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Attainable numbers and the Lagrange spectrum

by

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1. Introduction. In this paper we study the properties of the *Lagrange spectrum* $\mathbb{L}$ which is defined as the set of all values of *Lagrange constants*

$$\mu(\alpha)=\Big(\liminf_{p\in\mathbb{Z},\,q\in\mathbb{N}}|q(q\alpha-p)|\Big)^{-1}$$

for irrational α.

Let A be a doubly infinite sequence of positive integers $\ldots a_{-1},a_0,a_1,\ldots.$ For an arbitrary integer i define

$$\lambda_i(A)=[a_i;a_{i-1},\ldots]+[0;a_{i+1},a_{i+2},\ldots]$$

and set

$$L(A)=\limsup_{i\to\infty}\lambda_i(A),\quad M(A)=\sup_i\lambda_i(A).$$

It is well known that the Lagrange spectrum can also be defined as the set of $L(A)$ for all sequences A of positive integers. The sequence A is called *eventually periodic* if there exists an integer n such that both sequences $a_n,a_{n+1},\ldots$ and $a_{-n},a_{-n-1},\ldots$ are periodic. Here we do not suppose that the two periods coincide.

The Lagrange spectrum is a closed set [3] with minimal point $\sqrt{5}$. All the numbers of $\mathbb{L}$ which are less than 3 form a discrete set. They were described by Markov. Markov's results are discussed in detail in [4] and [2] (see also a recent paper by Bombieri [1]). The complement of $\mathbb{L}$ is a countable union of *maximal gaps* of the spectrum. These maximal gaps are open intervals which contain no elements of the Lagrange spectrum, and their endpoints are in $\mathbb{L}$. In this paper we are mostly interested in the properties of left endpoints of maximal gaps in the Lagrange spectrum. In [5] Dietz stated

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that if (a, b) is a maximal gap in the Lagrange spectrum then there exists an eventually periodic doubly infinite sequence A such that

$$a = M(A) = \lambda_0(A).$$

However, it was pointed by Cusick and Flahive [4, p. 63] that Dietz's proof is incorrect. Here we give a correct proof of Dietz's statement.

Next, we consider the quadratic irrationality

$$\lambda_0 = [3; 3, 3, 2, 1, \overline{1, 2}] + [0; 2, 1, \overline{1, 2}] = \frac{62976 - 1498\sqrt{3}}{16357} \approx 3.6914708.$$

We prove that for any α such that $\mu(\alpha) = \lambda_0$ the inequality

$$\left|\alpha - \frac{p}{q}\right| \le \frac{1}{\mu(\alpha)q^2} \tag{1.1}$$

does not have infinitely many solutions. If an irrational α is such that the inequality (1.1) has infinitely many solutions, then, following Malyshev [8], we call α *attainable*. Here we should note that for any $\lambda < 3$ in $\mathbb{L}$ there exists an attainable α such that $\mu(\alpha) = \lambda$. Malyshev [8] claimed that for any $\lambda \in \mathbb{L}$ there exists an irrational α such that $\mu(\alpha) = \lambda$ and α is attainable. Thus, our result gives a counterexample to Malyshev's statement.

Let us discuss an equivalent definition of attainable numbers. If (1.1) holds then the fraction p/q should be some convergent of α as $\mu(\alpha) \ge \sqrt{5}$. Let α have the continued fraction expansion

$$\alpha = [a_0; a_1, a_2, \ldots].$$

For any positive integer i define

$$\lambda_i(\alpha) = [a_i; a_{i+1}, a_{i+2}, \ldots] + [a_i; a_{i-1}, a_{i-2}, \ldots, a_1].$$

Denote by p_n/q_n the nth convergent of α. As

$$\left|\alpha - \frac{p_n}{q_n}\right| = \frac{1}{\lambda_{n+1}(\alpha)q_n^2}, \tag{1.2}$$

one can easily see that

$$\limsup \lambda_i(\alpha) = \mu(\alpha). \tag{1.3}$$

From (1.1)–(1.3) one can easily deduce that α is attainable if and only if $\lambda_i(\alpha) \ge \mu(\alpha)$ for infinitely many i.

2. Main results.

In our first theorem we establish a counterexample to Malyshev's statement.

Theorem 1. *The quadratic irrationality*

$$\lambda_0 = [3; 3, 3, 2, 1, \overline{1, 2}] + [0; 2, 1, \overline{1, 2}]$$

belongs to $\mathbb{L}$, *but if* α *is such that* $\mu(\alpha) = \lambda_0$ *then* α *is not attainable.*

It is known that λ_0 is the left endpoint of a gap in the Markov spectrum $\mathbb{M}$ (see [4, Table 2, p. 62]). As $\mathbb{L} \subset \mathbb{M}$, Theorem 1 implies that λ_0 is the left endpoint of a gap in the Lagrange spectrum. Our next theorem shows that each counterexample to Malyshev's statement is also the left endpoint of some gap in the Lagrange spectrum.

THEOREM 2. *If $\lambda \in \mathbb{L}$ is not the left endpoint of any maximal gap in the Lagrange spectrum then there exists an attainable α such that $\mu(\alpha) = \lambda$.*

The next theorem was formulated by Dietz [5].

THEOREM 3. *If (a, b) is a maximal gap in $\mathbb{L}$ then there exists an eventually periodic doubly infinite sequence A such that $a = \lambda_0(A) = M(A)$. Thus a can be represented by a sum of two quadratic irrationalities.*

As we mentioned in the previous section, Cusick and Flahive found an error in Dietz's proof. In this paper we give a new proof of this statement.

We prove Theorem 1 in Sections 3–4, Theorem 3 in Section 5, and Theorem 2 in Section 6.

