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Multiplicative maps into the spectrum

by

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Abstract. We consider the converse of a famous result of W. Żelazko et al. which characterizes multiplicative functionals amongst the dual space members of a complex unital Banach algebra A. Specifically, we investigate when a continuous multiplicative map $\phi : A \to \mathbb{C}$, with values $\phi(x)$ belonging to the spectrum of x, is automatically linear. Our main result states that if A is a $C^\star$-algebra, then ϕ always generates a corresponding character ψ_ϕ of A. It is then shown that ϕ shares many linear properties with its induced character. Moreover, if A is a von Neumann algebra, then it turns out that ϕ itself is linear, and that it corresponds to its induced character.

1. Introduction. Let A be a complex Banach algebra with identity element $\mathbf{1}$, invertible group $G(A)$, and spectrum $\sigma_A(x) := \{\lambda \in \mathbb{C} : \lambda\mathbf{1} - x \notin G(A)\}$ for each $x \in A$. Denote by A' the dual space of continuous linear functionals on A. A function $\phi : A \to \mathbb{C}$ is said to be *multiplicative* if $\phi(xy) = \phi(x)\phi(y)$ for all $x, y \in A$. A nonzero multiplicative linear functional on a Banach algebra A is commonly called a *character* of A. However, in contrast to linear functionals, the existence of characters on Banach algebras is generally not guaranteed, but they are always present when A is commutative. A classical result (see [4, 6, 10]) from the late sixties identifies characters amongst the members of A':

Theorem 1.1 (Gleason, Kahane, Żelazko). *Let A be a Banach algebra. Then $\phi \in A'$ is a character of A if and only if $\phi(x) \in \sigma(x)$ for each $x \in A$.*

For an elementary (nonanalytic) proof of Theorem 1.1, due to M. Roitman and Y. Sternfeld, see [1, Theorem 4.1.1]. In a more general vein, S. Kowalski and Z. Słodkowski [8, Theorem 1.2] give a characterization of characters on a Banach algebra A amongst the collection of all functions on A which map A to $\mathbb{C}$:

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Theorem 1.2 (Kowalski, Słodkowski). *Let A be a complex Banach algebra, and let $f : A \to \mathbb{C}$ satisfy*

(i) $f(0) = 0$,
(ii) $f(x) - f(y) \in \sigma(x - y)$ *for every* $x, y \in A$.

Then f is multiplicative and linear.

By a functional on a Banach algebra A we shall mean any function that maps A to its scalar field; specifically, for the purpose of this paper, a functional is not necessarily linear. A natural question which surfaces in the context of Theorem 1.1 is namely to what extent one can obtain linearity, given the hypothesis that the functional in question is multiplicative. There seems to be very little, in terms of published literature, which specifically addresses the aforementioned question (and one should perhaps also compare this to the large number of publications on "linear preserver problems" *versus* the number of results on "multiplicative preservers"). So it is (empirically) evident that "multiplicative problems" are much harder than "linear problems". Closest to the current paper appears to be an article of Maouche's [9] (see the comments following Corollary 3.4), and, to a lesser extent, the study of pseudodeterminants (see for instance [7]) as well as early work of L. Carleson [2]. In particular, the following result, which has its origin in [2], shows that continuous nonzero multiplicative functionals on Banach algebras do exhibit some good analytic behaviour:

Theorem 1.3 (Carleson). *Let ϕ be a nonzero continuous multiplicative functional on A, and let $x \in A$ be arbitrary. Then the map*

$$\lambda \mapsto \log|\phi(x - \lambda\mathbf{1})|$$

is harmonic on the unbounded connected component of $\mathbb{C} \setminus \sigma(x)$.

Remark 1.4. Carleson's definition of a multiplicative function requires the map to be real valued which necessitates the use of $|\phi|$ rather than ϕ in the statement of Theorem 1.3.

Problem 1.5. *Let A be a Banach algebra. When is a multiplicative functional ϕ on A satisfying $\phi(x) \in \sigma(x)$ for each $x \in A$ linear?*

To see that the answer to Problem 1.5 cannot be "always", consider the following simple example: Let $A = \mathbb{C}^2$ with the usual pointwise operations and any algebra norm. Define $\phi : A \to \mathbb{C}$ as follows:

$$\phi((\alpha, \gamma)) = \begin{cases} \gamma, & \alpha \neq 0, \\ 0, & \alpha = 0. \end{cases}$$

It is clear that ϕ is not linear but that it is multiplicative (and also homogeneous). Moreover, $\phi(x) \in \sigma(x)$ for each $x \in A$. Observe also that ϕ here is not continuous on A.

2. General Banach algebras. Throughout the paper we shall denote the connected component of $G(A)$ containing $\mathbf{1}$ by $G_{\mathbf{1}}(A)$. If A is a commutative Banach algebra then Δ denotes the collection of characters of A.

