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ABSOLUTE CONTINUITY OF SOLUTIONS TO THE AFFINE STOCHASTIC EQUATION

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Abstract. We study local regularity of perpetuities, i.e. solutions X to the stochastic equation $X \stackrel{d}{=} AX + B$. We give some nontrivial conditions on the law of (A, B) that imply absolute continuity of the law P_X of X.

1. Introduction. In this paper we study local regularity of solutions X to the stochastic equation

$$X \stackrel{d}{=} AX + B, \tag{1.1}$$

where (A, B) is an $\mathbb{R}^2$-valued random variable, X is independent of (A, B) and equality is understood in law. Random perpetuities X were introduced by Kesten [9] in the 1970's and have been widely used; however, not much is known about the absolute continuity of their distributions. By now, only examples of random perpetuities with an absolutely continuous or singular distributions have been known, and there is no general theory.

Our aim is to give some nontrivial conditions on the law μ of (A, B) that imply absolute continuity of X. Clearly, if either μ is absolutely continuous, or A, B are independent and one of them is absolutely continuous, then so is the law of X. On the other hand, there are discrete laws of (A, B) that imply absolute continuity of X, the best known being Bernoulli convolution.

Bernoulli convolution is a particular case of (1.1). Namely, if $B = 1, -1$ with probability $1/2$ and $A = \lambda$ for a $\lambda \in (0, 1)$ then the unique solution of (1.1) is called *Bernoulli convolution*. If $0 < \lambda < 1/2$ then the law P_X of X is continuous singular, and if $\lambda = 1/2$ then P_X is the uniform distribution on $[-2, 2]$. The latter follows from uniqueness of solution; for $\lambda = 1/2$ the proof is elementary (see e.g. [7]). However, when $\lambda > 1/2$, P_X may be absolutely continuous. Bernoulli convolutions were studied already by Erdős in

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the 1930's and they were the first nontrivial example of a random perpetuity with an absolutely continuous distribution.

It was proved by Solomyak [13] that the law P_X is absolutely continuous for almost every $\lambda \in (1/2, 1)$. Some generalizations of this result were given by Peres and Solomyak [10], [11]. Other examples were produced by Pratsiovytyi and Khvorostina [12] and Brieussel and Tanaka [5].

In all the examples mentioned above the distributions of the random vector (A, B) were discrete and the "infinite convolution" phenomenon played a crucial role in possible regularity of X. Indeed,

$$X = \sum_{n=1}^{\infty} A_1 \cdots A_{n-1} B_n,$$

where (A_n, B_n) is an i.i.d sequence of random variables with the same law as (A, B).

In this paper we establish some conditions under which finite convolution is enough to secure absolute continuity of P_X. In particular, we study spread-out measures, i.e. those whose k-fold convolutions have an absolutely continuous component.

The first part of the paper considers distributions of the random vector (A, B) supported by a curve and having a density on it. The recursion

$$X_n = \frac{-1}{A_n(1 + A_n)} X_{n-1} + \frac{1}{A_n} \tag{1.2}$$

studied in [12] fits into this setting provided A has a density. Then it follows from Theorem 3.1 that P_X is absolutely continuous. In [12] an integer valued A was considered, and it was proved that P_X is absolutely continuous if and only if $\mathbb{P}\{A = k\} = 1/(k(k+1))$, which is a much deeper result. We are grateful to Alexander Iksanov for pointing out this example to us.

In the second part we assume that μ is minorized by a product measure $\eta \times \sigma$ with B-component σ having a sufficiently good characteristic function. Again convolution improves properties of $(\eta \times \sigma)^{*p}$, and absolute continuity of P_X follows. Finally, in each case we consider, the chain X_n is strongly mixing with geometrical rate.

2. Elementary observations. Let (A, B) be an $\mathbb{R}^2$-valued random variable, and suppose that there exists a unique solution X to the stochastic equation

$$X \stackrel{d}{=} AX + B. \tag{2.1}$$

Sufficient and necessary conditions for the existence and uniqueness of the solution are described in [4]. Throughout this paper, we will assume that $\mathbb{E}\log^+|A| < 0$ and $\mathbb{E}\log^+|B| < \infty$.

