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General Stieltjes moment problems for rapidly decreasing smooth functions

by

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Abstract. We give (necessary and sufficient) conditions on a sequence $\{f_n\}_{n=0}^{\infty}$ of functions under which every generalized Stieltjes moment problem

$$\int_0^\infty f_n(x)\phi(x)\,dx = a_n, \qquad n \in \mathbb{N},$$

has solutions $\phi \in \mathcal{S}(\mathbb{R})$ with $\operatorname{supp}\phi \subseteq [0,\infty)$. Furthermore, we consider more general problems of this kind for measure or distribution sequences $\{f_n\}_{n=0}^{\infty}$. We also study vector moment problems with values in Fréchet spaces and multidimensional moment problems.

1. Introduction. The problem of moments, as its generalizations, is an important mathematical problem which has attracted much attention for more than a century. It was first raised and solved by Stieltjes for non-negative measures [24, 25]. Boas [1] and Pólya [21] showed later that given an arbitrary sequence $\{a_n\}_{n=0}^{\infty}$ there is always a function of bounded variation F such that

$$\int_0^\infty x^n\,dF(x) = a_n, \qquad n \in \mathbb{N}. \tag{1.1}$$

A major improvement to this result was achieved by Durán [4], who was able to show the existence of regular solutions to (1.1). He proved that every Stieltjes moment problem

$$\int_0^\infty x^n \phi(x)\,dx = a_n, \qquad n \in \mathbb{N}, \tag{1.2}$$

admits a solution $\phi \in \mathcal{S}(0,\infty)$, that is, a solution in the Schwartz class $\mathcal{S}(\mathbb{R})$ of rapidly decreasing smooth functions with $\operatorname{supp}\phi \subseteq [0,\infty)$. The

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corresponding generalization for the strong moment problem has been given in [6]. Extensions of these results to vector-valued Stieltjes moment problems are also well-known [5, 7, 14]. Chung, Chung, and Kim [3], and more recently, Lastra and Sanz [17, 18] have considered moment problems with solutions in Gelfand–Shilov classes.

The problem that we are concerned with in this article is a general Stieltjes moment problem in which we replace the sequence $\{x^n\}_{n=0}^{\infty}$ of monomials in (1.2) by a rather general sequence $\{f_n\}_{n=0}^{\infty}$ of functions (or distributions). We are interested in conditions on $\{f_n\}_{n=0}^{\infty}$ that ensure the existence of solutions $\phi \in \mathcal{S}(0,\infty)$ to every infinite system of equations

$$\int_0^{\infty} f_n(x)\phi(x)\,dx = a_n, \qquad n \in \mathbb{N}, \tag{1.3}$$

for a given arbitrary sequence $\{a_n\}_{n=0}^{\infty}$. In particular, we shall show that the following conditions on the asymptotic behavior of the primitives of the sequence suffice for the solvability of every (1.3):

THEOREM 1.1. *Let $\{f_n\}_{n=0}^{\infty}$ be a sequence of locally integrable functions on $[0,\infty)$ having at most polynomial growth at infinity. Then every generalized moment problem* (1.3) *has solutions ϕ in $\mathcal{S}(0,\infty)$ if the sequence $\{f_n\}_{n=0}^{\infty}$ fulfills:*

(i) $\int_0^x f_n(t)\,dt = o(\int_0^x f_{n+1}(t)\,dt)$ *as $x \to \infty$, $n \in \mathbb{N}$.*
(ii) *For every $\alpha > 0$ there are $N = N_\alpha$ and $\sigma = \sigma_\alpha > 1$ such that*

$$\inf_{a\in[1,\sigma]} \left| \int_0^{ax} f_N(t)\,dt \right| = \Omega(x^\alpha) \qquad \text{as } x \to \infty. \tag{1.4}$$

Theorem 1.1 is a general version of Durán's theorem quoted above. It also covers the case $f_n(x) = x^{\alpha_n}$ or generalized moment problems such as

$$\int_0^{\infty} x^{\alpha_n} \sin\left(\frac{1}{x^\beta}\right)\phi(x)\,dx = a_n, \qquad n \in \mathbb{N} \quad (\beta > 0), \tag{1.5}$$

where $-1 < \Re e\,\alpha_0 < \Re e\,\alpha_1 < \cdots < \Re e\,\alpha_n \to \infty$. The Ω in (1.4) stands for the Hardy–Littlewood symbol, namely, the negation of Landau's little o symbol: $f(x) = \Omega(g(x))$ (as $x \to \infty$) means that there is a constant $C > 0$ such that the inequality $|f(x)| \geq C|g(x)|$ holds infinitely often for arbitrarily large values of x.

The assumptions on the sequence $\{f_n\}_{n=0}^{\infty}$ from Theorem 1.1 can be greatly relaxed. As we show, there is a corresponding result that applies for sequences of functions that might not even be locally Lebesgue integrable. In fact, we study in Section 3 the general moment problem for distribution sequences $\{f_n\}_{n=0}^{\infty}$. We provide in Theorem 3.1 a complete characterization of those distribution sequences for which all moment problems $a_n = \langle f_n, \phi\rangle$,

$n \in \mathbb{N}$, have solutions $\phi \in \mathcal{S}(0,\infty)$. The notion of Cesàro admissibility, introduced in Section 3, plays a key role in our criterion for the existence of solutions in $\mathcal{S}(0,\infty)$ to generalized moment problems. In Section 4, we specialize our results to the case of function sequences.

In Section 5 we consider measure weighted moment problems of the form

$$\int_0^\infty x^{\alpha_n}\phi(x)\,dF(x) = a_n, \qquad n \in \mathbb{N}, \tag{1.6}$$

for a (fixed) non-negative measure dF and a sequence of real exponents $\{\alpha_n\}_{n=0}^\infty$. Theorem 5.2 gives necessary and sufficient conditions for the solvability of every (1.6) in $\mathcal{S}(0,\infty)$. Interestingly, our general considerations apply to show the existence of solutions to moment problems that could arise in quite different terms. For example, if $\{\alpha_n\}_{n=0}^\infty$ is an increasing sequence of real numbers tending to ∞, then, as follows from our results, every discrete moment problem

$$\sum_{p \text{ prime}} p^{\alpha_n}\phi(p) = a_n, \qquad n \in \mathbb{N},$$

admits solutions $\phi \in \mathcal{S}(0,\infty)$.

Section 7 is devoted to vector moment problems with values in a Fréchet space. Our analysis of the vector moment problem is based upon some results on the density of the set of solutions to moment problems, which will be obtained in Section 6. We conclude the article by studying moment problems in several variables in Section 8.

2. Preliminaries. We use the standard notation from distribution theory [2, 11, 28]. The symbol $\mathcal{S}'_+$ denotes [28] the space of all tempered distributions with supports in the interval $[0,\infty)$. It can be canonically identified with the dual space of $\mathcal{S}_+$, where

$$\mathcal{S}_+ = \{\psi \in C^\infty[0,\infty) : \psi = \varphi_{|[0,\infty)} \text{ for some } \varphi \in \mathcal{S}(\mathbb{R})\}.$$

Notice that $\mathcal{S}(0,\infty)$ is a closed subspace of $\mathcal{S}_+$, where, as in the Introduction, $\mathcal{S}(0,\infty)$ consists of those $\psi \in \mathcal{S}_+$ such that $\psi^{(m)}(0) = 0$ for every $m \in \mathbb{N}$.

We denote as $\mathcal{N}_0$ the annihilator of $\mathcal{S}(0,\infty)$ in $\mathcal{S}'_+$. It is then clear that $\mathcal{N}_0$ consists of *delta sums* at the origin, that is, finite linear combinations of the Dirac delta δ and its derivatives.

We shall employ the notion of Cesàro behavior of distributions, introduced in [8] (see also [11, 13, 20]). We start with primitives of distributions [28]. Given $f \in \mathcal{S}'_+$ and $m \in \mathbb{N}$, we denote as $f^{(-m)}$ the m-primitive of f that satisfies $f^{(-m)} \in \mathcal{S}'_+$. It can be expressed [28] as the convolution

$$f^{(-m)} = f * \frac{x_+^{m-1}}{(m-1)!}.$$

The Cesàro order growth symbols for distributions are defined as follows. Let $\alpha \in \mathbb{R} \setminus \{-1, -2, \dots\}$ and $m \in \mathbb{N}$. For $f \in \mathcal{S}'_+$, we write

$$f(x) = O(x^\alpha) \ (\mathrm{C}, m), \qquad x \to \infty,$$

if $f^{(-m)}$ is a regular distribution (locally Lebesgue integrable) for large arguments and there is a polynomial P of degree at most $m-1$ such that

$$f^{(-m)}(x) = P(x) + O(x^{\alpha+m}) \quad \text{as } x \to \infty. \tag{2.1}$$

Observe that if $\alpha > -1$ then the polynomial in (2.1) is irrelevant. The little o symbol is defined in a similar fashion. When f is locally Lebesgue integrable, the relation (2.1) reads

$$\frac{1}{x}\int_0^x f(t)\left(1 - \frac{t}{x}\right)^{m-1} dt = \frac{Q(x)}{x^m} + O(x^\alpha)$$

for some polynomial Q of degree at most $m-1$. We also introduce the Hardy–Littlewood symbol $\Omega(x^\alpha)$ in the Cesàro sense. Thus, we define

$$f(x) = \Omega(x^\alpha) \ (\mathrm{C}, m), \qquad x \to \infty,$$

as the negation of $f(x) = o(x^\alpha)$ (C, m), $x \to \infty$; in particular, if $\alpha > -1$ and $f^{(-m)}(x)$ is a function for large x, it just means that $f^{(-m)}(x) = \Omega(x^{\alpha+m})$. Analogous definitions apply as $x \to 0^+$.

If we do not have to emphasize the role of m in a Cesàro order relation, we simply write (C), which stands for (C, m) for some m. The growth order symbols can be used to define asymptotic relations and distributional evaluations in the Cesàro sense; see [11] for details.

3. General moment problems for distribution sequences.

We study in this section the following general moment problem. Let $\{f_n\}_{n=0}^\infty \subset \mathcal{S}'_+$ be a sequence of distributions. We seek conditions on $\{f_n\}_{n=0}^\infty$ such that every generalized moment problem, for a given arbitrary sequence $\{a_n\}_{n=0}^\infty$,

$$\langle f_n, \phi \rangle = a_n, \qquad n \in \mathbb{N}, \tag{3.1}$$

admits a solution $\phi \in \mathcal{S}(0, \infty)$.

We start with a natural condition on $\{f_n\}_{n=0}^\infty$. First, notice that if (3.1) is solvable for arbitrary sequences, we must necessarily have

(**P1′**) The distibutions $f_0, f_1, \dots$ are linearly independent.

