COLLOQUIUM MATHEMATICUM

VOL. 149 2017 NO. 2

HOMOGENEOUS ROTA–BAXTER OPERATORS ON THE 3-LIE ALGEBRA A_ω (II)

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Abstract. We study k-order homogeneous Rota–Baxter operators of weight 1 on the simple 3-Lie algebra A_ω (over a field $\mathbb{F}$ of characteristic zero), which is realized by an associative commutative algebra A equipped with a derivation Δ and an involution ω (Lemma 2.3). A k-order homogeneous Rota–Baxter operator on A_ω, where $k \in \mathbb{Z}$, is a Rota–Baxter operator R satisfying $R(L_m) = f(m+k)L_{m+k}$ for all generators $\{L_m \mid m \in \mathbb{Z}\}$ of A_ω and a map $f : \mathbb{Z} \to \mathbb{F}$. We prove that R is a k-order homogeneous Rota–Baxter operator on A_ω of weight 1 with $k \neq 0$ if and only if $R = 0$ (Theorem 3.2), and R is a 0-order homogeneous Rota–Baxter operator on A_ω of weight 1 if and only if R is one of the thirty-eight possibilities which are described in Theorems 3.5, 3.7, 3.9, 3.10, 3.18, 3.21 and 3.22.

1. Introduction. Rota–Baxter operators are closely related to many fields in mathematics and mathematical physics. They have played an important role in the Hopf algebra approach to renormalization of perturbative quantum field theory [3, 4, 9, 10], as well as in the application of the renormalization method in solving diverse problems in number theory [16, 18]. They are also of importance in many fields such as symplectic geometry, integrable systems, quantum groups and quantum field theory [1, 2, 8, 9, 12–17, 19, 20].

Bai, Guo and Li [5] investigated Rota–Baxter operators on n-Lie algebras [11] and the structure of Rota–Baxter 3-Lie algebras. They provided a method to obtain Rota–Baxter 3-Lie algebras from Rota–Baxter Lie algebras, Rota–Baxter pre-Lie algebras and Rota–Baxter commutative associative algebras and derivations. Bai and Zhang [7] discussed 0-order homogeneous Rota–Baxter operators of weight zero on an infinite-dimensional simple 3-Lie algebra A_ω over a field $\mathbb{F}$ of characteristic zero. A 0-order homogeneous Rota–Baxter operator on A_ω is a linear map R satisfying $R(L_m) = f(m)L_m$ for all generators $\{L_m \mid m \in \mathbb{Z}\}$ of A_ω and a map

2010 *Mathematics Subject Classification*: Primary 17B05; Secondary 17D99.
Key words and phrases: 3-Lie algebra, homogeneous Rota–Baxter operator, Rota–Baxter 3-algebra.
Received 1 June 2016; revised 6 October 2016 and 17 April 2017.
Published online 8 May 2017.

DOI: 10.4064/cm7000-4-2017

$f : \mathbb{Z} \to \mathbb{F}$. It is proved that R is a homogeneous Rota–Baxter operator on A_ω if and only if R is one of the five possibilities $R_{0_1}, R_{0_2}, R_{0_3}, R_{0_4}$ and R_{0_5}. By means of homogeneous Rota–Baxter operators, new 3-Lie algebras $(A, [\ ,\ ,\ ]_i)$ for $1 \le i \le 5$ have been constructed with R_{0_i} being their homogeneous Rota–Baxter operators, respectively.

In this paper we investigate k-order homogeneous Rota–Baxter operators of weight 1 on the simple 3-Lie $\mathbb{F}$-algebra A_ω, where $\mathbb{F}$ is a field of characteristic zero. Throughout this paper, by an algebra we mean an $\mathbb{F}$-algebra.

2. Preliminaries. We recall that a 3-*Lie algebra* over a field $\mathbb{F}$ is an $\mathbb{F}$-vector space A endowed with a ternary multi-linear skew-symmetric operation $[\ ,\ ,\ ]$ such that for all $x_1, x_2, x_3, y_2, y_3 \in A$,

$$[[x_1, x_2, x_3], y_2, y_3] = [[x_1, y_2, y_3], x_2, x_3] + [[x_2, y_2, y_3], x_3, x_1] + [[x_3, y_2, y_3], x_1, x_2].$$

DEFINITION 2.1. Let $\lambda \in \mathbb{F}$ be fixed. A *Rota–Baxter* 3-*algebra* over $\mathbb{F}$ is a 3-algebra $(A, \langle\ ,\ ,\ \rangle)$ with an $\mathbb{F}$-linear map $R : A \to A$ such that

$$\begin{aligned}\langle R(x_1), R(x_2), R(x_3)\rangle = R\big(&\langle R(x_1), R(x_2), x_3\rangle + \langle R(x_1), x_2, R(x_3)\rangle \\ &+ \langle x_1, R(x_2), R(x_3)\rangle + \lambda\langle R(x_1), x_2, x_3\rangle + \lambda\langle x_1, R(x_2), x_3\rangle \\ &+ \lambda\langle x_1, x_2, R(x_3)\rangle + \lambda^2\langle x_1, x_2, x_3\rangle\big).\end{aligned} \tag{2.1}$$

LEMMA 2.2. *Let $(A, \langle\ ,\ ,\ \rangle)$ be a 3-algebra over $\mathbb{F}$, $R : A \to A$ a linear map and $\lambda \in \mathbb{F}$, $\lambda \ne 0$. Then $(A, \langle\ ,\ ,\ \rangle, R)$ is a Rota–Baxter 3-algebra of weight λ if and only if $(A, \langle\ ,\ ,\ \rangle, \lambda^{-1}R)$ is a Rota–Baxter 3-algebra of weight 1.*

Proof. Apply (2.1). ∎

LEMMA 2.3 ([6]). *Let A be an $\mathbb{F}$-vector space with a basis $\{L_n \mid n \in \mathbb{Z}\}$. Then $(A, [\ ,\ ,\ ])$ is a simple 3-Lie algebra, where for all $l, m, n \in \mathbb{Z}$,*

$$[L_l, L_m, L_n] = \begin{vmatrix} (-1)^l & (-1)^m & (-1)^n \\ 1 & 1 & 1 \\ l & m & n \end{vmatrix} L_{l+m+n-1}. \tag{2.2}$$

NOTATION. In the following, the 3-Lie algebra A in Lemma 2.3 is denoted by A_ω, and we set

$$D(l, m, n) := \begin{vmatrix} (-1)^l & (-1)^m & (-1)^n \\ 1 & 1 & 1 \\ l & m & n \end{vmatrix}.$$

LEMMA 2.4 ([7]). *$D(l, m, n) = 0$ if and only if for all $l, m, n, k, s, t \in \mathbb{Z}$, either*

- $(l-m)(l-n)(m-n)=0$, *or*
- $l=2k+1$, $m=2s+1$, $n=2t+1$, *or*
- $l=2k, m=2s, n=2t$.

3. Homogeneous Rota–Baxter operators of weight 1 on A_ω. By Definition 2.1, if $(A,[\ ,\ ,\ ],R)$ is a Rota–Baxter 3-Lie algebra of weight 1, then the $\mathbb{F}$-linear map $R:A\to A$ satisfies, for all $x_1,x_2,x_3\in A$,

$$\begin{aligned}[R(x_1),R(x_2),R(x_3)] = R\big(&[R(x_1),R(x_2),x_3]+[R(x_1),x_2,R(x_3)]\\ &+[x_1,R(x_2),R(x_3)]+[R(x_1),x_2,x_3]+[x_1,R(x_2),x_3]\\ &+[x_1,x_2,R(x_3)]+[x_1,x_2,x_3]\big).\end{aligned}\tag{3.1}$$

Definition 3.1. Let R be a Rota–Baxter operator on A_ω. If there is a map $f:\mathbb{Z}\to\mathbb{F}$ and $k\in\mathbb{Z}$ such that

$$R(L_m)=f(m+k)L_{m+k},\qquad \forall m\in\mathbb{Z},\tag{3.2}$$

then R is called a *k-order homogeneous Rota–Baxter operator*, and denoted by R_k.

