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Deformed Fourier models with formal parameters

by

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Abstract. The deformed Fourier matrices $H = F_M \otimes_Q F_N$ with $Q \in \mathbb{T}^{MN}$ produce a matrix model $C(S_{MN}^+) \to M_{MN}(C(\mathbb{T}^{MN}))$. When $Q \in \mathbb{T}^{MN}$ is generic, the corresponding fiber can be investigated via algebraic techniques, and the main character law is asymptotically free Poisson. We present an alternative point of view on these questions, using formal parameters instead of generic parameters, and analytic tools.

Introduction. It is well-known that the unitary representations of a discrete group Γ are in one-to-one correspondence with the representations of the group algebra $C^*(\Gamma)$. Now given a discrete subgroup $\Gamma \subset U_N$, we obtain a representation $\pi : C^*(\Gamma) \to M_N(\mathbb{C})$. This representation is in general not faithul, its target algebra being finite-dimensional. On the other hand, this representation "reminds" Γ. We say that π is inner faithful.

The inner faithful representations can in fact be axiomatized in the general discrete quantum group context. Given such a quantum group Γ, and a representation $\pi : C^*(\Gamma) \to B$, one can construct a biggest quotient $\Gamma \to \Lambda$ producing a factorization $\pi : C^*(\Gamma) \to C^*(\Lambda) \to B$, and π is called inner faithful when $\Gamma = \Lambda$ (see [3]).

This construction is of particular interest when formulated from a dual viewpoint, with $\Gamma = \widehat{G}$, and with $B = M_K(C(X))$ being a random matrix algebra. To be more precise, given a compact quantum group G, and a matrix model $\pi : C(G) \to M_K(C(X))$, one can construct a biggest closed subgroup $H \subset G$ producing a factorization $\pi : C(G) \to C(H) \to M_K(C(X))$, and π is called inner faithful when $G = H$ (see [3]).

Generally speaking, an inner faithful model $\pi : C(G) \to M_K(C(X))$ can be regarded as being a source of interesting information about G, of both algebraic and analytic nature. Thus, we have here a new method for

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investigating the compact quantum groups. This method is alternative to the pure algebraic geometric point of view ("easiness").

A number of tools for dealing with the inner faithful models have been developed, some of them being algebraic [3], [4], [10], [12], and some other, analytic [6], [11], [20], [24]. However, at the level of concrete examples, only a few models have been succesfully investigated so far. Among them is the model $C(S_{MN}^+) \to M_{MN}(\mathbb{C})$ coming from a deformed Fourier matrix $H = F_M \otimes_Q F_N$, with parameter $Q \in \mathbb{T}^{MN}$.

The story with these latter models is long and twisted, and involved many people. As a brief summary, the development of the subject was as follows:

(1) Given an arbitrary inner faithful model $C(G) \to M_K(\mathbb{C})$, an abstract formula for the Haar integration over G, based on [15], was found in [6].
(2) The representations $C(S_{MN}^+) \to M_{MN}(\mathbb{C})$ coming from deformed Fourier matrices with generic parameters were studied in [4], using algebraic techniques.
(3) Some applications of the integration formula in [6], to the deformed Fourier matrix representations, were found short afterwards, in [2].
(4) In the meantime, the algebraic methods in [4] were substantially extended, so as to cover certain nongeneric parameters $Q \in \mathbb{T}^{MN}$, in [10].
(5) In the meantime as well, a generalization of the integration formula in [6], covering the models $C(G) \to M_K(C(X))$, was found in [24].
(6) The integration formula in [24] was applied to certain related representations, of type $C(S_{N^2}^+) \to M_{N^2}(C(U_N))$, in the recent paper [7].

The purpose of this paper is to study the deformed Fourier models, using analytic techniques. We will take advantage of the recent formula in [24], and investigate the full parametric model $\pi : C(S_{MN}^+) \to M_{MN}(C(\mathbb{T}^{MN}))$, instead of its individual fibers. The formula in [24] will turn out to apply well, and to lead to concrete results. As in [4], our main result will state that the main character becomes free Poisson in the $M = tN \to \infty$ limit. We will also discuss a number of further properties of the main character.

These results can be deduced from [4] as well, since in the probabilistic picture for the moments, the nongeneric parameters do not count. However, we believe that having a fully analytic proof is a good thing. In short, following [7], we now have a second concrete application of the integration formula in [6], [24]. Our hope is that this formula can be applied to some other situations, and could eventually become a serious alternative to the Weingarten formula [5], [13], and to the "easiness" methods in general [8], [18].

The paper is organized as follows: 1–2 are preliminary sections, in 3–4 we study the truncated moments of the main character, in 5–6 we compute the plain moments of the main character, in 7–8 we work out a number of moment estimates, and in 9–10 we state and prove our main results, and we end with a few concluding remarks.

1. Quantum groups. We use the quantum group formalism of Woronowicz [25], [26], with the extra axiom $S^2 = \mathrm{id}$. That is, we consider pairs (A, u) consisting of a C^*-algebra A and a unitary matrix $u \in M_N(A)$ such that the following formulae define morphisms of C^*-algebras:

$$\Delta(u_{ij}) = \sum_k u_{ik} \otimes u_{kj}, \qquad \varepsilon(u_{ij}) = \delta_{ij}, \qquad S(u_{ij}) = u_{ji}^*.$$

These morphisms are called *comultiplication*, *counit* and *antipode*. The abstract spectrum $G = \mathrm{Spec}(A)$ is called a *compact quantum group*, and we write $A = C(G)$.

The example we are interested in, due to Wang [23], is as follows:

DEFINITION 1.1. $C(S_N^+)$ is the universal C^*-algebra generated by the entries of an $N \times N$ matrix $u = (u_{ij})$ which is *magic*, in the sense that its entries are projections ($p = p^* = p^2$) summing up to 1 along each row and each column of u.

This algebra satisfies Woronowicz's axioms, and the underlying noncommutative space S_N^+ is therefore a quantum group, called the *quantum permutation group*. We have an inclusion $S_N \subset S_N^+$, which is an isomorphism for $N = 1, 2, 3$, but not for $N \geq 4$ (see [23]).

Now back to the general case, we have the following key notion, from [3]:

DEFINITION 1.2. Let $\pi : C(G) \to M_K(C(T))$ be a C^*-algebra representation.

(1) The *Hopf image* of π is the smallest quotient Hopf algebra $C(G) \to C(H)$ producing a factorization $\pi : C(G) \to C(H) \to M_K(C(T))$.
(2) When the inclusion $H \subset G$ is an isomorphism, i.e. when there is no nontrivial factorization as above, we say that π is *inner faithful.*

As a basic example, when $G = \widehat{\varGamma}$ is a group dual, π must come from a group representation $\varGamma \to C(T, U_K)$, and the factorization in (1) is the one obtained by taking the image, $\varGamma \to \varGamma' \subset C(T, U_K)$. Thus π is inner faithful when $\varGamma \subset C(T, U_K)$.