Since we are interested in solutions to (3.1) in $\mathcal{S}(0, \infty)$, one should assume that none of the f_n is a delta sum at the origin; therefore, we assume the following relation between the sequence and the annihilator of $\mathcal{S}(0, \infty)$:

(**P1**) No element of $\mathcal{N}_0$ is a linear combination of $f_0, f_1, \dots$.

Naturally (**P1**) implies (**P1′**) since one considers the zero distribution as a delta sum with zero coefficients.

A key property in our criterion for the existence of solutions to (3.1) is the notion of Cesàro admissibility, defined as follows. We shall say that the sequence $\{f_n\}_{n=0}^{\infty}$ is *Cesàro admissible* or that it *satisfies property* (**P2**) if: There is an increasing sequence of integers $\{m_j\}_{j=0}^{\infty}$ such that for every $j \in \mathbb{N}$ and every $\alpha > 0$ ($\alpha \notin \mathbb{Z}$) there exists $\nu = \nu_{j,\alpha} \in \mathbb{N}$ such that if $N \geq \nu$ then

$$\left.\begin{aligned} &\sum_{n=0}^{N} b_n f_n(x) = O(x^{\alpha})\ (\mathrm{C}, m_j), \quad x \to \infty \\ &\Big(\sum_{n=0}^{N} b_n f_n^{(-m_j)}\Big)_{|(0,\infty)} \in C(0,\infty), \\ &\sum_{n=0}^{N} b_n f_n(x) = O(x^{-\alpha})\ (\mathrm{C}, m_j), \quad x \to 0^+ \end{aligned}\right\} \Rightarrow\ b_\nu = b_{\nu+1} = \cdots = b_N = 0.$$

Note that (**P2**) is equivalent to: For each $m \in \mathbb{N}$ and $\alpha \in \mathbb{R}_+ \setminus \mathbb{N}$ there exists $\nu = \nu_{\alpha,m} \in \mathbb{N}$ such that if $N \geq \nu$ and $b_N \neq 0$, then either

$$\begin{aligned} &\sum_{n=0}^{N} b_n f_n(x) = \Omega(x^{\alpha})\ (\mathrm{C}, m), \quad x \to \infty, \text{ or} \\ &\Big(\sum_{n=0}^{N} b_n f_n^{(-m)}\Big)_{|(0,\infty)} \notin C(0,\infty), \quad \text{or} \\ &\sum_{n=0}^{N} b_n f_n(x) = \Omega(x^{-\alpha})\ (\mathrm{C}, m), \quad x \to 0^+. \end{aligned}$$

For example, the sequence of monomials $f_n(x) = x^n$ is Cesàro admissible, but so are $f_n(x) = \mathrm{Pf}(x^{-n})$ and $f_n(x) = \delta^{(n)}(x-1)$, $n \in \mathbb{N}$.

The following theorem is the main result of this section.

Theorem 3.1. *Every generalized moment problem* (3.1) *has a solution* $\phi \in \mathcal{S}(0,\infty)$ *if and only if the distribution sequence* $\{f_n\}_{n=0}^{\infty}$ *satisfies properties* (**P1**) *and* (**P2**).

In the proof of Theorem 3.1, we shall employ a result of Silva.

Lemma 3.2 (Silva [23]). *Let* E *be a* Silva space, *that is,* E *is the inductive limit of an increasing sequence* $\{E_n\}_{n=0}^{\infty}$ *of Banach spaces where each inclusion mapping* $E_n \to E_{n+1}$ *is compact. Then a linear subspace* $X \subset E$ *is closed if and only if* $X \cap E_n$ *is closed in each* E_n.

We point out that the class of Silva spaces is precisely that of (DFS)-spaces (strong duals of Fréchet–Schwartz spaces).

Proof of Theorem 3.1. Let $\mathbb{C}[[\xi]]$ be the space of formal power series in one indeterminate with the topology of convergence in each coefficient. Its dual is the space of polynomials in one variable, denoted here as $\mathcal{P}$. That every generalized moment problem (3.1) has a solution $\phi \in \mathcal{S}(0,\infty)$ is equivalent to the surjectivity of the continuous linear mapping

$$\Lambda : \mathcal{S}(0,\infty) \to \mathbb{C}[[\xi]]$$

given by

$$\Lambda(\psi) = \sum_{n=0}^{\infty} \mu_n(\psi)\xi^n, \quad \text{where} \quad \mu_n(\psi) = \langle f_n, \psi \rangle.$$

By the well-known criterion for surjectivity of continuous linear mappings between Fréchet spaces [26, Thm. 37.2, p. 382], the mapping Λ is surjective if and only if its transpose

$$\Lambda^{\top} : \mathcal{P} \to \mathcal{S}'(0,\infty) = \mathcal{S}'_+/\mathcal{N}_0$$

is injective and has weakly* closed range. Since $\mathcal{S}'_+/\mathcal{N}_0$ is Montel, it is reflexive, and there is therefore no distinction between weak* closedness and strong closedness for its linear subspaces. The transpose of Λ is easily seen to be given by

$$\Lambda^{\top}\Big(\sum_{n=0}^{N} b_n \xi^n\Big) = \sum_{n=0}^{N} b_n f_{n\ |\mathcal{S}(0,\infty)}.$$

It is then obvious that $\Lambda^{\top}$ is injective if and only if (**P1**) holds. Write π for the quotient mapping $\pi : \mathcal{S}'_+ \to \mathcal{S}'_+/\mathcal{N}_0$.

We first show the sufficiency of (**P1**) and (**P2**) for $\Lambda^{\top}$ to have closed range. Let $\mathcal{M}$ be the linear span of the $f_0, f_1, \dots$ in $\mathcal{S}'_+$. The range of $\Lambda^{\top}$ is $\pi(\mathcal{M})$. We now use the fact that $\mathcal{S}'_+$ is a Silva space. For its defining inductive sequence, we use the choice as in [28]. For each $n \in \mathbb{N}$, let $\mathcal{S}_{+,n}$ be the completion of $\mathcal{S}_+$ in the norm

$$\|\psi\|_n = \sup_{x\in[0,\infty),\, j\le n} (1+x^2)^{n/2}|\psi^{(j)}(x)|.$$

Clearly, $\mathcal{S}_{+,n}$ consists of all $\psi \in C^n[0,\infty)$ such that $\lim_{x\to\infty} x^n\psi^{(j)}(x) = 0$ for $0 \le j \le n$. Then, the injection $\mathcal{S}'_{+,n} \to \mathcal{S}'_{+,n+1}$ is compact and $\mathcal{S}'_+ = \bigcup_{n=0}^{\infty} \mathcal{S}'_{+,n}$, topologically. Moreover, denoting $\mathcal{S}'_{0,n} = \pi(\mathcal{S}'_{+,n})$, we have $\mathcal{S}'(0,\infty) = \bigcup_{n=0}^{\infty} \mathcal{S}'_{0,n}$ topologically and each $\mathcal{S}'_{0,n} \to \mathcal{S}'_{0,n+1}$ is also compact. By Lemma 3.2, $\pi(\mathcal{M})$ is closed if and only if $\pi(\mathcal{M}) \cap \mathcal{S}'_{0,n}$ is closed in $\mathcal{S}'_{0,n}$ for each n. Since finite-dimensional subspaces are always closed, the latter will be a consequence of the following claim:

CLAIM 1. *Cesàro admissibility implies that $\pi(\mathcal{M}) \cap \mathcal{S}'_{0,n}$ is finite-dimensional for each $n \in \mathbb{N}$.*

Indeed, for each $k \in \mathbb{N}$, set

$$\mathcal{X}_k = \Big\{ \sum_{n=0}^{k} b_n f_n : b_n \in \mathbb{C} \Big\} \oplus \mathcal{N}_0 \subset \mathcal{M} \oplus \mathcal{N}_0.$$

Let $\{m_j\}_{j=0}^{\infty}$ be the sequence from (**P2**). We show that for $p = m_j - 2 > 0$, there is $k = k_j$ such that $(\mathcal{M} \oplus \mathcal{N}_0) \cap \mathcal{S}'_{+,p} \subseteq \mathcal{X}_k$, whence the claim would follow. Suppose that $g = \sum_{n=0}^{N} (b_n f_n + c_n \delta^{(n)}) \in \mathcal{S}'_{+,p}$. Find ν such that (**P2**) holds for j and $\alpha = 2p + 3/2$. Observe that $\varphi_x(u) = (x-u)_+^{p+1} \in \mathcal{S}_{+,p}$. Thus,

$$g^{(-m_j)}(x) = g^{(-p-2)}(x) = g * \frac{x_+^{p+1}}{(p+1)!} = \frac{1}{(p+1)!} \langle g(u), \varphi_x(u) \rangle$$

is a continuous function (on the whole $[0,\infty)$) and

$$\begin{aligned} |g^{(-p-2)}(x)| &\le \frac{\|g\|_{\mathcal{S}'_{+,p}} \|\varphi_x\|_p}{(p+1)!} = O\Big(x^{p+1} \sup_{u \in [0,x),\, j \le p} (|u|^p + 1)(1 - u/x)^{p+1-j} \Big) \\ &= O(x^{2p+1}). \end{aligned}$$

Hence, $\sum_{n=0}^{N} b_n f_n(x) = O(x^{\alpha})$ (C, m_j), $x \to \infty$, $\sum_{n=0}^{N} b_n f_n^{(-m_j)}(x)$ is continuous for $x \in (0,\infty)$, and $\sum_{n=0}^{N} b_n f_n(x) = O(x^{-\alpha})$ (C, m_j), $x \to 0^+$. We deduce from (**P2**) that $b_n = 0$ for $n \ge \nu$. Thus, $g \in \mathcal{X}_{\nu-1}$. The claim has been established.

Conversely, assume that $\Lambda^{\top}$ is injective and has weakly* closed range. As already pointed out, the range $\Lambda^{\top}(\mathcal{P})$ is thus strongly closed because $\mathcal{S}'_+/\mathcal{N}_0$ is reflexive (in fact a (DFS)-space). We have already noticed that (**P1**) must necessarily hold. To show (**P2**), we first establish that $\Lambda^{\top}$ is an isomorphism into its image. Pták's theory [16, 22] applies to show that $\Lambda^{\top}\mathcal{P} \to \Lambda^{\top}(\mathcal{P})$ is open if we verify that $\mathcal{P}$ is fully complete (B-complete in the sense of Pták) and that $\Lambda^{\top}(\mathcal{P})$ is barreled. It is well-known [22, p. 123] that the strong dual of a reflexive Fréchet space is fully complete, so $\mathcal{P}$, as a (DFS)-space, is fully complete. Now, a closed subspace of a (DFS)-space must itself be a (DFS)-space. Since $\mathcal{S}'_+/\mathcal{N}_0$ is a (DFS)-space, we find that $\Lambda^{\top}(\mathcal{P})$ is also a (DFS)-space and hence barreled.

