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A quantitative form of the Erdős–Birch theorem

by

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1. Introduction. In 1959, Birch [3] proved the following theorem, which confirmed a conjecture posed by P. Erdős.

Theorem A. *For any coprime integers p, q greater than 1, there exists an integer B such that every integer $n \geq B$ is expressible as a sum of the form $n = p^{a_1}q^{b_1} + p^{a_2}q^{b_2} + \cdots + p^{a_k}q^{b_k}$, where the (a_i, b_i) are distinct pairs of nonnegative integers.*

Cassels [1] proved a more general theorem which includes Birch's theorem. Davenport pointed out that for any coprime integers p and q greater than 1, there exists an integer K such that the sequence $Y_K = \{p^a q^b \mid a \geq 0, 0 \leq b \leq K\}$ is complete. Let $K(p,q)$ be the least such k. It is very difficult to determine the exact value of $K(p,q)$.

In 2000, Hegyvári [6] obtained an effective upper bound for $K(p,q)$:

$$K(p,q) \leq 2p^{2d^{2^{2q^{4p+3}}}},$$

where $d = 1152 \log_2 p \log_2 q$ and $\log_2$ means the logarithm to base 2. This was improved by [4] and [5] to

$$K(p,q) \leq d^{2^{q^{2p+3}}},$$

where $d = 1152 \log_2 p \log_2 q$.

In this paper, the following result is proved.

Theorem 1.1. *For any coprime integers $p, q > 1$, there exist positive integers K and B with*

$$\log_2 \log_2 K < q^{2p}, \quad \log_2 \log_2 \log_2 B < q^{2p}$$

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such that every integer $n \geq B$ can be expressed as the sum of distinct terms taken from

$$\{p^a q^b \mid a \geq 0,\ 0 \leq b \leq K,\ a+b > 0,\ a, b \in \mathbb{Z}\}.$$

Recently, Bergelson and Simmons [2] proved that $K(p,q) \leq 4p-5$. It seems that their method could not be used to give an explicit bound for B. Currently, we are not able to prove that Theorem 1.1 is true with $K = 4p-5$.

2. Proof of Theorem 1.1. Before the proof of Theorem 1.1, we introduce the following notation and definitions. Denote the first quadrant $\mathbb{Z}^2_{\geq 0}$ by $\mathscr{Q}$. For $\mathscr{E} \subseteq \mathscr{Q}$, let $\phi(\mathscr{E}) = \phi_{p,q}(\mathscr{E}) = \sum_{(a,b)\in\mathscr{E}} p^a q^b$, and $\phi_{p,q}(\mathscr{E}) = 0$ if $\mathscr{E} = \emptyset$. For two integers A_0 and B_0, we write $(a,b) \in \mathscr{E} - (A_0, B_0)$ whenever $(a + A_0, b + B_0) \in \mathscr{E}$. For $T = \{t_1 < t_2 < \cdots\}$, define

$$P(T) = \Big\{\sum \varepsilon_i t_i \;\Big|\; \varepsilon_i \in \{0,1\},\ \sum \varepsilon_i < \infty\Big\}.$$

Here $0 \in P(T)$. Denote by $\lfloor x \rfloor$ the largest integer not exceeding x, and by $\lceil x \rceil$ the smallest integer greater than or equal to x.

Lemma 2.1. *For any coprime integers $p, q > 1$, there are disjoint nonempty sets*

$$\mathscr{E}_1, \mathscr{E}_2 \subseteq \{(a,b) \in \mathbb{Z}^2 \mid 1 \leq a \leq 4\log_2 q,\ 1 \leq b \leq 4\log_2 p\}$$

such that $\phi_{p,q}(\mathscr{E}_1) = \phi_{p,q}(\mathscr{E}_2)$.