Also, given a compact group G and elements $g_1, \dots, g_K \in G$, we can consider the representation $\pi = \oplus_i \mathrm{ev}_{g_i} : C(G) \to \mathbb{C}^K$. The minimal factorization of π is then via $C(G')$ with $G' = \overline{\langle g_1, \dots, g_K \rangle}$. Thus π is inner faithful when $G = \overline{\langle g_1, \dots, g_K \rangle}$.

We recall that an *Hadamard matrix* is a square matrix $H \in M_N(\mathbb{C})$ whose entries are on the unit circle, and whose rows are pairwise orthogonal. Given a parametric family of such matrices, $\{H^x \mid x \in T\}$, we can consider the corresponding element $H \in M_N(C(T))$, which we call an Hadamard matrix as well. The relation with S_N^+ comes from:

DEFINITION 1.3. Associated to every Hadamard matrix $H \in M_N(C(T))$ is the representation

$$\pi : C(S_N^+) \to M_N(C(T)), \qquad \pi(u_{ij}) : x \to \mathrm{Proj}(H_i^x/H_j^x),$$

where $H_1^x, \dots, H_N^x \in \mathbb{T}^N$ are the rows of H^x, and the quotients are taken inside $\mathbb{T}^N$.

Here the fact that the projections on the right form a magic matrix, and hence produce a representation of $C(S_N^+)$, follows from the Hadamard matrix condition.

The problem is to compute the Hopf image of the above representation. There is only one basic example here, namely the one coming from the Fourier coupling $F_G \in M_{G\times\widehat{G}}(\mathbb{C})$ of a finite abelian group G. Here the representation constructed above factorizes as $\pi : C(S_G^+) \to C(S_G) \to C(G) \to M_N(\mathbb{C})$, and the Hopf image is $C(G)$.

In order to approach the problem, we use tools from [6], [24]. Let us first go back to the general context of Definition 1.2, and assume that T is a measured space, so that we have a trace $\mathrm{tr} : M_K(C(T)) \to \mathbb{C}$, given by $\mathrm{tr}(M) = (1/K)\sum_{i=1}^K \int_X M_{ii}(x)\,dx$.

We have then the following key result, from [6], [24]:

PROPOSITION 1.4. *Given an inner faithful model* $\pi : C(G) \to M_K(C(T))$, *we have*

$$\int_G = \lim_{k\to\infty} \frac{1}{k}\sum_{r=1}^k (\mathrm{tr}\circ\pi)^{*r}$$

in moments, with the convolutions on the right being given by $\phi*\psi = (\phi\otimes\psi)\Delta$.

Proof. This was proved in [6] in the case $X = \{\cdot\}$, using the theory from [15], the idea being that the Haar state can be obtained by starting with an arbitrary positive linear functional, and then convolving. The general case was established in [24]. ∎

In the case where G has a fundamental corepresentation $u = (u_{ij})$, the above result has a more concrete formulation, of linear algebra flavor:

PROPOSITION 1.5. *Given an inner faithful model* $\pi : C(G) \to M_K(C(T))$, *mapping* $u_{ij} \mapsto U_{ij}$, *the moments of* $\chi = \sum_i u_{ii}$ *with respect to* $\int_G^r = (\mathrm{tr}\otimes\pi)^{*r}$ *are the numbers*

$$c_p^r = \mathrm{Tr}(T_p^r) : \ (T_p)_{i_1\dots i_p, j_1\dots j_p} = \mathrm{tr}(U_{i_1j_1}\dots U_{i_pj_p}),$$

and these numbers converge as $r \to \infty$ *to the moments of* χ *with respect to* $\int_G$.

Proof. By evaluating $\int_G^r = (\mathrm{tr} \otimes \pi)^{*r}$ on a product of coefficients, we obtain

$$\int_G^r u_{i_1 j_1} \ldots u_{i_p j_p} = (T_p^r)_{i_1 \ldots i_p, j_1 \ldots j_p}.$$

Now summing over $i_x = j_x$ gives the formula in the statement (see [6]). ■

We can apply Proposition 1.5 to Hadamard representations:

THEOREM 1.6. *For the representation coming from $H \in M_N(C(T))$ we have*

$$c_p^r = \frac{1}{N^{(p+1)r}} \int_{T^r} \sum_{i_1^1, \ldots, i_p^r} \sum_{j_1^1, \ldots, j_p^r} \frac{H^{x_1}_{i_1^1 j_1^1} H^{x_1}_{i_1^2 j_2^1}}{H^{x_1}_{i_1^1 j_2^1} H^{x_1}_{i_1^2 j_1^1}} \cdots \frac{H^{x_1}_{i_p^1 j_p^1} H^{x_1}_{i_p^2 j_1^1}}{H^{x_1}_{i_p^1 j_1^1} H^{x_1}_{i_p^2 j_p^1}} \cdots$$
$$\cdots \frac{H^{x_r}_{i_1^r j_1^r} H^{x_r}_{i_1^1 j_2^r}}{H^{x_r}_{i_1^r j_2^r} H^{x_r}_{i_1^1 j_1^r}} \cdots \frac{H^{x_r}_{i_p^r j_p^r} H^{x_r}_{i_p^1 j_1^r}}{H^{x_r}_{i_p^r j_1^r} H^{x_r}_{i_p^1 j_p^r}} \, dx,$$

and these numbers converge as $r \to \infty$ to the moments of χ with respect to $\int_G$.

Proof. We have indeed the following computation:

$$\begin{aligned} c_p^r &= \sum_{i_1^1, \ldots, i_p^r} (T_p)_{i_1^1 \ldots i_p^1, i_1^2 \ldots i_p^2} \cdots\cdots (T_p)_{i_1^r \ldots i_p^r, i_1^1 \ldots i_p^1} \\ &= \int_{T^r} \sum_{i_1^1, \ldots, i_p^r} \mathrm{tr}(U^{x_1}_{i_1^1 i_1^2} \ldots U^{x_1}_{i_p^1 i_p^2}) \cdots\cdots \mathrm{tr}(U^{x_r}_{i_1^r i_1^1} \ldots U^{x_r}_{i_p^r i_p^1}) \, dx \\ &= \frac{1}{N^r} \int_{T^r} \sum_{i_1^1, \ldots, i_p^r} \sum_{j_1^1, \ldots, j_p^r} (U^{x_1}_{i_1^1 i_1^2})_{j_1^1 j_2^1} \ldots (U^{x_1}_{i_p^1 i_p^2})_{j_p^1 j_1^1} \ldots (U^{x_r}_{i_1^r i_1^1})_{j_1^r j_2^r} \ldots (U^{x_r}_{i_p^r i_p^1})_{j_p^r j_1^r} \, dx. \end{aligned}$$

In terms of H, this gives the formula in the statement (see [2]). ■

2. Fourier models. As mentioned in Section 1, the "simplest" matrix model is the one coming from the Fourier matrix $F_G \in M_{G \times \widehat{G}}(\mathbb{C})$ of a finite abelian group G, where the associated quantum group is G itself. Our purpose here will be to investigate the "next simplest" models. These appear by deforming the Fourier matrices, or rather the tensor products of such matrices, $F_{G \times H} = F_G \otimes F_H$, via the following construction, due to Diţă [14]:

PROPOSITION 2.1. *The matrix $\mathcal{F}_{G \times H} \in M_{G \times H}(\mathbb{T}^{G \times H})$ given by*

$$(\mathcal{F}_{G \times H})_{ia,jb}(Q) = Q_{ib}(F_G)_{ij}(F_H)_{ab}$$

is complex Hadamard, and its fiber at $Q = (1_{ib})$ is the Fourier matrix $F_{G \times H}$.