Suppose now that (**P2**) were false. Then there are j and $\alpha > 0$ such that $g_n(x) = \sum_{\nu=0}^{n} b_{\nu,n} f_\nu(x) = O(x^{\alpha})$ (C, j), $x \to \infty$, $g_n(x) = O(x^{-\alpha})$ (C, j), $x \to 0^+$, and $g_n^{(-j)}(x) \in C(0,\infty)$, with $b_{n,n} \ne 0$ for infinitely many n. If $p \ge \max\{\alpha + j + 2\}$, we conclude that there is an increasing sequence $n_0 < n_1 < \cdots$ such that $\{\pi(g_{n_k})\}_{k=0}^{\infty} \subset \mathcal{S}'_{0,p}$. Let $h_{n_k} = \sum_{\nu=0}^{n_k} (b_{\nu,n_k} / \|\pi(g_{n_k})\|_{\mathcal{S}'_{0,p}}) \pi(f_\nu)$ $= \sum_{\nu=0}^{n_k} a_{\nu,n_k} \pi(f_\nu)$ with $a_{n_k,n_k} \ne 0$. Then $\{h_{n_k}\}_{k=0}^{\infty}$ is bounded in $\mathcal{S}'_0$ because of the continuity of the inclusion mapping $\mathcal{S}'_{0,p} \to \mathcal{S}'(0,\infty)$. We then deduce that $\{h_{n_k}\}_{k=0}^{\infty}$ is a bounded set of $\Lambda^{T}(\mathcal{P})$, and since $\Lambda^{\top}$ is open, the set $\{\sum_{\nu=0}^{n_k} a_{\nu,n_k} \xi^{\nu}\}_{k=0}^{\infty}$ of polynomials is bounded in $\mathcal{P}$ as well. But the latter can only hold if there is k_0 such that $a_{\nu,n_k} = 0$ for all $n_k \geq n_{k_0}$ and $\nu \geq n_{k_0}$, which produces a contradiction. ■

Let us discuss a simple example to illustrate Theorem 3.1.

EXAMPLE 3.3 (The generalized Borel problem). Let $\{k_n\}_{n=0}^{\infty}$ be a sequence of natural numbers with $k_n \to \infty$, and let $\{x_n\}_{n=0}^{\infty}$ be a sequence of positive real numbers such that all pairs (k_n, x_n) are distinct. The distribution sequence $\{f_n\}_{n=0}^{\infty}$ given by $f_n(x) = \delta^{(k_n)}(x - x_n)$ satisfies (**P1**) and (**P2**). Consequently, Theorem 3.1 implies that

$$\phi^{(k_n)}(x_n) = a_n, \quad n \in \mathbb{N}, \tag{3.2}$$

is always solvable in $\mathcal{S}(0,\infty)$. If $\{x_n\}_{n=0}^{\infty}$ stays on a fixed compact subset of $(0,\infty)$, multiplying by a cut-off function, we can in fact find solutions to (3.2) that belong to $\mathcal{D}(0,\infty)$. In particular, choosing $x_n = x_0$ to be constant and $k_n = n$ the sequence of all natural numbers, we recover the well-known fact that

$$\phi^{(n)}(x_0) = a_n, \quad n \in \mathbb{N}, \tag{3.3}$$

has a solution $\phi \in \mathcal{D}(\mathbb{R})$. Performing a translation, one easily sees that one may take an arbitrary $x_0 \in \mathbb{R}$ in (3.3), that is, every classical Borel problem has a solution $\phi \in \mathcal{D}(\mathbb{R})$.

Theorem 3.1 can be generalized to two-sided distribution sequences $\{f_n\}_{n\in\mathbb{Z}}$. In fact, consider

$$\langle f_n, \phi\rangle = a_n, \quad n \in \mathbb{Z}. \tag{3.4}$$

If we set, for $n \in \mathbb{N}$,

$$c_n = \begin{cases} a_{n/2} & \text{if } n \text{ is even,} \\ a_{-(n+1)/2} & \text{if } n \text{ is odd} \end{cases} \quad \text{and} \quad g_n = \begin{cases} f_{n/2} & \text{if } n \text{ is even,} \\ f_{-(n+1)/2} & \text{if } n \text{ is odd,} \end{cases}$$

then the moment problem (3.4) is equivalent to $\langle g_n, \phi\rangle = c_n$, $n \in \mathbb{N}$, while (**P1**) and (**P2**) for $\{g_n\}_{n=0}^{\infty}$ translate into:

(**P1***) no element of $\mathcal{N}_0$ is a linear combination of $\ldots, f_{-2}, f_{-1}, f_0, f_1, f_2, \ldots,$ and

(**P2***) there is an increasing sequence $\{m_j\}_{j=0}^{\infty}$ of integers such that for each $j \in \mathbb{N}$ and each $\alpha > 0$ there exists $\nu = \nu_{j,\alpha} \in \mathbb{N}$ such that from $N \geq \nu$ it follows that if $\sum_{n=-N}^{N} b_n f_n(x) = O(x^{\alpha})$ (C, m_j), $x \to \infty$, $\sum_{n=-N}^{N} b_n f_n^{(-m_j)}(x)$ is continuous for $x \in (0,\infty)$ and $\sum_{n=-N}^{N} b_n f_n(x) = O(x^{-\alpha})$ (C, m_j), $x \to 0^+$, then $b_n = 0$ for every $|n| \geq \nu$.

We thus obtain:

COROLLARY 3.4. *Let $\{f_n\}_{n\in\mathbb{Z}} \subset \mathcal{S}'_+$. Then every* (3.4) *admits a solution $\phi \in \mathcal{S}(0,\infty)$ if and only if conditions* (**P1***) *and* (**P2***) *are satisfied.*

We end this section with another example.

EXAMPLE 3.5. The so-called *strong moment problem*,

$$a_n = \int_0^\infty x^n \phi(x)\, dx, \quad n \in \mathbb{Z},$$

was studied and solved in [6] for $\phi \in \mathcal{S}(0,\infty)$. Using Corollary 3.4, we can strengthen the main result of [6] as follows. Let $\{\alpha_n\}_{n\in\mathbb{Z}}$ be a two-sided sequence. Then every strong moment problem

$$a_n = \int_0^\infty x^{\alpha_n} \phi(x)\, dx, \quad n \in \mathbb{Z},$$

is solvable in $\mathcal{S}(0,\infty)$ if and only if all elements of the sequence are distinct and $|\Re e\, \alpha_n| \to \infty$ as $|n| \to \infty$.

REMARK 3.6. Since the simple reduction explained above for moment problems with two-sided sequences also applies to all results from the next sections, we will omit any comment concerning two-sided sequences in what follows.

4. Moment problems with function sequences. We now focus our attention on function sequences. Throughout this section, $\{f_n\}_{n=0}^\infty \subset \mathcal{S}'_+$ stands for a sequence such that each f_n is a non-identically zero locally integrable (1) function with continuous primitives (2) on $(0,\infty)$. Note that we allow f_n to be non-integrable near $x = 0$. The distributional evaluation $\langle f_n, \varphi\rangle$ for $\phi \in \mathcal{S}(0,\infty)$ can always be written [27] as a Cesàro integral and thus we may rewrite in this case the moment problem (3.1) as

$$\int_0^\infty f_n(x)\phi(x)\, dx = a_n \ (\mathrm{C}), \quad n \in \mathbb{N}. \tag{4.1}$$

Naturally, if each f_n has at most polynomial growth, then the Cesàro integrals in (4.1) can be replaced by ordinary integrals.

Theorem 1.1 gives already a complete characterization of those $\{f_n\}_{n=0}^\infty$ for which every moment problem is solvable in $\mathcal{S}(0,\infty)$. Note that (**P1′**) for $\{f_n\}_{n=0}^\infty$ becomes equivalent to (**P1**). On the other hand, (**P2**) forces the linear combinations of the primitives of the sequence to have the ensuing

(1) All results from this section are valid for locally distributionally integrable functions in the sense of [12], and in particular for sequences of locally Denjoy–Perron–Henstock or Lebesgue integrable functions on $(0,\infty)$.

(2) If f_n is locally Denjoy–Perron–Henstock or Lebesgue integrable, its primitive is of course continuous. The primitives of distributionally integrable functions are Łojasiewicz functions, but in general they may be discontinuous [12], whence this assumption.

growth property: For each $m \in \mathbb{N}$ and $\alpha \in \mathbb{R}_+ \setminus \mathbb{N}$ there exists $\nu = \nu_{\alpha,m} \in \mathbb{N}$ such that if $N \geq \nu$ and $b_N \neq 0$, then either

$$\sum_{n=0}^{N} b_n f_n^{(-m)}(x) = \Omega(x^{\alpha}) \qquad \text{as } x \to \infty, \text{ or}$$
$$\sum_{n=0}^{N} b_n f_n^{(-m)}(x) = \Omega(x^{-\alpha}) \qquad \text{as } x \to 0^+. \tag{4.2}$$

Since by our standing assumption all primitives of f_n are continuous on $(0,\infty)$, the latter property is actually equivalent to (**P2**).

The next theorem tells us that if for a fixed m one slightly strengthens these two conditions, then one obtains, together with linear independence, a useful criterion for the solvability of arbitrary moment problems (4.1).

THEOREM 4.1. *Let $m \geq 1$. Suppose that $\{f_n\}_{n=0}^{\infty}$ satisfies* (**P1′**) *and has the following property:*

(**P3**) *For any given $\alpha > 0$ there is $\nu = \nu_\alpha \in \mathbb{N}$ such that if $N \geq \nu$ and $b_N \neq 0$, then one can find $\sigma = \sigma_N > 1$ such that either*

$$\inf_{a\in[1,\sigma]} \Big|\sum_{n=0}^{N} b_n f_n^{(-m)}(ax)\Big| = \Omega(x^{\alpha}) \qquad \textit{as } x \to \infty, \textit{ or}$$
$$\inf_{a\in[1,\sigma]} \Big|\sum_{n=0}^{N} b_n f_n^{(-m)}(ax)\Big| = \Omega(x^{-\alpha}) \qquad \textit{as } x \to 0^+. \tag{4.3}$$

Then every generalized moment problem (4.1) *is solvable in* $\mathcal{S}(0,\infty)$. *If additionally each f_n is a Darboux function* ([3]) *(i.e., has the intermediate value property), then* (4.3) *might be replaced by*

$$\inf_{a\in[1,\sigma]} \Big|\sum_{n=0}^{N} b_n f_n(ax)\Big| = \Omega(x^{\alpha}) \qquad \textit{as } x \to \infty, \textit{ or}$$
$$\inf_{a\in[1,\sigma]} \Big|\sum_{n=0}^{N} b_n f_n(ax)\Big| = \Omega(x^{-\alpha}) \qquad \textit{as } x \to 0^+. \tag{4.4}$$

Proof. In view of Theorem 1.1, we only need to show that (**P3**) ensures the validity of (**P2**). Suppose that (**P3**) is satisfied but (**P2**) does not hold. Then one can find $\alpha > j \geq 1$ ($\alpha \notin \mathbb{N}$) and a sequence $g_k = \sum_{n=0}^{N_k} b_{n,N_k} f_n$ with $b_{N_k,N_k} \neq 0$, and either $g_k^{(-j-m)}(x) = o(x^{\alpha+j})$ as $x \to \infty$, or $g_k^{(-j-m)}(x) =$

([3]) Every continuous function is of course a Darboux function. More generally, if each f_n is a Łojasiewicz function, it must have the Darboux property [12, 19].