We will moreover assume that A and B satisfy the following conditions:

$$\mathbb{P}(Ax + B = x) < 1 \quad \text{for every } x \in \mathbb{R}, \tag{2.2}$$

$$\mathbb{P}(A = 0) = 0. \tag{2.3}$$

Then the following result holds:

PROPOSITION 2.1. *The distribution P_X of the r.v. X does not have atoms and is of pure type, i.e. it is either absolutely continuous, or singular with respect to the Lebesgue measure.*

For the proof, see [2] or [6]. For two measures μ_1 and μ_2 on $\mathbb{R}$ or on $\mathbb{R}^2$ we write $\mu_1 \ll \mu_2$ if μ_1 is absolutely continuous with respect to μ_2, and $\mu_1 \le \mu_2$ if $\mu_1(U) \le \mu_2(U)$ for every Borel set U.

Suppose that there is a nonzero measure ν such that $\nu \le P_X$ and $\nu \ll \lambda$, where λ denotes the Lebesgue measure. Then by Proposition 2.1, $P_X \ll \lambda$.

We now define two types of convolutions.

DEFINITION 2.2. For two measures μ, ν on $\mathbb{R}^2$, let $\mu \star \nu$ denote the measure on $\mathbb{R}^2$ such that for any $f \in C_0(\mathbb{R}^2)$,

$$\int_{\mathbb{R}^2} f(a,b)\, d(\mu \star \nu)(a,b) = \int_{\mathbb{R}^2 \times \mathbb{R}^2} f(a_1 a_2, a_1 b_2 + b_1)\, d\mu(a_1, b_1)\, d\nu(a_2, b_2).$$

DEFINITION 2.3. For measures μ on $\mathbb{R}^2$ and P on $\mathbb{R}$, let $\mu \star P$ be the measure on $\mathbb{R}$ such that that for any $f \in C_0(\mathbb{R})$,

$$\int_{\mathbb{R}} f(x)\, d(\mu \star P)(x) = \int_{\mathbb{R}^2 \times \mathbb{R}} f(ax + b)\, d\mu(a,b)\, dP(x).$$

Actually, the former definition is simply convolution of measures on the semigroup $\mathbb{R}^2$ with "$ax + b$" multiplication:

$$(a_1, b_1)(a_2, b_2) = (a_1 a_2, a_1 b_2 + b_1),$$

and the latter refers to the action of this semigroup on $\mathbb{R}$. We use the same notation for both convolutions but this will not lead to any confusion. We have the following associativity: $(\mu \star \nu) \star P = \mu \star (\nu \star P)$. We start with an easy but useful observation.

PROPOSITION 2.4. *Let λ^2 be the Lebesgue measure on $\mathbb{R}^2$. Assume that $\mu \ll \lambda^2$ and P is a measure on $\mathbb{R}$. Then $\mu \star P \ll \lambda$.*

Proof. Let $\phi = d\mu/d\lambda^2$. Then for any $f \in C_0(\mathbb{R})$ we have

$$\begin{aligned}\int_{\mathbb{R}} f(x)\, d(\mu \star P)(x) &= \int_{\mathbb{R}\times\mathbb{R}^2} f(ax+b)\, d\mu(a,b)\, dP(x) \\ &= \int_{\mathbb{R}\times\mathbb{R}\times\mathbb{R}} f(ax+b)\phi(a,b)\, da\, db\, dP(x) \\ &= \int_{\mathbb{R}} f(\xi)\Big[\int_{\mathbb{R}\times\mathbb{R}} \phi(a,\xi-ax)\, dP(x)\, da\Big]\, d\xi \\ &= \int_{\mathbb{R}} f(\xi)\Phi(\xi)\, d\xi.\end{aligned}$$

Here the function $\Phi(\xi) = \int_{\mathbb{R}\times\mathbb{R}} \phi(a,\xi-ax)\, dP(x)\, da$ is the density of the measure $\mu \star P$. ■

From now on let μ denote the law of (A,B). Then (2.1) is equivalent to

$$P_X = \mu \star P_X.$$

We observe that

$$P_X = \mu \star P_X = \cdots = \mu^{\star p} \star P_X \tag{2.4}$$

for any $p \in \mathbb{N}$. Equivalently, by iterating (2.1) we get

$$X \stackrel{d}{=} A_1 \dots A_p X + \sum_{k=1}^{p} A_p \dots A_{k+1} B_k = \tilde{A}X + \tilde{B} \tag{2.5}$$

where $((A_k,B_k))_{k\in\mathbb{N}}$ is a sequence of independent copies of (A,B). Then the vector $(\tilde{A},\tilde{B})$ has the law $\mu^{\star p}$, and without loss of generality we may study (2.5) instead of (2.1), i.e., we can always consider $\mu^{\star p}$ instead of μ.