Proof. Since p, q are coprime and greater than 1, it follows that $p \geq 3$ or $q \geq 3$. Without loss of generality, we may assume that $p \geq 3$. Let

$$S = \{(a,b) \mid 1 \leq a \leq 4\log_2 q,\ 1 \leq b \leq 4\log_2 p\}.$$

Since

$$\begin{aligned}\sum_{\substack{1 \leq a \leq 4\log_2 q \\ 1 \leq b \leq 4\log_2 p}} p^a q^b &= \frac{p^{\lfloor 4\log_2 q\rfloor+1} - p}{p-1}\,\frac{q^{\lfloor 4\log_2 p\rfloor+1} - q}{q-1} \\ &< \frac{pq}{(p-1)(q-1)} p^{4\log_2 q} q^{4\log_2 p} \\ &= \frac{pq}{(p-1)(q-1)} 2^{8\log_2 p \log_2 q} < 2^{8\log_2 p\log_2 q + 3}\end{aligned}$$

and

$$\begin{aligned}\#\{(a,b) \in \mathbb{Z}^2 \mid 1 \leq a \leq 4\log_2 q,\ 1 \leq b \leq 4\log_2 p\} &= \lfloor 4\log_2 q\rfloor \lfloor 4\log_2 p\rfloor \\ &> (4\log_2 q - 1)(4\log_2 p - 1) = 16\log_2 p\log_2 q - 4\log_2 q - 4\log_2 p + 1 \\ &= 8\log_2 p\log_2 q + 4\log_2 q(\log_2 p - 1) + 4\log_2 p(\log_2 q - 1) + 1 \\ &\geq 8\log_2 p\log_2 q + 4(\log_2 p - 1) + 1 = 8\log_2 p\log_2 q + 4\log_2 p - 3 \\ &> 8\log_2 p\log_2 q + 3,\end{aligned}$$

it follows that

$$\sum_{(a,b)\in S} p^a q^b < 2^{|S|}.$$

Thus

$$\#\Big\{ \sum_{(a,b)\in \mathscr{E}} p^a q^b \ \Big|\ \mathscr{E} \subseteq S \Big\} < 2^{|S|}.$$

Hence there are different sets $\mathscr{E}_1', \mathscr{E}_2' \subseteq S$ such that $\phi_{p,q}(\mathscr{E}_1') = \phi_{p,q}(\mathscr{E}_2')$. Let

$$\mathscr{E}_1 = \mathscr{E}_1' \setminus (\mathscr{E}_1' \cap \mathscr{E}_2'), \quad \mathscr{E}_2 = \mathscr{E}_2' \setminus (\mathscr{E}_1' \cap \mathscr{E}_2').$$

Then $\mathscr{E}_1$ and $\mathscr{E}_2$ are disjoint nonempty subsets of S such that $\phi_{p,q}(\mathscr{E}_1) = \phi_{p,q}(\mathscr{E}_2)$. ∎

LEMMA 2.2. *Let $T = \{t_1 < t_2 < \cdots\}$ be a set of positive integers and $P(T) = \{y_1 < y_2 < \cdots\}$. Then*

$$y_{n+1} - y_n \le \sup_k \Big(t_k - \sum_{i=1}^{k-1} t_i \Big).$$

Proof. Let

$$P(\{t_1, \dots, t_l\}) = \{y_{1,l} < \cdots < y_{s_l,l}\}, \quad l = 1, 2, \dots.$$

It is clear that

$$y_{1,l} = 0, \quad y_{s_l,l} = \sum_{i=1}^{l} t_i.$$

For any positive integer n, there exists an integer $l \ge 1$ such that $y_n, y_{n+1} \in P(\{t_1, \dots, t_l\})$. Thus, it is enough to prove that, for any integer $l \ge 1$,

$$y_{m+1,l} - y_{m,l} \le \sup_k \Big(t_k - \sum_{i=1}^{k-1} t_i \Big), \quad m = 1, \dots, s_l - 1.$$

For $l = 1$, we have $P(\{t_1\}) = \{0, t_1\}$, and the conclusion is clear. Suppose that the conclusion is true for l. Since

$$P(\{t_1, \dots, t_l, t_{l+1}\}) = \{y_{1,l} < \cdots < y_{s_l,l}\} \cup \{y_{1,l} + t_{l+1} < \cdots < y_{s_l,l} + t_{l+1}\}$$

and

$$y_{s_l,l} = \sum_{i=1}^{l} t_i, \quad t_{l+1} = y_{1,l} + t_{l+1},$$

the inductive hypothesis implies that the conclusion is true for $l + 1$. ∎

LEMMA 2.3. *Let p and q be coprime integers greater than 1 and let*

$$P(\{p^a q^{2b} \mid a \ge 0,\ 0 \le b \le p\}) = \{y_1 < y_2 < \cdots\}.$$

Then, for every positive integer n, we have $y_{n+1} - y_n < q^{2p} - q^{2p-2}$.