$o(x^{j-\alpha})$ as $x \to 0^+$. The latter implies that, for each $\phi \in \mathcal{S}(0,\infty)$,

(4.5)
$$\lim_{\lambda\to\infty} \frac{1}{\lambda^\alpha} \int_0^\infty g_k^{(-m)}(\lambda x)\phi(x)\,dx = 0 \quad \text{or} \quad \lim_{\lambda\to 0^+} \lambda^\alpha \int_0^\infty g_k^{(-m)}(\lambda x)\phi(x)\,dx = 0.$$

Let ν be the integer corresponding to α in (**P3**). Fix k and $\sigma > 1$. Taking a non-negative test function ϕ with $\operatorname{supp}\phi \subset [1,\sigma]$ and $\int_1^\sigma \phi(x)\,dx = 1$ in (4.5) and applying the mean-value theorem [12, Sect. 11], we deduce that for each λ one can find $a_\lambda \in (1,\sigma)$ such that either

$$\lim_{\lambda\to\infty} \frac{g_k^{(-m)}(\lambda a_\lambda)}{\lambda^\alpha} = 0 \quad \text{or} \quad \lim_{\lambda\to\infty} \frac{g_k^{(-m)}(a_\lambda/\lambda)}{\lambda^\alpha} = 0. \tag{4.6}$$

On the other hand, (**P3**) tells us that if $N_k > \nu$, then there are $\sigma > 1$, a sequence $\lambda_n \to \infty$, and $C > 0$ such that either $|g_k^{(-m)}(\lambda_n a)| \geq C\lambda_n^\alpha$ or $|g_k^{(-m)}(a/\lambda_n)| \geq C\lambda_n^\alpha$ for all $a \in [1,\sigma]$ and $n \in \mathbb{N}$, contradicting (4.6).

Note that integration by parts in (4.5) leads to

$$\text{either} \quad \lim_{\lambda\to\infty} \frac{1}{\lambda^{\alpha-m}} \int_0^\infty g_k(\lambda x)\phi(x)\,dx = 0,$$
$$\text{or} \quad \lim_{\lambda\to 0^+} \lambda^{\alpha+m} \int_0^\infty g_k(\lambda x)\phi(x)\,dx = 0.$$

If we assume that each f_n is a Darboux function, then so is each g_k. The mean-value theorem still applies ([4]) to prove that assuming that (**P2**) fails is contradictory with (4.4). The result has been established. ■

It should be noticed that Theorem 1.1, stated in the Introduction, immediately follows by taking $m = 1$ in Theorem 4.1.

Next, we are interested in weighted moment problems of the form

$$\int_0^\infty x^{\alpha_n} f(x)\phi(x)\,dx = a_n \text{ (C)}, \qquad n \in \mathbb{N}, \tag{4.7}$$

where $f \in \mathcal{S}'_+$ is a (non-identically zero) locally integrable function (with continuous primitive) on $(0,\infty)$. We point out that (**P1**) holds if and only if $\{\alpha_n\}_{n=0}^\infty$ consists of distinct complex numbers. We will prove that (**P2**) forces $\{\alpha_n\}_{n=0}^\infty$ to stay off vertical strips except for finitely many terms.

THEOREM 4.2. *Let $\{\alpha_n\}_{n=0}^\infty$ be a sequence of distinct complex numbers with the property that there is n_0 such that $\Re e\,\alpha_n \neq \Re e\,\alpha_m$ for all distinct $n, m > n_0$. Then every moment problem (4.7) is solvable in $\mathcal{S}(0,\infty)$ if and*

([4]) See [12, proof of Prop. 11.1].

only if

$$\lim_{n\to\infty} |\Re e\, \alpha_n| = \infty \tag{4.8}$$

and:

(I) *if* $\lim_{n\to\infty} \Re e\, \alpha_n = \infty$, *then for each* $m \in \mathbb{N}$ *the* m*-primitive of* f *satisfies*

$$-\infty < \limsup_{x\to\infty} \frac{\log |f^{(-m)}(x) - P(x)|}{\log x} \tag{4.9}$$

for each polynomial P *of degree at most* $m-1$*;*

(II) *if* $\lim_{n\to\infty} \Re e\, \alpha_n = -\infty$, *then for each* $m \in \mathbb{N}$,

$$\infty < \limsup_{x\to 0^+} \frac{\log |f^{(-m)}(x) - P(x)|}{|\log x|} \tag{4.10}$$

for each polynomial P *of degree at most* $m-1$*;*

(III) *if* $\liminf_{n\to\infty} \Re e\, \alpha_n = -\infty$ *and* $\limsup_{n\to\infty} \Re e\, \alpha_n = \infty$, *then for each* $m \in \mathbb{N}$ *both* (4.9) *and* (4.10) *hold for each polynomial* P *of degree at most* $m-1$.

Proof. For each n, find $f_n \in \mathcal{S}'_+$ such that $f_n(x) = x^{\alpha_n} f(x)$ for $x \in (0,\infty)$. It is clear that the sequence $\{f_n\}_{n=0}^{\infty}$ is linearly independent on $(0,\infty)$, and consequently $\mathcal{N}_0$ has trivial intersection with the linear span of $\{f_n\}_{n=0}^{\infty}$.

Assume that every moment problem (4.7) is solvable in $\mathcal{S}(0,\infty)$. Fix $M > 0$ and set $A_M = \{n \in \mathbb{N} : -M \le \Re e\, \alpha_n \le M\}$. Since $f \in \mathcal{S}'_+$, there are $m \ge 1$ and $\theta > m$ such that $f^{(-m)} \in C[0,\infty)$, $f^{(-m)}(x) = O(x^{\theta})$ as $x \to \infty$, and $f^{(-m)}(x) = O(x^{M+1/2})$ as $x \to 0^+$. On the other hand (cf. [11, (6.53), p. 302] or [2, Lemma 1.3]),

$$f_n^{(-m)} = \sum_{k=0}^{m} (-1)^k k! \binom{m}{k}\binom{\alpha_n}{k} (x^{\alpha_n - k} f^{(-m)})^{(-k)} + P \quad \text{on } (0,\infty), \tag{4.11}$$

where $P(x)$ is a polynomial of degree at most $m-1$. So, if $n \in A_M$, one sees that $f_n^{(-m)}(x) = O(x^{\theta+M})$ as $x \to \infty$ and $f_n^{(-m)}(x) = Q(x) + O(x^{1/2})$ as $x \to 0^+$ for some polynomial Q of degree at most $m-1$. In particular $f_n^{(-m)} \in C(0,\infty)$, $f_n(x) = O(x^{\alpha})$ (C, m), $x \to \infty$, and $f_n = O(x^{-\alpha})$ (C, m), $x \to 0^+$, for $\alpha \ge \max\{\theta + M - m, m - 1/2\}$. Property (**P2**) implies that this cannot hold for infinitely many n, so that A_M must be finite.

Suppose that $\lim_{n\to\infty} \Re e\, \alpha_n = \infty$ but (4.9) is false for some m, i.e., $f^{(-m)}(x) = Q(x) + o(x^{-\alpha_n})$ as $x \to \infty$ for every $n > 0$, where Q is a polynomial of degree less than m. Since

$$0 = \sum_{k=0}^{m} (-1)^k k! \binom{m}{k}\binom{\alpha_n}{k} (x^{\alpha_n - k} Q)^{(m-k)} \quad \text{on } (0,\infty),$$

formula (4.10) then yields $f_n^{(-m)} \in C(0,\infty)$ and $f_n(x) = O(1)$ (C, m) as $x \to \infty$ for each n. Furthermore, since $\Re e\, \alpha_n$ is bounded from below, we also get $f_n(x) = O(x^{-\alpha})$ (C, m) for some α (and a possibly enlarged m), whence we deduce that (4.2) (and hence (**P2**)) cannot hold. Cases (II) and (III) can be treated in a similar fashion.

An analogous argument, with the aid of the relation (cf. [2, Lemma 1.3])

$$f^{(-m)} = \sum_{k=0}^{m} (-1)^k k! \binom{m}{k} \left(\frac{g_N^{(-m)}}{F_N^{(k)}} \right)^{(-k)} + Q,$$

where $F_N(x) = \sum_{n=0}^{N} b_n x^{\alpha_n}$, $g_N(x) = \sum_{n=0}^{N} b_n f_n$, and Q is a certain polynomial (depending on F_N) of degree at most $m-1$, shows that the conditions are sufficient for (**P2**). ∎

Employing the same method as in the proof of Theorem 4.1, one deduces:

PROPOSITION 4.3. *Let $\{\alpha_n\}_{n=0}^{\infty}$ be a sequence of distinct complex numbers with the property that there is n_0 such that $\Re e\, \alpha_n \neq \Re e\, \alpha_m$ for all distinct $n, m > n_0$. Suppose that* (4.8) *holds and there are $m \geq 1$ and $\sigma > 1$ such that:*

(I) *if $\lim_{n\to\infty} \Re e\, \alpha_n = \infty$, then*

$$-\infty < \limsup_{x\to\infty} \frac{\log \inf_{a\in[1,\sigma]} |f^{(-m)}(ax) - P(ax)|}{\log x} \tag{4.12}$$

for each polynomial P of degree at most $m-1$;

(II) *if $\lim_{n\to\infty} \Re e\, \alpha_n = -\infty$, then*

$$\liminf_{x\to 0^+} \frac{\log \inf_{a\in[1,\sigma]} |f^{(-m)}(ax) - P(ax)|}{|\log x|} < \infty \tag{4.13}$$

for each polynomial P of degree at most $m-1$;

(III) *if $\liminf_{n\to\infty} \Re e\, \alpha_n = -\infty$ and $\limsup_{n\to\infty} \Re e\, \alpha_n = \infty$, then both* (4.12) *and* (4.13) *hold for each polynomial P of degree at most $m-1$.*

Then every moment problem (4.7) *has solutions $\phi \in \mathcal{S}(0,\infty)$. If f is a Darboux function, one may take $m = 0$ (so that $P = 0$) in* (4.12) *and* (4.13).

5. The measure weighted moment problem. We consider the measure weighted moment problem

$$\int_0^{\infty} x^{\alpha_n} \phi(x)\, dF(x) = a_n, \qquad n \in \mathbb{N}, \tag{5.1}$$

where we assume that $\{\alpha_n\}_{n=0}^{\infty}$ is sequence of *real* numbers and F is a function of local bounded variation on $(0,\infty)$ that satisfies

$$\int_0^1 x^{\sigma_1}\,|dF|(x) < \infty, \tag{5.2}$$

$$\int_1^\infty x^{-\sigma_1}\,|dF|(x) < \infty \tag{5.3}$$

for some $\sigma_1 > 0$, where $|dF|$ is the total variation measure of dF. This ensures $dF \in \mathcal{S}'(0,\infty)$.