It may happen that $\mu^{\star p}$ has an absolutely continuous component with respect to the Lebesgue measure while μ does not. We say that the measure μ is *$\star$-spread-out* if it is spread-out in terms of $\star$-convolution, i.e. $\mu^{\star p}$ has an absolutely continuous component μ_1 for some $p \in \mathbb{N}$. Our strategy in the next section will be to show that μ is $\star$-spread-out. Then, from Proposition 2.1, we can immediately deduce the absolute continuity of P_X because

$$\mu_1 \star P_X \le \mu^{\star p} \star P_X = P_X.$$

A simple observation (made already e.g. in [6]) shows that less than existence of an absolutely continuous component of $\mu^{\star p}$ is sufficient. Namely, it is enough if $\mu^{\star p}$ dominates a product measure having an absolutely continuous horizontal or vertical component. We include the proof for completeness.

PROPOSITION 2.5. *Let μ denote the joint distribution of (A,B). Assume that there exist $p \in \mathbb{N}$ and measures η and σ such that $\mu^{\star p} \ge \eta \times \sigma$ and either $\eta \ll \lambda$ or $\sigma \ll \lambda$. Then the distribution P_X of X is absolutely continuous with respect to λ.*

Proof. Suppose that $\eta \ll \lambda$; the proof for the other case is identical. Let $\tilde{P}$ be a nonzero measure such that $\tilde{P} \le P_X$ and $0 \notin \operatorname{supp}(\tilde{P})$; in view of Proposition 2.1 such a measure exists. Since $\mu^{\star p} \ge \eta \times \sigma$, there exists a measure ρ such that $\mu^{\star p} = \eta \times \sigma + \rho$. By (2.4), for any integrable function f,

$$\int_{\mathbb{R}} f(x)\, dP_X(x) = \int_{\mathbb{R}\times\mathbb{R}^2} f(ax+b)\, dP_X(x)\, d\mu^{\star p}(a,b)$$
$$= \int_{\mathbb{R}\times\mathbb{R}\times\mathbb{R}} f(ax+b)\, dP_X(x)\, d\eta(a)\, d\sigma(b) + \int_{\mathbb{R}\times\mathbb{R}^2} f(ax+b)\, dP_X(x)\, d\rho(a,b)$$
$$= I_1 + I_2.$$

In view of the Radon–Nikodym theorem, there is a function $\phi = d\eta/d\lambda$. Then we can rewrite the first integral as

$$I_1 = \int_{\mathbb{R}\times\mathbb{R}\times\mathbb{R}} f(ax+b)\phi(a)\, da\, d\sigma(b)\, dP_X(x)$$
$$\ge \int_{\mathbb{R}\times\mathbb{R}\times\mathbb{R}} f(\xi)\phi\left(\frac{\xi - b}{x}\right)\frac{1}{x}\, d\xi\, d\sigma(b)\, d\tilde{P}(x)$$
$$= \int_{\mathbb{R}} f(\xi)\left[\int_{\mathbb{R}\times\mathbb{R}} \phi\left(\frac{\xi - b}{x}\right)\frac{1}{x}\, d\sigma(b)\, d\tilde{P}(x)\right] d\xi = \int_{\mathbb{R}} f(\xi)\Phi(\xi)\, d\xi.$$

The condition $0 \notin \operatorname{supp}(\tilde{P})$ implies that despite x appearing in the denominator, the integrand is bounded in the support of the measure, which allows us to use Fubini's theorem. The function Φ on the right hand side is a density of some measure. This means that P_X dominates an absolutely continuous measure, and hence, by Proposition 2.1, is itself absolutely continuous with respect to λ. ∎

3. Densities on curves. A natural question is whether we get a result analogous to Proposition 2.5 assuming absolute continuity on slant lines, or on curves. The answer is affirmative.

Before we formulate our main result, let us specify what is meant by a measure ν having a density on a curve. We say that ν has a *density on a curve* if the support of ν is not included in the line $y = px - p$ and there are a differentiable function g and an integrable function ϕ defined on an open interval I such that either

$$\int_{\mathbb{R}^2} f(a,b)\, d\nu(a,b) = \int_I f(a, g(a))\phi(a)\, da,$$

or

$$\int_{\mathbb{R}^2} f(a,b)\, d\nu(a,b) = \int_I f(g(b), b)\phi(b)\, db$$

for $f \in C_0(\mathbb{R}^2)$. Our main result is as follows.