Proof. The fact that the rows of $F_G \otimes_Q F_H = \mathcal{F}_{G \times H}(Q)$ are pairwise orthogonal follows from definitions (see [14]). With $1 = (1_{ij})$ we have

$(F_G \otimes_1 F_H)_{ia,jb} = (F_G)_{ij}(F_H)_{ab}$, and we recognize here the formula $F_{G\times H} = F_G \otimes F_H$, in double index notation. ∎

The fibers $F_G \otimes_Q F_H = \mathcal{F}_{G\times H}(Q)$ were investigated in [4], and then in [10], by using algebraic techniques. Our purpose here is to obtain some related results, regarding the matrix $\mathcal{F}_{G\times H}$ itself, by using analytic techniques. We have:

THEOREM 2.2. *For the representation coming from $\mathcal{F}_{G\times H}$ we have*

$$c_p^r = \frac{1}{M^{r+1}N}\#\left\{\begin{matrix} i_1,\dots,i_r,a_1,\dots,a_p \in \{0,\dots,M-1\}, \\ b_1,\dots,b_p \in \{0,\dots,N-1\}, \\ [(i_x+a_y,b_y),(i_{x+1}+a_y,b_{y+1}) \mid y=1,\dots,p] \\ = [(i_x+a_y,b_{y+1}),(i_{x+1}+a_y,b_y) \mid y=1,\dots,p], \forall x \end{matrix}\right\}$$

where $M = |G|$, $N = |H|$, and the sets between brackets are sets with repetitions.

Proof. We use the formula of Theorem 1.6. With $K = F_G$ and $L = F_H$ we have

$$c_p^r = \frac{1}{(MN)^r}\int_{T^r} A(Q)\,dQ,$$

where

$$\begin{aligned} A(Q) &\\ = \sum_{i_1^1,\dots,i_p^r}\sum_{b_1^1,\dots,b_p^r} &\frac{Q^1_{i_1^1b_1^1}Q^1_{i_1^2b_2^1}}{Q^1_{i_1^1b_2^1}Q^1_{i_1^2b_1^1}}\cdots\frac{Q^1_{i_p^1b_p^1}Q^1_{i_p^2b_1^1}}{Q^1_{i_p^1b_1^1}Q^1_{i_p^2b_p^1}}\cdots\cdots\frac{Q^r_{i_1^rb_1^r}Q^r_{i_1^1b_2^r}}{Q^r_{i_1^rb_2^r}Q^r_{i_1^1b_1^r}}\cdots\frac{Q^r_{i_p^rb_p^r}Q^r_{i_p^1b_1^r}}{Q^r_{i_p^rb_1^r}Q^r_{i_p^1b_p^r}} \\ \times\frac{1}{M^{pr}}\sum_{j_1^1,\dots,j_p^r} &\frac{K_{i_1^1j_1^1}K_{i_1^2j_2^1}}{K_{i_1^1j_2^1}K_{i_1^2j_1^1}}\cdots\frac{K_{i_p^1j_p^1}K_{i_p^2j_1^1}}{K_{i_p^1j_1^1}K_{i_p^2j_p^1}}\cdots\cdots\frac{K_{i_1^rj_1^r}K_{i_1^1j_2^r}}{K_{i_1^rj_2^r}K_{i_1^1j_1^r}}\cdots\frac{K_{i_p^rj_p^r}K_{i_p^1j_1^r}}{K_{i_p^rj_1^r}K_{i_p^1j_p^r}} \\ \times\frac{1}{N^{pr}}\sum_{a_1^1,\dots,a_p^r} &\frac{L_{a_1^1b_1^1}L_{a_1^2b_2^1}}{L_{a_1^1b_2^1}L_{a_1^2b_1^1}}\cdots\frac{L_{a_p^1b_p^1}L_{a_p^2b_1^1}}{L_{a_p^1b_1^1}L_{a_p^2b_p^1}}\cdots\cdots\frac{L_{a_1^rb_1^r}L_{a_1^1b_2^r}}{L_{a_1^rb_2^r}L_{a_1^1b_1^r}}\cdots\frac{L_{a_p^rb_p^r}L_{a_p^1b_1^r}}{L_{a_p^rb_1^r}L_{a_p^1b_p^r}}. \end{aligned}$$

Since we are in the Fourier matrix case, $K = F_G$, $L = F_H$, we can perform the sums over j, a. To be more precise, the last two averages appearing above are respectively

$$\varDelta(i) = \prod_x\prod_y \delta(i_y^x + i_{y-1}^{x+1}, i_y^{x+1} + i_{y-1}^x),$$

$$\varDelta(b) = \prod_x\prod_y \delta(b_y^x + b_{y-1}^{x+1}, b_y^{x+1} + b_{y-1}^x).$$

We therefore obtain the following formula for the truncated moments of the main character, where $\varDelta$ is the product of Kronecker symbols constructed above:

$$c_p^r = \frac{1}{(MN)^r}\int_{T^r}\sum_{\Delta(i)=\Delta(b)=1}\frac{Q^1_{i^1_1b^1_1}Q^1_{i^2_1b^1_2}}{Q^1_{i^1_1b^1_2}Q^1_{i^2_1b^1_1}}\cdots\frac{Q^1_{i^1_pb^1_p}Q^1_{i^2_pb^1_1}}{Q^1_{i^1_pb^1_1}Q^1_{i^2_pb^1_p}}\cdots$$
$$\cdots\frac{Q^r_{i^r_1b^r_1}Q^r_{i^1_1b^r_2}}{Q^r_{i^r_1b^r_2}Q^r_{i^1_1b^r_1}}\cdots\frac{Q^r_{i^r_pb^r_p}Q^r_{i^1_pb^r_1}}{Q^r_{i^r_pb^r_1}Q^r_{i^1_pb^r_p}}\,dQ,$$

Now by integrating with respect to $Q \in (\mathbb{T}^{G\times H})^r$, we are led to counting the multi-indices i, b satisfying $\Delta(i) = \Delta(b) = 1$, along with the following conditions, where the sets in brackets are by definition sets with repetitions:

$$\begin{gathered}[i^1_1b^1_1,\dots,i^1_pb^1_p,i^2_1b^1_2,\dots,i^2_pb^1_1] = [i^1_1b^1_2,\dots,i^1_pb^1_1,i^2_1b^1_1,\dots,i^2_pb^1_p],\\ \vdots\\ [i^r_1b^r_1,\dots,i^r_pb^r_p,i^1_1b^r_2,\dots,i^1_pb^r_1] = [i^r_1b^r_2,\dots,i^r_pb^r_1,i^1_1b^r_1,\dots,i^1_pb^r_p].\end{gathered}$$