We start with a necessary condition for the solvability of (5.1).

PROPOSITION 5.1. *Suppose that every moment problem* (5.1) *has solutions* $\phi \in \mathcal{S}(0,\infty)$. *Then the numbers* α_n *are distinct,* $|\alpha_n| \to \infty$, *and there is* $\sigma_0 > 0$ *such that:*

(I) *if* $\lim_{n\to\infty} \alpha_n = \infty$, *then*

$$\int_1^\infty x^{-\sigma_0}\,|dF|(x) = \infty; \tag{5.4}$$

(II) *if* $\lim_{n\to\infty} \alpha_n = -\infty$, *then*

$$\int_0^1 x^{\sigma_0}\,|dF|(x) = \infty; \tag{5.5}$$

(III) *if* $\liminf_{n\to\infty} \alpha_n = -\infty$ *and* $\limsup_{n\to\infty} \alpha_n = \infty$, *then both* (5.4) *and* (5.5) *hold.*

Proof. Let f_n be an extension of $x^{\alpha_n}\,dF(x)$ to $\mathcal{S}'_+$. Since (**P1**) holds, the α_n's should be distinct. If $|\alpha_n| \le M$ for all $n \in \mathbb{N}$, then using the assumptions (5.2) and (5.3), we would have

$$\Big|\sum_{n=0}^{N} b_n \int_1^x t^{\alpha_n}\,dF(t)\Big| = O(x^{M+\sigma_1}) \qquad \text{as } x \to \infty$$

and

$$\Big|\sum_{n=0}^{N} b_n \int_1^x t^{\alpha_n}\,dF(t)\Big| = O(x^{-M-\sigma_1}) \qquad \text{as } x \to 0^+, \tag{5.6}$$

contradicting (**P2**) for $\{f_n\}_{n=0}^{\infty}$. If $\alpha_n \to \infty$, a bound (5.6) must hold because α_n is bounded from below. If (5.4) were not valid, then we would have $\int_1^\infty t^{\alpha_n}\,|dF|(t) < \infty$ for all $n \in \mathbb{N}$, leading to

$$\Big|\sum_{n=0}^{N} b_n \int_1^x t^{\alpha_n}\,dF(t)\Big| = O(1), \qquad x \ge 1.$$

This again contradicts (**P2**). Cases (II) and (III) can be treated similarly. ∎

Note that either (5.4) or (5.5) always shows that the support of dF is an infinite subset in $(0,\infty)$. If additionally the α_n's are distinct, the latter implies (**P1**) for any distribution sequence $\{f_n\}_{n=0}^{\infty}$ in $\mathcal{S}'_+$ that extends $x^{\alpha_n}\,dF(x)$, that is, $f_n(x)=x^{\alpha_n}\,dF(x)$ on $(0,\infty)$.

We give the converse of Proposition 5.1 for non-negative measures. In fact, the next theorem gives a complete characterization of those non-negative measures dF for which (5.1) is always solvable in $\mathcal{S}(0,\infty)$.

THEOREM 5.2. *Let F be non-decreasing on $(0,\infty)$ satisfying (5.2) and (5.3), and let $\{\alpha_n\}_{n=0}^{\infty}$ be a sequence of distinct real numbers. Then every moment problem (5.1) has solutions $\phi\in\mathcal{S}(0,\infty)$ if and only if $|\alpha_n|\to\infty$ and there is $\sigma_0>0$ such that:*

(I) *if $\lim_{n\to\infty}\alpha_n=\infty$, then (5.4) holds;*
(II) *if $\lim_{n\to\infty}\alpha_n=-\infty$, then (5.5) holds;*
(III) *if $\liminf_{n\to\infty}\alpha_n=-\infty$ and $\limsup_{n\to\infty}\alpha_n=\infty$, then both (5.4) and (5.5) hold.*

Proof. We only need to show the converse. We assume $\lim_{n\to\infty}\Re e\,\alpha_n=\infty$, the proofs in the remaining two cases being analogous. We may rearrange the sequence in increasing order $\alpha_0<\alpha_1<\cdots<\alpha_n\to\infty$. We prove that (5.4) implies (**P2**), for an extension sequence of the $x^{\alpha_n}\,dF(x)$ to $\mathcal{S}'_+$, by contraposition. Note that the O-bound (5.6) holds as $x\to0^+$ with $-M=\alpha_0$; so, if (**P2**) fails, there are $m,\alpha>0$, and an infinite sequence of indices N_k such that

$$\int_1^x\Big(t^{\alpha_{N_k}}+\sum_{n=0}^{N_k-1}b_{n,k}t^{\alpha_n}\Big)\Big(1-\frac{t}{x}\Big)^m\,dF(t)=O(x^{\alpha}),\qquad x\ge1,$$

for some constants $b_{n,k}$. We have $\sum_{n=0}^{N_k-1}b_{n,k}t^{\alpha_n}=o(t^{\alpha_{N_k}})$. Therefore,

$$G(t)=\int_1^x t^{\alpha_{N_k}}\,dF(t)\le 2^m\int_1^{2x}\Big(1-\frac{t}{2x}\Big)^m t^{\alpha_{N_k}}\,dF(t)=O(x^{\alpha}),\qquad x\ge1.$$

Integrating by parts leads to

$$\int_1^x t^{\alpha_{N_k}-\alpha-1}\,dF(t)=\frac{G(x)}{x^{\alpha+1}}+(\alpha+1)\int_1^x\frac{G(t)}{t^{\alpha+2}}\,dt=O(1),$$

which implies that (5.4) cannot hold because $\alpha_{N_k}\to\infty$. ■

If the measure dF vanishes on $(0,\lambda_0)$, then (5.5) cannot hold, which excludes cases (II) and (III) from Proposition 5.1 and Theorem 5.2. Also, the abscissa of absolute convergence of the Mellin transform of $\int_0^\infty x^{-s}\,dF(x)$, denoted as σ_a, is the infimum of those σ_1 for which (5.3) holds. Under our assumption (5.3), $\sigma_a<\infty$, but it may be equal to $-\infty$.

Corollary 5.3. *Let F be a non-decreasing function of at most polynomial growth that vanishes on $(-\infty,\lambda)$ for some $\lambda>0$, and let $\{\alpha_n\}_{n=0}^{\infty}$ be a sequence of distinct real numbers. Then every moment problem* (5.1) *has solutions $\phi\in\mathcal{S}(0,\infty)$ if and only if $\alpha_n\to\infty$ and the abscissa of convergence of the Mellin transform of dF is finite, that is, $\sigma_a>-\infty$.*

Example 5.4. Let $\{\lambda_k\}_{k=1}^{\infty}$ be a non-decreasing sequence of positive real numbers and let $\{c_k\}_{k=0}^{\infty}$ be a non-negative sequence. According to Corollary 5.3, every moment problem

$$a_n=\sum_{k=1}^{\infty}c_k\lambda_k^{\alpha_n}\phi(k),\qquad n\in\mathbb{N},$$

has solutions $\phi\in\mathcal{S}(0,\infty)$ if and only if $\alpha_n\to\infty$ and the Dirichlet series $F(s)=\sum_{k=1}^{\infty}c_k\lambda^{-s}$ has finite abscissa of convergence. In particular, if $\alpha_n\to\infty$, moment problems such as

$$a_n=\sum_{k=1}^{\infty}\phi(k)k^{\alpha_n},\quad n\in\mathbb{N},\quad\text{and}\quad a_n=\sum_{p\text{ prime}}\phi(p)p^{\alpha_n},\quad n\in\mathbb{N},$$

are always solvable in $\mathcal{S}(0,\infty)$.

6. Density of the set of solutions of moment problems. In order to study vector-valued moment problems, it is convenient to consider first several results on the density of some linear manifolds in a general topological vector space.

Let E be a locally convex topological vector space. It is well-known that a linear functional $f:E\to\mathbb{C}$ is continuous if and only if the linear subspace $\ker f=\{x\in E:f(x)=0\}$ is closed, or equivalently, f is discontinuous if and only if $\ker f$ is dense in E. We would like to consider the corresponding situation when not one but several linear functionals are given. In the following we shall employ the notation $\ker(f_1,\dots,f_n)=\bigcap_{k=1}^{n}\ker f_k$.

Definition 6.1. Let $f_1,\dots,f_n$ be linear functionals on a locally convex topological vector space E. We say that they are *completely discontinuous* if the only linear combination $\sum_{k=1}^{n}c_kf_k$ that is continuous is the one with $c_k=0$ for all k.

If we denote by π the projection from the *algebraic* dual space E'_{alg} onto E'_{alg}/E', then $f_1,\dots,f_n$ are completely discontinuous if and only if $\pi(f_1)$, $\dots,\pi(f_n)$ are linearly independent.

Proposition 6.2. *Let $f_1,\dots,f_n$ be n linearly independent linear functionals on the locally convex topological vector space E. Let k be the dimension of the vector subspace G of $\mathbb{C}^n$ formed by those vectors $(c_1,\dots,c_n)$ such that*

$\sum_{i=1}^{n} c_i f_i$ *is continuous. Then*

$$\operatorname{codim}_E \overline{\ker(f_1,\ldots,f_n)} = k.$$

In particular, $\ker(f_1,\ldots,f_n)$ *is dense in* E *if and only if* $f_1,\ldots,f_n$ *are completely discontinuous.*

Proof. Indeed, denote by m the codimension of $\overline{\ker(f_1,\ldots,f_n)}$ in E. We shall first show that $m \geq k$. This is obvious, of course, if $k=0$, so suppose that $k>0$. Then if $\mathbf{c}_j = (c_{j,i})_{i=1}^n$, $1 \leq j \leq k$, are a basis of G, then the k functionals $g_j = \sum_{i=1}^{n} c_{j,i} f_i$, $1 \leq j \leq k$, are continuous, and they are also linearly independent, because the f_j's are. Since $\ker(f_1,\ldots,f_n) \subset \ker(g_1,\ldots,g_k)$, and the latter space is closed, we obtain $\overline{\ker(f_1,\ldots,f_n)} \subset \ker(g_1,\ldots,g_k)$, and thus $m \geq k$.