3. The Lagrange spectrum contains λ_0.

Define a doubly infinite sequence $A_0 = \ldots, a_{-1}, a_0, a_1, \ldots$ by

$$A_0 = (\overline{2,1}, 1, 2, 3, 3^*, 3, 2, 1, \overline{1,2}),$$

where * indicates the element a_0.

LEMMA 3.1.

$$\sup_{i \in \mathbb{Z}} \lambda_i(A_0) = \lambda_1(A_0) = \lambda_{-1}(A_0) = \lambda_0.$$

Proof. One can easily see that $\lambda_1(A) = \lambda_{-1}(A) = \lambda_0$. Also

$$\lambda_0(A) = 3 + 2[0; 3, 2, 1, \overline{1,2}] = \frac{246 + \sqrt{3}}{69} \approx 3.59032 < \lambda_1(A_0).$$

Now let $i \neq -1, 0$ or 1. Then

$$(3.1) \qquad [0; a_{i+1}, a_{i+2}, \ldots] \leq [0; \overline{1,3}], \quad [a_i; a_{i-1}, a_{i-2}, \ldots] \leq [2; \overline{1,3}],$$

because $a_k \leq 3$ for any integer k. Inequalities (3.1) give

$$\lambda_i(A_0) \leq 2 + 2[0; \overline{1,3}] = \sqrt{21} - 1 < 3.6 < \lambda_1(A_0).$$

Thus the supremum of $\lambda_i(A_0)$ is reached only at $i = -1$ and $i = 1$. ∎

LEMMA 3.2. *We have $\lambda_0 \in \mathbb{L}$.*

Proof. For any positive integer n let C^n denote the finite sequence

$$C^n = ((2,1)_n, 1, 2, 3, 3, 3, 2, 1, (1,2)_n),$$

where the subscript n on a sequence means that the sequence is repeated n times. Define an irrational number α_0 by

$$\alpha_0 = [0; C^1, C^2, \ldots].$$

We show that

$$\limsup \lambda_i(\alpha_0) = \lambda_0. \tag{3.2}$$

Denote by i_n the index of the last element in C^n which is equal to 3. Then

$$\lambda_{i_n}(\alpha_0) = \lambda_{i_n-2}(\alpha_0) = [3; 3, 3, 2, 1, (1, 2)_n, \ldots] + [0; 2, 1, (1, 2)_n, \ldots].$$

Therefore,

$$\lim_{n\to\infty} \lambda_{i_n}(\alpha_0) = \lambda_0 = \lim_{n\to\infty} \lambda_{i_n-2}(\alpha_0).$$

On the other hand, if i is not equal to i_n or i_{n-2} for any n then $\lambda_i(\alpha_0) < 3.6$, as one can easily deduce from the proof of Lemma 3.1. Thus (3.2) holds, and the proof is complete. ■

4. No α such that $\mu(\alpha) = \lambda_0$ is attainable. We will frequently use the following classical lemma.

LEMMA 4.1 ([4]). *Suppose $\alpha = [a_0; a_1, \ldots, a_n, b_1, \ldots]$ and $\beta = [a_0; a_1, \ldots, a_n, c_1, \ldots]$, where $n \geq 0$, a_0 is an integer, and $a_1, \ldots, a_n, b_1, b_2, \ldots, c_1, c_2, \ldots$ are positive integers with $b_1 \neq c_1$. Then for n odd, $\alpha > \beta$ if and only if $b_1 > c_1$; for n even, $\alpha > \beta$ if and only if $b_1 < c_1$. Also,*

$$|\alpha - \beta| < 2^{-(n-1)}.$$

Denote $\varepsilon_n = 2^{-(n-1)}$.

LEMMA 4.2. *If*

$$\limsup \lambda_i(\alpha) = \lambda_0 \tag{4.1}$$

for some $\alpha = [a_0, a_1, \ldots]$, then no pattern from the list

$$(1, 3), (3, 1), (2, 2, 3), (3, 2, 2), (3, 2, 3), (1, 2, 3, 2, 1) \tag{4.2}$$

occurs in the sequence $a_1, a_2, \ldots$ infinitely many times. Also, $a_i \leq 3$ for almost all $i \in \mathbb{N}$.

Proof. If $a_i \geq 4$ for infinitely many i's, then

$$\limsup \lambda_i(\alpha) \geq 4,$$

contradicting (4.1).

It follows from the definition of λ_i that

$$\limsup \lambda_i([0; a_1, a_2, \ldots, a_n, \ldots]) = \limsup \lambda_i([0; a_n, a_{n+1} \ldots])$$

for any positive integer n. Thus, without loss of generality, one can assume that $a_i \leq 3$ for any $i \in \mathbb{N}$.

Denote the infinite sequence $a_1, a_2, \ldots$ by A. If the pattern $(3,1)$ occurs in A infinitely many times, choose an arbitrary n such that $a_n = 3,\ a_{n+1} = 1$. Then

$$\lambda_n(\alpha) = [3; 1, a_{n+2}, \ldots] + [0; a_{n-1}, \ldots, a_1].$$

As $a_i \le 3$ for any natural i, we use Lemma 4.1 to estimate the first term:

$$[3; 1, a_{n+2}, \ldots] \ge [3; 1, \overline{1, 3}].$$

The second term does not exceed the finite continued fraction having length $n-1$ which is equal to $[0; 3, 1, 3, 1, \ldots]$. The least partial quotient equals 3 if n is even, and equals 1 otherwise. Denote the finite continued fraction $[0; 3, 1, 3, 1, \ldots]$ by α_n. As α_n is a convergent of $[0; \overline{3, 1}]$, we have

$$\alpha_n \ge [0; \overline{3, 1}] - \varepsilon_n.$$

Since $\lim_{n\to\infty} \varepsilon_n = 0$, we obtain

$$\lambda_n(\alpha) \ge [3; 1, \overline{1, 3}] + [0; \overline{3, 1}] - \varepsilon_n = \frac{39 + 4\sqrt{21}}{15} - \varepsilon_n \approx 3.82202 - \varepsilon_n > \lambda_0 + 10^{-4}$$

for sufficiently large n, contrary to (4.1).