3.1. k-Order homogeneous Rota–Baxter operator on A_ω with $k\neq 0$. Let R_k be a k-order homogeneous Rota–Baxter operator on A_ω with $k\neq 0$. By (3.2) and (2.2), for all $x,y,z\in A_\omega$,

$$\begin{aligned}[R_k(L_l),R_k(L_m),R_k(L_n)] &= [f(l+k)L_{l+k},f(m+k)L_{m+k},f(n+k)L_{n+k}]\\ &= f(l+k)f(m+k)f(n+k)D(l+k,m+k,n+k)L_{l+m+n+3k-1},\end{aligned}$$

$$\begin{aligned}&R_k\big([L_l,R_k(L_m),R_k(L_n)]+[R_k(L_l),L_m,R_k(L_n)]+[R_k(L_l),R_k(L_m),L_n]\\ &\quad+[R_k(L_l),L_m,L_n]+[L_l,R_k(L_m),L_n]+[L_l,L_m,R_k(L_n)]+[L_l,L_m,L_n]\big)\\ &=R_k([L_l,f(m+k)L_{m+k},f(n+k)L_{n+k}]+[f(l+k)L_{l+k},L_m,L_n]\\ &\quad+[f(l+k)L_{l+k},f(m+k)L_{m+k},L_n])+[f(l+k)L_{l+k},L_m,f(n+k)L_{n+k}]\\ &\quad+[L_l,f(m+k)L_{m+k},L_n]+[L_l,L_m,f(n+k)L_{n+k}]+[L_l,L_m,L_n])\\ &=f(l+k)f(l+m+n+2k-1)D(l+k,m,n)L_{l+m+n+2k-1}\\ &\quad+f(m+k)f(l+m+n+2k-1)D(l,m+k,n)L_{l+m+n+2k-1}\\ &\quad+f(n+k)f(l+m+n+2k-1)D(l,m,n+k)L_{l+m+n+2k-1}\\ &\quad+f(l+m+n+k-1)D(l,m,n)L_{l+m+n+k-1}\\ &\quad+f(m+k)f(n+k)f(l+m+n+3k-1)D(l,m+k,n+k)L_{l+m+n+3k-1}\\ &\quad+f(l+k)f(n+k)f(l+m+n+3k-1)D(l+k,m,n+k)L_{l+m+n+3k-1}\\ &\quad+f(l+k)f(m+k)f(l+m+n+3k-1)D(l+k,m+k,n)L_{l+m+n+3k-1}.\end{aligned}$$

We deduce that if $k\neq 0$, then $R_k([L_l,L_m,L_n])=0$ for all $l,m,n\in\mathbb{Z}$. Since $A_\omega=[A_\omega,A_\omega,A_\omega]$, we have $R_k(A_\omega)=0$.

This shows the following result.

THEOREM 3.2. *A linear map $R_k : A_\omega \to A_\omega$ defined as in* (3.2) *is a k-order homogeneous Rota–Baxter operator of weight* 1 *if and only if $R_k = 0$.*

3.2. 0-Order homogeneous Rota–Baxter operators of weight 1. In the following we discuss 0-order homogeneous Rota–Baxter operators of weight 1 on A_ω. Then (3.2) reduces to

$$R(L_m) = f(m)L_m, \quad \forall m \in \mathbb{Z}. \tag{3.3}$$

For convenience, throughout this paper a 0*-order homogeneous Rota–Baxter operator of weight* 1 *on A_ω is simply called* a homogeneous Rota–Baxter operator on A_ω.

Let R be an $\mathbb{F}$-linear map on A_ω defined as in (3.3). Denote

$$\begin{aligned} W_1 &= \{2m \mid m \in \mathbb{Z} \setminus \{0\},\ f(2m) \neq 0\}, \\ U_1 &= \{2m+1 \mid m \in \mathbb{Z} \setminus \{0\},\ f(2m+1) \neq 0\}, \\ W_2 &= \{2m \mid m \in \mathbb{Z} \setminus \{0\},\ f(2m) = 0\}, \\ U_2 &= \{2m+1 \mid m \in \mathbb{Z} \setminus \{0\},\ f(2m+1) = 0\}. \end{aligned}$$

LEMMA 3.3. *Let $R : A_\omega \to A_\omega$ be a linear map defined as in* (3.3)*. Then R is a homogeneous Rota–Baxter operator on A_ω if and only if the map f has the property that for all $l, m, n \in \mathbb{Z}$ with $m \neq n$,*

$$\begin{aligned} f(2n+1)f(2m+1)f(2l) = \big(&f(2n+1)f(2m+1) + f(2n+1)f(2l) \\ &+ f(2m+1)f(2l) + f(2n+1) + f(2m+1) + f(2l) + 1\big) f(2n+2m+2l+1), \end{aligned} \tag{3.4}$$

$$\begin{aligned} f(2l+1)f(2m)f(2n) = \big(&f(2l+1)f(2m) + f(2l+1)f(2n) \\ &+ f(2m)f(2n) + f(2l+1) + f(2m) + f(2n) + 1\big) f(2l+2m+2n). \end{aligned} \tag{3.5}$$

Proof. By (3.1) and (3.3), R is a homogeneous Rota–Baxter operator on A_ω if and only if the map $f : \mathbb{Z} \to \mathbb{F}$ in (3.3) satisfies, for all $l, m, n \in \mathbb{Z}$,

$$\begin{aligned} f(l)f(m)f(n)D(l,m,n) = \big(&f(l)f(m) + f(l)f(n) + f(m)f(n) \\ &+ f(l) + f(m) + f(n) + 1\big) f(l+m+n-1)D(l,m,n). \end{aligned}$$

Therefore, (3.4) and (3.5) hold. ∎

If $l = n = 0$ and $m \neq 0, 1$, then by (3.4) and (3.5),

$$(f(m)f(1) + f(0)f(m) + f(0) + f(1) + f(m) + 1)f(m) = 0.$$

Hence

$$(f(0) + f(1) + 1)f(m)(f(m) + 1) = 0. \tag{3.6}$$

We consider several cases according to the value $f(0) + f(1) + 1$.

3.2.1. *The case of $f(0) + f(1) + 1 \neq 0$.* In this section we discuss homogeneous Rota–Baxter operators in (3.3) which satisfy $f(0) + f(1) + 1 \neq 0$.

LEMMA 3.4. *Let R be a homogeneous Rota–Baxter operator on A_ω with $f(0)+f(1)+1\neq 0$. Then*

$$f(m)(f(m)+1)=0,\qquad \forall m\in\mathbb{Z}\setminus\{0,1\}. \tag{3.7}$$

Proof. Apply (3.6). ∎

THEOREM 3.5. *Let $R: A_\omega\to A_\omega$ be a linear map defined as in (3.3) with $f(0)+f(1)+1\neq 0$. If at least one of the four subsets W_i, U_i, $i=1,2$, is finite, then R is a homogeneous Rota–Baxter operator on A_ω if and only if the map $f:\mathbb{Z}\to\mathbb{F}$ in (3.3) satisfies one of the following conditions, for all $m\in\mathbb{Z}$:*

(1) $f(m)=0$.
(2) $f(m)=-1$.
(3) $f(2m)=0, f(2m+1)=-1$ *if* $m\neq 0$, *and* $f(0)(f(1)+1)=0$.
(4) $f(2m)=-1, f(2m+1)=0$ *if* $m\neq 0$ *and* $f(1)(f(0)+1)=0$.

Proof. If f satisfies one of conditions (1)–(4), then R satisfies (3.4) and (3.5). Therefore, R is a homogeneous Rota–Baxter operator on A_ω.

Conversely, let R be a homogeneous Rota–Baxter operator on A_ω.

We first prove that if W_i (or U_i) is finite, where $i=1$ or 2, then it is empty.

Without loss of generality, we can suppose that $|W_1|=s<\infty$.

If $s\geq 1$, then without loss of generality, we can suppose that $W_1=\{2m_0,\dots,2m_{s-1}\}$ and $|U_1|\neq 0$. Then there is $n_0\neq 0$ such that $f(2n_0+1)=-1$. If $|U_2|=\infty$, then there exist distinct $2m,2n\in W_2$ and $2l+1\in U_2$ such that $2m+2n+2l=2m_0$. We arrive at the contradiction $f(2m_0)=f(2m)f(2n)f(2l+1)=0$. Therefore, $|U_2|<\infty$, and $|U_1|=\infty$. Then there exist distinct $2l+1, 2n+1\in U_1$ and $2m\in W_2$ such that $2m+2n+2l=2n_0$. We get the contradiction $f(2n_0+1)=f(2m)f(2n+1)f(2l+1)=0$.