To show that $m \leq k$ we may assume that $m>0$. Since $\overline{\ker(f_1,\ldots,f_n)}$ is a closed subspace of codimension m, we can find m linearly independent continuous functionals $g_1,\ldots,g_m$ such that $\overline{\ker(f_1,\ldots,f_n)} = \ker(g_1,\ldots,g_m)$. The fact that $\ker(f_1,\ldots,f_n) \subset \ker g_j$ for any j implies that g_j is a linear combination of $f_1,\ldots,f_n$, say $g_j = \sum_{i=1}^{n} c_{j,i} f_i$; then the vectors $\mathbf{c}_j = (c_{j,i})_{i=1}^n$ for $1 \leq j \leq m$ are linearly independent in G, and thus $m \leq k$. ∎

Observe, furthermore, that if $f_1,\ldots,f_n$ are linearly independent, then the dimension of the vector space generated by $\pi(f_1),\ldots,\pi(f_n)$ is precisely $n-k$.

If $f_1,\ldots,f_n$ are linearly independent, then the map from E to $\mathbb{C}^n$ given by $x \mapsto (\langle f_j, x\rangle)_{j=1}^n$ is surjective. Therefore we obtain the following result on *finite* moment problems.

COROLLARY 6.3. *If* $f_1,\ldots,f_n$ *are completely discontinuous, then for any vector* $(a_j)_{j=1}^n$ *in* $\mathbb{C}^n$*, the set of solutions of the moment problem*

$$\langle f_j, x\rangle = a_j, \qquad 1 \leq j \leq n,$$

is dense in E.

Sometimes it is necessary to employ the following simple result.

LEMMA 6.4. *Let* E *be a locally convex topological vector space and let* F *be a closed subspace of finite codimension. Then linear functionals* $f_1,\ldots,f_n$ *are completely discontinuous in* E *if and only if their restrictions to* F *are.*

In general the intersection of an infinite sequence of dense linear manifolds does not have to be dense, so that Corollary 6.3 does not hold for infinite moment problems. We shall show that if $E = \mathcal{S}(0,\infty)$ and the linear functionals are Cesàro admissible, then the density of the set of solutions holds in many cases, but before we do this we give an example when the set of solutions is not dense.

Example 6.5. Let E be the normed space formed by the polynomials in one variable, with norm $\|p\| = \max_{|t|\le 1} p(t)$. Consider the sequence of functionals $\{f_k\}_{k=1}^{\infty}$, $f_k(t) = \delta^{(k)}(t)$. For each n, $f_1, \dots, f_n$ are completely discontinuous, and as the corollary predicts, the set $S_n = \{p \in E : \langle f_j, p\rangle = a_j,\ 1 \le j \le n\}$ is dense in E for any constants $a_1, \dots, a_n$. However, for many infinite sequences $\{a_k\}_{k=1}^{\infty}$ the set $\{p \in E : \langle f_j, p\rangle = a_j,\ j \ge 1\}$ is empty, and when it is not, it is an affine subspace of dimension 1. Thus $\bigcap_{n=1}^{\infty} S_n$ is never dense in E.

Suppose now that E is a Fréchet space whose topology is given by the basis of increasing seminorms $\{p_k\}_{k=1}^{\infty}$. Consider a family $\{f_k\}_{k=1}^{\infty}$ of continuous linear functionals. We shall say that the sequence $\{\{f_j\}_{j=N_k}^{N_k+1}\}_{k=0}^{\infty}$ of blocks is *strictly admissible* with respect to the sequence $\{p_k\}_{k=1}^{\infty}$ of seminorms if $\{N_k\}_{k=0}^{\infty}$ is an increasing sequence of integers with $N_0 = 1$ such that:

(1) $\{f_1, \dots, f_{N_k}\}$ are continuous with respect to p_k;
(2) $\{f_{N_k+1}, f_{N_k+2}, \dots\}$ are completely discontinuous with respect to p_k.

We can then prove the density of the set of solutions of certain infinite moment problems.

Proposition 6.6. *Let $\{\{f_j\}_{j=N_k}^{N_k+1}\}_{k=0}^{\infty}$ be strictly admissible with respect to $\{p_k\}_{k=1}^{\infty}$. Let $\{a_k\}_{k=N_1+1}^{\infty}$ be an arbitrary sequence of complex numbers. Then the set of solutions of the moment problem*

$$\langle f_j, x\rangle = a_j, \quad j \ge N_1 + 1, \tag{6.1}$$

is dense in the seminormed space (E, p_1). Actually for any $\varepsilon > 0$ and any $x_0 \in E$ there is a solution of (6.1) *such that*

$$p_1(x - x_0) \le \varepsilon, \quad \langle f_j, x\rangle = \langle f_j, x_0\rangle, \quad 1 \le j \le N_1.$$

Proof. Let $\{\varepsilon_k\}_{k=1}^{\infty}$ be a sequence with $\varepsilon_k > 0$ for all k and with $\sum_{k=1}^{\infty} \varepsilon_k = \varepsilon$. Considering the set $f_{N_1+1}, \dots, f_{N_2}$, which is completely discontinuous with respect to p_1, we see that Corollary 6.3 and Lemma 6.4 show the existence of x_1 such that $p_1(x_1) < \varepsilon_1$, $\langle f_j, x_1\rangle = 0$ for $1 \le j \le N_1$, while $\langle f_j, x_1\rangle = a_j - \langle f_j, x_0\rangle$ for $N_1 + 1 \le j \le N_2$. Proceeding in a recursive fashion, for each $k \ge 2$ we can find x_k such that $p_k(x_k) < \varepsilon_k$, $\langle f_j, x_k\rangle = 0$ for $1 \le j \le N_k$, while $\langle f_j, x_k\rangle = a_j - \sum_{i=0}^{k-1}\langle f_j, x_i\rangle$ for $N_k + 1 \le j \le N_{k+1}$.

The series $\sum_{k=0}^{\infty} x_k$ converges in E since for any q we have

$$\sum_{k=0}^{\infty} p_q(x_k) \le \sum_{k=0}^{q-1} p_q(x_k) + \sum_{k=q}^{\infty} p_q(x_k) \le \sum_{k=0}^{q-1} p_q(x_k) + \sum_{k=q}^{\infty} \varepsilon_k < \infty.$$

Let $x=\sum_{k=0}^{\infty}x_k$. Then

$$p_1(x-x_0)\le\sum_{k=1}^{\infty}p_1(x_k)\le\sum_{k=1}^{\infty}\varepsilon_k=\varepsilon,$$

while by continuity $\langle f_j,x\rangle=\sum_{i=0}^{\infty}\langle f_j,x_i\rangle$, which is actually a finite sum; this gives $\langle f_j,x\rangle=\langle f_j,x_0\rangle$ for $1\le j\le N_1$, and if $k\ge 1$, then

$$\langle f_j,x\rangle=\sum_{i=0}^{k}\langle f_j,x_i\rangle=\langle f_j,x_k\rangle+\sum_{i=0}^{k-1}\langle f_j,x_i\rangle=a_j$$

for $N_k+1\le j\le N_{k+1}$, as required. ∎

Let us now go back to moment problems in the space $E=\mathcal{S}(0,\infty)$. If a sequence $\{f_k\}_{k=0}^{\infty}$ of functionals is the restriction to $(0,\infty)$ of a sequence that is Cesàro admissible, and $\{p_k\}_{k=1}^{\infty}$ is an increasing sequence of continuous seminorms of E that gives the topology, in general it is not possible to arrange $\{f_k\}_{k=0}^{\infty}$ in blocks to obtain strict admissibility. However, we can construct sequences $\{\widetilde{p}_k\}_{k=1}^{\infty}$, $\{g_k\}_{k=0}^{\infty}$, and $\{N_k\}_{k=0}^{\infty}$ such that:

(1) $\{\widetilde{p}_k\}_{k=1}^{\infty}$ is also an increasing sequence of continuous seminorms that gives the topology of E;

(2) $\big\{\{g_j\}_{j=N_k}^{N_k+1}\big\}_{k=0}^{\infty}$ is strictly admissible with respect to $\{\widetilde{p}_k\}_{k=1}^{\infty}$;

(3) there is a linear bijective map T from $\mathbb{C}^{\mathbb{N}}$ to itself ([5]) with $T\big(\{f_j\}_{j=0}^{\infty}\big)=\{g_j\}_{j=0}^{\infty}$. Actually there are increasing sequences $m_0=0<m_1<m_2<\cdots$ such that the vector $(g_j)_{j=m_k+1}^{m_{k+1}}$ is obtained by multiplying $(f_j)_{j=m_k+1}^{m_{k+1}}$ by an invertible matrix.

The construction is as follows. Naturally we set $N_0=1$. Next, take $\widetilde{p}_1=p_1$. Since $\{f_k\}_{k=0}^{\infty}$ is Cesàro admissible, we can find integers n_1 and l_1 with $n_1\le l_1$ such that $f_0,\dots,f_{n_1}$ are continuous with respect to $\widetilde{p}_1$ while $f_{l_1+1},f_{l_1+2},f_{l_1+3},\dots$ are completely discontinuous with respect to $\widetilde{p}_1$; we can take them so that n_1 is the maximum while l_1 is the minimum with this property. If $l_1=n_1$, we take $N_1=n_1$. If $l_1=n_1+q$, $q\ge 1$, we can find linear combinations

$$g_{n_1+j}=\sum_{i=1}^{q}\alpha_{j,i}f_{n_1+i}+\sum_{k=n_1+q+1}^{m_1}\beta_{j,i}f_k$$

for $1\le j\le q$ such that the matrix $(\alpha_{j,i})_{j,i=1}^{q}$ is invertible, and for some N_1 with $n_1<N_1\le l_1$ the functionals $f_0,\dots,f_{n_1},g_{n_1+1},\dots,g_{N_1}$ are continuous with respect to $\widetilde{p}_1$, while $g_{N_1+1},\dots,g_{l_1},f_{l_1+1},f_{l_1+2},f_{l_1+3},\dots$ are completely discontinuous with respect to $\widetilde{p}_1$. Set $g_j=f_j$ for $0\le j\le n_1$ and for l_1+1

([5]) The map T is actually bicontinuous with respect to the topology of pointwise convergence in $\mathbb{C}^{\mathbb{N}}$.

$\leq j \leq m_1$. The vector $(g_j)_{j=1}^{m_1}$ is obtained by multiplying $(f_j)_{j=1}^{m_1}$ by an invertible matrix.

We then find a seminorm $\widetilde{p}_2$ among the p_k's for $k \geq r_1$ such that $g_0, \ldots, g_{m_1}$ and $f_{m_1+1}, \ldots, f_{n_2}$ are continuous with respect to $\widetilde{p}_2$ while f_{l_2+1}, $f_{l_2+2}, f_{l_2+3}, \ldots$ are completely discontinuous with respect to $\widetilde{p}_2$ for some integers n_2 and l_2, with $m_1+1 \leq n_2 \leq l_2$. Then we employ the same procedure to find N_2 and m_2 such that $n_2 \leq N_2 \leq l_2 \leq m_2$ and linear combinations g_j, $m_1+1 \leq j \leq m_2$, of the f_k's, $m_1+1 \leq k \leq m_2$, so that $(g_j)_{j=m_1+1}^{m_2}$ is obtained by multiplying $(f_j)_{j=m_1+1}^{m_2}$ by an invertible matrix, $g_0, \ldots, g_{N_2}$ are continuous with respect to $\widetilde{p}_2$, while $g_{N_2+1}, g_{N_2+2}, g_{N_2+3}, \ldots$ are completely discontinuous with respect to $\widetilde{p}_2$.