In a more compact notation, the moment formula is therefore as follows:

$$c_p^r = \frac{1}{(MN)^r}\#\{i,b \mid \Delta(i)=\Delta(b)=1,\ [i^x_yb^x_y, i^{x+1}_yb^x_{y+1}] = [i^x_yb^x_{y+1}, i^{x+1}_yb^x_y],\ \forall x\}.$$

Now observe that the Kronecker type conditions $\Delta(i) = \Delta(b) = 1$ tell us that the arrays of indices $i = (i^x_y), b = (b^x_y)$ must be of the following special form:

$$\begin{pmatrix} i^1_1 & \dots & i^1_p\\ \dots & \dots & \dots\\ i^1_r & \dots & i^r_p\end{pmatrix} = \begin{pmatrix} i_1+a_1 & \dots & i_1+a_p\\ \dots & \dots & \dots\\ i_r+a_1 & \dots & i_r+a_p\end{pmatrix},$$

$$\begin{pmatrix} b^1_1 & \dots & b^1_p\\ \dots & \dots & \dots\\ b^1_r & \dots & b^r_p\end{pmatrix} = \begin{pmatrix} j_1+b_1 & \dots & j_1+b_p\\ \dots & \dots & \dots\\ j_r+b_1 & \dots & j_r+b_p\end{pmatrix}.$$

Here all the new indices i_x, j_x, a_y, b_y are uniquely determined, up to a choice of i_1, j_1. By replacing i^x_y, b^x_y with these new indices i_x, j_x, a_y, b_y, with an MN factor added, which accounts for the choice of i_1, j_1, we obtain

$$c_p^r = \frac{1}{(MN)^{r+1}}\#\left\{i,j,a,b\ \middle|\ \begin{aligned}&[(i_x+a_y,j_x+b_y),(i_{x+1}+a_y,j_x+b_{y+1})]\\ &=[(i_x+a_y,j_x+b_{y+1}),(i_{x+1}+a_y,j_x+b_y)],\forall x\end{aligned}\right\}.$$

Now observe that we can delete if we want the j_x indices, which are irrelevant. Thus, we obtain the formula in the statement. ∎

Summarizing, the Haar integration formula in [24] leads to a combinatorial interpretation of the moments of the main character. In what follows we will investigate these moments, first with some exact computations, and then with analytic techniques.

3. Exact computations. In this section and in the next one we study the numbers c_p^r found in Theorem 2.2, with a number of exact computations.

Observe first that these numbers depend only on $M = |G|$ and $N = |H|$. In what follows we denote them by $c_p^r(M,N)$.

As an illustration, here are a few trivial computations:

PROPOSITION 3.1. *The numbers $c_p^r(M,N)$ have the following properties:*

(1) $c_p^r(1,N) = N^{p-1}$.
(2) $c_p^r(M,1) = M^{p-1}$.
(3) $c_1^r(M,N) = 1$.
(4) $c_p^1(M,N) = (MN)^{p-1}$.

Proof. In all the cases under investigation, the conditions on the sets with repetitions in Theorem 2.2 are trivially satisfied, and this gives the above formulae. ■

We have in fact the following result, including all the "obvious" information:

PROPOSITION 3.2. *The normalized quantities*

$$d_p^r(M,N) = \frac{1}{(MN)^{p-1}} \cdot c_p^r(M,N)$$

all belong to $[0,1]$, *and are equal to* 1 *if* $M=1$, $N=1$, $p=1$ *or* $r=1$.

Proof. According to Theorem 2.2, the rescaled moments are given by

$$d_p^r(M,N) = \frac{1}{M^{p+r}N^p} \# \left\{ \begin{array}{l} i_1,\dots,i_r,a_1,\dots,a_p \in \{0,\dots,M-1\}, \\ b_1,\dots,b_p \in \{0,\dots,N-1\}, \\ [(i_x+a_y,b_y),(i_{x+1}+a_y,b_{y+1})] \\ = [(i_x+a_y,b_{y+1}),(i_{x+1}+a_y,b_y)],\ \forall x \end{array} \right\}.$$

Thus $d_p^r(M,N) \in [0,1]$, and the other assertions follow from Proposition 2.1. ■

Let us now perform some computations. The formulae look better for the numbers $d_p^r(M,N)$ of Proposition 3.2, so we will use the latter. First, we have:

PROPOSITION 3.3. *When one of i,a,b consists of equal indices, the conditions defining $d_p^r(M,N)$ are trivially satisfied. The corresponding contribution is*

$$\alpha_p^r(M,N) = 1 - \frac{(M^p-M)(M^r-M)(N^p-N)}{M^{p+r}N^p},$$

and this quantity equals $d_p^r(M,N)$ if $M=1$, $N=1$, $r=1$, or $p \le 2$.

Proof. Assume that one of i,a,b consists of equal indices. By translation we can assume that this common index is 0, and the conditions defining $d_p^r(M,N)$ read:

$$\begin{aligned}
i_x = 0: &\quad [(a_y, b_y), (a_y, b_{y+1})] = [(a_y, b_{y+1}), (a_y, b_y)],\\
a_y = 0: &\quad [(i_x, b_y), (i_{x+1}, b_{y+1})] = [(i_x, b_{y+1}), (i_{x+1}, b_y)],\\
b_y = 0: &\quad [(i_x + a_y, 0), (i_{x+1} + a_y, 0)] = [(i_x + a_y, 0), (i_{x+1} + a_y, 0)].
\end{aligned}$$

Thus the conditions are trivially satisfied when $i_x = 0$ or $b_y = 0$, and the same happens when $a_y = 0$, by performing a cyclic permutation on the y indices.

The number of situations where one of i, a, b consists of equal indices is

$$K = M^{p+r}N^p - (M^p - M)(M^r - M)(N^p - N).$$

Dividing by $M^{p+r}N^p$, we obtain the formula in the statement.

The assertions about $M = 1$, $N = 1$, $p = 1$, $r = 1$ are clear, because in all these cases the product in the definition of $\alpha_p^r(M, N)$ vanishes, and so $\alpha_p^r(M, N) = 1$.