THEOREM 2.1. *Let A be a Banach algebra. Then a multiplicative functional ϕ on A satisfying $\phi(x) \in \sigma(x)$ for each $x \in A$ is linear if and only if for each $x \in A$ the map*

$$\lambda \mapsto \phi(\lambda\mathbf{1} - x) \tag{2.1}$$

is an entire function on $\mathbb{C}$.

Proof. If ϕ is linear, then $\phi(\lambda\mathbf{1}-x) = \lambda-\phi(x)$, which is obviously entire. Assume, conversely, that ϕ is multiplicative on A, $\phi(x) \in \sigma(x)$, and for each $x \in A$ the function $\tau_x(\lambda) := \phi(\lambda\mathbf{1} - x)$ is an entire function on $\mathbb{C}$. Now observe that, by spectral translation, we have $\tau_x(\lambda) - \lambda \in \sigma(-x)$ for each $\lambda \in \mathbb{C}$. Since

$$[\phi(\mathbf{1})]^2 = \phi(\mathbf{1}) \in \sigma(\mathbf{1}) = \{1\},$$

it follows that $\phi(\mathbf{1}) = 1$. A simple argument then implies that $\phi(\lambda x) = \lambda\phi(x)$ for all $x \in A$ and $\lambda \in \mathbb{C}$, so ϕ is homogeneous. Now, because $\sigma(-x)$ is bounded, and the map

$$\lambda \mapsto \tau_x(\lambda) - \lambda$$

is entire, it must be constant. In particular, since ϕ is homogeneous, we see that $\tau_x(\lambda)-\lambda = -\phi(x)$ for each $\lambda \in \mathbb{C}$, from which $\phi(\lambda\mathbf{1}-x) = \lambda-\phi(x)$ for each $x \in A$ and $\lambda \in \mathbb{C}$. We are now in a position to prove that ϕ is additive: Let $x, y \in A$ and fix $\lambda \in \mathbb{C}$ such that $|\lambda| > \|y\|$. Then

$$\begin{aligned}\phi(y + x) &= \phi\big((\lambda\mathbf{1} + y) + (x - \lambda\mathbf{1})\big)\\ &= \phi\big(\mathbf{1} + (x - \lambda\mathbf{1})(\lambda\mathbf{1} + y)^{-1}\big)\phi(\lambda\mathbf{1} + y)\\ &= \big[1 + \phi\big((x - \lambda\mathbf{1})(\lambda\mathbf{1} + y)^{-1}\big)\big]\phi(\lambda\mathbf{1} + y)\\ &= \phi(\lambda\mathbf{1} + y) + \phi(x - \lambda\mathbf{1}) = \phi(y) + \phi(x). \ \blacksquare\end{aligned}$$

We can improve Theorem 2.1 by replacing the assumption on (2.1) by the substantially weaker requirement that the maps

$$\lambda \mapsto |\phi(x - \lambda\mathbf{1}) + \lambda|$$

are subharmonic on $\mathbb{C}$ for each $x \in A$:

COROLLARY 2.2. *Let A be a Banach algebra. Then a multiplicative functional ϕ on A satisfying $\phi(x) \in \sigma(x)$ for each $x \in A$ is linear if and only if for each $x \in A$ the map*

$$\lambda \mapsto |\phi(x - \lambda\mathbf{1}) + \lambda| \tag{2.2}$$

is subharmonic on $\mathbb{C}$.

Proof. Assume (2.2) is subharmonic for each $x \in A$ (the reverse implication is clear). Since $|\phi(x-\lambda\mathbf{1})+\lambda| \le \|x\|$, it follows immediately that $|\phi(x-\lambda\mathbf{1})+\lambda|$ is constant on $\mathbb{C}$, whence

$$|\phi(x-\lambda\mathbf{1})+\lambda| = |\phi(x)| \tag{2.3}$$

for all $x \in A$ and $\lambda \in \mathbb{C}$. By substituting $x+\lambda\mathbf{1}$ for x in (2.3) we find that $|\phi(x)+\lambda| = |\phi(x+\lambda\mathbf{1})|$ for all $x \in A$ and $\lambda \in \mathbb{C}$, and so

$$|\phi(x-\lambda\mathbf{1})| = |\phi(x)-\lambda| \tag{2.4}$$

for all $x \in A$ and $\lambda \in \mathbb{C}$. With $\lambda = \phi(x)$ in (2.4) we then have $\phi(x-\phi(x)\mathbf{1}) = 0$, from which (2.3) again gives

$$\big|\phi\big((x-\phi(x)\mathbf{1})-\lambda\mathbf{1}\big)+\lambda\big| = |\phi(x-\phi(x)\mathbf{1})| = 0 \tag{2.5}$$

for all $x \in A$ and $\lambda \in \mathbb{C}$. If we replace λ by $-\phi(x)+\lambda$ in (2.5), then we arrive at

$$\phi(x-\lambda\mathbf{1}) = \phi(x)-\lambda,$$

valid for all $x \in A$ and $\lambda \in \mathbb{C}$. Theorem 2.1 then implies that ϕ is linear. ∎