Proof. Let

$$T = \{p^a q^{2b} \mid a \geq 0,\ 0 \leq b \leq p\} = \{t_1 < t_2 < \cdots\}.$$

By Lemma 2.2, it is enough to prove that, for all $k \geq 1$,

$$t_k - \sum_{i=1}^{k-1} t_i < q^{2p} - q^{2p-2}. \tag{2.1}$$

Fix $k \geq 1$. If $t_k > q^{2p}$, then

$$\sum_{\substack{p^t q^{2s} < t_k \\ t \geq 0,\, 0 \leq s \leq p}} p^t q^{2s} \geq \sum_{s=0}^{p} q^{2s} \sum_{p^t < t_k/q^{2s}} p^t \geq \sum_{s=0}^{p} q^{2s} \frac{t_k/q^{2s} - 1}{p-1}$$
$$\geq \frac{1}{p-1}\left(pt_k + t_k - \frac{q^2 t_k}{q^2 - 1}\right) > t_k,$$

hence

$$t_k - \sum_{i=1}^{k-1} t_i < 0 < q^{2p} - q^{2p-2}.$$

If $q^{2p-2} < t_k \leq q^{2p}$, then (2.1) holds. If $t_k \leq q^{2p-2}$, then

$$t_k - \sum_{i=1}^{k-1} t_i \leq q^{2p-2} < q^{2p} - q^{2p-2},$$

so (2.1) also holds. ∎

COROLLARY 2.4. *Let p and q be coprime integers greater than* 1 *and let*

$$P(\{p^a q^{2b} \mid a \geq 0,\ 0 \leq b \leq p\}) = \{y_1 < y_2 < \cdots\}.$$

Then, for any nonnegative integer x, there exists an integer n such that $0 \leq x - y_n < q^{2p} - q^{2p-2}$.

Proof. For any x as above, there exists an integer n such that $y_n \leq x < y_{n+1}$. Then $0 \leq x - y_n < y_{n+1} - y_n$, and the corollary follows from Lemma 2.3. ∎

LEMMA 2.5. *Let p and q be coprime integers greater than* 1 *and let*

$$c_1 = 4\lceil \log_2 p \rceil, \quad c_2 = 4\lceil \log_2 q \rceil, \quad c = 32\lceil \log_2 p \rceil \lceil \log_2 q \rceil.$$

Let $U_0 = 1$ and $V_0 = 2$. For $n = 1, 2, \ldots,$ define

$$U_n = c_2 U_{n-1} V_{n-1}, \quad V_n = c_1 U_{n-1} V_{n-1}.$$

Then

$$U_n = c_1^{-1} c^{2^{n-1}}, \quad V_n = c_2^{-1} c^{2^{n-1}}, \quad n = 1, 2, \ldots.$$

Proof. Since $U_n V_n = c_1 c_2 (U_{n-1} V_{n-1})^2$, we have

$$U_n V_n = (c_1 c_2)^{1+2+\cdots+2^{n-1}} (U_0 V_0)^{2^n} = (c_1 c_2)^{2^n - 1} 2^{2^n} = (c_1 c_2)^{-1} c^{2^n}.$$

Thus by the definitions of U_n and V_n,

$$U_n = c_2 U_{n-1} V_{n-1} = c_1^{-1} c^{2^{n-1}}, \quad V_n = c_1 U_{n-1} V_{n-1} = c_2^{-1} c^{2^{n-1}}. \ \blacksquare$$

LEMMA 2.6. *There exists a nonnegative integer $R \le W p^U q^V$ such that*

$$mp^U q^V + R \in P(\{p^a q^{2b+1} \mid a \ge 0,\ 0 \le b \le V + p - 1\}), \quad m = 0, 1, \ldots,$$

where

$$U = U_{x_0}, \quad V = V_{x_0}, \quad W = 2^{cU}, \quad x_0 = q^{2p} - q^{2p-2}.$$

Proof. For the convenience of the reader, first we outline the proof and then give the details.