Using the same argument, one can easily deduce that the inverse pattern $(1,3)$ does not occur in A infinitely many times. Without loss of generality, we can say that the patterns $(3,1)$ and $(1,3)$ do not occur in A.

If $(3,2,2)$ occurs in A infinitely many times, let n be such that $a_n = 3$, $a_{n+1} = 2$, $a_{n+2} = 2$. Then

$$\lambda_n(\alpha) = [3; 2, 2, a_{n+3}, \ldots] + [0; a_{n-1}, \ldots, a_1].$$

Since all $a_i \le 3$ and the patterns $(3,1)$ and $(1,3)$ do not occur in A, we obtain

$$[3; 2, 2, a_{n+3}, \ldots] \ge [3; 2, 2, \overline{3, 2}], \quad [0; a_{n-1}, \ldots, a_1] \ge [0; \overline{3, 2}] - \varepsilon_n.$$

Thus,

$$\begin{aligned} \lambda_n(\alpha) &\ge [3; 2, 2, \overline{3, 2}] + [0; \overline{3, 2}] - \varepsilon_n = \frac{39 + 10\sqrt{15}}{21} - \varepsilon_n \approx 3.70142 - \varepsilon_n \\ &> \lambda_0 + 10^{-4} \end{aligned}$$

for sufficiently large n, contradicting (4.1). Using the same argument, one can easily deduce that the inverse pattern $(2,2,3)$ does not occur in A infinitely many times.

If $(3,2,3)$ occurs in A infinitely many times, pick n such that $a_n = 3$, $a_{n+1} = 2$, $a_{n+2} = 3$. Then

$$\begin{aligned} \lambda_n(\alpha) &= [3; 2, 3, a_{n+3}, \ldots] + [0; a_{n-1}, \ldots, a_1] \\ &> [3; 2, 2, a_{n+3}, \ldots] + [0; a_{n-1}, \ldots, a_1]. \end{aligned}$$

Now we use the estimates from the previous paragraph.

Finally, assume that $(1,2,3,2,1)$ occurs in A infinitely many times. Choose n such that $a_{n-2}=1, a_{n-1}=2, a_n=3, a_{n+1}=2, a_{n+2}=1$. Then

$$\lambda_n(\alpha)=[3;2,1,a_{n+3},\ldots]+[0;2,1,a_{n-3},\ldots,a_1].$$

As $(3,1)$ and $(1,3)$ do not occur in A, and of course no element in A exceeds 3, we obtain

$$[3;2,1,a_{n+3},\ldots]\geq[3;2,1,\overline{2,1}],\quad [0;2,1,a_{n-3},\ldots,a_1]\geq[0;\overline{2,1}]-\varepsilon_n.$$

Thus,

$$\lambda_{i_n}(\alpha)\geq[3;2,1,\overline{2,1}]+[0;\overline{2,1}]-\varepsilon_n=2+\sqrt{3}-\varepsilon_n\approx 3.73205-\varepsilon_n>\lambda_0+10^{-4}$$

for sufficiently large n, contrary to (4.1). ∎

LEMMA 4.3. *If an irrational* $\alpha=[0,a_1,a_2,\ldots]$ *is such that*

$$\limsup\lambda_i(\alpha)=\lambda_0,$$

then there exist $N\in\mathbb{N}$ *such that for any integer* $n>N$ *the inequality*

$$\lambda_n(\alpha)>3.691 \tag{4.3}$$

holds only when a_n *is the first or the last* 3 *from the pattern* $(1,2,3,3,3,2,1)$ *in the infinite sequence* $A=(a_1,a_2,\ldots)$.

Proof. Lemma 4.2 implies that there exists an integer $N>100$ such that in the infinite sequence $a_{N-10},a_{N-9},\ldots$ no element exceeds 3 and no pattern from the list (4.2) occurs. We will call the patterns from the list (4.2) *prohibited*. Now let $n>N$ be such that (4.3) holds. One can easily see that $\varepsilon_n<10^{-25}$.

Of course, $a_n=3$, because if $a_n\leq 2$, then

$$\lambda_n(\alpha)\leq 2+2[0;\overline{1,2}]+\varepsilon_n=2\sqrt{3}+\varepsilon_n\approx 3.464101<3.691.$$

As $a_n=3$, Lemma 4.2 implies that only two cases are possible: either $a_{n-1}=a_{n+1}=3$, or of the two numbers a_{n-1} and a_{n+1}, one is 2 and the other 3. Consider the first case

$$\lambda_n(\alpha)=3+[0;3,a_{n-2},\ldots,a_1]+[0;3,a_{n+2},\ldots]. \tag{4.4}$$

Since the sequence A does not contain the patterns $(1,3)$ or $(3,1)$, we obtain an upper estimate of (4.4), using Lemma 4.1:

$$[0;3,a_{n+2},\ldots]\leq[0;3,\overline{3,2}],\quad [0;3,a_{n-2},\ldots,a_1]\leq[0;3,\overline{3,2}]+\varepsilon_n.$$