Finally, for $p = 2$, the equations defining $d_2^r(M, N)$ are as follows:

$$\begin{aligned}
&[(i_x + a_1, b_1), (i_x + a_2, b_2), (i_{x+1} + a_1, b_2), (i_{x+1} + a_2, b_1)]\\
&\quad = [(i_x + a_1, b_2), (i_x + a_2, b_1), (i_{x+1} + a_1, b_1), (i_{x+1} + a_2, b_2)], \qquad \forall x.
\end{aligned}$$

We already know that these conditions are satisfied when $a_1 = a_2$ or $b_1 = b_2$. So, assume $a_1 \neq a_2, b_1 \neq b_2$. The element $(i_x + a_1, b_1)$ must appear somewhere on the right, and the only possible choice is $(i_x + a_1, b_1) = (i_{x+1} + a_1, b_1)$, which gives $i_x = i_{x+1}$. Thus, all the i_x indices must be equal, and we are done. ■

In general, the situation is more complicated. As a first remark, we have:

PROPOSITION 3.4. *We have $d_p^r(M, N) \geq \delta_p(M, N)$, where*

$$\delta_p(M, N) = \frac{1}{(MN)^p} \times \#\left\{ \begin{matrix} a_1, \dots, a_p \in \{0, \dots, M-1\} \\ b_1, \dots, b_p \in \{0, \dots, N-1\} \end{matrix} \;\middle|\; \begin{matrix} [(a_1, b_1), (a_2, b_2), \dots, (a_p, b_p)] \\ = [(a_1, b_p), (a_2, b_1), \dots, (a_p, b_{p-1})] \end{matrix} \right\},$$

where the sets in square brackets are as usual sets with repetitions.

Proof. This is clear from the fact that the conditions defining $\delta_p^r(M, N)$ are trivially satisfied when the indices a, b satisfy $[(a_y, b_y)] = [(a_y, b_{y+1})]$. ■

We can merge and extend Propositions 3.3 and 3.4, as follows:

THEOREM 3.5. *When the index i consists of equal indices, or when $[(a_y, b_y)] = [(a_y, b_{y+1})]$, the conditions defining $d_p^r(M, N)$ are trivially satisfied. The corresponding contribution is*

$$\beta_p^r(M, N) = \delta_p(M, N) + \frac{1}{M^{r-1}}(1 - \delta_p(M, N)),$$

and this quantity equals $d_p^r(M, N)$ if $M = 1$, $N = 1$, $r = 1$, or $p \leq 3$.

Proof. The first assertion is clear, and by definition of $\delta_p(M,N)$, the corresponding contribution is the one in the statement. Since if $M=1$, $N=1$, $r=1$ or $p\leq 2$ we have $\beta_p^r(M,N)=\alpha_p^r(M,N)$, the results here follow from Proposition 3.3.

It remains to discuss the case $p=3$. Here the equations are as follows:

$$\begin{aligned}&[(i_x+a_1,b_1),(i_x+a_2,b_2),(i_x+a_3,b_3),\\&\qquad\qquad(i_{x+1}+a_1,b_2),(i_{x+1}+a_2,b_3),(i_{x+1}+a_3,b_1)]\\&\quad=[(i_x+a_1,b_2),(i_x+a_2,b_3),(i_x+a_3,b_1),\\&\qquad\qquad(i_{x+1}+a_1,b_1),(i_{x+1}+a_2,b_2),(i_{x+1}+a_3,b_3)].\end{aligned}$$

We must prove that all the solutions are trivial, in the sense that either all the i_x are equal, or the following condition is satisfied:

$$[(a_1,b_1),(a_2,b_2),(a_3,b_3)]=[(a_1,b_2),(a_2,b_3),(a_3,b_1)].$$

So, assume that we are in the nontrivial case, and pick x such that $i_x\neq i_{x+1}$. Let us look now at the first element appearing on the left in the above equation, namely (i_1+a_1,b_1). Since this element must appear on the right as well, we have six cases to investigate. Observe now that in these six cases we must have, respectively:

$$b_1=b_2,\quad b_1=b_3,\quad a_1=a_3,\quad i_x=i_{x+1},\quad b_1=b_2,\quad b_1=b_3.$$

Thus, we have one case which is impossible, namely the one needing $i_x=i_{x+1}$, and in the other five cases, we always obtain a relation of type $a_i=a_j$ or $b_i=b_j$, with $i\neq j$.

So, assume $a_i=a_j$ with $i\neq j$. By a cyclic permutation of the indices, we can assume that $a_2=a_3$. Now our equations simplify as follows:

$$\begin{aligned}&[(i_x+a_1,b_1),(i_x+a_2,b_2),\underline{(i_x+a_2,b_3)},\\&\qquad\qquad(i_{x+1}+a_1,b_2),\underline{(i_{x+1}+a_2,b_3)},(i_{x+1}+a_2,b_1)]\\&\quad=[(i_x+a_1,b_2),\underline{(i_x+a_2,b_3)},(i_x+a_2,b_1),\\&\qquad\qquad(i_{x+1}+a_1,b_1),(i_{x+1}+a_2,b_2),\underline{(i_{x+1}+a_2,b_3)}].\end{aligned}$$

The condition $[(a_y,b_y)]\neq[(a_y,b_{y+1})]$ simplifies as well:

$$[(a_1,b_1),(a_2,b_2),\underline{(a_2,b_3)}]\neq[(a_1,b_2),\underline{(a_2,b_3)},(a_2,b_1)].$$

Summarizing, the simplifications make the variables a_3,b_3 disappear, and so we are led to a $p=2$ problem, where the solutions are already known to be trivial.

In the case $b_i=b_j$, with $i\neq j$, the situation is similar. By a cyclic permutation we can assume $b_1=b_3$, and our equations simplify as follows:

$$\begin{aligned}&[(i_x+a_1,b_1),(i_x+a_2,b_2),\underline{(i_x+a_3,b_1)},\\ &\qquad\qquad (i_{x+1}+a_1,b_2),(i_{x+1}+a_2,b_1),\underline{(i_{x+1}+a_3,b_1)}]\\ &\quad=[(i_x+a_1,b_2),(i_x+a_2,b_1),\underline{(i_x+a_3,b_1)},\\ &\qquad\qquad (i_{x+1}+a_1,b_1),(i_{x+1}+a_2,b_2),\underline{(i_{x+1}+a_3,b_1)}].\end{aligned}$$

The condition $[(a_y,b_y)]\neq[(a_y,b_{y+1})]$ simplifies to

$$[(a_1,b_1),(a_2,b_2),\underline{(a_3,b_1)}]\neq[(a_1,b_2),(a_2,b_1),\underline{(a_3,b_1)}].$$

Thus, once again we are led to a $p=2$ problem, whose solutions are trivial. ■

4. Higher truncations. We know from Theorem 3.5 above that for small values of the truncation parameter, namely $p=1,2,3$, the numbers $d_p^r(M,N)$ come only from “trivial contributions”.

For $p=4$ and higher the situation becomes considerably more complex, involving the arithmetic of M,N, and this even in the simplest case, $r=2$.