Therefore, W_1 is empty, that is, $f(2m)=0$ for all $m\in\mathbb{Z}\setminus\{0\}$.

Now we need to discuss the following three cases.

• *U_2 is non-empty.* There is $2n_0+1\in U_2$ such that $f(2n_0+1)=0$, and for all $m\in\mathbb{Z}\setminus\{0,-n_0\}$ and $s\in\mathbb{Z}$,

$$f(2n_0+1)f(2m)f(-2n_0-2m)=f(0)=0,$$
$$f(2n_0+1)f(1)f(-2n_0)=(f(1)+1)f(1)=0,$$
$$f(2n_0+1)f(1)f(2s)=f(2n_0+2s+1)=0.$$

Therefore, $f(0)=f(1)=0$, and for all $l\in\mathbb{Z}\setminus\{-n_0\}$ we have $f(2l+1)=0$. We obtain (1).

• *U_2 is empty.* Then for all $l\in\mathbb{Z}\setminus\{0\}$, $f(2l+1)=-1$, and $f(0)(f(1)+1)=0$. We obtain (3).

• *W_2 is empty.* Then for all $m\in\mathbb{Z}\setminus\{0\}$, we have $f(2m)=-1$.

If U_1 is empty, then we obtain (4). If U_2 is empty, then we obtain (2). The proof is complete. ∎

Now we discuss the case $|W_i| = |U_i| = \infty$, $i = 1, 2$.

LEMMA 3.6. *Let R be a homogeneous Rota–Baxter operator on A_ω with $f(0) + f(1) + 1 \neq 0$. If $W_1 = \{2m_0 < 2m_1 < \cdots\}$, then $U_1 = \{2l_0 + 1 < 2l_1 + 2 < \cdots\}$, where $l_0 \geq -m_1$ and $l_1 \geq -m_0$.*

Proof. By (3.4) and (3.5), for all $2l+1 \in U_1$, we have $f(2m_0+2m_1+2l) = -1$. Then $2l+2m_0+2m_1 \geq m_0$, and $l \geq -m_1$. Therefore, U_1 is as stated. ∎

THEOREM 3.7. *Let R be a homogeneous Rota–Baxter operator on A_ω with $f(0) + f(1) + 1 \neq 0$. Then the map $f : \mathbb{Z} \to \mathbb{F}$ in (3.3) satisfies one of the following conditions:*

(1) *There exist $m_0, m_1 \in \mathbb{Z}$ with $m_0 < m_1$ such that $f(2m_0) = f(2m_1 + 2k(m_1 - m_0)) = -1$, $f(-2m_1 + 1) = f(-2m_0 + 2k(m_1 - m_0) + 1) = -1$ for all $k \in \mathbb{Z}_{\geq 0}$, and $f(m) = 0$ for the remaining $m \in \mathbb{Z}$.*
(2) *$f(2k) = f(-1) = f(2k+1) = -1$ for all $k \in \mathbb{Z}_{>0}$, and $f(m) = 0$ for the remaining $m \in \mathbb{Z}$.*
(3) *There is $m_0 \in \mathbb{Z}_{>0}$ such that $f(2km_0) = f(-2m_0 + 1) = f(2km_0 + 1) = -1$ for all $k \in \mathbb{Z}_{\geq 0}$, and $f(m) = 0$ for the remaining $m \in \mathbb{Z}$.*
(4) *There is $m_0 \in \mathbb{Z}_{<0}$ such that $f(2m_0) = f(2km_0) = f(2km_0 + 1) = -1$ for all $k \in \mathbb{Z}_{\leq 0}$, and $f(m) = 0$ for the remaining $m \in \mathbb{Z}$.*
(5) *$f(0) = f(1) = -1$, and there is $m_0 \in \mathbb{Z}_{>1}$ such that $f(m) = -1$ for all $m \in \mathbb{Z}_{\geq m_0}$, and $f(m) = 0$ for the remaining $m \in \mathbb{Z}$.*
(6) *$f(0) = f(1) = -1$ and there exist distinct $m_0, l_0 \in \mathbb{Z}_{>0}$ such that for all $m, n \in \mathbb{Z}$, if $m \geq m_0$ and $n \geq l_0$, then $f(2m) = f(2n+1) = -1$, and $f(m) = 0$ for the remaining $m \in \mathbb{Z}$.*

Proof. Suppose $W_1 = \{2m_0 < 2m_1 < \cdots\}$, $U_1 = \{2l_0 + 1 < 2l_1 + 1 < \cdots\}$. By Lemma 3.6, we have $l_0 \geq -m_1$ and $m_0 \geq -l_1$.

(i) *The case $l_0 = -m_1$.* By (3.4) and (3.5), for all $i \in \mathbb{Z}_{\geq 1}$,

$$m_i = m_1 + (i-1)(m_1 - m_0), \quad l_1 = -m_0, \quad l_i = -m_0 + (i-1)(m_1 - m_0).$$

We obtain (1).

(ii) *The case $-m_1 < l_0 < -m_0$.* Since $2(m_0 + l_0 + m_1) \in W_1$ and $m_0 + l_0 + m_1 < m_1$, we see that $m_0 + l_0 + m_1 = m_0$. We obtain the contradiction $l_0 < -m_1$. Therefore, this case does not occur.

(iii) *The case $-m_1 < l_0 = -m_0$.* Since

$$f(0) = f(0)f(2m_0)f(2l_0+1) = f(0)f(2m_0)f(-2m_0+1) = -f(0)^2,$$
$$f(1) = f(1)f(2m_0)f(2l_0+1) = f(1)f(2m_0)f(-2m_0+1) = -f(1)^2,$$

we have either $f(0) = f(1) = 0$, or $f(0) = f(1) = -1$.

• If $f(0) = f(1) = 0$, then for all $k, l \in \mathbb{Z}_{>0}$,

$$f(2m_0 - 2k)f(-2m_0 - 2l + 1)f(0) = f(-2(k + l)) = 0,$$
$$f(2m_0 - 2k)f(-2m_0 - 2l + 1)f(1) = f(-2(k + l) + 1) = 0.$$

We obtain $m_0 \geq 1$, $-m_0 = l_0 \geq -1$. If there is $k_0 > 1$ such that $f(2k_0) = 0$, then $f(-2k_0 - 2 + 1) = 0$. We get the contradiction

$$f(1)f(2k_0)f(-2k_0 - 2 + 1) = f(-2 + 1) = f(2l_0 + 1) = 0.$$

Therefore, $W_1 = \{2k \mid k \in \mathbb{Z}_{>0}\}$ and $U_1 = \{-1, 2k + 1 \mid k \in \mathbb{Z}_{>0}\}$. This yields case (2).

• If $f(0) = f(1) = -1$, then for $l, m, n, s \in \mathbb{Z} \setminus \{0\}$, from

$$f(2l + 1) = f(2n + 1) = f(2m) = f(2s) = -1,$$

we have $f(2l + 2n + 1) = f(2m + 2s) = f(2l + 2m) = f(2l + 2m + 1) = -1$. Then $2m_1 + 2l_0 = 2m_1 - 2m_0 \in W_1$ and $2l_1 + 2l_0 + 1 = 2l_1 - 2m_0 + 1 \in U_1$.

If $m_0 > 0$, then by Lemma 3.6, we have $m_1 - m_0 > 0$, $l_1 - m_0 < l_1$, and $m_1 = 2m_0$, $l_1 = m_0$. Inductively, suppose $m_k = (k + 1)m_0$, $l_k = km_0$. Since

$$m_{k-1} = km_0 = m_k - m_0 < m_{k+1} - m_0 < m_{k+1},$$

we have

$$m_{k+1} = (k + 2)m_0, \quad l_{k-1} = (k - 1)m_0 = l_k - m_0 < l_{k+1} - m_0 < l_{k+1},$$

and $l_{k+1} = (k + 1)m_0$. Then

$$W_1 = \{2km_0 \mid k \in \mathbb{Z}_{>0}\}, \quad U_1 = \{-2m_0 + 1, 2km_0 + 1 \mid k \in \mathbb{Z}_{>0}\}.$$

We obtain case (3).