Thus

$$\begin{aligned}\lambda_n(\alpha)&=3+[0;3,a_{n-2},\ldots,a_1]+[0;3,a_{n+2},\ldots]\leq 3+2[0;3,\overline{3,2}]+\varepsilon\\&=\frac{33-2\sqrt{15}}{7}+\varepsilon\approx 3.60772<3.691,\end{aligned}$$

contradicting (4.3). Thus, either $a_{n-1}=3,\ a_{n+1}=2$, or $a_{n-1}=2,\ a_{n+1}=3$. We consider the first case only; the second can be treated in exactly the same way using symmetry. As $(3,2,3)$ and $(3,2,2)$ are prohibited, we have $a_{n+2} = 1$. As $(1,3)$ is prohibited, a_{n-2} equals 2 or 3. If $a_{n-2} = 2$, then $a_{n-3} = 1$ because $(3,2,3)$ and $(3,2,2)$ are prohibited. We have

$$\lambda_n(\alpha) = [3;3,2,1,a_{n-2},\ldots,a_1] + [0;2,1,a_{n+2},\ldots].$$

Since the patterns $(1,3)$ and $(3,1)$ are prohibited, we estimate the first term as follows:

$$[3;3,2,1,a_{n-2},\ldots,a_1] \le [3;3,2,1,\overline{2,1}] + \varepsilon_n.$$

The second term is estimated in a similar way:

$$[0;2,1,a_{i_k+2},\ldots] \le [0;2,1,\overline{1,2}]. \tag{4.5}$$

Eventually

$$\begin{aligned}\lambda_n(\alpha) &= 3 + [0;3,2,1,a_{n-2},\ldots,a_1] + [0;2,1,a_{n+2},\ldots]\\ &\le [3;3,2,1,\overline{2,1}] + [0;2,1,\overline{1,2}] + \varepsilon_n = 4 - \frac{2\sqrt{3}}{11} + \varepsilon_n \approx 3.68508,\end{aligned}$$

contradicting (4.3). Hence $a_{n-2} = 3$. As the pattern $(1,3)$ is prohibited, a_{n-3} equals 2 or 3. If $a_{n-3} = 3$, then

$$\lambda_n(\alpha) = [3;3,3,3,a_{n-4},\ldots,a_1] + [0;2,1,a_{n+2},\ldots].$$

The first term is estimated as follows:

$$[3;3,3,3,a_{n-4},\ldots,a_1] \le [3;3,3,3,3,2,1,\overline{1,2}] + \varepsilon_n.$$

We use the estimate (4.5) of the second term and finally obtain

$$\begin{aligned}\lambda_n(\alpha) &= [3;3,3,3,a_{n-4},\ldots,a_1] + [0;2,1,a_{n+2},\ldots]\\ &\le [3;3,3,3,3,2,1,\overline{1,2}] + [0;2,1,\overline{1,2}] + \varepsilon_n\\ &= \frac{681609 - 16103\sqrt{3}}{177122} + \varepsilon_n \approx 3.69078,\end{aligned}$$

contrary to (4.3). Hence $a_{n-3} = 2$; as the patterns $(2,2,3)$ and $(3,2,3)$ are prohibited, we have $a_{n-4} = 1$, and the lemma is proved. ∎

Lemma 4.4. *If an irrational* $\alpha = [0,a_1,\ldots,a_n,\ldots]$ *is such that*

$$\limsup \lambda_i(\alpha) = \lambda_0,$$

then there exist $N \in \mathbb{N}$ *such that* $\lambda_n(\alpha) < \lambda_0$ *for any* $n > N$.

Proof. Suppose otherwise. Then there exists an increasing sequence $k(j)$ such that

$$\lambda_{k(j)}(\alpha) \ge \lambda_0 \quad \forall j \in \mathbb{N}.$$

Lemma 4.3 implies that there exists an integer J such that for each $j > J$, $a_{k(j)}$ is either the first or the last 3 from the pattern $(1,2,3,3,3,2,1)$ in the sequence $A = a_1,a_2,\ldots$. Pick $j_0 > J+2$. Define $n = k(j_0)$. Let a_n be the last

3 from the pattern $(1, 2, 3, 3, 3, 2, 1)$. Then $a_{n-4} = 1, a_{n-3} = 2, a_{n-2} = 3$, $a_{n-1} = 3, a_n = 3, a_{n+1} = 2, a_{n+2} = 1$. In this case we have

$$\lambda_n(\alpha) = [3; 3, 3, 2, 1, \ldots, a_1] + [0; 2, 1, \ldots].$$

We show that the first term satisfies

$$[3; 3, 3, 2, 1, \ldots, a_1] < [3; 3, 3, 2, 1, \overline{1, 2}]. \tag{4.6}$$

Suppose not. We note that $[3; 3, 3, 2, 1, \ldots, a_1]$ is not a convergent of the quadratic irrationality $[3; 3, 3, 2, 1, \overline{1, 2}]$: in the first continued fraction $a_{k(j_0-1)} = 3$, while in the second the corresponding partial quotient equals 2 or 1. Denote the partial quotients of the first continued fraction by $[b_0; b_1, \ldots, b_{n-1}]$ and those of the second by $[c_0; c_n, \ldots, c_{n-1}, \ldots]$. Consider the minimal r such that $b_r \neq c_r$. Of course, $r > 4$. If r is even, using Lemma 4.1 we obtain $b_r > c_r$. Since $c_r = 2$, we have $b_r = 3$. As r is minimal, $c_{r-1} = b_{r-1} = 1$, and we have the prohibited pattern $(1, 3)$ in $[3; 3, 3, 2, 1, \ldots, a_1]$, which contradicts Lemma 4.2. If r is odd, Lemma 4.1 implies that $b_r < c_r$, but $c_r = 1$ and we obtain a contradiction.