Here we have the following result, which we will not use in what follows, but which might be interesting for instance in connection with the speculations in [1]:

THEOREM 4.1. *We have the formula*

$$d_4^2(M,N)=\beta_4^2(M,N)+\delta_{2|M}\frac{(M-2)(N-1)}{M^4N^3},$$

where $\delta_{2|M}\in\{0,1\}$ *is equal to* 0 *when* M *is odd, and to* 1 *when* M *is even.*

Proof. We have two equations, the one at $x=1$ being

$$\begin{aligned}&[(i_1+a_1,b_1),\dots,(i_1+a_4,b_4),(i_2+a_1,b_2),\dots,(i_2+a_4,b_1)]\\ &\qquad=[(i_1+a_1,b_2),\dots,(i_1+a_4,b_1),(i_2+a_1,b_1),\dots,(i_2+a_4,b_4)].\end{aligned}$$

The equation for $x=2$ is

$$\begin{aligned}&[(i_2+a_1,b_1),\dots,(i_2+a_4,b_4),(i_1+a_1,b_2),\dots,(i_1+a_4,b_1)]\\ &\qquad=[(i_2+a_1,b_2),\dots,(i_2+a_4,b_1),(i_1+a_1,b_1),\dots,(i_1+a_4,b_4)].\end{aligned}$$

Since these equations are equivalent, we are left with the $x=1$ equation.

In order to compute the nontrivial contributions, we can assume $i_1\neq i_2$. Let us look at the first element appearing on the left, (i_1+a_1,b_1). Since it must appear on the right as well, we have eight cases to investigate. In these eight cases, we must have in turn

$$\begin{aligned}&b_1=b_2,\ a_1=a_2,\ b_1=b_4,\ a_1=a_4,\ i_1=i_2,\\ &\qquad\qquad b_1=b_2,\ (i_1+a_1,b_1)=(i_2+a_3,b_3),\ b_1=b_4.\end{aligned}$$

Thus one case is impossible, six cases reduce to the case $p = 3$ by using a cyclic reduction as in the proof of Theorem 3.5, and there is one case left, $(i_1 + a_1, b_1) = (i_2 + a_3, b_3)$.

The same argument applies to the other seven elements appearing on the left, and we conclude that the nontrivial solutions could only come from

$$(i_1 + a_x, b_x) = (i_2 + a_{x+2}, b_{x+2}), \qquad (i_2 + a_x, b_{x+1}) = (i_1 + a_{x+2}, b_{x+3}).$$

Thus i, a, b must be of the following special form, with $2i = 0$:

$$\begin{cases} i = (i_1, i + i_1), \\ a = (a_1, a_2, i + a_1, i + a_2), \\ b = (b_1, b_2, b_1, b_2). \end{cases}$$

In order to find the nontrivial solutions, we must now assume that $i \neq 0$ and $[(a_y, b_y)] \neq [(a_y, b_{y+1})]$. But, by translating by i_1, this latter condition reads

$$[(a_1, b_1), (a_2, b_2), (i + a_1, b_1), (i + a_2, b_2)] \\ \neq [(a_1, b_2), (a_2, b_1), (i + a_1, b_2), (i + a_2, b_1)].$$

Thus we must have $b_1 \neq b_2$, and $a_1 \neq a_2$, $a_1 \neq i + a_2$ as well.

We can now compute the nontrivial contribution. It is given by

$$K = \frac{1}{M^6 N^4} \cdot M\delta_{2|M} \cdot M(M-2) \cdot N(N-1).$$

To be more precise, $\frac{1}{M^6 N^4}$ is the normalization factor from the definition of $d_4^2(M, N)$; then $M\delta_{2|M}$ comes from the choice of i_1 and of $i \neq 0$ satisfying $2i = 0$; then $M(M-2)$ comes from the choice of a_1 and of $a_2 \neq a_1, i + a_1$; and finally $N(N-1)$ comes from the choice of $b_1 = b_2$. This gives the formula in the statement. ∎

As a conclusion, the exact computation of $d_p^r(M, N)$ is an interesting problem. In what follows we will only study the asymptotics of these numbers, with the result that the estimate $d_p^r(M, N) \geq \beta_p^r(M, N)$ from Theorem 3.5 becomes an equality as $r \to \infty$.

5. Limiting moments. Let us now go back to the numbers $\delta_p(M, N)$ from Proposition 3.4.

These numbers are known from [4] to be the rescaled moments of the main character for the matrix model associated to $\mathcal{F}_{G\times H}(Q)$, where $|G| = M$, $|H| = N$, and where $Q \in \mathbb{T}^{G\times H}$ is generic. We will prove now that our moments are precisely these numbers:

$$\lim_{r\to\infty} d_p^r(M, N) = \delta_p(M, N).$$

For this purpose, observe that both $d_p^r(M, N)$ and $\delta_p(M, N)$ count, modulo some normalizations, the solutions of certain equations on the indices

$a_1,\ldots,a_p\in\{0,\ldots,M-1\}$ and $b_1,\ldots,b_p\in\{0,\ldots,N-1\}$. We will prove the convergence componentwise, with respect to these pairs (a,b) of multi-indices. We use the following simple fact:

PROPOSITION 5.1. *We have $[a_y]=[b_y]$ in a finite abelian group G precisely when*

$$\sum_y\chi(a_y)=\sum_y\chi(b_y)$$

as an equality of complex numbers, for any character $\chi\in\widehat{G}$.

Proof. By linearity, we have the following equivalences:

$$\begin{aligned}[a_y]=[b_y]\ &\Leftrightarrow\ \sum_y a_y=\sum_y b_y\text{ inside }C^*(G)\\ &\Leftrightarrow\ \varphi\Big(\sum_y a_y\Big)=\varphi\Big(\sum_y b_y\Big),\ \forall\varphi\in C(G)\\ &\Leftrightarrow\ \chi\Big(\sum_y a_y\Big)=\chi\Big(\sum_y b_y\Big),\ \forall\chi\in\widehat{G}.\ \blacksquare\end{aligned}$$

Now back to our question, since only the cardinalities $M=|G|$ and $N=|H|$ are revelant, we can assume $G=\mathbb{Z}_M$ and $H=\mathbb{Z}_N$. We first have the following technical result:

PROPOSITION 5.2. *For a pair (a,b) of multi-indices, the following are equivalent:*

(1) $[(a_y,b_y)]=[(a_y,b_{y+1})]$.
(2) $[(i+a_y,b_y),(a_y,b_{y+1})]=[(i+a_y,b_{y+1}),(a_y,b_y)]$, *for any $i\in\mathbb{Z}_M$.*

Proof. Observe that (1)⇒(2) is clear. For (2)⇒(1), we use Proposition 5.1. By using the identification $\widehat{\mathbb{Z}_M\times\mathbb{Z}_N}\simeq\widehat{\mathbb{Z}_M}\times\widehat{\mathbb{Z}_N}$, we have, for any $i\in\mathbb{Z}_M$, $\eta\in\widehat{\mathbb{Z}_M}$, $\rho\in\widehat{\mathbb{Z}_N}$,

$$\begin{aligned}&[(i+a_y,b_y),(a_y,b_{y+1})]=[(i+a_y,b_{y+1}),(a_y,b_y)]\\ &\quad\Leftrightarrow\ \sum_y\eta(i+a_y)\rho(b_y)+\eta(a_y)\rho(b_{y+1})=\sum_y\eta(i+a_y)\rho(b_{y+1})+\eta(a_y)\rho(b_y)\\ &\quad\Leftrightarrow\ \eta(i)\sum_y\eta(a_y)\rho(b_y)-\eta(a_y)\rho(b_{y+1})=\sum_y\eta(a_y)\rho(b_y)-\eta(a_y)\rho(b_{y+1})\\ &\quad\Leftrightarrow\ \sum_y\eta(a_y)\rho(b_y)-\eta(a_y)\rho(b_{y+1})=0,\ \forall\eta,\rho\ \Leftrightarrow\ [(a_y,b_y)]=[(a_y,b_{y+1})].\ \blacksquare\end{aligned}$$