Theorem 3.1. *Suppose that* (2.2) *and* (2.3) *are satisfied and there is* p *such that* $\mu^{\star p}$ *admits a decomposition* $\mu^{\star p} = \nu + \rho$, *where* ν *has a density on a curve. Then* P_X *is absolutely continuous.*

Theorem 3.1 follows from Theorem 3.2 below.

Theorem 3.2. *Suppose that the measure* μ *on* $\mathbb{R}^2$ *admits two decompositions:* $\mu = \mu_1 + \rho_1$ *and* $\mu = \mu_2 + \rho_2$, *where* μ_1, μ_2 *have densities on the curves* $(a, g(a))$ *and* $(h(b), b)$ *respectively. Assume moreover that* $g \in C^1(I_1)$ *and* $h \in C^1(I_2)$ *for some open intervals. Then* $\mu \star \mu$ *has an absolutely continuous component.*

Remark 3.3. In fact, it is sufficient to have only one, nonlinear function $g(x)$. Then it is not constant, so it has a nonzero derivative at some point x_0. From the inverse function theorem we deduce that g is invertible in some neigbourhood of x_0. Hence we can restrict g to that neighbourhood and take $h = g^{-1}$.

Proof of Theorem 3.2. For a function $f : \mathbb{R}^2 \to \mathbb{R}$, we have

$$\int_{\mathbb{R}^2} f(a,b)\, d(\mu \star \mu)(a,b) = \int_{\mathbb{R}^2} f(a,b)\, d(\mu_1 \star \mu_2)(a,b) + \int_{\mathbb{R}^2} f(a,b)\, d\rho(a,b)$$

where $\rho = \mu_1 \star \rho_2 + \rho_1 \star \mu_2 + \rho_1 \star \rho_2$. Now, we will take a closer look at the first integral on the right hand side having in mind that $\int_{\mathbb{R}^2} f(a,b)\, \mu_1(a,b) = \int_{I_1} f(a, g(a))\psi_1(a)\, da$ and $\int_{\mathbb{R}^2} f(a,b)\mu_2(a,b) = \int_{I_2} f(h(b), b)\psi_2(b)\, db$:

$$\begin{aligned}
&\int_{\mathbb{R}^2} f(a,b)\, d(\mu_1 \star \mu_2)(a,b) \\
&\qquad = \int_{\mathbb{R}^2 \times \mathbb{R}^2} f(a_1 a_2, a_1 b_2 + a_2)\, d\mu_1(a_1, b_1)\, d\mu_2(a_2, b_2) \\
&\qquad = \int_{I_1 \times I_2} f\big(a_1 h(b_2), a_1 b_2 + g(a_1)\big)\psi_1(a_1)\psi_2(b_2)\, da_1\, db_2 \\
&\qquad = \int_{F(I_1 \times I_2)} f(\alpha, \beta)\psi_1\big(a_1(\alpha,\beta)\big)\psi_2\big(b_2(\alpha,\beta)\big) J(\alpha,\beta)\, d\alpha\, d\beta.
\end{aligned}$$

To write the last equality, we need the inverse function theorem and so we possibly have to take smaller intervals I_1, I_2. Let us explain that more carefully. The Jacobian determinant of the map $(a_1, b_2) \mapsto (\alpha, \beta)$ is $\tilde{J}(a_1, b_2) = |h(b_2)a_1 - a_1 h'(b_2)(b_2 + g'(a_1))|$. In Proposition 2.5 we proved the result for vertical lines, hence now we may assume that $h'(b_2) \neq 0$.

Observe that $\tilde{J}$ vanishes for every (a_1, b_2) if and only if $\frac{h(b_2)}{h'(b_2)} - b_2 = g'(a_1)$. The left hand side depends only on b_2, and the right hand side only on a_1. Hence the only possibility is that they are both constant. For the right hand side, this means exactly that g is a linear function, say $g(x) = px + q$.

Then, $\tilde{J}(a_1, b_2) = 0$ if and only if $\frac{h'(b_2)}{h(b_2)} = \frac{1}{b_2+p}$, which we can write in the form $[\log h(b_2)]' = [\log(b_2 + p)]'$. The solutions h to the last equality are of course linear functions.