We may then proceed inductively, constructing seminorms $\widetilde{p}_k$, integers n_k, N_k, l_k, and m_k with $m_{k-1}+1 \leq n_k \leq N_k \leq l_k \leq m_k$, define $(g_j)_{j=m_{k-1}+1}^{m_k}$ by multiplying $(f_j)_{j=m_{k-1}+1}^{m_k}$ by an appropriate invertible matrix, so that $g_1, \ldots, g_{N_k}$ are continuous with respect to $\widetilde{p}_k$, while g_{N_k+1}, g_{N_k+2}, $g_{N_k+3}, \ldots$ are completely discontinuous with respect to $\widetilde{p}_k$. The three conditions above are then clearly satisfied.

Notice that condition (**3**) in our construction implies that the moment problems

$$\langle f_j, \phi \rangle = a_j, \quad j \in \mathbb{N},$$

have solutions for *all* sequences $\{a_k\}_{k=0}^{\infty}$ if and only if the same is true of all the moment problems

$$\langle g_j, \phi \rangle = b_j, \quad j \in \mathbb{N},$$

for arbitrary sequences $\{b_k\}_{k=0}^{\infty}$. We can also use the density of the set of solutions of one moment problem to obtain the corresponding density of the set of solutions of the other, as we explain in precise terms in the next proposition.

PROPOSITION 6.7. *Let $\{f_k\}_{k=0}^{\infty}$ be a sequence of distributions on the space $\mathcal{S}_+$ that satisfies conditions* (**P1**) *and* (**P2**). *Let p be any continuous seminorm in $\mathcal{S}(0,\infty)$. Then there exists $m \in \mathbb{N}$ such that if $\{a_k\}_{k=m}^{\infty}$ is an arbitrary sequence of complex numbers, then the set of solutions of the moment problem*

$$\langle f_j, \phi \rangle = a_j, \quad j \geq m, \tag{6.2}$$

is dense in the seminormed space $(\mathcal{S}(0,\infty), p)$. If $\{f_k\}_{k=0}^{\infty}$ is completely discontinuous with respect to p, then we can take $m = 0$.

Proof. Let $\{p_k\}_{k=1}^{\infty}$ be an increasing sequence of continuous seminorms in $\mathcal{S}(0,\infty)$ that gives the topology, with $p_1 = p$. We can then construct a sequence $\{g_j\}$ such that conditions (**1**)–(**3**) are satisfied. Observe that for any k the set of solutions of the moment problem $\langle f_j, \phi \rangle = 0$ for $j \geq m_k$ is exactly the same as the set of solutions of $\langle g_j, \phi\rangle = 0$ for $j \geq m_k$, and since $\big\{\{g_j\}_{j=N_k}^{N_k+1}\big\}_{k=0}^{\infty}$ is strictly admissible with respect to $\{\widetilde{p}_k\}_{k=1}^{\infty}$, Proposition 6.6 shows that such a set of solutions is dense in $\mathcal{S}(0,\infty)$ with the topology given by the seminorm $\widetilde{p}_k$. Hence, if we take $m = m_1$, then the set of solutions of $\langle f_j, \phi\rangle = 0$ for $j \geq m$ is dense in $(\mathcal{S}(0,\infty), p)$. Therefore, if $\{a_k\}_{k=m}^{\infty}$ is an arbitrary sequence of complex numbers, then by translating by a particular solution of (6.2)—a particular solution that exists because (**P1**) and (**P2**) are satisfied—we conclude that the set of solutions of (6.2) is likewise dense in $(\mathcal{S}(0,\infty), p)$. That we can take $m = 0$ if $\{f_k\}_{k=0}^{\infty}$ is completely discontinuous with respect to p should be clear from our construction. ∎

The next related result, which will be needed in our analysis of the vector moment problems, follows by the same arguments.

PROPOSITION 6.8. *Let $\{f_k\}_{k=0}^{\infty}$ be a sequence of distributions on $\mathcal{S}_+$ that satisfies* (**P1**) *and* (**P2**). *Let p be any continuous seminorm in $\mathcal{S}(0,\infty)$. Then there exists $m \in \mathbb{N}$ such that if $\{a_k\}_{k=m}^{\infty}$ is an arbitrary sequence of complex numbers, then for each $\varepsilon > 0$ the moment problem*

$$\langle f_j, \phi\rangle = a_j, \quad j \geq m, \qquad \langle f_j, \phi\rangle = 0, \quad 0 \leq j < m,$$

has a solution with $p(\phi) < \varepsilon$.

7. Vector moment problems. Let $\mathcal{X}$ be a Fréchet space. Let $\|\ \|_1 \leq \|\ \|_2 \leq \cdots$ be a sequence of seminorms of $\mathcal{X}$ that generate its topology. Let $\{\mathbf{a}_k\}_{k=0}^{\infty}$ be an arbitrary sequence of elements of $\mathcal{X}$, and let $\{f_j\}_{j=0}^{\infty}$ be a sequence of distributions on the space $\mathcal{S}_+$. We wish to study the problem of finding a rapidly decreasing smooth function $\phi : \mathbb{R} \to \mathcal{X}$ with support in $[0,\infty)$ such that

$$\langle f_j, \phi\rangle = \mathbf{a}_j, \quad j = 0, 1, \ldots. \tag{7.1}$$

Notice that asking ϕ to be a rapidly decreasing smooth function means that $\phi \in \mathcal{S}(\mathbb{R}, \mathcal{X}) \cong \mathcal{S}(\mathbb{R}) \mathbin{\widehat{\otimes}} \mathcal{X}$. In general [26], ψ belongs to $\mathcal{S}(\mathbb{R}^n, \mathcal{X})$ if and only if for all $\mathbf{k}, \mathbf{m} \in \mathbb{N}^n$ the set $\{\mathbf{x}^{\mathbf{k}}\mathbf{D}^{\mathbf{m}}\psi(\mathbf{x}) : \mathbf{x} \in \mathbb{R}^n\}$ is bounded in $\mathcal{X}$. For a Fréchet space this means that

$$\|\mathbf{D}^{\mathbf{m}}\psi(\mathbf{x})\|_q = O(|\mathbf{x}|^{-k}) \quad \text{as } |\mathbf{x}| \to \infty \tag{7.2}$$

for each $\mathbf{m} \in \mathbb{N}^n$ and $q, k \in \mathbb{N}$. Notice also that $\mathcal{S}((0,\infty), \mathcal{X}) \cong \mathcal{S}(0,\infty) \mathbin{\widehat{\otimes}} \mathcal{X}$ consists of the elements of $\mathcal{S}(\mathbb{R}, \mathcal{X})$ with support in $[0,\infty)$.

If $\{f_j\}_{j=0}^{\infty}$ is a sequence of distributions on $\mathcal{S}_+$ that satisfies conditions (**P1**) and (**P2**), then we can find functions $\rho_k \in \mathcal{S}(0,\infty)$ such that

$$\langle f_j, \rho_k\rangle = \delta_{j,k}, \quad j = 0, 1, \ldots, \tag{7.3}$$

where $\delta_{j,k}$ is the Kronecker delta. One is tempted to try to solve (7.1) by setting $\phi(x) = \sum_{k=0}^{\infty} \rho_k(x)\mathbf{a}_k$. However, this series could be divergent, and even if convergent the sum might not belong to $\mathcal{S}((0,\infty),\mathcal{X})$. However, as we shall show, if we choose the ρ_k's carefully, then the series converges and gives an element of $\mathcal{S}((0,\infty),\mathcal{X})$.

THEOREM 7.1. *Let $\mathcal{X}$ be a Fréchet space and let $\{\mathbf{a}_k\}$ be an arbitrary sequence of elements of $\mathcal{X}$. Let $\{f_k\}_{k=0}^{\infty}$ be a sequence of distributions on $\mathcal{S}_+$ that satisfies* (**P1**) *and* (**P2**). *Then the moment problem*

$$\langle f_j,\phi\rangle=\mathbf{a}_j,\qquad j\in\mathbb{N}, \tag{7.4}$$

has solutions $\phi\in\mathcal{S}((0,\infty),\mathcal{X})$.

Proof. Let

$$Q_n=\|\mathbf{a}_n\|_n, \tag{7.5}$$

so that $\|\mathbf{a}_j\|_n\le Q_j$ if $j\ge n$. Choose $\varepsilon_n>0$ so that $\sum_{n=1}^{\infty}\varepsilon_nQ_n<\infty$. Let $\{p_k\}_{k=1}^{\infty}$ be an increasing sequence of continuous seminorms on $\mathcal{S}(0,\infty)$ that gives the topology.

Employing Proposition 6.8 we can find a sequence $\{m_i\}_{i=1}^{\infty}$ of positive integers, which we may suppose increasing, and for $k\ge m_i$ solutions $\rho_k^{\{i\}}$ of the moment problem (7.3) with $p_i(\rho_k^{\{i\}})\le\varepsilon_k$. If we now write $\rho_k=\rho_k^{\{i\}}$ for $m_i\le k\le m_{i+1}-1$, and take any solutions ρ_k for $k<m_1$, then the series $\sum_{k=0}^{\infty}\rho_k(x)\mathbf{a}_k$ converges in $\mathcal{S}(0,\infty)\widehat{\otimes}\mathcal{X}\cong\mathcal{S}((0,\infty),\mathcal{X})$. Indeed, convergence of the series in the tensor product would follow as in [26] if we show that for all N and M the series $\sum_{k=0}^{\infty}p_N(\rho_k)\|\mathbf{a}_k\|_M$ converges. But by taking $K=\max\{M,m_N\}$ we obtain

$$\begin{aligned}\sum_{k=0}^{\infty}p_N(\rho_k)\|\mathbf{a}_k\|_M&\le\sum_{k=0}^{K-1}p_N(\rho_k)\|\mathbf{a}_k\|_M+\sum_{k=K}^{\infty}p_N(\rho_k)\|\mathbf{a}_k\|_M\\&\le\sum_{k=0}^{K-1}p_N(\rho_k)\|\mathbf{a}_k\|_M+\sum_{k=K}^{\infty}\varepsilon_kQ_k<\infty.\end{aligned}$$

If we now set $\phi(x)=\sum_{k=0}^{\infty}\rho_k(x)\mathbf{a}_k$, then we obtain

$$\langle f_j,\phi\rangle=\sum_{k=0}^{\infty}\langle f_j,\rho_k(x)\rangle\mathbf{a}_k=\sum_{k=0}^{\infty}\delta_{j,k}\mathbf{a}_k=\mathbf{a}_j,$$

because of the convergence and the continuity of the f_j's. ∎

Needless to say, conditions (**P1**) and (**P2**) are also necessary for the solvability of the vector moment problem, as follows from Theorem 3.1.