With the above result in hand, we can prove the estimate we need:

Proposition 5.3. *Assuming* $[(a_y,b_y)]\neq[(a_y,b_{y+1})]$, *the number*

$$K_p^r(a,b)=\frac{1}{M^r}\#\left\{i_1,\ldots,i_r\leq M\;\middle|\;\begin{matrix}[(i_x+a_y,b_y),(i_{x+1}+a_y,b_{y+1})]\\ =[(i_x+a_y,b_{y+1}),(i_{x+1}+a_y,b_y)],\forall x\end{matrix}\right\}$$

goes to 0 *as* $r\to\infty$.

Proof. Observe that the problem is already solved for $p\leq 3$, because by Theorem 3.5 all the i_x indices must be equal, and so the number in the statement is

$$K_2^r(a,b)=\frac{1}{M^{r-1}}\to 0.$$

In general, consider the set $S\subset\{0,\ldots,M-1\}$ consisting of the solutions i of the equation

$$[(i+a_y,b_y),(a_y,b_{y+1})]=[(i+a_y,b_{y+1}),(a_y,b_y)].$$

In terms of this set, the quantity in the statement is

$$K_p^r(a,b)=\frac{1}{M^r}\#\{i_1,\ldots,i_r\leq M\mid i_2-i_1,\ldots,i_r-i_1\in S\}.$$

Now by ignoring the last condition, we have M choices for i_1, then $|S|$ choices for i_2, $|S|$ choices for i_3, and so on, up to $|S|$ choices for i_r. Thus,

$$K_p^r(a,b)\leq\frac{1}{M^r}\cdot M\cdot|S|\cdot\ldots\cdot|S|=\left(\frac{|S|}{M}\right)^{r-1}.$$

On the other hand, by Proposition 5.2, our assumption $[(a_y,b_y)]\neq[(a_y,b_{y+1})]$ implies $S\neq\{0,\ldots,M-1\}$. In particular, $|S|\leq M-1$, and this gives the result. ∎

With the above estimate in hand, we can now prove:

Theorem 5.4. *We have*

$$\lim_{r\to\infty}d_p^r(M,N)=\delta_p(M,N)\qquad\text{for any } p\geq 1 \text{ and any } M,N\in\mathbb{N}.$$

Proof. Our claim is that, for any pair (a,b) of multi-indices,

$$\lim_{r\to\infty}K_p^r(a,b)=\delta_{[(a_y,b_y)],[(a_y,b_{y+1})]}.$$

Indeed, when $[(a_y,b_y)]\neq[(a_y,b_{y+1})]$, this is exactly what we found in Proposition 5.3. If $[(a_y,b_y)]=[(a_y,b_{y+1})]$, this is trivial, because here the equations defining $K_p^r(a,b)$ are all trivial, and so $K_p^r(a,b)=1$ for any $r\in\mathbb{N}$. ∎

Summarizing, we have proved that the law of the main character for $\mathcal{F}_{G,H}$ coincides with that computed in [4] for the matrix $\mathcal{F}_{G\times H}(Q)$ with $Q\in\mathbb{T}^{G\times H}$ generic. As a consequence, all the findings in [4] apply. In what follows we will review these results by using an analytic approach, and by bringing in some technical improvements.

6. Gram matrices. We now study the behavior of the limiting moments $\delta_p(M,N)$ as $p\to\infty$. For this purpose, let us first recall the following result from [4]:

PROPOSITION 6.1. *We have the formula*

$$\delta_p(M,N)=\frac{1}{(MN)^p}\int_{\mathbb{T}^{MN}}\mathrm{Tr}(G(Q)^p)\,dQ,$$

where $G\in M_M(C(\mathbb{T}^{MN}))$ is given by letting be the $G(Q)$ Gram matrix of the rows of Q.

Proof. If $R_1,\dots,R_M\in\mathbb{T}^N$ are the rows of $Q\in\mathbb{T}^{MN}$, we have

$$\begin{aligned}\delta_p(M,N)&=\frac{1}{(MN)^p}\sum_{a_1,\dots,a_p}\sum_{b_1,\dots,b_p}\delta_{[a_1b_1,\dots,a_pb_p],[a_1b_p,\dots,a_pb_{p-1}]}\\&=\frac{1}{(MN)^p}\int_{\mathbb{T}^{MN}}\sum_{a_1,\dots,a_p}\sum_{b_1,\dots,b_p}\frac{Q_{a_1b_1}\dots Q_{a_pb_p}}{Q_{a_1b_p}\dots Q_{a_pb_{p-1}}}\,dQ\\&=\frac{1}{(MN)^p}\int_{\mathbb{T}^{MN}}\sum_{a_1,\dots,a_p}\langle R_{a_1},R_{a_2}\rangle\langle R_{a_2},R_{a_3}\rangle\dots\langle R_{a_p},R_{a_1}\rangle dQ\end{aligned}$$

This gives the formula in the statement. ■

In the case $M=2$ some simplifications appear, and we have:

PROPOSITION 6.2. *We have the formula*

$$\delta_p(2,N)=\frac{1}{2^{p-1}}\sum_{k\ge0}\binom{p}{2k}\int_{\mathbb{T}^N}\left|\frac{q_1+\dots+q_N}{N}\right|^{2k}dq$$

with the integral taken with respect to the uniform measure on $\mathbb{T}^N$.

Proof. We use the formula of Proposition 6.1. If we denote by $R_1,R_2\in\mathbb{T}^N$ the rows of Q then, with $q=R_1/R_2\in\mathbb{T}^N$, the Gram matrix we are interested in is

$$G(Q)=\begin{pmatrix}N&q_1+\dots+q_N\\\bar q_1+\dots+\bar q_N&N\end{pmatrix}.$$

Thus, with $S=(q_1+\dots+q_N)/N$, we have $G(Q)=NA(q)$, where

$$A(q)=\begin{pmatrix}1&S\\\bar S&1\end{pmatrix}.$$

Now since $q\in\mathbb{T}^N$ is uniform when $Q\in\mathbb{T}^{2N}$ is, we deduce that

$$\delta_p(2,N)=\frac{1}{2^p}\int_{\mathbb{T}^N}\sum_{a_1,\dots,a_p}A(q)_{a_1a_2}A(q)_{a_2a_3}\dots A(q)_{a_pa_1}\,dq.$$