We argue in the following two steps:

STEP 1: By using Lemma 2.1, we prove that there are sequences $\{F_k - (A_k, B_k)\}$ and $\{Y_k - (A_k, B_k)\}$ such that the sets $\mathscr{Q}$, $F_k - (A_k, B_k)$, $Y_k - (A_k, B_k)$ $(k = 1, 2, \ldots)$ are all disjoint,

$$F_k, Y_k, \{(A_k, B_k)\} \subseteq \{(a, 2b) \mid 1 \le a \le U_k,\ 1 \le b \le V_k/2\}$$

and

$$\phi_{p,q}(F_k - (A_k, B_k)) - \phi_{p,q}(Y_k - (A_k, B_k)) = 1.$$

STEP 2: By Corollary 2.4, each nonnegative integer m can be written as $m = y + z$ with

$$y \in P(\{p^a q^{2b} \mid a \ge 0,\ 0 \le b \le p\}), \quad 0 \le z < x_0.$$

Let

$$r = \sum_{i=1}^{x_0} \phi_{p,q}(Y_i - (A_i, B_i)), \quad A = \max_{1 \le i \le x_0} A_i, \quad B = \max_{1 \le i \le x_0} B_i.$$

Then

$$m + r = y + z + r = y + \sum_{i=1}^{z} \phi_{p,q}(F_i - (A_i, B_i)) + \sum_{i=z+1}^{x_0} \phi_{p,q}(Y_i - (A_i, B_i)).$$

Thus

$$mp^A q^B + rp^A q^B \in \{p^a q^{2b} \mid a \ge 1,\ 1 \le b \le 2V_{x_0} + p\}.$$

Replacing m by $mp^{U-A+1} q^{V-B+1}$, we get

$$mp^U q^V + R \in P(\{p^a q^{2b+1} \mid a \ge 0,\ 0 \le b \le 2V + p - 1\}), \quad m = 0, 1, \ldots,$$

where $R = rp^{A-1} q^{B-1}$.

Now we give the details of the proof. Let

$$p_k = p^{U_{k-1}}, \quad q_k = q^{V_{k-1}}, \quad k = 1, 2, \ldots.$$

By Lemma 2.1 there are disjoint nonempty sets $\mathscr{E}_{1,k}, \mathscr{E}_{2,k}$ with

$$\mathscr{E}_{1,k}, \mathscr{E}_{2,k} \subseteq \{(a, b) \in \mathbb{Z}^2 \mid 1 \le a \le 4 \log_2 q_k,\ 1 \le b \le 4 \log_2 p_k\}$$

such that $\phi_{p_k,q_k}(\mathscr{E}_{1,k}) = \phi_{p_k,q_k}(\mathscr{E}_{2,k})$. Take (A'_k, B'_k) in $\mathscr{E}_{1,k} \cup \mathscr{E}_{2,k}$ so that $A'_k + B'_k$ is as large as possible. Say $(A'_k, B'_k) \in \mathscr{E}_{2,k}$. Thus

$$p_k^{A'_k} q_k^{B'_k} = \phi_{p_k,q_k}(F'_k) - \phi_{p_k,q_k}(Y'_k),$$

where

$$F'_k = \mathscr{E}_{1,k}, \quad Y'_k = \mathscr{E}_{2,k} \setminus \{(A'_k, B'_k)\}.$$

Dividing both sides by $p_k^{A'_k} q_k^{B'_k}$, we have

$$1 = \phi_{p_k,q_k}(F'_k - (A'_k, B'_k)) - \phi_{p_k,q_k}(Y'_k - (A'_k, B'_k)), \tag{2.2}$$

where the sets $\mathscr{Q}$, $F'_k - (A'_k, B'_k)$ and $Y'_k - (A'_k, B'_k)$ are disjoint. Rewrite (2.2) as

$$1 = \phi_{p,q}(F_k - (A_k, B_k)) - \phi_{p,q}(Y_k - (A_k, B_k)), \tag{2.3}$$

where

$$F_k = \{(aU_{k-1}, bV_{k-1}) \mid (a, b) \in F'_k\}, \qquad Y_k = \{(aU_{k-1}, bV_{k-1}) \mid (a, b) \in Y'_k\},$$
$$A_k = A'_k U_{k-1}, \qquad B_k = B'_k V_{k-1}.$$