We have shown that the nonlinearity of any of the functions g or h yields $\tilde{J}(a_1, b_2) \not\equiv 0$. Now, let us take a closer look at the linear case, keeping the notation $g(x) = px + q$. Again, in view of Proposition 2.5, we may assume that $p \neq 0$. Note that then g is invertible and we can take $h = g^{-1}$, namely $h(x) = \frac{1}{p}x - \frac{1}{q}$. Then the condition $\tilde{J} \equiv 0$ is equivalent to $p = -q$, which we excluded while defining densities on curves.

We have shown that there is a point in $\mathbb{R}^2$ where $\tilde{J}$ is nonzero. Then, applying the inverse function theorem, we can find some neighbourhood on which $\tilde{J}$ is nonzero and there exists an inverse map $(\alpha, \beta) \mapsto (a_1, b_2)$ with Jacobian $J(\alpha, \beta) = [\tilde{J}(a_1, b_2)]^{-1}$.

Now we see that all the functions appearing in the last integral are well-defined when we restrict to some open subset $\Omega \subseteq \mathbb{R}^2$. There exist intervals I_1 and I_2 such that $I_1 \times I_2 \subseteq \Omega$, and without losing generality we can assume that the measures μ_1 and μ_2 are supported in I_1 and I_2 respectively. The choice of these intervals depends only on the functions g and h.

We can write the last integral in the form $\int_{\mathbb{R}\times\mathbb{R}} f(\alpha, \beta)\Phi(\alpha, \beta)\, d\alpha\, d\beta$, where Φ is a density of the measure $\mu_1 \star \mu_2$. Now, $\mu \star \mu \geq \mu_1 \star \mu_2$ and the proof is complete. ∎

Remark. Notice that due to the assumption (2.2), μ cannot be supported on the line $y = px - p$, because then taking $x = -p$ we obtain $Ax + B = -Ap + Ap - p = -p = x$ for any value of A. So the assumption in Theorem 3.1 is not very restrictive.

4. Fourier transform in "B" direction. In this section we consider a measure such that $\mu^{\star k}$ has a "good" B component for some k. More precisely, we assume that there are measures η and σ such that

$$\mu^{\star k} \geq \eta \times \sigma$$

and the Fourier transform $\hat{\sigma}$ of σ is in L^p for some $p > 0$. Then convolution powers of σ as well as of $\mu^{\star k}$ have better properties and we may deduce absolute continuity of P_X.

There are nontrivial singular measures σ such that σ^{*p} has a density. The construction of Salem sets in [8] uses a measure defined as the image of a positive measure under the Wiener function. The Fourier transform of the random measure obtained that way has a.s. the asymptotics at infinity defined by some properties of the initial measure. Therefore, it is possible to obtain a measure whose Fourier transform is in L^p for a fixed $p > 1$, and not in L^1. The construction of Kahane was pointed out to us by Jacek Zienkiewicz.

Recall that the formula for the Fourier transform of a measure ν on $\mathbb{R}$ is

$$\hat{\nu}(\xi) = \int_{\mathbb{R}} e^{ix\xi}\, d\nu(x).$$

The Fourier transform of a bounded (in particular probability) measure is a bounded function. One of the basic properties of this transformation is the convolution theorem, which can be formulated in a simple way: $\widehat{\mu * \nu} = \hat{\mu}\cdot\hat{\nu}$. Here $*$ denotes the standard convolution, which should not be confused with the $\star$-convolution defined in the previous section.

If we take a measure $\nu \ll \lambda$, we get $\hat{\nu}(\xi) = \int_{\mathbb{R}} e^{ix\xi}\, d\nu(x) = \int_{\mathbb{R}} \phi(x) e^{ix\xi}\, dx$, where $d\nu/d\lambda = \phi$ is the density function. In fact we obtain the Fourier transform of ϕ. Assuming that $\hat{\nu}$ is integrable, we can apply the inverse Fourier transform, and by the uniqueness theorem, we will get the function ϕ. This implies the following

Corollary 4.1. *Let ν be a bounded measure on $\mathbb{R}$. If $\hat{\nu} \in L^1$, then $\nu \ll \lambda$ and the density function $d\nu/d\lambda$ is the inverse Fourier transform of $\hat{\nu}$.*

Now we return to the distribution μ of the random vector (A, B).