If $m_0 < 0$, then $W_1 = \{2m_0, -2km_0 \mid k \in \mathbb{Z}_{>0}\}$ and $U_1 = \{2km_0 + 1 \mid k \in Z_{>0}\}$. We obtain case (4).

(iv) *The case* $l_0 > -m_0$. If there is $m' > m_0$ such that $f(2m') = 0$, then from $m' > m_0$ and $-m' < -m_0 < l_0$, we have $f(-2m' + 1) = 0$, and

$$f(0)f(2m')f(-2m' + 1) = (f(0) + 1)f(0) = 0,$$
$$f(1)f(2m')f(-2m' + 1) = (f(1) + 1)f(1) = 0.$$

Thus either $f(0) = f(1) = 0$, or $f(0) = f(1) = -1$.

• If $f(0) = f(1) = -1$, then from $f(2m_0 + 2l_0) = f(2m_0 + 2l_0 + 1) = -1$, we obtain $m_0, l_0 > 0$.

If $m_0 = l_0$, then from $f(k2m_0) = -1$, we have $l_0 = m_0 > 1$, and

$$\{2km_0 \mid k \in \mathbb{Z}_{>0}\} \subseteq W_1, \quad \{2km_0 + 1 \mid k \in \mathbb{Z}_{>0}\} \subseteq U_1.$$

If there exist $r, k \in \mathbb{Z}_{>0}$ such that $r < m_0$ and $f(2m_0k + 2r) = 0$, then from $f(-2r) = f(-2km_0 + 1) = 0$, we get the contradiction

$$0 = f(2m_0k + 2r)f(-2r)f(-2km_0 + 1) = f(0) = -1.$$

Therefore, for all $m \geq m_0$, we have $f(2m) \neq 0$. We obtain case (5).

If $l_0 \neq m_0$, then from $f(2l_0 + 2m_0) = f(2m_0 + 2l_0 + 1) = -1$, we have

$$\{2km_0 + 2ln_0 \mid k \in \mathbb{Z}_{>0},\ l \in \mathbb{Z}_{\geq 0}\} \subseteq W_1,$$
$$\{2km_0 + 2ln_0 + 1 \mid k \in \mathbb{Z}_{\geq 0},\ l \in \mathbb{Z}_{>0}\} \subseteq U_1.$$

We obtain $W_1 = \{2m \mid m \in \mathbb{Z}_{\geq m_0}\}$, $U_1 = \{2n+1 \mid n \in \mathbb{Z}_{\geq l_0}\}$, and $f(l) = -1$ for all $l \in W_1 \cup U_1$. This gives (6).

• Now we prove that the case $f(0) = f(1) = 0$ does not occur.

If $f(0) = f(1) = 0$, then from $l_0 > -m_0 > -m'$ and $l_0 > -m' + 1$, we have $f(2m') = 0$, and

$$f(0)f(2m')f(-2m' + 2 + 1) = (f(0) + 1)f(2) = 0,$$
$$f(1)f(2m')f(-2m' + 2 + 1) = (f(0) + 1)f(3) = 0.$$

Then $f(2) = f(3) = 0$. If $f(2k) = f(2k + 1) = 0$ for $k \in \mathbb{Z}_{>0}$, then by (3.4) and (3.5), we have

$$f(0)f(2k)f(2 + 1) = (f(0) + 1)f(2k + 2) = f(2k + 2) = 0,$$
$$f(1)f(2k)f(2 + 1) = (f(0) + 1)f(2k + 2 + 1) = f(2k + 2 + 1) = 0.$$

Therefore, $f(2k) = f(2k+2+1) = 0$ for all $k \in \mathbb{Z}_{\geq 0}$. We get the contradiction $|U_1| = \infty$. Thus the case $f(0) = f(1) = 0$ does not occur. The proof is complete. ∎

LEMMA 3.8. *Let R be a homogeneous Rota–Baxter operator with $f(0) + f(1) + 1 \neq 0$, and $W_1 = \{2m_0 > 2m_1 > \cdots\}$. Then $U_1 = \{2l_0 + 1 > 2l_1 + 1 > \cdots\}$ with $l_0 \leq -m_1$ and $l_1 \leq -m_0$.*

Proof. By (3.5), $f(2m_0 + 2m_1 + 2l) = -1$ for all $2l + 1 \in U_1$. Then $2l+2m_0+2m_1 \leq 2m_0$ and $l \leq -m_1$. Therefore, $U_1 = \{2l_0+1 > 2l_1+1 > \cdots\}$ with $l_0 \leq -m_1$. Thanks to (3.4), $m_0 \leq -l_1$. ∎

THEOREM 3.9. *Let R be a homogeneous Rota–Baxter operator with $f(0) + f(1) + 1 \neq 0$. Then the map $f : \mathbb{Z} \to \mathbb{F}$ in (3.3) satisfies one of the following conditions:*

(1) *There exist $m_0, m_1 \in \mathbb{Z}$ with $m_0 > m_1$ such that*

$$f(2m_0) = f(2m_1 + 2k(m_0 - m_1)) = -1, \quad k \in \mathbb{Z}_{\leq 0},$$
$$f(-2m_1 + 1) = f(-2m_0 + 2k(m_0 - m_1) + 1) = -1, \quad k \in \mathbb{Z}_{\leq 0},$$

and $f(m) = 0$ for the remaining $m \in \mathbb{Z}$.

(2) *$f(2) = f(2k) = f(2k + 1) = -1$ for all $k \in \mathbb{Z}_{<0}$, and $f(m) = 0$ for the remaining $m \in \mathbb{Z}$.*

(3) *There is $m_0 \in \mathbb{Z}_{<0}$ such that $f(2km_0) = f(-2m_0 + 1) = f(2km_0 + 1) = -1$ for all $k \in \mathbb{Z}_{\geq 0}$, and $f(m) = 0$ for the remaining $m \in \mathbb{Z}$.*

(4) *There is $m_0 \in \mathbb{Z}_{>0}$ such that $f(2m_0) = f(2km_0) = f(2km_0 + 1) = -1$ for all $k \in \mathbb{Z}_{\leq 0}$, and $f(m) = 0$ for the remaining $m \in \mathbb{Z}$.*

(5) *$f(0) = f(1) = -1$, and there is $m_0 \in \mathbb{Z}_{<-1}$ such that $f(l) = -1$ for all $l \le 2m_0 + 1$, and $f(m) = 0$ for the remaining $m \in \mathbb{Z}$.*
(6) *$f(0) = f(1) = -1$, and there exist distinct $m_0, l_0 \in \mathbb{Z}_{<0}$ such that $f(2m) = f(2l+1) = -1$ for all $m \le m_0$ and $l \le l_0$, and $f(m) = 0$ for the remaining $m \in \mathbb{Z}$.*

Proof. Apply the arguments used in the proof of Theorem 3.7. ∎

THEOREM 3.10. *If $\inf W_i = \inf U_i = -\infty$ and $\sup W_i = \sup U_i = \infty$, then R is a homogeneous Rota–Baxter operator on A_ω with $f(0) + f(1) + 1 \neq 0$ if and only if the map $f : \mathbb{Z} \to \mathbb{F}$ in (3.3) satisfies one of the following conditions:*

(1) *There is $m_0 \in \mathbb{Z} \setminus \{0\}$ such that $f(2km_0) = f(2m_0k + 1) = 0$ for all $k \in \mathbb{Z}$, and $f(m) = -1$ for the remaining $m \in \mathbb{Z}$.*
(2) *There is $m_0 \in \mathbb{Z} \setminus \{0\}$ such that $f(2km_0) = f(2m_0k + 1) = -1$ for all $k \in \mathbb{Z}$, and $f(m) = 0$ for the remaining $m \in \mathbb{Z}$.*

Proof. Let R be a homogeneous Rota–Baxter operator on A_ω. Suppose $W_2 = \{2m_i, 2m_i' \mid i \in \mathbb{Z}_{\ge 0}\}$, $U_2 = \{2l_i + 1, 2l_i' + 1 \mid i \in \mathbb{Z}_{\ge 0}\}$, where for $i \in \mathbb{Z}_{\ge 0}$,

$$m_{i+1}' < m_i', \quad m_i < m_{i+1}, \quad m_0' < 0 < m_0, \quad l_{i+1}' < l_i', \quad l_i < l_{i+1}, \quad l_0' < 0 < l_0.$$

If $f(0) = b \neq 0, -1$, then for distinct $l, k \in \mathbb{Z}$ with $f(2l + 1) = f(2k + 1) = 0$, we have $f(2l + 2k + 1) = f(2l_0 + 2l_0' + 1) = 0$. Since $2l_0' + 1 < 2l_0 + 2l_0' + 1 < 2l_0 + 1$, we see that $f(1) = 0$. From $f(2m_0 + 2m_0') = 0$ and $2m_0' < 2m_0 + 2m_0' < 2m_0$, we get the contradiction $f(0) = b = 0$. Therefore, either $f(0) = 0$, or $f(0) = -1$.