Let $(a_k, b_k) \in F_k \cup Y_k \cup \{(A_k, B_k)\}$. Then there is $(a'_k, b'_k) \in \mathbb{Z}^2$ with

$$1 \le a'_k \le 4\log_2 q_k, \quad 1 \le b'_k \le 4\log_2 p_k, \quad a_k = a'_k U_{k-1}, \quad b_k = b'_k V_{k-1}.$$

It is clear that

$$\begin{aligned} U_{k-1} \le a_k = a'_k U_{k-1} &\le (4\log_2 q_k) U_{k-1} \\ &= (4\log_2 q) U_{k-1} V_{k-1} \le 4\lceil \log_2 q \rceil U_{k-1} V_{k-1} = U_k \end{aligned} \tag{2.4}$$

and

$$\begin{aligned} V_{k-1} \le b_k = b'_k V_{k-1} &\le (4\log_2 p_k) V_{k-1} \\ &= (4\log_2 p) U_{k-1} V_{k-1} \le 4\lceil \log_2 p \rceil U_{k-1} V_{k-1} = V_k. \end{aligned} \tag{2.5}$$

In particular,

$$U_{k-1} \le A_k \le U_k, \quad V_{k-1} \le B_k \le V_k. \tag{2.6}$$

Now we show that the sets $\mathscr{Q}$, $F_k - (A_k, B_k)$, $Y_k - (A_k, B_k)$ $(k = 1, 2, \dots)$ are all disjoint.

By the definition of (A_k, B_k), for any

$$(a, b) \in (F_k - (A_k, B_k)) \cup (Y_k - (A_k, B_k)),$$

we have $a < 0$ or $b < 0$. So $(a, b) \notin \mathscr{Q}$. By the constructions of F_k and Y_k, the sets $F_k - (A_k, B_k)$ and $Y_k - (A_k, B_k)$ are disjoint.

We now prove that, for any $k < l$, the sets $(F_k - (A_k, B_k)) \cup (Y_k - (A_k, B_k))$ and $(F_l - (A_l, B_l)) \cup (Y_l - (A_l, B_l))$ are disjoint. Assume the contrary. Then there exist

$$(a_k, b_k) \in F_k \cup Y_k, \quad (a_l, b_l) \in F_l \cup Y_l$$

such that

$$a_k - A_k = a_l - A_l, \quad b_k - B_k = b_l - B_l. \tag{2.7}$$

Since $(a_k, b_k) \in F_k \cup Y_k$, it follows from (2.4)–(2.6) that

$$|a_k - A_k| \le U_k - U_{k-1} < U_k \le U_{l-1}, \tag{2.8}$$

$$|b_k - B_k| \le V_k - V_{k-1} < V_k \le V_{l-1}. \tag{2.9}$$

Noting that $(a_l, b_l) \in F_l \cup Y_l$, by the definitions of F_l and Y_l, we know that $U_{l-1} \mid a_l$ and $V_{l-1} \mid b_l$. By the definitions of A_l and B_l, we have $U_{l-1} \mid A_l$ and $V_{l-1} \mid B_l$. So

$$U_{l-1} \mid a_l - A_l, \quad V_{l-1} \mid b_l - B_l.$$

It follows from (2.7) that

$$U_{l-1} \mid a_k - A_k, \quad V_{l-1} \mid b_k - B_k.$$

By (2.8) and (2.9), we have $a_k - A_k = 0$ and $b_k - B_k = 0$, contrary to $(a_k, b_k) \ne (A_k, B_k)$.

Thus, we have proved that the sets $\mathscr{Q}$, $F_k - (A_k, B_k)$, $Y_k - (A_k, B_k)$ $(k = 1, 2, \dots)$ are all disjoint.