8. Moment problems in several variables. In this section we consider moment problems for functions of several variables. Indeed, let V be an open cone with vertex at the origin in $\mathbb{R}^d$, that is, V is an open set such that $\lambda\mathbf{x}\in V$ whenever $\lambda>0$ and $\mathbf{x}\in V$. Let $\mathcal{X}$ be a Fréchet space. We denote by $\mathcal{S}(V,\mathcal{X})$ the space of elements of $\mathcal{S}(\mathbb{R}^d,\mathcal{X})$ with support contained in $\overline{V}$. If $\{F_n\}_{n=0}^{\infty}$ is a sequence of distributions in $\mathcal{S}'(\mathbb{R}^d)$ and $\{\mathbf{a}_n\}_{n=0}^{\infty}$ is a sequence of vectors in $\mathcal{X}$, we would like to study the existence of solutions of the vector moment problem

$$\langle F_n(\mathbf{x}),\phi(\mathbf{x})\rangle = \mathbf{a}_n, \qquad n\in\mathbb{N}, \tag{8.1}$$

in the space $\mathcal{S}(V,\mathcal{X})$.

We shall denote by $\mathbb{S}$ the unit sphere of $\mathbb{R}^d$, and if U is an open subset of $\mathbb{S}$, then $\mathcal{D}(U)$ will denote the set of smooth functions defined in $\mathbb{S}$ whose support is contained in U.

If $F\in\mathcal{S}'(\mathbb{R}^d)$ and ψ is a smooth function defined in $\mathbb{S}$, then we can define the distribution $f(r)=\langle F(r\omega),\psi(\omega)\rangle_\omega$ in the space $\mathcal{S}'(0,\infty)$ by

$$\langle f(r),\rho(r)\rangle_r = \langle F_n(\mathbf{x}),|\mathbf{x}|^{1-d}\rho(|\mathbf{x}|)\psi(\mathbf{x}/|\mathbf{x}|)\rangle, \qquad \rho\in\mathcal{S}(0,\infty).$$

In general this equation cannot be applied if $\rho\in\mathcal{S}_+$, so that f does not belong to $\mathcal{S}'_+$, but there are always extensions ([6]) of f to $\mathcal{S}'_+$.

Proposition 8.1. *Let $\{F_n\}_{n=0}^{\infty}$ be a sequence of distributions in $\mathcal{S}'(\mathbb{R}^d)$. Suppose there exists a smooth function $\psi\in\mathcal{D}(V\cap\mathbb{S})$ such that the generalized functions of one variable $f_n(r)=\langle F_n(r\omega),\psi(\omega)\rangle_\omega$ have extensions* ([7]) *in $\mathcal{S}'_+$ that satisfy* (**P1**) *and* (**P2**). *Then for every sequence $\{\mathbf{a}_n\}_{n=0}^{\infty}$ of vectors in a Fréchet space $\mathcal{X}$ the moment problem* (8.1) *has solutions in $\mathcal{S}(V,\mathcal{X})$.*

Proof. Indeed, the moment problem $\langle f_n(r),\varphi(r)\rangle = \mathbf{a}_n$ has solutions $\varphi\in\mathcal{S}((0,\infty),\mathcal{X})$. If we set

$$\phi(\mathbf{x}) = |\mathbf{x}|^{1-d}\varphi(|\mathbf{x}|)\psi(\mathbf{x}/|\mathbf{x}|),$$

then $\phi\in\mathcal{S}(V,\mathcal{X})$ and it satisfies the moment problem (8.1). ∎

Proposition 8.1 implies that if the F_n's are radial distributions, that is, they depend only on $|\mathbf{x}|$, $F_n(\mathbf{x})=f_n(|\mathbf{x}|)$, and the distributions ([8]) f_n satisfy (**P1**) and (**P2**), then (8.1) has solutions in $\mathcal{S}(V,\mathcal{X})$ for *any* cone V: we may just take as ψ any element of $\mathcal{S}(V\cap\mathbb{S})$ whose integral over $V\cap\mathbb{S}$ does not vanish. Actually we can improve this result:

Corollary 8.2. *Let $\mathcal{X}$ be a Fréchet space and let $\{\mathbf{a}_k\}$ be an arbitrary sequence of elements of $\mathcal{X}$. Let $\{f_n\}_{n=0}^{\infty}$ be a sequence of distributions in $\mathcal{S}'_+$ for which* (**P1**) *and* (**P2**) *hold, and let $\{g_n\}_{n=0}^{\infty}$ be a distribution sequence in $\mathcal{D}'(\mathbb{S})$. If $\{F_n\}_{n=0}^{\infty}$ is any sequence of distributions in $\mathcal{S}'(\mathbb{R}^d)$ such that*

$$\langle F_n(\mathbf{x}),\phi(\mathbf{x})\rangle = \langle f_n(r)\otimes g_n(\omega), r^{d-1}\phi(r\omega)\rangle \quad \textit{for each } \phi\in\mathcal{S}(V),$$

([6]) Interestingly [9], there are no *continuous* extension operators from $\mathcal{S}'(0,\infty)$ to $\mathcal{S}'_+$.

([7]) These distributions belong to the spaces $\mathcal{R}_d$ introduced in [15].

([8]) The distributions f_n of one variable are not unique [10].

then (8.1) *is solvable in* $\mathcal{S}(V, \mathcal{X})$ *for any open cone* V *provided that there is an open set* U *of* $\mathbb{S}$ *such that* $\overline{U} \subset V \cap \mathbb{S}$ *and* $g_n \neq 0$ *on* U.

Proof. In fact, using Proposition 8.1, it suffices to check that there is a smooth function ψ with $\operatorname{supp}\psi \subseteq \overline{U}$ such that $\langle g_n(\omega), \psi(\omega)\rangle \neq 0$ for every $n \in \mathbb{N}$. To show this, consider the Fréchet space $\mathcal{Y} = \{\psi \in C^\infty(\mathbb{S}) : \operatorname{supp}\psi \subseteq \overline{U}\}$ and the sequence of closed subspaces $\mathcal{Y}_n = \ker_{\mathcal{Y}} g_n = \{\psi \in \mathcal{Y} : \langle g_n, \psi\rangle = 0\}$. Since each g_n is non-identically zero on U, we see that $\mathcal{Y}_n \neq \mathcal{Y}$ is of the first category. The Baire theorem implies that $\mathcal{Y} \neq \bigcup_{n=0}^{\infty} \mathcal{Y}_n$. ∎

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References

[1] R. P. Boas, *The Stieltjes moment problem for functions of bounded variation*, Bull. Amer. Math. Soc. 45 (1939), 399–404.

[2] J. Campos Ferreira, *Introduction to the Theory of Distributions*, Longman, Harlow, 1997.

[3] J. Chung, S.-Y. Chung and D. Kim, *Every Stieltjes moment problem has a solution in Gel'fand–Shilov spaces*, J. Math. Soc. Japan 55 (2003), 909–913.

[4] A. J. Durán, *The Stieltjes moments problem for rapidly decreasing functions*, Proc. Amer. Math. Soc. 107 (1989), 731–741.

[5] A. J. Durán, *The Stieltjes moment problem with complex exponents*, Math. Nachr. 158 (1992), 175–194.

[6] A. L. Durán and R. Estrada, *Strong moment problems for rapidly decreasing smooth functions*, Proc. Amer. Math. Soc. 120 (1994), 529–534.

[7] R. Estrada, *Vector moment problems for rapidly decreasing smooth functions of several variables*, Proc. Amer. Math. Soc. 126 (1998), 761–768.

[8] R. Estrada, *The Cesàro behaviour of distributions*, R. Soc. Lond. Proc. Ser. A Math. Phys. Eng. Sci. 454 (1998), 2425–2443.

[9] R. Estrada, *The nonexistence of regularization operators*, J. Math. Anal. Appl. 286 (2003), 1–10.

[10] R. Estrada, *On radial functions and distributions and their Fourier transforms*, J. Fourier Anal. Appl. 20 (2014), 301–320.

[11] R. Estrada and R. P. Kanwal, *A Distributional Approach to Asymptotics. Theory and Applications*, 2nd ed., Birkhäuser Boston, Boston, MA, 2002.

[12] R. Estrada and J. Vindas, *A general integral*, Dissertationes Math. 483 (2012).

[13] R. Estrada and J. Vindas, *On Tauber's second Tauberian theorem*, Tohoku Math. J. 64 (2012), 539–560.

[14] F. Galindo, F. López and J. Sanz, *A generalized moment problem for rapidly decreasing smooth vector functions of several variables*, J. Math. Anal. Appl. 263 (2001), 655–665.

[15] L. Grafakos and G. Teschl, *On Fourier transforms of radial functions and distributions*, J. Fourier Anal. Appl. 19 (2013), 167–179.

[16] G. Köthe, *Topological Vector Spaces. II*, Springer, New York, 1979.

[17] A. Lastra and J. Sanz, *Linear continuous operators for the Stieltjes moment problem in Gelfand–Shilov spaces*, J. Math. Anal. Appl. 340 (2008), 968–981.

[18] A. Lastra and J. Sanz, *Stieltjes moment problem in general Gelfand–Shilov spaces*, Studia Math. 192 (2009), 111–128.

[19] S. Łojasiewicz, *Sur la valeur et la limite d'une distribution en un point*, Studia Math. 16 (1957), 1–36.

[20] S. Pilipović, B. Stanković and J. Vindas, *Asymptotic Behavior of Generalized Functions*, Ser. Anal. Appl. Comput. 5, World Sci., Hackensack, NJ, 2012.

[21] G. Pólya, *Sur l'indétermination d'un problème voisin du problème des moments*, C. R. Acad. Sci. Paris 207 (1938), 708–711.

[22] A. P. Robertson and W. Robertson, *Topological Vector Spaces*, 2nd ed., Cambridge Univ. Press, London, 1973.

[23] J. Sebastião e Silva, *Su certe classi di spazi localmente convessi importanti per le applicazioni*, Rend. Mat. Appl. (5) 14 (1955), 388–410.

[24] J. A. Shohat and J. D. Tamarkin, *The Problem of Moments*, Amer. Math. Soc. Math. Surveys 1, Amer. Math. Soc., New York, 1943.

[25] T. J. Stieltjes, *Recherches sur les fractions continues*, Reprint of the 1894 original, Ann. Fac. Sci. Toulouse Math. (6) 4 (1995), J.1–J.35, J.36–J.75.

[26] F. Trèves, *Topological Vector Spaces, Distributions and Kernels*, Academic Press, New York, 1967.

[27] J. Vindas and R. Estrada, *On the support of tempered distributions*, Proc. Edinburgh Math. Soc. 53 (2010), 255–270.

[28] V. S. Vladimirov, *Methods of the Theory of Generalized Functions*, Anal. Methods Spec. Funct. 6, Taylor & Francis, London, 2002.

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