If $f(0) = 0$, then from $2l_0' + 1 < 2l_0 + 2l_0' + 1 < 2l_0 + 1$ and (3.4), we have $f(0)f(2l_0 + 1)f(2l_0' + 1) = f(2l_0 + 2l_0' + 1) = 0$. Therefore, $l_0' = -l_0$ and $f(1) = 0$. We deduce that $m_i = -m_i'$, $l_i = -l_i'$ for all $i \in \mathbb{Z}_{\ge 0}$. Since $0 < 2m_1 - 2m_0 = 2m_1 + 2m_0' < 2m_1$, we have $m_1 = 2m_0$. Inductively, we obtain $m_i = (i + 1)m_0$, $m_i' = -(i + 1)m_0$, $l_i = (i + 1)l_0$ and $l_i' = -(i + 1)l_0$ for all $i \in \mathbb{Z}_{\ge 0}$.

If $m_0 \neq l_0$, then from $2m_0 - 2l_0 = 2m_0 + 2l_0' < 2m_0$, we have $2m_0 - 2l_0 \in W_2$, $2l_0' + 1 < 2m_0 - 2l_0 + 1$, and $2m_0 - 2l_0 + 1 \in U_2$, and we get the contradiction $2m_0 - 2l_0 < 0$ and $2m_0 - 2l_0 > 0$. Therefore, $m_0 = l_0$. This gives case (1).

Similarly, we obtain (2) for $f(0) = -1$. The proof is complete. ∎

3.2.2. *The case $f(0) + f(1) + 1 = 0$ and $f(0) = a \neq 0$.* In this section we discuss homogeneous Rota–Baxter operators on A_ω with $f(0) + f(1) + 1 = 0$ and $f(0) = a \in \mathbb{F} \setminus \{0\}$.

LEMMA 3.11. *Let R be a homogeneous Rota–Baxter operator on A_ω with $f(0) + f(1) + 1 = 0$ and $f(0) = a \in \mathbb{F} \setminus \{0\}$. Then f satisfies the following* *equations, for* $l, m, n \in \mathbb{Z}$*:*

(1) $af(2l+1)f(2m+1) = \big((a+1)f(2l+1) + (a+1)f(2m+1) + a + 1 + f(2l+1)f(2m+1)\big)f(2l+2m+1)$ *if* $l \neq m$.

(2) $(a+1)f(2m+1)f(2n) = \big(af(2m+1) + af(2n) - f(2m+1)f(2n) - a\big)f(2m+2n+1)$ *if* $m \neq 0$.

(3) $af(2l+1)f(2m) = \big((a+1)f(2l+1) + (a+1)f(2m) + a + 1 + f(2l+1)f(2m)\big)f(2l+2m)$ *if* $m \neq 0$.

(4) $(a+1)f(2m)f(2n) = \big(af(2m) + af(2n) - f(2m)f(2n) - a\big)f(2m+2n)$ *if* $m \neq n$.

Proof. This follows from (3.4) and (3.5). ∎

THEOREM 3.12. *Let* R *be a homogeneous Rota–Baxter operators on* A_ω *with* $f(0) + f(1) + 1 = 0$ *and* $f(0) = a \in \mathbb{F} \setminus \{0\}$*, then the map* $f : \mathbb{Z} \to \mathbb{F}$ *in* (3.3) *satisfies*

$$f(1-m) + f(m) + 1 = 0, \qquad \forall m \in \mathbb{Z}. \tag{3.8}$$

Proof. From Lemma 3.11, for all $m, n \in \mathbb{Z} \setminus \{0\}$, we have

$$\begin{aligned}-f(2m+1)f(2n) &= f(2m+2n+1)\big(-af(2m+1) - af(2n) - a + f(2m+1)f(2n)\big)\\ &\quad + f(2m+2n)\big((a+1)f(2m+1)+(a+1)f(2n)+f(2m+1)f(2n)+a+1\big).\end{aligned}$$

Therefore, $f(2m+1) + f(-2m) + 1 = 0$ for all $m \in \mathbb{Z}$. ∎

THEOREM 3.13. *Let* R *be a homogeneous Rota–Baxter operator on* A_ω *with* $f(0) + f(1) + 1 = 0$ *and* $f(0) = a \in \mathbb{F} \setminus \{0\}$*. If*

$$f(2k)f(2l)f(2m+1)f(2n+1) \neq 0 \quad \text{for } k, l, m, n \in \mathbb{Z} \setminus \{0\},$$

then

(1) $f(2k+2l) \neq 0$; (2) $f(2k+2m) \neq 0$; (3) $f(2k+2m+1) \neq 0$; (4) $f(2m+2n+1) \neq 0$; (5) $f(2m+2n+2k+1) \neq 0$; (6) $f(2m+2k+2l) \neq 0$; (7) $f(1-2k+2m) \neq 0$, $m \neq -k$; (8) $f(4k) \neq 0$; (9) $f(1-2k-2m)+1 \neq 0$; (10) $f(2k-2m)+1 \neq 0$; (11) $f(1-4k)+1 \neq 0$.

Proof. (1), (2), (3) and (4) follow from (4), (2), (3) and (1) in Lemma 3.11, respectively. (5) and (6) follow from (3.4) and (3.5).

(7) follows from Lemma 3.11(1) and $l = 0$, $m \neq -k$.

(8) follows from Lemma 3.11(3) and $k \neq 0$.

(9), (10) and (11) follow from (2), (7) and (10), respectively. ∎

COROLLARY 3.14. *Let* R *be a linear map on* A_ω *defined by* (3.3) *with* $f(0) + f(1) + 1 = 0$ *and* $f(0) = a \in \mathbb{F} \setminus \{0\}$*. If at least one of the subsets* W_i, U_i*,* $i = 1, 2$ *is finite, then* R *is not a homogeneous Rota–Baxter operator.*

Proof. This follows from Theorem 3.13. ∎

THEOREM 3.15. *If R is a homogeneous Rota–Baxter operator on A_ω with $f(0)+f(1)+1=0$ and $f(0)=a\in\mathbb{F}\setminus\{0\}$, then* $\inf W_i=\inf U_i=-\infty$, $\sup W_i=\sup U_i=\infty$, *and there is $m_0\in\mathbb{Z}\setminus\{0\}$ such that*

$$W_1=\{2m_0k\mid k\in\mathbb{Z}\setminus\{0\}\},\qquad U_1=\{2m_0k+1\mid k\in\mathbb{Z}\}.$$

Proof. Without lost of generality, suppose that there is $m_0\in\mathbb{Z}$ such that $f(2m_0)\neq 0$, and $m\geq m_0$ (or $m\leq m_0$) for all $2m\in W_1$. By Theorem 3.13, $f(2m+2m_0)\neq 0$ and $f(4m_0)\neq 0$ for all $2m+1\in U_1$. We obtain $2m+2m_0\geq 2m_0$ and $m_0>0$. Then there is $l_0\in\mathbb{Z}_{>0}$ such that $2l+1\in U_1$ for all $2l+1\geq 2l_0+1$. From Theorem 3.13(7), $f(1+2l_0-2m_0)\neq 0$. We get the contradiction $2l_0+1\leq 1+2l_0-2m_0<1+2l_0$. Therefore, $\inf W_i=\inf U_i=-\infty$ and $\sup W_i=\sup U_i=\infty$. Then we can suppose $W_1=\{2m_i,2m_i'\mid i\in\mathbb{Z}_{\geq 0}\}$ and $U_1=\{2l+1,2l_i'+1\mid i\in\mathbb{Z}_{\geq 0}\}$, where

$$m_{i+1}'<m_i'<0<m_i<m_{i+1},\qquad l_{i+1}'<l_i'<0<l_i<l_{i+1},\qquad i\geq 0.$$

Thanks to Theorem 3.13, $2m_0+2m_0'\in W_1$, $m_0'<m_0+m_0'<m_0$, and $m_0'=-m_0$.