A similar argument shows that

$$[0; 2, 1, \ldots, a_1] < [0; 2, 1, \overline{1, 2}]. \tag{4.7}$$

Using (4.6) and (4.7) we obtain

$$\begin{aligned} \lambda_n(\alpha) &= [3; 3, 3, 2, 1, \ldots, a_1] + [0; 2, 1, \ldots] \\ &< [3; 3, 3, 2, 1, \overline{1, 2}] + [0; 2, 1, \overline{1, 2}] = \lambda_0, \end{aligned}$$

a contradiction. If a_n is the first 3 from the pattern $(1, 2, 3, 3, 3, 2, 1)$, a similar argument also leads to a contradiction. ■

Now Theorem 1 immediately follows from Lemmas 3.2 and 4.4.

5. Proof of Theorem 3. Freiman's famous theorem [7] states that $[c_f, \infty) \subset \mathbb{L}$, where

$$c_f = 4 + [0; 3, 2, 1, 1, \overline{3, 1, 3, 1, 2, 1}] + [0; 4, 3, 2, 2, \overline{3, 1, 3, 1, 2, 1}] \approx 4.52783.$$

However, for our purpose it is sufficient to use a weaker result of Freiman [6] and Shecker [9] which states that $(\sqrt{21}, \infty) \subset \mathbb{L}$.

Corollary 5.1. *If a is the left endpoint of a gap of the Lagrange spectrum and $\alpha = [0; a_1, a_2, \ldots]$ is an irrational number such that $\mu(\alpha) = a$, then there exists N such that $a_n \leq 4$ for all $n > N$.*

Let $\alpha = [a_0; a_1, \ldots, a_n, b_1, \ldots]$ and $\beta = [a_0; a_1, \ldots, a_n, c_1, \ldots]$ be real numbers such that $b_1 \neq c_1$. Lemma 4.1 establishes an upper estimate of $|\alpha - \beta|$. If all partial quotients of α and β are bounded, a lower estimate can also be established. Denote $\delta_n = 5^{-2(n+2)}$.

Lemma 5.1. *Suppose α and β satisfy the hypothesis of Lemma* 4.1 *and no partial quotient of α or β exceeds* 4. *Then*

$$\delta_n < |\alpha - \beta| < \varepsilon_n.$$

Proof. Denote by p_n/q_n and p'_n/q'_n the nth convergents of α and β respectively. One can easily see that $q_n < 5^n$ and $q'_n < 5^n$. As $p_n/q_n \neq p'_n/q'_n$, either p_{n+1}/q_{n+1} or p_{n+2}/q_{n+2} lies between α and β. If p_{n+1}/q_{n+1} lies between α and β then so does p'_{n+2}/q'_{n+2}. Since $p_{n+1}/q_{n+1} \neq p'_{n+2}/q'_{n+2}$, we have

$$\left|\frac{p_{n+1}}{q_{n+1}} - \frac{p'_{n+2}}{q'_{n+2}}\right| > \frac{1}{q_{n+1}q'_{n+2}} > \delta_n.$$

The case when p_{n+2}/q_{n+2} lies between α and β is treated in exactly the same way. ■

Lemma 5.2. *Let $n > 0$ be even. Denote $N = N(n) = (2n+1)(4^{2n+1}+1)$. If $a_1, \ldots, a_N$ is an arbitrary integer sequence of length N such that $1 \leq a_i \leq 4$ for all $1 \leq i \leq N$, then there exist integers n_1, n_2 such that $a_{n_1+i} = a_{n_2+i}$ for all $0 \leq i \leq 2n$ and $n_1 \equiv n_2 \pmod 2$.*

Proof. There exist only 4^{2n+1} distinct sequences of length $2n+1$ with elements $1, 2, 3, 4$. Consider the $4^{2n+1}+1$ sequences

$$(a_1, \ldots, a_{2n+1}), (a_{2n+2}, \ldots, a_{4n+2}), \ldots, \\ (a_{(2n+1)4^{2n+1}+1}, \ldots, a_{(2n+1)4^{2n+1}+2n+2}).$$

At least two of them coincide, say $(a_{n_1}, \ldots, a_{n_1+2n})$ and $(a_{n_2}, \ldots, a_{n_2+2n})$. As n is even, we have $n_1 \equiv n_2 \pmod 2$, which finishes the proof. ■

Lemma 5.3. *Let $\gamma = [0; c_1, c_2, \ldots]$ be an irrational number, not a quadratic irrationality. Set $C_N = (c_1, \ldots, c_N)$ and let n_1 and n_2 be as in Lemma* 5.2. *Define two sequences of positive integers by*

$$C_N^1 = (c_1, \ldots, c_{n_1-1}, c_{n_2}, c_{n_2+1}, \ldots, c_N),$$
$$C_N^2 = (c_1, \ldots, c_{n_1-1}, c_{n_1}, \ldots, c_{n_2-1}, c_{n_1}, \ldots, c_{n_2-1}, c_{n_2}, c_{n_2+1}, \ldots, c_N).$$

Also define two irrational numbers by

$$\gamma^1 = [0; c_1, \ldots, c_{n_1-1}, c_{n_2}, c_{n_2+1}, \ldots, c_N, c_{N+1} \ldots] = [0; C_N^1, c_{N+1}, \ldots],$$
$$\gamma^2 = [0; c_1, \ldots, c_{n_1-1}, c_{n_1}, \ldots, c_{n_2-1}, c_{n_1}, \ldots, c_{n_2-1}, c_{n_2}, c_{n_2+1}, \ldots, c_N, \ldots] \\ = [0; C_N^2, c_{N+1}, \ldots].$$

Then $\max(\gamma^1, \gamma^2) > \gamma$.