COROLLARY 2.3. *Let A be a Banach algebra such that $\sigma(x)$ is totally disconnected for each $x \in A$. Then a multiplicative functional ϕ on A, satisfying $\phi(x) \in \sigma(x)$ for each $x \in A$, is linear if and only if ϕ is continuous on A.*

Proof. Since characters are automatically continuous, the forward implication is trivial. We prove the reverse implication: Since, by assumption, $\sigma(-x)$ is totally disconnected, the continuous function $\tau_x(\lambda)-\lambda$ (which takes its values in $\sigma(-x)$) must be constant on $\mathbb{C}$. The result follows from Theorem 2.1. ∎

Theorem 2.1, and the example in Section 1, lead to the following question:

PROBLEM 2.4. *Let A be a Banach algebra. Is a continuous multiplicative functional ϕ on A satisfying $\phi(x) \in \sigma(x)$ for each $x \in A$ necessarily linear?*

The question seems to be nontrivial; despite the simplicity of the example in Section 1, we could not find an example of a continuous multiplicative map with values in the spectrum which is not a character.

3. $C^\star$-algebras. We investigate Problem 2.4 for the case where A is a $C^\star$-algebra; in particular we show that if A is a von Neumann algebra then ϕ is a character of A. We start with

LEMMA 3.1. *Let A be a commutative Banach algebra, and let $\phi : A \to \mathbb{C}$ be a continuous multiplicative map such that $\phi(x) \in \sigma(x)$ for each $x \in A$. Then, corresponding to each $x \in A$ and $\lambda \in \mathbb{C}$, there exist characters χ_λ depending on λ and x, and characters χ_x and $\tilde{\chi}_x$ depending on x such that:*

(i) $\phi(e^{\lambda x}) = e^{\mathrm{Re}(\lambda)\chi_x(x)}e^{\mathrm{Im}(\lambda)\tilde{\chi}_x(x)i} = e^{\lambda\chi_\lambda(x)}$,
(ii) $\mathrm{Re}(\lambda)[\mathrm{Re}\,\chi_x(x) - \mathrm{Re}\,\chi_\lambda(x)] = \mathrm{Im}(\lambda)[\mathrm{Im}\,\tilde{\chi}_x(x) - \mathrm{Im}\,\chi_\lambda(x)]$,
(iii) $\mathrm{Re}(\lambda)[\mathrm{Im}\,\chi_x(x) - \mathrm{Im}\,\chi_\lambda(x)] = \mathrm{Im}(\lambda)[\mathrm{Re}\,\chi_\lambda(x) - \mathrm{Re}\,\tilde{\chi}_x(x)]$.

Proof. Suppose that A is a commutative Banach algebra, $x \in A$, and $\sigma(x)$ lies in the fundamental strip $-\pi < \mathrm{Im}(z) \le \pi$. Let $m \in \mathbb{N}$. Then there exists $\chi_m \in \Delta$ (depending on x and m) such that

$$\phi(e^{x/m}) = \chi_m(e^{x/m}) = e^{\frac{1}{m}\chi_m(x)}.$$

Taking mth powers and using the fact that ϕ is multiplicative we see that

$$\phi(e^x) = e^{\chi_m(x)}.$$

On the other hand, there exists $\chi_x \in \Delta$ such that

$$\phi(e^x) = e^{\chi_x(x)}.$$

So, since $\sigma(x)$ is in the fundamental strip, we have

$$\phi(e^{x/m}) = \chi_x(e^{x/m}) = e^{\frac{1}{m}\chi_x(x)}$$

for all $m \in \mathbb{N}$. That is, the dependence of the character on m and x reduces to dependence on only x. It is then clear that for each $n \in \mathbb{N}$ we have

$$\phi(e^{nx/m}) = e^{\frac{n}{m}\chi_x(x)}.$$

By continuity of ϕ, and the density of $\mathbb{Q}$ in $\mathbb{R}$, it follows that if $r > 0$ is a real number, then

$$\phi(e^{rx}) = e^{r\chi_x(x)}.$$

This extends to $r = 0$, since $\phi(x) \in \sigma(x)$ for each $x \in A$. Moreover, the spectral condition also implies that for $y \in A$ invertible we have $\phi(y^{-1}) = \phi(y)^{-1}$, whence we may conclude that

$$\phi(e^{rx}) = e^{r\chi_x(x)} \quad \text{for all } r \in \mathbb{R}. \tag{3.1}$$

In the same manner we can show that there exists $\tilde{\chi}_x \in \Delta$ such that

$$\phi(e^{six}) = e^{si\tilde{\chi}_x(x)} \quad \text{for all } s \in \mathbb{R}. \tag{3.2}$$

If $\sigma(x)$ is not in the fundamental strip, then $\sigma(x/M)$ is in the strip for some $M \in \mathbb{N}$ sufficiently large (which we now keep fixed). Following the same argument as before we can find $\chi_{x,M} \in \Delta$ such that

$$\phi(e^{rx/M}) = e^{r\chi_{x,M}(x/M)}$$

for all $r \in \mathbb{R}$. Raising to Mth powers throughout we find that

$$\phi(e^{rx}) = e^{r\chi_{x,M}(x)}$$

for all $r \in \mathbb{R}$. This argument suffices to prove that (3.1) and (3.2) extend to $x \in A$ even if $\sigma(x)$ is not in the fundamental strip.