From $0<2m_1+m_0'=2m_1-2m_0<2m_1$, we have $m_1=2m_0$. Inductively, we get $m_i=(i+1)m_0$, $m_i'=-(i+1)m_0$, $l_i=(i+1)l_0$ and $l_i'=-(i+1)l_0$ for all $i\geq 0$. Then there exist positive $s,t\in\mathbb{Z}$ such that $2l_0+2m_0=2sm_0=2tl_0$. Therefore, $l_0=m_0$. The proof is complete. ∎

Let R be a homogeneous Rota–Baxter operator on A_ω. The subset $T_{m_0}=W_1\cup U_1$ is called the m_0-*supporter* of R. Then for all $m\in\mathbb{Z}\setminus\{0,1\}$, $f(m)\neq 0$ if and only if $m\in T_{m_0}$.

COROLLARY 3.16. *Let R be a homogeneous Rota–Baxter operator with $f(0)+f(1)+1=0$ and $f(0)=a\in\mathbb{F}\setminus\{0\}$. Then the map $f:\mathbb{Z}\to\mathbb{F}$ in* (3.3) *has the property that for $k\in\mathbb{Z}$, if $f(2m_0k)\neq 0$, then $f(2m_0k)\neq -1$, $f(1+2m_0k)\neq 0,-1$, and*

$$\frac{1}{f(2m_0k)}+\frac{1}{f(-2m_0k)}+\frac{1}{f(2m_0k)f(-2m_0k)}=\frac{1+2a}{a^2}. \tag{3.9}$$

Proof. From Theorem 3.13(9)&(10), if $f(2m_0k)\neq 0$, then $f(2km_0)\neq -1$, $f(1+2km_0)\neq 0,-1$. By Lemma 3.11(4),

$$(1+2a)f(2m_0k)f(-2m_0k)=a^2\big(f(2m_0k)+f(-2m_0k)+1\big)$$

for $m=-n=2m_0k\neq 0$. This yields (3.9). ∎

COROLLARY 3.17. *Let R be a homogeneous Rota–Baxter operator with $f(0)+f(1)+1=0$, $f(0)=a\in\mathbb{F}\setminus\{0\}$ and m_0-supporter T_{m_0}. Then the map $f:\mathbb{Z}\to\mathbb{F}$ in* (3.3) *has the property that for all $k_1,k_2,k_3\in\mathbb{Z}$ with $k_2\neq k_3$,*

$$\frac{1}{f(2m_0k_2)}+\frac{1}{f(2m_0k_3)f(2m_0k_2)}+\frac{1}{f(2m_0k_3)}=\frac{1}{f(2m_0k_1)} + \frac{1}{f(2m_0k_1)f(2m_0(-k_1+k_2+k_3))}+\frac{1}{f(2m_0(-k_1+k_2+k_3))}. \tag{3.10}$$

Proof. Apply the arguments used in the proof of Corollary 3.16. ∎

THEOREM 3.18. *Let R be a homogeneous Rota–Baxter operator on A_ω with $f(0)+f(1)+1=0$ and $f(0)=a\in\mathbb{F}\setminus\{0\}$. Then there is $m_0\in\mathbb{Z}\setminus\{0\}$ such that the map $f:\mathbb{Z}\to\mathbb{F}$ in (3.3) satisfies one of the following conditions:*

(1) *$f(2m_0k)=a$, $f(2m_0k+1)=-1-a$ for all $k\in\mathbb{Z}$, and $f(m)=0$ for the remaining $m\in\mathbb{Z}$.*
(2) *If there is $k_0\in\mathbb{Z}\setminus\{0\}$ such that $f(2m_0k_0)\neq a$, then $a\neq -1,-1/2$, and $f(4m_0k)=a$, $f(4m_0k+1)=-1-a$, $f(4m_0k+2)=\frac{-a}{1+2a}$, $f(4m_0k+3)=-\frac{1+a}{1+2a}$ for all $k\in\mathbb{Z}$, and $f(m)=0$ for the remaining $m\in\mathbb{Z}$.*

Proof. By Theorems 3.13 and 3.15, there is $m_0\in\mathbb{Z}\setminus\{0\}$ such that $f(l)\neq 0$ if and only if either $l=2m_0k$ or $l=2m_0k+1$. Thanks to (3.10), for $k\neq 0$,

$$f(2m_0k)=f(-2m_0k). \tag{3.11}$$

Then for all distinct $l,k\in\mathbb{Z}\setminus\{0\}$,

$$f(1+2m_0k)=f(1-2m_0k)=-1-f(2m_0k),$$
$$(f(2m_0k)-f(2m_0l))(f(2m_0k)+2f(2m_0k)f(2m_0l)+f(2m_0l))=0,$$
$$(f(2m_0k)-a)(f(2m_0k)+2af(2m_0k)+a)=0.$$

We deduce that if $f(2m_0l)\neq a$, then $a\neq -1,-1/2$, and

$$f(2m_0l)=f(-2m_0l)=\frac{-a}{1+2a},\qquad f(2m_0l+1)=f(-2m_0l+1)=-\frac{1+a}{1+2a}.$$

If there exist $n_0,k_0\in\mathbb{Z}\setminus\{0\}$ such that $f(2m_0k_0)\neq a$ and $f(2m_0n_0)=a$, then $k_0\neq\pm n_0$ and $f(2m_0(n_0+k_0))\neq a$. By (3.5), for $k_1\neq k_0$ and $n_1\neq n_0$, if $f(2m_0n_1)=a$ and $f(2m_0k_1)\neq a$, then $f(2m_0(k_0+k_1))=f(2m_0(n_0+n_1))$ $=a$. Without loss of generality, we can suppose $m_0>0$. Then for positive $k_0,n_0\in\mathbb{Z}$ with $f(2m_0k)\neq a$ and $f(2m_0s)=a$, we have $k\geq k_0, s\geq n_0$ and $f(2m_0(k_0-n_0))\neq a$. Since $k_0-n_0<k_0$, we get $k_0=1$.

If $n_0>2$, then $f(2m_02)\neq a$, and $f(2m_0(1+2))=a$. We obtain $n_0=3$. Therefore, $f(2m_03)=a$, $f(2m_0(2+3))\neq a$, $f(2m_0(2+5))=a$, and $f(2m_0(1+3))\neq a$. We get the contradiction $f(2m_0(3+4))\neq a$. Therefore, $n_0=2$, and $f(2m_0k)=a$ for $k=2l$, and $f(2m_0k)\neq a$ for $k=2l+2$. This gives (2).

If for all $k\in\mathbb{Z}$, $f(2m_0k)=a$, then by (3.8), $f(2m_0k+1)=-1-f(2m_0k)=-1-a$. This yields (1). ∎

3.2.3. *The case $f(0) = 0$ and $f(1) = -1$.* In this section we discuss homogeneous Rota–Baxter operators R with $R(L_0) = 0$ and $R(L_1) = -L_1$, that is, the map $f : \mathbb{Z} \to \mathbb{F}$ in (3.3) has the property that $f(0) = 0$ and $f(1) = -1$.

Lemma 3.19. *Let R be a homogeneous Rota–Baxter operator on A_ω with $f(0) = 0$ and $f(1) = -1$. Then the map $f : \mathbb{Z} \to \mathbb{F}$ in* (3.3) *has the property that for all $l, m, n \in \mathbb{Z}$:*

(1) $(f(2l+1)+1)(f(2m+1)+1)f(2l+2m+1) = 0$ *if* $l \neq m$.
(2) $f(2m+1)f(2n)(1+f(2m+2n+1)) = 0$ *if* $m \neq 0$.
(3) $(f(2l+1)+1)(f(2m)+1)f(2l+2m) = 0$ *if* $m \neq 0$.
(4) $f(2m)f(2n)(1+f(2m+2n)) = 0$ *if* $m \neq n$.

