^{\|\cdot\|}$ with $\|U\| \le 1$ such that $\sup_{z \in K} \|Jz - Uz\| \le \varepsilon/2$, and let $S_0 \in \mathcal{A}(X)$ be such that $\|S_0 J - U\| \le \varepsilon/2$. Then

$$\sup_{x \in K} \|S_0 x - x\| \le \sup_{z \in K} \|S_0 J z - Uz\| + \sup_{z \in K} \|Uz - Jz\| \le \varepsilon.$$

Hence X has the $\mathcal{A}$-AP. ■

The following corollary extends Theorem 5.2.

Corollary 6.5. *Let X be a Banach space. Suppose that $\mathcal{A}(X)$ is a linear subspace of $\mathcal{K}(X)$ containing $\mathcal{F}(X)$ such that $S_1 S_2 \in \mathcal{A}(X)$ for all $S_1, S_2 \in \mathcal{A}(X)$. Then X^* has the $\mathcal{A}^{\mathrm{a}}$-AP if and only if for every separable reflexive Banach space Z and every $T \in \mathcal{K}(X, Z)$, $T\mathcal{A}(X)$ is an ideal in $\langle T\mathcal{A}(X) \cup \{T\} \rangle$.*

Proof. By Theorem 5.2 we only need to show the “if” part. Let K be a compact subset of the unit ball of X^* and let $\varepsilon > 0$. Then there exist a separable reflexive Banach space Z, which is a linear subspace of X^*, such that the inclusion map $J : Z \to X^*$ is a norm one compact operator and K is a compact subset of B_Z. By Proposition 6.3(b) we have

$$\mathcal{F}(Z, X^*) \subset \overline{\mathcal{F}^{\mathrm{a}}(X)J}^{\|\cdot\|} \subset \overline{\mathcal{A}^{\mathrm{a}}(X)J}^{\|\cdot\|}.$$

Now, let $J_X : X \to Z^*$ be such that $J_X^{\mathrm{a}} = J$. By the assumption $J_X\mathcal{A}(X)$ is an ideal in $\langle J_X\mathcal{A}(X) \cup \{J_X\}\rangle$. Then as in the proof of Theorem 5.1, we see that $\mathcal{A}^{\mathrm{a}}(X)J$ is an ideal in $\langle \mathcal{A}^{\mathrm{a}}(X)J \cup \{J\}\rangle$. As in the proof of Corollary 6.4, we complete the proof. ■

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