Proof. Let r be minimal positive such that $c_{n_1+r} \neq c_{n_2+r}$. As γ is not a quadratic irrationality, the r exists. Suppose that $\gamma^1 < \gamma$. The first different partial quotient in these two continued fractions is c_{n_2+r} for γ^1 and c_{n_1+r} for γ. Now compare γ^2 and γ. The first different partial quotient in these two continued fractions is c_{n_1+r} for γ^2 and c_{n_2+r} for γ. As $n_1 \equiv n_2 \pmod 2$, either γ^1 or γ^2 is greater than γ. ■

Denote $\max(\gamma^1, \gamma^2)$ by γ^+. Denote the corresponding sequence of partial quotients (C_N^1 or C_N^2) by C_N^+.

If all partial quotients of γ are bounded, one can easily obtain a quantitative version of Lemma 5.3 using Lemma 5.1.

Corollary 5.2. *Let* $\gamma = [0; c_1, c_2, \ldots]$ *be an irrational number, not a quadratic irrationality. Suppose that no partial quotient of* γ *exceeds* 4. *Define* γ^+ *as above. Then*

$$\gamma^+ - \gamma > \delta_{N+r}.$$

Lemma 5.4. *Let* $\gamma = [0; c_1, c_2, \ldots]$ *and* $\gamma' = [0; c'_1, c'_2, \ldots]$ *be irrational numbers with partial quotients not exceeding* 4. *Suppose that every sequence of partial quotients of length* $2n+1$ *which occurs in the sequence* $(c'_1, c'_2, \ldots)$ *infinitely many times also occurs in* $(c_1, c_2, \ldots)$ *infinitely many times. Then* $\mu(\gamma') < \mu(\gamma) + 2\varepsilon_n$.

Proof. Let $k(j)$ be an increasing sequence of positive integers such that

$$\lim_{j\to\infty} \lambda_{k(j)}(\gamma') = \mu(\gamma').$$

As the partial quotients of γ' are uniformly bounded, there exists a subsequence $k'(j)$ such that $c'_{k'(j_1)+i} = c'_{k'(j_2)+i}$ for any natural j_1, j_2 and $-n \le i \le n$. Denote the sequence

$$c'_{k'(j)-n}, c'_{k'(j)-n+1}, \ldots, c'_{k'(j)}, \ldots, c'_{k'(j)+n}$$

by D. The index is omitted because these sequences coincide for all j's. The pattern D has length $2n+1$ and occurs in the sequence $(c'_1, c'_2, \ldots)$ infinitely many times. Hence there exists an increasing sequence of indices $l(j)$ such that

$$c'_{k'(j)+i} = c_{l(j)+i}$$

for any $j \in \mathbb{N}$ and $-n \le i \le n$. Lemma 5.1 implies that

$$|\lambda_{k'(j)}(\gamma') - \lambda_{l(j)}(\gamma)| < 2\varepsilon_n. \tag{5.1}$$

Note that

$$\lim_{j\to\infty} \lambda_{k'(j)}(\gamma') = \lim_{j\to\infty} \lambda_{k(j)}(\gamma') = \mu(\gamma') \tag{5.2}$$

and

$$\limsup_{j\to\infty} \lambda_{l(j)}(\gamma) \le \mu(\gamma). \tag{5.3}$$

The statement of the lemma now follows from (5.1)–(5.3). ■

Lemma 5.5. *Let* n *be a positive integer. Define* $N = N(n)$ *as in Lemma* 5.2 *and consider an arbitrary sequence* C_N *of length* N. *Let* $\gamma = [0; c_1, c_2, \ldots]$

be an irrational such that the pattern C_N *occurs in the sequence* $C = (c_1, c_2, \ldots)$ *infinitely many times. Define an infinite sequence* C^+ *by replacing all patterns* C_N *in* C *by* C_N^+. *Define* $\gamma' = [0; C^+]$. *Then* $\mu(\gamma') < \mu(\gamma) + 2\varepsilon_n$.

Proof. C_N^+ equals C_N^1 or C_N^2. One can easily see that in both cases the conditions of Lemma 5.4 hold. ∎

Proof of Theorem 3. Let (a, b) be a maximal gap in the Lagrange spectrum. Consider an even n such that $\varepsilon_n < (b-a)/2$ and $N = N(n)$ as defined in Lemma 5.2. Let $\gamma = [0; c_1, \ldots]$ be an irrational such that $\mu(\gamma) = a$. Without loss of generality, we may assume that $c_i \le 3$ for all $i \in \mathbb{N}$. There exists a monotonic sequence $k(j)$ such that

$$\lim_{j\to\infty} \lambda_{k(j)}(\gamma) = a$$

and $k(j+1) - k(j)$ tends to ∞ as $j \to \infty$. For each integer j consider the finite sequence

$$c_{k(j)-N}, c_{k(j)-N+1}, \ldots, c_{k(j)+N}.$$

As all partial quotients of α are bounded, there exists an infinite sequence of indices j_m such that $c_{k(j_{m_1})+i} = c_{k(j_{m_2})+i}$ for all $m_1, m_2 \in \mathbb{N}$ and $-N \le i \le N$. Without loss of generality, we can assume that $c_{k(j_1)+i} = c_{k(j_2)+i}$ for all $j_1, j_2 \in \mathbb{N}$ and $-N \le i \le N$. As the sequence $(c_{k(j)+1}, c_{k(j)+2}, \ldots, c_{k(j)+N})$ is independent of j, we denote it by C_N. Denote $c_{k(j)}$ by c, as $c_{k(j)}$ is also independent of j. Define an infinite continued fraction

$$\eta_j(\gamma) = [c_{k(j)}; c_{k(j)+1}, \ldots, c_{k(j)+N}, c_{k(j)+N+1}, \ldots] = [c_{k(j)}; C_N, c_{k(j)+N+1}, \ldots].$$

Lemma 5.2 implies that there exist positive integers n_1 and n_2 such that

$$c_{k(j)+n_1+i} = c_{k(j)+n_2+i}, \quad \forall j \in \mathbb{N},\ 0 \le i \le n-1.$$

Let $r(j)$ be the minimal positive number such that $c_{k(j)+n_1+r(j)} \ne c_{k(j)+n_2+r(j)}$. Of course, $r(j) \ge n$. Now consider two cases:

(1) There exists a constant M such that $r(j) = M$ for infinitely many j's.
(2) $r(j) \to \infty$ as $j \to \infty$.