Proof. This follows from (3.4) and (3.5). ∎

Corollary 3.20. *Let R be a homogeneous Rota–Baxter operator on A_ω with $f(0) = 0$ and $f(1) = -1$. Then the map $f : \mathbb{Z} \to \mathbb{F}$ in* (3.3) *has the property that for $k, l, m, n \in \mathbb{Z} \setminus \{0\}$:*

(1) *If $f(2k) \neq 0$ and $f(2l) \neq 0$, $k \neq \pm l$, then $f(2k+2l) = -1$.*
(2) *If $f(2k) \neq 0$ and $f(2m+1) \neq 0$, then $f(2k+2m+1) = -1$.*
(3) *If $f(2k) = 0$ and $f(2n+1) = 0$, then $f(2k+2n) = 0$.*
(4) *If $f(2m+1) = f(2n+1) = 0$ with $m \neq \pm n$, then $f(2m+2n+1) = 0$.*
(5) $f(2k)f(-2k) = 0$.
(6) $(f(2m+1)+1)(f(-2m+1)+1) = 0$.
(7) $|W_2| = |U_1| = \infty$.

Proof. This follows from Lemma 3.19. ∎

Theorem 3.21. *If $|W_1| < \infty$, then R is a homogeneous Rota–Baxter operator on A_ω with $f(0)=0$ and $f(1)=-1$ if and only if the map $f : \mathbb{Z} \to \mathbb{F}$ in* (3.3) *satisfies one of the following conditions:*

(1) *$|W_1| = |U_2| = 0$, and $f(2m) = 0$, $f(2m+1) = -1$ for all $m \in \mathbb{Z}$.*
(2) *$|W_1| = |U_2| = 0$, and there is $n_0 \in \mathbb{Z} \setminus \{0\}$ such that $f(2n_0+1) \neq 0, -1$ and $f(2m) = 0$, $f(2n+1) = -1$ for all $m, n \in \mathbb{Z}$ with $n \neq n_0$.*
(3) *$|W_1| = 0$, $|U_2| = 1$, and there is $n_0 \in \mathbb{Z} \setminus \{0\}$ such that $f(2n_0+1) = 0$ and $f(2m) = 0$, $f(2n+1) = -1$ for all $m, n \in \mathbb{Z}$ with $n \neq n_0$.*
(4) *$|W_1| = 1$, $|U_2| = 0$, and there is $m_0 \in \mathbb{Z} \setminus \{0\}$ such that $f(2m_0) \neq 0$ and $f(2m) = 0$, $f(2n+1) = -1$ for all $m, n \in \mathbb{Z}$ with $m \neq m_0$.*

Proof. Apply the arguments used in the proof of Theorem 3.5. ∎

From Theorem 3.21, if R is a homogeneous Rota–Baxter operator, then $|W_1| \neq 0$ and $|U_2| \neq 0$ if and only if $|W_1| = |U_2| = \infty$. So in the rest of this section, we discuss the case $|W_1| = |U_2| = \infty$.

THEOREM 3.22. *If $|W_1| = \infty$, then R is a homogeneous Rota–Baxter operator with $f(0) = 0$ and $f(1) - 1$ if and only if the map $f : \mathbb{Z} \to \mathbb{F}$ in (3.3) satisfies one of the following conditions:*

(1) *There exist $m_0 \in \mathbb{Z}_{>0}$ and $n_0 \in \mathbb{Z}_{<0}$ such that $f(2n+1) = f(2m) = -1$ for all $n > n_0$ and $m \geq m_0$, and $f(m) = 0$ for the remaining $m \in \mathbb{Z}$.*
(2) *There exist $m_0 \in \mathbb{Z}_{>0}$ and $c, d \in \mathbb{F}\setminus\{0, -1\}$ such that $f(2m) = f(2n+1) = -1$, $f(-1) = c$ and $f(-3) = d$ for all $m \in \mathbb{Z}_{\geq m_0}$ and $n \in \mathbb{Z}_{\geq 0}$, and $f(m) = 0$ for the remaining $m \in \mathbb{Z}$.*
(3) *There exist $m_0 \in \mathbb{Z}_{>0}$, and $c' \in \mathbb{F}\setminus\{0, -1\}$ such that $f(2m) = f(2n+1) = f(-1) = f(-3) = -1$ and $f(3) = c'$ for all $m \geq m_0$, $n \geq 0$ and $n \neq 1$, and $f(m) = 0$ for the remaining $m \in \mathbb{Z}$.*
(4) *There exist $m_0 \in \mathbb{Z}_{>0}$, and $g \in \mathbb{F}\setminus\{0, -1\}$ such that $f(2m) = f(2n+1) = -1$ and $f(-1) = g$ for all $m \geq m_0$, $n \geq 0$, and $f(m) = 0$ for the remaining $m \in \mathbb{Z}$.*
(5) *There exist $m_1, m_0, n_0 \in \mathbb{Z}$ with $m_1 \geq m_0 > 0$ and $n_0 < 0$, and $h \in \mathbb{F}\setminus\{0, -1\}$ such that $f(2m_1) = h$, $f(2m) = f(2n+1) = -1$ for all $m \geq m_0$, $m \neq m_1$ and $n > n_0$ and $f(m) = 0$ for the remaining $m \in \mathbb{Z}$.*
(6) *There exist $m_1, m_0, n_1, n_0 \in \mathbb{Z}$ with $m_1 \geq m_0 > 0$, $n_0 < n_1$, $n_0 < 0$, and $h, h' \in \mathbb{F}\setminus\{0, -1\}$ such that $f(2m_1) = h$, $f(2n_1 + 1) = h'$ and $f(2m) = f(2n+1) = -1$ for all $m \geq m_0$, $n > n_0$ with $m \neq m_1$ and $n \neq n_1$, and $f(m) = 0$ for the remaining $m \in \mathbb{Z}$.*
(7) *There exist $m_0, m_1, m_2 \in \mathbb{Z}_{>0}$ with $m_1, m_2 \geq m_0$, $m_1 \neq m_2$, $n_0 \in \mathbb{Z}_{<0}$ and $g, r \in \mathbb{F}\setminus\{0, -1\}$ such that $f(2m_1) = g$, $f(2m_2) = r$, $f(2n+1) = f(2m) = -1$ for all $n > n_0$, $m \geq m_0$ with $m \neq m_1$ and $m \neq m_2$, and $f(m) = 0$ for the remaining $m \in \mathbb{Z}$.*

$(1)'$ *There exist $m_0 \in \mathbb{Z}_{<0}$ and $n_0 \in \mathbb{Z}_{>0}$ such that $f(2n+1) = f(2m) = -1$ for all $n < n_0$, $m \leq m_0$, and $f(m) = 0$ for the remaining $m \in \mathbb{Z}$.*
$(2)'$ *There exist $m_0 \in \mathbb{Z}_{<0}$ and $c \in \mathbb{F}\setminus\{0, -1\}$ such that $f(2m) = f(2n+1) = -1$ and $f(3) = c$ for all $m \leq m_0$ and $n < 0$, and $f(m) = 0$ for the remaining $m \in \mathbb{Z}$.*
$(3)'$ *There exist $m_0 \in \mathbb{Z}_{<0}$ and $c', d' \in \mathbb{F}\setminus\{0, -1\}$ such that $f(-1) = c'$ and $f(-3) = d'$, and $f(2m) = f(2n+1) = f(1) = f(3) = -1$ for all $m \leq m_0$ and $n < -2$, and $f(m) = 0$ for the remaining $m \in \mathbb{Z}$.*
$(4)'$ *There exist $m_0 \in \mathbb{Z}_{<0}$ and $g \in \mathbb{F}\setminus\{0, -1\}$ such that $f(-1) = g$, $f(2m) = f(2n+1) = -1$ for all $m \geq m_0$, $n < 0$ with $n \neq -1$, and $f(m) = 0$ for the remaining $m \in \mathbb{Z}$.*
$(5)'$ *There exist $m_0, m_1, n_0 \in \mathbb{Z}$ with $m_1 \leq m_0 < 0$ and $n_0 > 0$ and $h \in \mathbb{F}\setminus\{0, -1\}$ such that $f(2m_1) = h$, $f(2m) = f(2n+1) = -1$ for all $m \geq m_0$ with $m \neq m_1$ and $n < n_0$, and $f(m) = 0$ for the remaining $m \in \mathbb{Z}$.*

(6)′ *There exist* $m_1, m_0, n_1, n_0 \in \mathbb{Z}$ *with* $m_1 \le m_0 < 0$, $n_1 < n_0$, $n_0 > 0$ *and* $h, h' \in \mathbb{F} \setminus \{0, -1\}$ *such that* $f(2m_1) = h$, $f(2n_1 + 1) = h'$ *and* $f(2m) = f(2n+1) = -1$ *for all* $m \le m_0$ *with* $m \ne m_1$ *and* $n \ne n_1$ *with* $n < n_0$, *and* $f(m) = 0$ *for the remaining* $m \in \mathbb{Z}$.