Theorem 4.2. *Assume that there are measures η and σ such that $\mu^{\star k} \geq \eta \times \sigma$ where $\hat{\sigma} \in L^p$ for some $p \in \mathbb{N}$. Then $P_X \ll \lambda$.*

Proof. Recall that $P_X = \mu^{\star kp} \star P_X = (\mu^{\star k})^{\star p} \star P_X \geq (\eta \times \sigma)^{\star p} \star P_X = \rho$. We are going to show that $\rho \ll \lambda$. By the definition of $\star$-convolution, for $f \in C_0(\mathbb{R})$ we have

$$\int_{\mathbb{R}} f(x)\, d\rho(x) = \int_{\mathbb{R}^{2p+1}} f\Big(a_1 \ldots a_p x + \sum_{k=0}^{p-1} a_1 \ldots a_k b_{k+1}\Big) \times d\eta(a_1)\, d\sigma(b_1) \ldots d\eta(a_p)\, d\sigma(b_p)\, dP_X(x).$$

Now take any set $E \subset \mathbb{R}$ of zero Lebesgue measure. Denote $a = a_1 \ldots a_p$ and $\underline{a} = (a_1, \ldots, a_p)$. Then

$$\begin{aligned}
\rho(E) &= \int_{\mathbb{R}} \mathbb{1}_E(x)\, d\rho(x) \\
&= \int_{\mathbb{R}^{2p+1}} \mathbb{1}_E\Big(ax + \sum_{k=0}^{p-1} a_1 \ldots a_k b_{k+1}\Big)\, d\eta(a_1)\, d\sigma(b_1) \ldots d\eta(a_p)\, d\sigma(a_p)\, dP_X(x) \\
&= \int_{\mathbb{R}^{p+1}} \Big[\int_{\mathbb{R}^p} \mathbb{1}_E\Big(ax + \sum_{k=0}^{p-1} a_1 \ldots a_k b_{k+1}\Big)\, d\sigma(b_1) \ldots d\sigma(b_p)\Big] \\
&\qquad\qquad \times d\eta(a_1) \ldots d\eta(a_p)\, dP_X(x).
\end{aligned}$$

Let us focus on the inner integral. We fix the values of $a_1, \ldots, a_p$ and x. Let σ_k denote the distribution of the random variable $a_1 \ldots a_{k-1} B$, where

$B \sim \sigma$, and let $\zeta_{p,\underline{a}} = \zeta_p = \sigma_1 * \ldots * \sigma_p$ [1]. Then

$$\int_{\mathbb{R}^p} \mathbb{1}_E\Big(ax + \sum_{k=0}^{p-1} a_1 \ldots a_k b_{k+1}\Big)\, d\sigma(b_1) \ldots d\sigma(b_p) = \int_{\mathbb{R}} \mathbb{1}_{E-ax}(b)\, d\zeta_p(b).$$

We observe that $\hat{\sigma}_k(\xi) = \hat{\sigma}(a_1 \ldots a_{k-1}\xi)$ belongs to L^p as a function of ξ, i.e. $\int_{\mathbb{R}} |\hat{\sigma}(a_1 \ldots a_{k-1}\xi)|^p\, d\xi < \infty$. Now, by Hölder's inequality,

$$\begin{aligned}\int_{\mathbb{R}} |\hat{\zeta}_p(\xi)|\, d\xi &= \int_{\mathbb{R}\times\cdots\times\mathbb{R}} |\hat{\sigma}(\xi) \cdot \ldots \cdot \hat{\sigma}(a_1 \ldots a_{k-1}\xi)|\, d\xi \\ &\le \Big(\int_{\mathbb{R}} |\hat{\sigma}(\xi)|^p\, d\xi\Big)^{1/p} \cdot \ldots \cdot \Big(\int_{\mathbb{R}} |\hat{\sigma}(a_1 \ldots a_{k-1}\xi)|^p\, d\xi\Big)^{1/p} < \infty.\end{aligned}$$

Hence $\hat{\zeta}_p \in L^1$ and, by Corollary 4.1, the measure ζ_p is absolutely continuous with respect to the Lebesgue measure. Note that if $\lambda(E) = 0$, then also $\lambda(E - ax) = 0$, so in fact we integrate a bounded function over a zero-measure set, and hence the integral is zero.