Now let $\lambda \in \mathbb{C}$. From the preceding paragraph observe that

$$\phi(e^{\lambda x}) = e^{\mathrm{Re}(\lambda)\chi_x(x)}e^{\mathrm{Im}(\lambda)i\tilde{\chi}_x(x)}. \tag{3.3}$$

On the other hand, the spectral condition implies that, for each $\lambda \in \mathbb{C}$, there exists $\chi_\lambda \in \Delta$ (depending on λ and x) such that

$$\phi(e^{\lambda x}) = e^{\lambda \chi_\lambda(x)}, \tag{3.4}$$

which proves (i).

We now show that the map

$$\lambda \mapsto \chi_\lambda(x)$$

is continuous on $\mathbb{C} \setminus \{0\}$. Let (λ_n) be a sequence of complex numbers such that $\lim \lambda_n = \lambda_0 \neq 0$. Since (λ_n) is bounded, there exists $M > 0$ such that, for each $n \in \mathbb{N} \cup \{0\}$, $\sigma(\lambda_n x/M)$ lies in the fundamental strip. By continuity of ϕ it follows that

$$\lim \phi(e^{\lambda_n x/M}) = \phi(e^{\lambda_0 x/M}) = e^{\lambda_0 \chi_{\lambda_0}(x/M)}.$$

If we use (3.4) on the other hand, then we also obtain

$$\lim e^{\lambda_n \chi_{\lambda_n}(x/M)} = e^{\lambda_0 \chi_{\lambda_0}(x/M)}.$$

So, since the exponential function is bijective on the fundamental strip and $\lambda_0 \neq 0$, we see that $\lim \chi_{\lambda_n}(x) = \chi_{\lambda_0}(x)$, which proves the continuity claim.

Returning to (3.3) and (3.4) we split the expressions into real and imaginary parts, and by comparison we obtain

$$\operatorname{Re}(\lambda)[\operatorname{Re} \chi_x(x) - \operatorname{Re} \chi_\lambda(x)] = \operatorname{Im}(\lambda)[\operatorname{Im} \tilde{\chi}_x(x) - \operatorname{Im} \chi_\lambda(x)] \tag{3.5}$$

and

$$\operatorname{Re}(\lambda)[\operatorname{Im} \chi_x(x) - \operatorname{Im} \chi_\lambda(x)] = \operatorname{Im}(\lambda)[\operatorname{Re} \chi_\lambda(x) - \operatorname{Re} \tilde{\chi}_x(x)] + h(\lambda)2\pi \tag{3.6}$$

where $h(\cdot)$ is a function from $\mathbb{C}$ to $\mathbb{Z}$. From (3.6), using the fact that $\lambda \mapsto \chi_\lambda(x)$ is continuous on $\mathbb{C} \setminus \{0\}$, it follows that $h(\lambda)$ is also continuous on $\mathbb{C}\setminus\{0\}$. So, since h is discrete, it must be constant. Moreover, since characters map into the spectrum, both the expressions in the square brackets in (3.6) are bounded sets as λ runs through $\mathbb{C}$. In particular, if $\rho(x)$ denotes the spectral radius of x, we have

$$|h(\lambda)| \leq (|\operatorname{Re}(\lambda)| + |\operatorname{Im}(\lambda)|)\rho(x)/\pi.$$

Hence $\lim_{\lambda \to 0} h(\lambda) = h(0) = 0$, from which it follows that h is continuous on $\mathbb{C}$. We may infer that $h = 0$ on $\mathbb{C}$. ∎

If A is a $C^\star$-algebra then Lemma 3.1 leads to:

Theorem 3.2. *Let A be a $C^\star$-algebra and let $\phi : A \to \mathbb{C}$ be a continuous multiplicative functional such that $\phi(x) \in \sigma(x)$ for each $x \in A$. Then, for each $x \in A$, the map*

$$\lambda \mapsto \phi(e^{\lambda x})$$

is an entire function. In particular, corresponding to each $x \in A$, there exists a unique $\alpha_x \in \sigma(x)$ such that

$$\phi(e^{\lambda x}) = e^{\lambda\alpha_x} \quad \textit{for all } \lambda \in \mathbb{C}.$$