Let

$$r = \sum_{i=1}^{x_0} \phi_{p,q}(Y_i - (A_i, B_i)).$$

We rewrite r as

$$r = \frac{R'}{p^A q^B},$$

a rational number with denominator $p^A q^B$. It follows from (2.6) that $A_1 \le \dots \le A_{x_0}$ and $B_1 \le \dots \le B_{x_0}$, hence

$$A = \max_{1 \le i \le x_0} A_i = A_{x_0}, \qquad B = \max_{1 \le i \le x_0} B_i = B_{x_0}.$$

Since $V_{x_0 - 1} \mid B_{x_0}$ and $x_0 > 1$, we infer from the definition of V_n that $4 \mid B$.

For any nonnegative integer m, by Corollary 2.4, there exist integers

$$y \in P(\{p^a q^{2b} \mid a \ge 0,\ 0 \le b \le p\}), \quad 0 \le z < x_0$$

such that $m = y + z = \phi(\mathscr{E}) + z$, where

$$\mathscr{E} \subseteq \{(a, 2b) \mid a \ge 0,\ 0 \le b \le p\}. \tag{2.10}$$

By (2.3),

$$z = \sum_{i=1}^{z} [\phi_{p,q}(F_i - (A_i, B_i)) - \phi_{p,q}(Y_i - (A_i, B_i))].$$

Thus

$$\begin{aligned} m+r &= \phi(\mathscr{E}) + z + \sum_{i=1}^{x_0} \phi_{p,q}(Y_i - (A_i, B_i)) \\ &= \phi(\mathscr{E}) + \sum_{i=1}^{z} [\phi_{p,q}(F_i - (A_i, B_i)) - \phi_{p,q}(Y_i - (A_i, B_i))] \\ &\quad + \sum_{i=1}^{x_0} \phi_{p,q}(Y_i - (A_i, B_i)) \\ &= \phi(\mathscr{E}) + \sum_{i=1}^{z} \phi_{p,q}(F_i - (A_i, B_i)) + \sum_{i=z+1}^{x_0} \phi_{p,q}(Y_i - (A_i, B_i)) \\ &= \phi(H - (A, B)), \end{aligned}$$

where

$$H = (\mathscr{E} + (A,B)) \cup \bigcup_{i=1}^{z} (F_i + (A - A_i, B - B_i)) \cup \bigcup_{i=z+1}^{x_0} (Y_i + (A - A_i, B - B_i))$$

is a union of disjoint subsets of $\mathscr{Q}$. It follows from (2.4)–(2.6) and (2.10) that

$$H \subseteq \{p^a q^{2b} \mid a \geq 1,\ 1 \leq b \leq 2V_{x_0} + p\}.$$

Hence, for any nonnegative integer m, we infer from

$$m + \frac{R'}{p^A q^B} = m + r = \phi(H - (A, B))$$

that

$$mp^A q^B + R' = \phi(H) \in P(\{p^a q^{2b} \mid a \geq 1,\ 1 \leq b \leq 2V_{x_0} + p\}). \tag{2.11}$$

By (2.6),

$$A = A_{x_0} \leq U_{x_0}, \quad B = B_{x_0} \leq V_{x_0}.$$

Replacing m by $mp^{U_{x_0} - A + 1} q^{V_{x_0} - B + 1}$ in (2.11), we obtain

$$mp^{U_{x_0}+1} q^{V_{x_0}+1} + R' \in P(\{p^a q^{2b} \mid a \geq 1,\ 1 \leq b \leq 2V_{x_0} + p\}), \quad m = 0, 1, \dots.$$

Since each integer in $\{p^a q^{2b} \mid a \geq 1,\ 1 \leq b \leq 2V_{x_0} + p\}$ is divisible by pq, it follows that $pq \mid R'$. Let $R = R'/(pq)$. Then

$$mp^{U_{x_0}} q^{V_{x_0}} + R \in P(\{p^a q^{2b+1} \mid a \geq 0,\ 0 \leq b \leq 2V_{x_0} + p - 1\}), \quad m = 0, 1, \dots,$$

that is,

$$mp^U q^V + R \in P(\{p^a q^{2b+1} \mid a \geq 0,\ 0 \leq b \leq 2V + p - 1\}), \quad m = 0, 1, \dots.$$