The inner integral is zero for any $a_1, \ldots, a_p$ and x, so the whole expression is zero. Thus for any set $E \subset \mathbb{R}$ such that $\lambda(E) = 0$, we also have $\rho(E) = 0$, which means that $\rho \ll \lambda$. ■

Until now, we have always assumed that the measure μ, or its $\star$-convolution power $\mu^{\star p}$, dominates some product measure. Now we are going to take a look at more general situation: instead of a product, we will consider a desintegration. Sufficient conditions for a measure to have a desintegration are not very restrictive. Let us recall the desintegration theorem.

THEOREM 4.3. *Let $(X, \mathcal{F})$ be any measurable space and $(Y, \mathcal{G})$ a compact metric space, where $\mathcal{G}$ is the Borel σ-field. Let ν be any finite measure on the product σ-field $\mathcal{F} \otimes \mathcal{G}$. Then there exists a desintegration $\{\nu_x : x \in E\}$ of the measure ν with respect to the marginal measure ν_X on X given by $\nu_X(A) = \nu(A \times Y)$ for $A \in \mathcal{F}$, where $E \subset X$ is a set with full measure ν_X, that is,*

$$\int_{X\times Y} f(x,y)\, d\nu(x,y) = \int_{X\times Y} f(x,y)\, d\nu_x(y)\, d\nu_X(x).$$

The theorem below is more general than Theorem 4.2, but due to simplicity of the conditions it was worth stating as well.

THEOREM 4.4. *Assume that $\mu^{\star k} \ge \tilde{\mu}$ and the measure $\tilde{\mu}$ admits a desintegration $\{\sigma_a\}$ with respect to η, i.e. for any integrable function f we have $\int_{\mathbb{R}^2} f(a,b)\, d\tilde{\mu}(a,b) = \iint_{\mathbb{R}^2} f(a,b)\, d\sigma_a(b)\, d\eta(a)$, where the measure σ_a depends*

[1] $\zeta_{p,\underline{a}}$ depends on $\underline{a}$, but we suppress the index $\underline{a}$ in further calculations for simplicity.

on a. Suppose that there is a set D such that $\eta(D) > 0$, $\hat{\sigma}_a$ is not identically zero and $\hat{\sigma}_a \in L^p$ for any $a \in D$. Then the measure P_X is absolutely continuous.

Proof. We rewrite the proof of Theorem 4.2 with a few modifications. Let ν be the measure given by

$$\int_{\mathbb{R}^2} f(a,b)\, d\nu(a,b) = \int_{\mathbb{R}^2} f(a,b)\, d\sigma_a(b)\, \mathbf{1}_D(a)\, d\eta(a).$$

By assumption, ν is not zero. First, observe that $P_X = (\mu^{\star k})^{\star p} \star P_X \geq \tilde{\mu}^{\star p} \star P_X \geq \nu^{\star p} \star P_X = \rho$. Take a set $E \subset \mathbb{R}$ with $\lambda(E) = 0$. We can write

$$\begin{aligned}\rho(E) &= \int_{\mathbb{R}} \mathbb{1}_E(x)\, d\rho(x)\\ &= \int_{\mathbb{R}^{p+1}} \Big[\int_{\mathbb{R}^p} \mathbb{1}_E\Big(ax + \sum_{k=0}^{p-1} a_1 \ldots a_k b_{k+1}\Big)\, d\sigma_{a_1}(b_1) \ldots d\sigma_{a_p}(b_p)\Big]\\ &\qquad\qquad \times \mathbf{1}_D(a_1) \ldots \mathbf{1}_D(a_p)\, d\eta(a_1) \ldots d\eta(a_p)\, dP_X(x).\end{aligned}$$

Again, we fix $a_1, \ldots, a_p$ and x, and we consider the inner integral. Let B_{a_k} denote a random variable such that $B_{a_k} \sim \sigma_{a_k}$. Moreover, let $\sigma_{k,\underline{a}}$ denote the distribution of $a_1 \ldots a_{k-1} B_{a_k}$. Then $\hat{\sigma}_{k,\underline{a}} \in L^p$ and we can write

$$\begin{aligned}\int_{\mathbb{R}^p} \mathbb{1}_E\Big(ax + \sum_{k=0}^{p-1} a_1 \ldots a_k b_{k+1}\Big)\, d\sigma_{a_1}(b_1) \ldots d\sigma_{a_p}(b_p)&\\ = \int_{\mathbb{R}} \mathbb{1}_{E-ax}(b)\, d(\sigma_{1,\underline{a}} * \ldots * \sigma_{p,\underline{a}})(b).&\end{aligned}$$