5.1. Case 1.

Again, without loss of generality, we may assume that $r(j) = M$ for all $j \in \mathbb{N}$. Denote the segment

$$(c_{k(j)+N+1}, \ldots, c_{k(j+1)})$$

by B_j and the initial segment

$$(c_1, \ldots, c_{k(1)-1})$$

by B_0. With this notation we have

$$\gamma = [0; B_0, c, C_N, B_1, c, C_N, \ldots].$$

Define

$$\gamma' = [0; B_0, c, C_N^+, B_1, c, C_N^+, \ldots].$$

We denote by $l(i+1)$ the index of the element c following B_i in the continued fraction expansion of γ'. Then

$$\lambda_{k(j)}(\gamma) = [c, C_N, B_j, C_N, \ldots] + [0; \overleftarrow{B_{j-1}}, \overleftarrow{C_N}, \ldots, \overleftarrow{B_0}],$$
$$\lambda_{l(j)}(\gamma') = [c, C_N^+, B_j, C_N, \ldots] + [0; \overleftarrow{B_{j-1}}, \overleftarrow{C_N^+}, \ldots, \overleftarrow{B_0}].$$

Here $\overleftarrow{D}$ denotes the finite sequence D in reverse order. Collorary 5.2 implies that $[c, C_N^+, B_j, C_N, \ldots] > [c, C_N, B_j, C_N, \ldots] + \delta_{N+r}$. As the length of B_{j-1} tends to infinity, we have

$$\lim_{j\to\infty} \lambda_{l(j)}(\gamma') \geq \lim_{j\to\infty} \lambda_{k(j)}(\gamma) + \delta_{N+r} \geq a + \delta_{N+r}.$$

Thus, $\mu(\gamma') > a$. On the other hand, Lemma 5.5 implies that $\mu(\gamma') < \mu(\gamma) + 2\varepsilon_n < b$, which contradicts the fact that (a, b) is a gap in the Lagrange spectrum.

5.2. Case 2. Without loss of generality, we may assume that $r(j) > n_2 - n_1$ for any j. Write $r(j) = (n_2 - n_1)q(j) + t(j)$, where $0 \leq t(j) < n_2 - n_1$. Then

$$c_{k(j)+n_1} = c_{k(j)+n_2} = c_{k(j)+2n_2-n_1} = \cdots = c_{k(j)+n_1+q(j)(n_2-n_1)},$$
$$c_{k(j)+n_1+1} = c_{k(j)+n_2+1} = c_{k(j)+2n_2+1-n_1} = \cdots = c_{k(j)+n_1+1+q(j)(n_2-n_1)},$$
$$\ldots$$
$$c_{k(j)+n_2-1} = c_{k(j)+2n_2-n_1-1} = c_{k(j)+3n_2-2n_1-1} = \cdots = c_{k(j)+n_2-1+q(j)(n_2-n_1)}.$$

Thus, the sequence $c_{k(j)+n_1}, c_{k(j)+n_1+1} \ldots, c_{k(j)+n_2-1}$ is repeated $q(j)$ times. Denote this sequence by P. It is independent of j. We have

$$[c; C_N, B_j, C_N, \ldots] = [c; c_{k(j)+1}, \ldots, c_{k(j)+n_1-1}, \underbrace{P, \ldots, P}_{q(j)\text{ times}}, \ldots].$$

Since $r(j)$ tends to infinity, so does $q(j)$, and hence

$$\lim_{j\to\infty} [c; C_N, B_j, C_N, \ldots] = [c; c_{k(j)+1}, \ldots, c_{k(j)+n_1-1}, \overline{P}].$$

The limit is a quadratic irrationality.

In exactly the same way we can prove that $[0; \overleftarrow{B_{j-1}}, \overleftarrow{C_N}, \ldots, \overleftarrow{B_0}]$ tends to a quadratic irrationality. We consider the sequence $C_N = (c_{k(j)-N}, c_{k(j)-N+1}, \ldots, c_{k(j)-1})$ of length N, find integers n_1 and n_2 from Lemma 5.2 and set $r(j)$ to be minimal such that

$$c_{k(j)-n_1-r(j)} \neq c_{k(j)-n_2-r(j)}.$$

The detailed proof is omitted. ∎

6. Proof of Theorem 2. Let Q be the set of quadratic irrationalities. It is well known that $\mathbb{L} = \overline{\mu(Q)}$. Thus, if λ is not the left endpoint of some maximal gap in the Lagrange spectrum, there are two options:

(1) There exists a quadratic irrationality γ such that $\mu(\gamma) = \lambda$.
(2) There exists a sequence of quadratic irrationalities γ_n such that $\lim_{n\to\infty} \mu(\gamma_n) = \lambda$ and the sequence $\mu(\gamma_n)$ decreases.