Now we estimate R from above. Since

$$r = \phi_{p,q}(Y_1 - (A_1, B_1)) + \phi_{p,q}(Y_2 - (A_2, B_2)) + \dots + \phi_{p,q}(Y_{x_0} - (A_{x_0}, B_{x_0})),$$

we can assume that

$$r = \frac{T_1}{p^{A_1} q^{B_1}} + \frac{T_2}{p^{A_2} q^{B_2}} + \cdots + \frac{T_{x_0}}{p^{A_{x_0}} q^{B_{x_0}}}. \tag{2.12}$$

For any integer k with $1 \le k \le x_0$, by the proof of Lemma 2.1 and the definitions of p_k and q_k we have

$$\begin{aligned} T_k &\le 2^{8 \log_2 p_k \log_2 q_k + 3} = 2^{8U_{k-1}V_{k-1} \log_2 p \log_2 q + 3} \\ &\le 2^{11U_{k-1}V_{k-1}\lceil \log_2 p\rceil \lceil \log_2 q\rceil} \le 2^{cU_k/2}. \end{aligned}$$

It follows from $r = R'/(p^A q^B)$, $R = R'/(pq)$ and (2.12) that

$$\begin{aligned} R \le R' &= \left(\frac{T_1}{p^{A_1} q^{B_1}} + \frac{T_2}{p^{A_2} q^{B_2}} + \cdots + \frac{T_{x_0}}{p^{A_{x_0}} q^{B_{x_0}}} \right) p^A q^B \\ &\le \sum_{k=1}^{x_0} 2^{cU_k/2} p^A q^B \le x_0 2^{cU_{x_0}/2} p^A q^B \\ &\le 2^{cU_{x_0}} p^{U_{x_0}} q^{V_{x_0}} = W p^U q^V. \end{aligned}$$

This completes the proof of Lemma 2.6. ■

Lemma 2.7 ([7, Lemma 2.1]). *Let m be a positive integer and A a multiset of m integers coprime to m. Then $P(A)$ contains every residue modulo m.*

Proof of Theorem 1.1. Let the notation be as in Lemma 2.6. Let n be an integer. Similar to the proof of [4, Lemma 2.7], let $m = p$ and $A = \{q^2, q^4, \ldots, q^{2p}\}$. Then by Lemma 2.7, $P(A)$ contains every residue modulo p. Hence

$$n - R \equiv q^{2l_{1,1}} + q^{2l_{1,2}} + \cdots + q^{2l_{1,s_1}} \pmod p, \quad 1 \le l_{1,1} < l_{1,2} < \cdots < l_{1,s_1} \le p.$$

If $n - R \equiv 0 \pmod p$, then $s_1 = 0$.

Write

$$n - R = \sum_{i=1}^{s_1} q^{2l_{1,i}} + M_1 p.$$

Continuing, we have

$$n - R = \sum_{i=1}^{s_1} q^{2l_{1,i}} + \sum_{i=1}^{s_2} q^{2l_{2,i}} p + \cdots + \sum_{i=1}^{s_U} q^{2l_{U,i}} p^{U-1} + M_U p^U,$$

where

$$1 \le l_{j,1} < l_{j,2} < \cdots < l_{j,s_j} \le p, \quad 1 \le j \le U.$$

Let $m = q^2$ and $A = \{p, p^2, \ldots, p^{q^2}\}$. Then by Lemma 2.7, $P(A)$ contains every residue modulo q^2. Similarly, noting that $2 \mid V$, we have

$$M_U = \sum_{i=1}^{t_1} p^{w_{1,i}} + \sum_{i=1}^{t_2} p^{w_{2,i}} q^2 + \cdots + \sum_{i=1}^{t_{V/2}} p^{w_{V/2,i}} (q^2)^{V/2-1} + N_{V/2} (q^2)^{V/2},$$

where

$$1 \le w_{j,1} < w_{j,2} < \cdots < w_{j,s_j} \le q^2, \quad 1 \le j \le V/2.$$

Let

$$\begin{aligned} y = {} & \sum_{i=1}^{s_1} q^{2l_{1,i}} + \sum_{i=1}^{s_2} q^{2l_{2,i}} p + \cdots + \sum_{i=1}^{s_U} q^{2l_{U,i}} p^{U-1} \\ & + p^U \Big(\sum_{i=1}^{t_1} p^{w_{1,i}} + \sum_{i=1}^{t_2} p^{w_{2,i}} q^2 + \cdots + \sum_{i=1}^{t_{V/2}} p^{w_{V/2,i}} (q^2)^{V/2-1} \Big). \end{aligned}$$