By Corollary 4.1, we conclude that $\zeta_{p,\underline{a}} = \sigma_{1,\underline{a}} * \ldots * \sigma_{p,\underline{a}} \ll \lambda$, and hence the integral is zero. This means that $\rho \ll \lambda$, and the proof is complete. ∎

5. Transition probability and mixing. Consider the stochastic recurrence equation $X_n = A_n X_{n-1} + B_n$, where (A_k, B_k) is an i.i.d. sequence with law μ. Clearly, the sequence $(X_n)_{n\geq 0}$ is a homogeneous Markov chain. Let P denote its transition kernel and P^k the k-step transition kernel. The P is defined by

$$\begin{aligned}P(x,C) &= \mathbb{P}(X_1 \in C \mid X_0 = x) = \mathbb{P}_x(X_1 \in C)\\ &= \int_{\mathbb{R}^2} \mathbb{1}_C(ax+b)\, d\mu(a,b).\end{aligned}$$

Similarly, for $n \geq 2$, the n-step transition probability kernel is given by

$$P^n(x,C) = \mathbb{P}_x(X_n \in C) = \int_{\mathbb{R}^2} \mathbb{1}_C(ax+b)\, d\mu^{\star n}(a,b). \tag{5.1}$$

In Sections 3 and 4 our goal was to show the absolute continuity of the stationary distribution P_X. The same calculations allow us to conclude that the transition kernel has an absolutely continuous component.

PROPOSITION 5.1. *Under the assumptions of Theorem* 3.1, $P^{2p}(x,\cdot)$ *has an absolutely continuous component for every* $x \in \mathbb{R}$. *Under the assumptions of Theorems* 4.2 *or* 4.4, $P^{kp}(x,\cdot)$ *has an absolutely continuous component for every* $x \in \mathbb{R}$.

Proof. In the first case the measure $\mu^{\star 2p}$ can be written as $\mu^{\star 2p} = \nu_1 + \nu_2$, where ν_1 is absolutely continuous with respect to the Lebesgue measure λ^2. This follows from Theorem 3.2. Let ϕ be the density of ν_1. Then

$$P^{2p}(x,C) \geq \int_{\mathbb{R}^2} \mathbb{1}_C(ax+b)\,d\nu_1(a,b) = \int_{\mathbb{R}^2} \mathbb{1}_C(b)\phi(a,b-ax)\,db\,da. \tag{5.2}$$

The measure defined by the right hand side of (5.2) is clearly absolutely continuous, and so the conclusion follows in the first case. In the second case $\mu^{\star kp} \geq \nu^{\star p}$, and so the calculations in the proof of Theorem 4.4 show that

$$P^{2p}(x,C) \geq \int_{\mathbb{R}\times\mathbb{R}^p} \mathbb{1}_{C-ax}(b)\,d\zeta_{p,\underline{a}}(b)\mathbb{1}_D(a_1)\ldots\mathbb{1}_D(a_p)\,d\eta(a_1)\ldots d\eta(a_p). \tag{5.3}$$

Again, since $\zeta_{p,\underline{a}}$ is absolutely continuous with respect to the Lebesgue measure, so is the measure defined by the integral on the right hand side above, and the conclusion follows. ∎

Proposition 5.1 implies that in both cases the chain X_n is P_X-irreducible. Indeed, condition (c) of [1, Theorem 2.1] is satisfied, and so irreducibility follows from [1, Corollary 2.3]. Eventually, we apply [3, Theorem 2.8] to conclude that (X_n) is geometrically ergodic, and hence strongly mixing with geometric rate. More precisely, the following statement is true:

THEOREM 5.2. *Suppose that* (2.2) *and* (2.3) *are satisfied,* $\mathbb{E}\log|A| < 0$ *and there is* $\varepsilon > 0$ *such that*

$$\mathbb{E}(|A|^{\varepsilon} + |B|^{\varepsilon}) < \infty.$$

Assume further that the assumptions of either Theorem 3.2, 4.2 *or* 4.4 *hold. Then* (X_n) *is geometrically ergodic, and hence strongly mixing with geometric rate.*

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