6.1. Case 1. Without loss of generality, we may assume that γ has purely periodic continued fraction expansion

$$\gamma = [0; \overline{P}],$$

where $P = (c_1, \dots, c_n)$. There exists $1 \le j \le n$ such that

$$\lim_{m\to\infty} \lambda_{j+mn}(\gamma) = \mu(\gamma) = [c_j; c_{j+1}, \dots, c_n, \overline{P}] + [0; c_{j-1}, \dots, c_1, \overline{P}].$$

Note that for any finite sequence R,

$$\mu(\gamma) = \mu([0; R, \overline{P}]).$$

Denote $[0; R, \overline{P}]$ by γ'. Denote the length of R by l. Then

$$\lambda_{j+mn+l}(\gamma') = [c_j; c_{j+1}, \dots, c_n, \overline{P}] + [0; c_{j-1}, \dots, c_1, \underbrace{P, \dots, P}_{m\,\text{times}}, \overleftarrow{R}].$$

Consider an arbitrary integer sequence R such that

$$[0; c_{j-1}, \dots, c_1, \overleftarrow{R}] > [0; c_{j-1}, \dots, c_1, \overline{P}].$$

Then

$$[0; c_{j-1}, \dots, c_1, \underbrace{P, \dots, P}_{2m\,\text{times}}, \overleftarrow{R}] > [0; c_{j-1}, \dots, c_1, \overline{P}]$$

for any natural m. Thus

$$\lambda_{j+2mn+l}(\gamma') > \mu(\gamma) = \mu(\gamma') \quad \forall m \in \mathbb{N}.$$

As

$$\lim_{m\to\infty} \lambda_{j+mn+l}(\gamma') = \mu(\gamma) = \mu(\gamma'),$$

the theorem is proved in the first case.

6.2. Case 2. Denote the period of the continued fraction expansion of γ_n by P_n. Without loss of generality, we assume that no element of any period P_n exceeds $\mu(\gamma) + 1$. As in the previous case, we may assume that all γ_n are purely periodic.

The following lemma easily follows from the general properties of continued fractions.

Lemma 6.1. *Let γ_n be a purely periodic quadratic irrationality with period $P_n = (c_1^n, c_2^n, \dots, c_{l_n}^n)$ of length l_n. Consider $1 \le j \le l_n$ such that*

$$\lim_{m\to\infty} \lambda_{j+ml_n}(\gamma_n) = \lim_{m\to\infty} [c_j^n; c_{j+1}^n, \dots, c_{l_n}^n, \overline{P}] + [0; c_{j-1}^n, \dots, c_1^n, \underbrace{P, \dots, P}_{m\ times}]$$
$$= \mu(\gamma_n).$$

Given any $\varepsilon > 0$ there exists $N = N(\varepsilon)$ such that for any finite or infinite sequences R, S we have

$$[c_j^n; c_{j+1}^n, \dots, c_{l_n}^n, \underbrace{P, \dots, P}_{N\ times}, R] + [0; c_{j-1}^n, \dots, c_1^n, \underbrace{P, \dots, P}_{N\ times}, S] > \mu(\gamma_n) - \varepsilon_n.$$

The following lemma shows that the lengths of the periods P_n tend to infinity.

Lemma 6.2. *Let γ_n be a sequence of purely periodic quadratic irrationalities with period $P_n = (c_1^n, c_2^n, \dots, c_{l_n}^n)$ of length l_n. Let λ be an irrational number, not a quadratic irrationality, such that $\lim_{n\to\infty} \mu(\gamma_n) = \lambda$ and the sequence $\mu(\gamma_n)$ decreases. Then $l_n \to \infty$ as $n \to \infty$. Moreover, there exists an infinite subsequence P'_n of P_n such that the first n elements of the periods $P'_n, P'_{n+1}, \dots$ coincide.*

Proof. Assume that l_n does not tend to infinity. Then there exists an integer M such that there are infinitely many periods of length M. As there are only finitely many sequences of elements of length M with elements bounded by $\mu(\gamma) + 1$, there exists a period P' of length M which occurs in the sequence $P_1, P_2, \dots$ infinitely many times. Let i_n be the indices such that $P_{i_n} = P'$. Then

$$\lambda = \lim_{n\to\infty} \mu(\gamma_n) = \lim_{n\to\infty} \mu([0; \overline{P_n}]) = \lim_{n\to\infty} \mu([0; \overline{P_{i_n}}]) = \mu([0; \overline{P'}]) > \lambda,$$

a contradiction.

Now we prove the "moreover" statement. As all elements of P_n are bounded, for any integer m there exists a sequence $c'_1, \dots, c'_m$ and an infinite set of indices i_n such that

$$c_1^{i_n} = c'_1, \quad c_2^{i_n} = c'_2, \quad \dots, \quad c_m^{i_n} = c'_m$$

for any integer n. Now the construction of the subsequence P'_n is clear; the detailed proof is omitted; ■

Now we are ready to prove Theorem 2. Consider the sequence P'_n from Lemma 6.2 and the corresponding sequence of quadratic irrationalities $\gamma'_n = [0; \overline{P'_n}]$. Define $\varepsilon_n = (\mu(\gamma_n) - \lambda)/3$ and $N(n) = N(\varepsilon)$. Consider the irrational number

$$\gamma' = [0; \underbrace{P'_1, \dots, P'_1}_{2N(1)+1\ \text{times}}, \underbrace{P'_2, \dots, P'_2}_{2N(2)+1\ \text{times}}, \dots, \underbrace{P'_n, \dots, P'_n}_{2N(n)+1\ \text{times}}, \dots].$$

Lemma 6.1 implies that there exist infinitely many j's such that $\lambda_j(\gamma') > \lambda$. Lemma 6.2 implies that

$$\mu(\gamma') = \lambda.$$

The theorem is proved.

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