Proof. With $x \in A$ fixed, write $x = u + iv$ where u and v are self-adjoint, so that $e^{\lambda x} = e^{\lambda u + \lambda vi}$. Using the Lie–Trotter formula (see [1, p. 67]), together with the assumptions that ϕ is multiplicative and continuous, we obtain

$$\begin{aligned}\phi(e^{\lambda x}) = \phi(e^{\lambda u+\lambda vi}) &= \phi\Big(\lim_n [e^{\lambda u/n} e^{i\lambda v/n}]^n\Big)\\ &= \lim_n \phi([e^{\lambda u/n} e^{\lambda vi/n}]^n) = \lim_n \phi(e^{\lambda u/n} e^{\lambda vi/n})^n\\ &= \lim_n \phi(e^{\lambda u/n})^n \lim_n \phi(e^{\lambda vi/n})^n = \phi(e^{\lambda u})\phi(e^{\lambda vi}).\end{aligned}$$

Now let $C_{\{u\}}$ be the bicommutant of $\{u\}$. Since $\sigma(u) \subset \mathbb{R}$, and since the spectrum is preserved in the bicommutant, it follows from Lemma 3.1 (with $A = C_{\{u\}}$) that there exists a unique real number $\alpha_u \in \sigma(u)$ such that $\phi(e^{\lambda u}) = e^{\lambda\alpha_u}$ for each $\lambda \in \mathbb{C}$. To see this, notice that we can write

$$\phi(e^{\lambda u}) = e^{\mathrm{Re}(\lambda)\alpha_1 + \mathrm{Im}(\lambda)\alpha_2 i}$$

where $\alpha_1, \alpha_2 \in \sigma(u)$ (Lemma 3.1(i)). But now, since u is self-adjoint, it follows from Lemma 3.1(ii) that

$$\mathrm{Re}(\lambda)[\alpha_1 - \chi_\lambda(u)] = \mathrm{Im}(\lambda)\cdot 0 = 0,$$

and from Lemma 3.1(iii) that

$$\mathrm{Im}(\lambda)[\chi_\lambda(u) - \alpha_2] = \mathrm{Re}(\lambda)\cdot 0 = 0.$$

Thus $\chi_\lambda(u) = \alpha_1 = \alpha_2$. A similar argument with v implies the existence of a unique real number α_v such that $\phi(e^{\lambda vi}) = e^{\lambda\alpha_v i}$ for each $\lambda \in \mathbb{C}$. Hence there exists a unique $\alpha_x := \alpha_u + \alpha_v i \in \mathbb{C}$ such that $\phi(e^{\lambda x}) = e^{\lambda\alpha_x}$ for each $\lambda \in \mathbb{C}$.

It remains to prove that $\alpha_x \in \sigma(x)$: Passing to $C_{\{x\}}$, there exists, for each $\lambda \in \mathbb{C}$, χ_λ in the character space of $C_{\{x\}}$ such that $\phi(e^{\lambda x}) = e^{\lambda\chi_\lambda(x)}$. If $|\lambda_0|$ is sufficiently small, then $\sigma(\lambda_0 x)$ lies in the fundamental strip, from which we obtain $\chi_{\lambda_0}(x) = \alpha_x$. But of course $\chi_{\lambda_0}(x) \in \sigma(x)$, which completes the proof. ■

COROLLARY 3.3. *Let A be a $C^\star$-algebra. Then every continuous multiplicative functional $\phi : A \to \mathbb{C}$ satisfying $\phi(x) \in \sigma(x)$ for each $x \in A$ generates a corresponding character ψ_ϕ on A.*

Proof. Define, corresponding to ϕ, the map $\psi_\phi : A \to \mathbb{C}$ by $\psi_\phi(x) = \alpha_x$ for each $x \in A$, with α_x as in Theorem 3.2. We prove that ψ_ϕ is linear: For $x, y \in A$ the Lie–Trotter formula again gives, for each $\lambda \in \mathbb{C}$,

$$e^{\lambda\alpha_{x+y}} = \phi(e^{\lambda(x+y)}) = \phi(e^{\lambda x})\phi(e^{\lambda y}) = e^{\lambda(\alpha_x+\alpha_y)}.$$

Since $\lambda \in \mathbb{C}$ is arbitrary, $\alpha_{x+y} = \alpha_x + \alpha_y$. To see that ψ_ϕ is homogeneous: If $\beta \in \mathbb{C}$ is fixed, then for each $\lambda \in \mathbb{C}$ we have

$$e^{\lambda\beta\alpha_x} = \phi(e^{(\lambda\beta)x}) = \phi(e^{\lambda(\beta x)}) = e^{\lambda\alpha_{\beta x}}.$$

Again, since λ is arbitrary, we have $\beta\alpha_x = \alpha_{\beta x}$. Thus ψ_ϕ is linear. Now, since $\psi_\phi(x) = \alpha_x \in \sigma(x)$ for each $x \in A$, it follows by Theorem 1.1 that ψ_ϕ is in fact a character. ∎

Corollary 3.4. *Let A be a $C^\star$-algebra and let $\phi : A \to \mathbb{C}$ be a continuous multiplicative functional such that $\phi(x) \in \sigma(x)$ for each $x \in A$. Then ϕ agrees with ψ_ϕ on $G_{\mathbf{1}}(A)$.*

Proof. Let $z \in G_{\mathbf{1}}(A)$. Then z can be expressed as

$$z = \prod_{j=1}^{n} e^{x_n}$$

for some $n \in \mathbb{N}$ and $x_j \in A$. The result follows from the fact that ϕ is multiplicative together with Theorem 3.2 and Corollary 3.3. ∎

We shall prove later that, in fact, ϕ agrees with ψ_ϕ on $G(A)$.