(7)′ *There exist* $m_0 \in \mathbb{Z}_{<0}$ *and distinct* $m_1, m_2 \in \mathbb{Z}_{\le m_0}$, $n_0 \in \mathbb{Z}_{>0}$ *and* $g, r \in \mathbb{F} \setminus \{0, -1\}$ *such that* $f(2m_1) = g, f(2m_2) = r$, *and* $f(2n + 1) = f(2m) = -1$ *for all* $m \in \mathbb{Z}_{\le m_0} \setminus \{m_1, m_2\}$ *and* $n < n_0$, *and* $f(m) = 0$ *for the remaining* $m \in \mathbb{Z}$.

Proof. Since $|W_1| = \infty$, without loss of generality we can suppose that there is $m \in \mathbb{Z}_{>0}$ with $f(2m) \ne 0$.

Then there is $m_0 \in \mathbb{Z}_{>0}$ such that if $2m \in W_1$ and $m > 0$, then $m \ge m_0$. We will prove that $W_1 = \{2m \mid m \in \mathbb{Z} \setminus \{m_0\}\}$ and $U_2 = \{2n + 1 \mid n \in \mathbb{Z} \setminus \{n_0\}\}$ for some $n_0 \in \mathbb{Z}_{<0}$.

If $f(2n + 1) \ne 0$ for all negative $n \in \mathbb{Z}$, then by Corollary 3.20 we have $f(2n + k2m_0 + 1) = -1$ for all $k \in \mathbb{Z}_{>0}$. We get the contradiction $|U_2| = 0$.

Therefore, there is $n_0 \in \mathbb{Z}_{<0}$ such that $f(2n_0+1) = 0$, and $f(2n+1) \ne 0$ for all $n \in \mathbb{Z}$ with $n_0 < n < 0$.

First, if there is $m < 0$ such that $2m \in W_1$, then there is $m_0' \in \mathbb{Z}_{<0}$ such that $2m \notin W_1$ with $m_0' < m < 0$. By Corollary 3.20, $2m_0' + 2m_0 \in W_1$. Since $2m_0' < 2m_0' + 2m_0 < 2m_0$, $m_0' = -m_0$. We get the contradiction $f(2m_0)f(-2m_0) = 0$. Therefore, $m \ge m_0$ for all $2m \in W_1$.

If there is $n > n_0$ such that $2n+1 \notin U_1$, then there is $2n' \in U_2$ such that $n' > n_0$, and $2n+1 \notin U_2$ for all $n_0 < n < n'$. From $f(2n'+2n_0+1) = 0$ and $n_0 < 0$, we get $2n'+2n_0 < 2n'$ and $2n'+2n_0 < 2n_0$. Therefore, $n' < 0$. We get the contradiction $n_0 > n'$. Therefore, $n > n_0$ for all $2n+1 \in U_1$. We deduce that $f(2n + 1) = f(2m) = 0$ for $m < m_0$ and $n \le n_0$, $f(2n + 1) = -1$ for $n > -n_0$, $f(2n+1) \ne 0$ for $n_0 < n < 0$, and $f(2m) = 0$ for all $0 < m < m_0$.

If there is $n \in \mathbb{Z}$ with $0 < n < -n_0$ such that $f(2n + 1) = 0$, then there is $n'' \in \mathbb{Z}$ with $f(2n'' + 1) = 0$, and $f(2n + 1) \ne 0$ for all $0 < n < n''$. Then $f(2n_0+2n''+1) = 0$, and we get the contradiction $2n_0+1 < 2n_0+2n''+1 < 2n'' + 1$. Therefore, $f(2n + 1) \ne 0$ for all $n \in \mathbb{Z}$ with $0 < n < -n_0$.

If there is $m \in \mathbb{Z}$ with $-m_0 < m < 0$ such that $f(2m) = 0$, then there is $m'' \in \mathbb{Z}$ with $-m_0 < m'' < 0$ such that $f(2m'') \ne 0$, and $f(2m) = 0$ for all $m'' < m < 0$. Then $f(2m_0 + 2m'') \ne 0$, and we get the contradiction $2m'' < 2m_0+2 < m'' < 2m_0$. Therefore, there exist $m_0 \in \mathbb{Z}_{>0}$ and $n_0 \in \mathbb{Z}_{<0}$ such that $W_1 = \{2m \mid m \in \mathbb{Z}_{\ge m_0}\}$, $W_2 = \{2m \mid m \in \mathbb{Z}_{<m_0}\}$, $U_1 = \{2n + 1 \mid n \in \mathbb{Z}_{>n_0}\}$, $U_2 = \{2n + 1 \mid n \in \mathbb{Z}_{\le n_0}\}$.

If there is $m \in \mathbb{Z}_{<0}$ such that $f(2m) \ne 0$, then there exist $m_0 \in \mathbb{Z}_{<0}$ and $n_0 \in \mathbb{Z}_{>0}$ such that $W_1 = \{2m \mid m \in \mathbb{Z}_{\le m_0}\}$, $W_2 = \{2m \mid m \in \mathbb{Z}_{>m_0}\}$, $U_1 = \{2n + 1 \mid n \in \mathbb{Z}_{<n_0}\}$, $U_2 = \{2n + 1 \mid n \in \mathbb{Z}_{\ge n_0}\}$.

For the case $m_0 > 0$ and $n_0 < 0$, by Corollary 3.20, for all $l, k, s \in \mathbb{Z}_{>0}$ with $l \neq k$,

$$(f(2m_0 + 2s) + 1)(f(2n_0 + 2k + 1) + 1)(f(2n_0 + 2l + 1) + 1) = 0, \tag{3.12}$$

$$(f(2n_0 + 2s + 1) + 1)(f(2m_0 + 2k) + 1)(f(2m_0 + 2l) + 1) = 0. \tag{3.13}$$

• If $f(2m) = f(2n + 1) = -1$ for all $m \geq m_0$ and $n > n_0$, then we obtain (1).

If $f(2m) = -1$ for all $m \geq m_0$, and there is $n_1 \in \mathbb{Z}_{>n_0}$ such that $f(2n_1 + 1) \neq -1$, then by Corollary 3.20, we have $n_0 \geq -3$ and either

$$f(2n+1) \begin{cases} = -1 & \text{if } n \geq -n_0, \\ \neq 0 & \text{if } n_0 < n < 0, \\ = -1 & \text{if } 0 < n < -n_0; \end{cases} \quad \text{or} \quad f(2n+1) \begin{cases} = -1 & \text{if } n_0 < n < 0, \\ \neq 0 & \text{if } 0 < n < -n_0. \end{cases}$$

Therefore, in the case $n_0 = -3$, we get (2) and (3). If $n_0 = -2$, we obtain (4).

• If there is a unique $m_1 \in \mathbb{Z}_{\geq m_0}$ such that $f(2m_1) \neq 0, -1$, then by (3.12), $f(2n+1) = -1$ for all $n > n_0$, and we obtain (5). If there is a unique $n_1 \in \mathbb{Z}_{>n_0}$ such that $f(2n_1 + 1) \neq 0, -1$ and $f(2n + 1) = -1$ for all $n > n_0$ with $n \neq n_1$, then we obtain (6).

• If the subset $S = \{m_k \mid m_k \in \mathbb{Z}_{\geq m_0},\ f(2m_k) \neq 0, -1,\ k \in \mathbb{Z}\}$ is non-empty, then by (3.12) and (3.13), either $S = \{m_1\}$ or $S = \{m_1, m_2\}$. Therefore, we obtain (6) and (7).

Similar to the above discussion, we obtain $(1')$–$(7')$ for the case $m_0 < 0$ and $n_0 > 0$. The proof is complete. ∎

Acknowledgements. The first author was supported in part by the Natural Science Foundation (11371245) and the Natural Science Foundation of Hebei Province (A2014201006).

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