Then $n - R = y + N_{V/2}(q^2)^{V/2} p^U = y + N_{V/2} p^U q^V$ and

$$y \in P(\{p^a q^{2b} \mid a \ge 0,\, 0 \le b \le V/2 + p - 1,\, a + b > 0\}). \tag{2.13}$$

It is clear that

$$\begin{aligned} y & \le (q^2 + q^4 + \cdots + q^{2p})(1 + p + \cdots + p^{U-1}) \\ & \quad + (p + p^2 + \cdots + p^{q^2})(1 + q^2 + q^4 + \cdots + q^V) p^U \\ & \le \frac{1}{2} q^{2p+1} p^U + \frac{1}{(p-1)(q^2-1)} p^{U+q^2+1} q^{V+2} \le p^{U+q^2+1} q^{V+2}. \end{aligned}$$

Thus

$$y + R \le p^{U+q^2+1} q^{V+2} + W p^U q^V < W p^{U+q^2+1} q^{V+2} = 2^{cU} p^{U+q^2+1} q^{V+2}.$$

If $n \ge 2^{cU} p^{U+q^2+1} q^{V+2}$, then $N_{V/2} > 0$. By Lemma 2.6 we have

$$N_{V/2} p^U q^V + R \in P(\{p^a q^{2b+1} \mid a \ge 0,\, 0 \le b \le 2V + p - 1\}). \tag{2.14}$$

It follows from (2.13) and (2.14) that

$$n \in P(\{p^a q^b \mid a \ge 0,\, 0 \le b \le 4V + 2p - 1,\, a + b > 0\}).$$

Let

$$B = 2^{cU} p^{U+q^2+1} q^{V+2}, \quad K = 4V + 2p - 1 = 4V_{x_0} + 2p - 1.$$

Then every integer $n \ge B$ belongs to

$$P(\{p^a q^b \mid a \ge 0,\, 0 \le b \le K,\, a + b > 0\}).$$

Finally, we estimate B and K. We have

$$\begin{aligned} B &= 2^{cU} p^{U+q^2+1} q^{V+2} = 2^{cU_{x_0}} p^{U_{x_0}+q^2+1} q^{V_{x_0}+2}, \\ K &= 4V + 2p - 1 = 4V_{x_0} + 2p - 1 < 5V_{x_0}, \end{aligned}$$

where $x_0 = q^{2p} - q^{2p-2}$ and

$$U_{x_0} = c_1^{-1} c^{2^{x_0 - 1}}, \quad V_{x_0} = c_2^{-1} c^{2^{x_0 - 1}},$$
$$c = 32\lceil \log_2 p \rceil \lceil \log_2 q \rceil, \quad c_1 = 4\lceil \log_2 p \rceil, \quad c_2 = 4\lceil \log_2 q \rceil.$$

Hence

$$\log_2 B \le cU_{x_0} + (U_{x_0} + q^2 + 1)\log_2 p + (V_{x_0} + 2)\log_2 q$$
$$\le cU_{x_0} + 3U_{x_0}\log_2 p + 2V_{x_0}\log_2 q \le \tfrac{1}{2}cU_{x_0}V_{x_0} = c^{2^{x_0}}.$$

Thus, noting that $pq \ge 6$, we have

$$\log_2\log_2 B \le 2^{x_0}\log_2 c = 2^{x_0}\log_2(32\lceil\log_2 p\rceil\lceil\log_2 q\rceil) \le 2^{x_0}(5+pq) < 2^{x_0}2^{pq},$$

that is,

$$\log_2\log_2\log_2 B < x_0 + pq = q^{2p} - q^{2p-2} + pq < q^{2p}.$$

For K, we have

$$\log_2 K \le \log_2(5V_{x_0}) < 1 + \log_2 c^{2^{x_0-1}} < (\log_2 c)2^{x_0} < 2^{x_0}2^{pq}.$$

Thus

$$\log_2\log_2 K \le x_0 + pq = q^{2p} - q^{2p-2} + pq < q^{2p}.$$

This completes the proof of Theorem 1.1. ∎

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