Remark 3.5. In [9] Maouche shows, using Theorem 1.2, that if A is any Banach algebra, and if ϕ is a multiplicative functional with values belonging to the spectrum, then there exists a corresponding character on A which agrees with ϕ on $G_{\mathbf{1}}(A)$. Maouche's result can easily be deduced using Lemma 3.1, the Lie–Trotter formula, the elementary result of [8, Lemma 2.1], and Theorem 1.1.

Lemma 3.6. *Let A be a $C^\star$-algebra, and let $\phi : A \to \mathbb{C}$ be a continuous multiplicative functional such that $\phi(x) \in \sigma(x)$ for each $x \in A$. Then, for each $x \in A$, the function*

$$\lambda \mapsto \phi(x - \lambda\mathbf{1})$$

is analytic on the unbounded component of $\mathbb{C} \setminus \sigma(x)$.

Proof. Suppose λ_0 belongs to the unbounded component of $\mathbb{C} \setminus \sigma(x)$. Then there exists an open disc $B_0 := B(\lambda_0, r_0)$ such that for each $\lambda \in B_0$ we have $x - \lambda\mathbf{1} \in G(A)$ and $\log(x - \lambda\mathbf{1})$ is a well-defined element of A. Moreover, the function $h : \lambda \to \log(x - \lambda\mathbf{1})$ is analytic from B_0 into A, and so it has a power series representation

$$h(\lambda) = \sum_{j=0}^{\infty} a_j(\lambda - \lambda_0)^j \qquad (a_j \in A),$$

valid for all $\lambda \in B_0$. Define, for each $n \in \mathbb{N}$ and $\lambda \in \mathbb{C}$,

$$h_n(\lambda) = \sum_{j=0}^{n} a_j(\lambda - \lambda_0)^j.$$

Then, since ϕ is multiplicative,

$$\phi(e^{h_n(\lambda)}) = \prod_{j=0}^{n} \phi(e^{a_j(\lambda-\lambda_0)^j}).$$

By Theorem 3.2, the functions $\tau_n : \lambda \mapsto \phi(e^{h_n(\lambda)})$ are analytic for each n, and the sequence τ_n converges pointwise to the continuous function $\tau(\lambda) := \phi(x - \lambda\mathbf{1})$ and must therefore be analytic on B_0 (see for instance [3, Theorem 2.1, p. 151]). Since λ_0 is an arbitrary number in the unbounded component of $\mathbb{C} \setminus \sigma(x)$, the result now follows. ∎

LEMMA 3.7. *Let A be a $C^\star$-algebra, and let $\phi : A \to \mathbb{C}$ be a continuous multiplicative functional such that $\phi(x) \in \sigma(x)$ for each $x \in A$. If $x \in A$ is self-adjoint, then*

$$\phi(x - \lambda\mathbf{1}) = \phi(x) - \lambda \quad \textit{for each } \lambda \in \mathbb{C}.$$

Proof. By Lemma 3.6, and the fact that $\sigma(x) \subset \mathbb{R}$, the function

$$\lambda \mapsto \phi(x - \lambda\mathbf{1}) + \lambda$$

is analytic everywhere except possibly on a compact subset of $\mathbb{R}$. But the range of this function is contained in the aforementioned compact set. So, by the Open Mapping Theorem for analytic functions, $\phi(x - \lambda\mathbf{1}) + \lambda$ must in fact be constant on $\mathbb{C} \setminus \sigma(x)$. By continuity it must then necessarily be constant on $\mathbb{C}$, and in particular $\phi(x - \lambda\mathbf{1}) = \phi(x) - \lambda$ for each $\lambda \in \mathbb{C}$. ∎

We may observe that Lemma 3.7 also holds under the more general condition that $\sigma(x)$ has empty interior.

COROLLARY 3.8. *Let A be a $C^\star$-algebra, and let $\phi : A \to \mathbb{C}$ be a continuous multiplicative functional such that $\phi(x) \in \sigma(x)$ for each $x \in A$. If $x \in A$ is self-adjoint, then $\phi(x) = \psi_\phi(x)$.*

Proof. Let $t \in \mathbb{R}$. Then, using Lemma 3.7, we have

$$\begin{aligned}\phi(e^{tx}) &= \phi(\mathbf{1} + tx + t^2x^2/2! + \cdots) = 1 + \phi(tx + t^2x^2/2! + \cdots)\\ &= 1 + \phi(tx(\mathbf{1} + tx/2! + \cdots)) = 1 + t\phi(x)\phi(\mathbf{1} + tx/2! + \cdots),\end{aligned}$$

from which an inductive argument, together with continuity of ϕ, yields

$$\phi(e^{tx}) = e^{t\phi(x)} \qquad (t \in \mathbb{R}).$$

But we also have $\phi(e^{tx}) = e^{t\psi_\phi(x)}$ for each $t \in \mathbb{R}$. Thus $\phi(x) = \psi_\phi(x)$ as advertised. ∎

Let $\mathcal{S}$ be the (real) Banach space of self-adjoint elements of A. We immediately have the following:

COROLLARY 3.9. *Let A be a $C^\star$-algebra, and let $\phi : A \to \mathbb{C}$ be a continuous multiplicative functional such that $\phi(x) \in \sigma(x)$ for each $x \in A$. Then ϕ is linear on $\mathcal{S}$.*

Corollary 3.10. *Let A be a $C^\star$-algebra, and let $\phi : A \to \mathbb{C}$ be a continuous multiplicative functional such that $\phi(x) \in \sigma(x)$ for each $x \in A$. If $u \in A$ is unitary then*

$$\phi(u - \lambda\mathbf{1}) = \psi_\phi(u) - \lambda = \phi(u) - \lambda \quad \text{for all } \lambda \in \mathbb{C}.$$

Proof. We shall prove the result for $\sigma(u) = S$ where S is the unit circle in $\mathbb{C}$; the case where the spectral containment is proper is easier. If $|\lambda| > 1$ then, by Corollary 3.4,

$$\phi(u - \lambda\mathbf{1}) = \psi_\phi(u) - \lambda,$$

which extends by continuity to $|\lambda| \geq 1$. If $|\lambda| < 1$ then $u - \lambda\mathbf{1}$ might not be in $G_{\mathbf{1}}(A)$. But, of course, for all λ with $|\lambda| < 1$ the elements $u - \lambda\mathbf{1}$ belong to a common coset of the quotient group $G(A)/G_{\mathbf{1}}(A)$, say $u - \lambda\mathbf{1} \in aG_{\mathbf{1}}(A)$ for some $a \in G(A)$. This means that $u - \lambda\mathbf{1}$ takes the form $u - \lambda\mathbf{1} = aw(\lambda)$ where w takes values in $G_{\mathbf{1}}(A)$. Since a is fixed and invertible, and the map $\lambda \mapsto u - \lambda\mathbf{1}$ analytic into A, it follows that $\lambda \mapsto w(\lambda)$ is analytic on the open unit disk. Then observe that

$$\phi(u - \lambda\mathbf{1}) = \phi(aw(\lambda)) = \phi(a)\phi(w(\lambda)) = \phi(a)\psi_\phi(w(\lambda)).$$

Since ψ_ϕ is linear, $\psi_\phi \circ w$ is analytic on $|\lambda| < 1$. Hence the function $\phi(u - \lambda\mathbf{1}) + \lambda$ is analytic on $|\lambda| < 1$. But this function maps into $\sigma(u)$, a set with empty interior, whence as before, from the Open Mapping Theorem, $\phi(u - \lambda\mathbf{1}) + \lambda$ must be constant on $|\lambda| < 1$. A continuity argument on the unit circle, together with the fact that $\phi(u - \lambda\mathbf{1}) = \psi_\phi(u) - \lambda$ when $|\lambda| \geq 1$, implies that $\phi(u - \lambda\mathbf{1}) = \psi_\phi(u) - \lambda$ for all $\lambda \in \mathbb{C}$. The rest of the proof is now clear. ■

Corollary 3.11. *Let A be a $C^\star$-algebra, and let $\phi : A \to \mathbb{C}$ be a continuous multiplicative functional such that $\phi(x) \in \sigma(x)$ for each $x \in A$. If u_1, u_2 are unitary, then*

$$\phi(\alpha u_1 + \beta u_2) = \alpha\phi(u_1) + \beta\phi(u_2) \quad \text{for all } \alpha, \beta \in \mathbb{C}.$$

Proof. We may assume $\alpha, \beta \neq 0$. Write

$$\alpha u_1 + \beta u_2 = \beta u_1(\alpha\beta^{-1}\mathbf{1} + u_1^{-1}u_2).$$

Then the result follows from the assumption that ϕ is multiplicative, together with Corollary 3.10 and the fact that the unitary elements is a group under the multiplication of A. ■

Corollary 3.11 seems interesting because if it could be extended to hold for linear combinations of three unitary elements, then ϕ would be a character. Indeed, let $0 \neq x \in A$. Then, by [5, Corollary 2], there exist $\alpha > 0$ and unitary elements $u_1, u_2, u_3 \in A$ such that $x = \alpha(u_1 + u_2 + u_3)$. So, by the extended Corollary 3.11, we would have

$$\begin{aligned}\phi(x) &= \phi(\alpha(u_1 + u_2 + u_3)) = \alpha(\phi(u_1) + \phi(u_2) + \phi(u_3))\\ &= \alpha(\psi_\phi(u_1) + \psi_\phi(u_2) + \psi_\phi(u_3)) = \psi_\phi(x).\end{aligned}$$

THEOREM 3.12. *Let A be a $C^\star$-algebra, and let $\phi : A \to \mathbb{C}$ be a continuous multiplicative functional such that $\phi(x) \in \sigma(x)$ for each $x \in A$. Then ϕ agrees with ψ_ϕ on the closure of $G(A)$. In particular, if $G(A)$ is dense in A, then ϕ is a character of A.*

Proof. Every invertible element x of A admits a polar decomposition $x = uh$ where u is unitary and h self-adjoint. Then, for each invertible $x \in A$,

$$\phi(x) = \phi(u)\phi(h) = \psi_\phi(u)\psi_\phi(h) = \psi_\phi(uh) = \psi_\phi(x).$$

By continuity, the equality of ϕ and ψ_ϕ extends to the boundary of $G(A)$. ∎

If A is a von Neumann algebra, then one can go further and deduce that ϕ must be a character of A:

THEOREM 3.13. *Let A be a von Neumann algebra, and let $\phi : A \to \mathbb{C}$ be a continuous multiplicative functional such that $\phi(x) \in \sigma(x)$ for each $x \in A$. Then ϕ is a character of A.*

Proof. Let ψ_ϕ be the character generated by ϕ, as in Corollary 3.3. We shall first prove that ϕ and ψ_ϕ are absolutely equal, that is, $|\phi(x)| = |\psi_\phi(x)|$ for each $x \in A$: Let $u \in A$ be any partial isometry. Since $\|u\| = 1$ and $\phi(u) \in \sigma(u)$, we have $|\phi(u)| \le 1$. If we had $0 < |\phi(u)| < 1$, then we are forced to conclude that $1 < |\phi(u^\star)|$, contradicting the fact that $u^\star$ is also a partial isometry. So $|\phi(u)| = 1$ or $\phi(u) = 0$. The same is obviously true for ψ_ϕ, i.e. $|\psi_\phi(u)| = 1$ or $\psi_\phi(u) = 0$. If $\phi(u) = 0$ then, since $u^\star u u^\star = u^\star$, we also have $\phi(u^\star) = 0$; if $|\phi(u)| = 1$ then, since $uu^\star u = u$, we also have $|\phi(u^\star)| = 1$. Thus, since projections belong to $\partial G(A)$, the boundary of $G(A)$, it follows from Theorem 3.12 that $|\phi(u)| = |\psi_\phi(u)|$. If x is arbitrary in a von Neumann algebra, then x has a representation $x = uh$ where u is a partial isometry and $h \ge 0$. So, since ϕ and ψ_ϕ agree on $\mathcal{S}$, it follows that $|\phi(x)| = |\psi_\phi(x)|$.

Next, let $x \in A$ have a representation of the form

$$x = \sum_{j=1}^{n} \alpha_j p_j + i \sum_{j=1}^{m} \beta_j q_j, \tag{3.7}$$

where $\alpha_j, \beta_j \in \mathbb{R}$, and $\{p_1, \ldots, p_n\}$ and $\{q_1, \ldots, q_m\}$ each consist of mutually orthogonal self-adjoint projections. To prove that $\psi_\phi(x) = \phi(x)$ we consider three cases: Suppose firstly that $\psi_\phi(p_l), \psi_\phi(q_k) \ne 0$ for some l and k. Observe that then $p_l x q_k = (\alpha_l + i\beta_k)p_l q_k$, from which it directly follows that $\psi_\phi(x) = \phi(x) = \alpha_l + i\beta_k$. Secondly, suppose $\psi_\phi(p_l) = \psi_\phi(q_k) = 0$ for all l and k. Then it is clear that $\psi_\phi(x) = 0$. So, since ϕ and ψ_ϕ are absolutely equal, we have $\phi(x) = 0$. Thirdly, suppose ψ_ϕ annihilates one of the sets $\{q_1, \ldots, q_m\}$ or $\{p_1, \ldots, p_n\}$, but not both. Without loss of generality, say $\psi_\phi(q_k) = 0$ for each k, and $\psi_\phi(p_l) \neq 0$. If we write $q := q_1 + \cdots + q_m$, then $\mathbf{1} - q$ is a projection such that $\psi_\phi(\mathbf{1} - q) = 1$. If we now consider $p_l x(\mathbf{1} - q) = \alpha_l p_l(\mathbf{1} - q)$, then an easy calculation shows $\psi_\phi(x) = \alpha_l = \phi(x)$.

Finally, since the collection of elements of the form (3.7) is dense in a von Neumann algebra, and since ϕ is continuous on A, it follows that $\phi = \psi_\phi$ everywhere on A. ∎

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