The point now is that the nontrivial factors in the above product, namely $S, \bar{S}$, will form together $|S|^k$ factors, with $k \geq 0$. To be more precise, in order to find the number of $|S|^{2k}$ summands, we have to count the circular configurations consisting of p numbers $1, 2$, such that both the 1 values and the 2 values are arranged into k nonempty intervals. By looking at the endpoints of these $2k$ intervals, we have $2\binom{p}{2k}$ choices, so the kth contribution is $C_k = 2\binom{k}{2p}|S|^{2k}$. Thus,

$$\delta_p(2,N) = \frac{1}{2^p}\sum_{k\geq 0} 2\binom{p}{2k}\int_{\mathbb{T}^N} |S|^{2k}\,dq.$$

This gives the formula in the statement. ■

We write $a_k \simeq b_k$ when $a_k/b_k \to 1$. We will need the following result, due to Richmond and Shallit [19]:

Proposition 6.3. *We have*

$$\int_{\mathbb{T}^N} \left|\frac{q_1 + \cdots + q_N}{N}\right|^{2k} dq \simeq \sqrt{\frac{N^N}{(4\pi k)^{N-1}}} \quad \text{as } k \to \infty.$$

Proof. This is a reformulation of the result in [19]. Observe first that

$$\begin{aligned}\int_{\mathbb{T}^N} |q_1 + \cdots + q_N|^{2k}\,dq &= \int_{\mathbb{T}^N} \sum_{i_1,\ldots,i_k}\sum_{j_1,\ldots,j_k} \frac{q_{i_1}\cdots q_{i_k}}{q_{j_1}\cdots q_{j_k}}\,dq \\ &= \#\left\{\begin{matrix} i_1,\ldots,i_k \in \{0,\ldots,N-1\} \\ j_1,\ldots,j_k \in \{0,\ldots,N-1\}\end{matrix}\;\middle|\;\begin{matrix}[i_1,\ldots,i_k] \\ = [j_1,\ldots,j_k]\end{matrix}\right\}.\end{aligned}$$

Let us now examine the numbers on the right. If we denote by $r_1,\ldots,r_N$ the number of occurrences of $0,\ldots,N-1$ in the set with repetitions $[i] = [j]$, then $r_1 + \cdots + r_N = k$, and the corresponding solutions of $[i] = [j]$ come by dividing, once for i, and once for j, the set $\{1,\ldots,k\}$ into subsets of sizes $r_1,\ldots,r_N$. Thus

$$\int_{\mathbb{T}^N} |q_1 + \cdots + q_N|^{2k} dq = \sum_{k=\sum r_i} \binom{k}{r_1,\ldots,r_N}^2.$$

By using the estimate in [19], we obtain the result. ■

We can now deduce a final estimate for $M = 2$:

Theorem 6.4. *We have*

$$\delta_p(2,N) \simeq \sqrt{\frac{N^N}{(\pi p)^{N-1}}} \quad \text{as } p \to \infty.$$

Proof. We use the formula of Proposition 6.2. Since for any $T > 0$ the values $k < T$ will not contribute to the $p \to \infty$ limit, we can use Proposition

6.3 to obtain

$$\delta_p(2,N) \simeq \sqrt{\frac{N^N}{(2\pi)^{N-1}}} \cdot \frac{1}{2^{p-1}} \sum_{k\geq 0} \binom{p}{2k} \frac{1}{\sqrt{(2k)^{N-1}}}.$$

Let A_{even} be the average of the 2^{p-1} terms on the right. This average is indexed by the integers $s=2k$ in an obvious way, and we can also consider the "complementary" quantity A_{odd}, indexed by the integers $s=2k+1$. By estimating $|A_{\mathrm{even}}-A_{\mathrm{odd}}|$ we deduce that $A_{\mathrm{even}} \simeq A_{\mathrm{odd}}$, and so $A_{\mathrm{even}} \simeq (A_{\mathrm{even}}+A_{\mathrm{odd}})/2$. Thus,

$$\delta_p(2,N) \simeq \sqrt{\frac{N^N}{(2\pi)^{N-1}}} \cdot \frac{1}{2^{p}} \sum_{s\geq 0} \binom{p}{s} \frac{1}{\sqrt{s^{N-1}}}.$$

On the other hand, by differentiating the binomial formula $(1+x)^p = \sum_{s\geq 0}\binom{p}{s}x^s$ several times, and then evaluating at $x=1$, we get

$$\frac{1}{2^p}\sum_{s\geq 0}\binom{p}{s}s^\alpha \simeq \left(\frac{p}{2}\right)^\alpha.$$

With $\alpha=(1-N)/2$, this gives

$$\delta_p(2,N) \simeq \sqrt{\frac{N^N}{(2\pi)^{N-1}}} \cdot \sqrt{\left(\frac{2}{p}\right)^{N-1}},$$

which is the formula in the statement. ∎

7. Partition decomposition. Our purpose now will be to estimate $\delta_p(M,N)$ when $M,N\in\mathbb{N}$ are arbitrary. The idea will be to decompose over partitions. First, we have:

Proposition 7.1. *We have*

$$\delta_p(M,N) = \frac{1}{(MN)^p}\sum_{\pi\rhd\sigma}\frac{M!}{(M-|\pi|)!}\cdot\frac{N!}{(N-|\sigma|)!}$$

where for $\pi,\sigma\in P(p)$ we write $\pi\rhd\sigma$ when $|\beta\cap\gamma|=|(\beta-1)\cap\gamma|$ for all $\beta\in\pi$ and all $\gamma\in\sigma$.

Proof. We know that $\delta_p(M,N)$ is the probability that $[(a_x,b_x)]=[(a_x,b_{x+1})]$. We can split this quantity over pairs of partitions, as follows:

$$\delta_p(M,N) = \frac{1}{(MN)^p} \times \sum_{\pi,\sigma\in P(p)} \#\left\{\begin{matrix} a_1,\ldots,a_p\in\{0,\ldots,M-1\} \\ b_1,\ldots,b_p\in\{0,\ldots,N-1\}\end{matrix} \;\middle|\; \begin{matrix}\ker a=\pi,\ \ker b=\sigma, \\ [(a_x,b_x)]=[(a_x,b_{x+1})]\end{matrix}\right\}.$$

Now observe that the validity of the condition $[(a_x,b_x)]=[(a_x,b_{x+1})]$ depends only on the partitions $\pi=\ker a$ and $\sigma=\ker b$. To be more precise, this condition is satisfied precisely when $\pi\triangleright\sigma$. Therefore

$$\delta_p(M,N)=\frac{1}{(MN)^p}\sum_{\pi\triangleright\sigma}\#\left\{\begin{matrix}a_1,\dots,a_p\in\{0,\dots,M-1\}\\ b_1,\dots,b_p\in\{0,\dots,N-1\}\end{matrix}\,\middle|\,\begin{matrix}\ker a=\pi\\ \ker b=\sigma\end{matrix}\right\},$$

which gives the stated formula. ∎

As an application, we can discuss what happens in the $M=tN\to\infty$ regime, which means $N\to\infty$ and $M=tN+o(1)$ with $t>0$ fixed. The result, from [4], is:

Proposition 7.2. *If $M=tN\to\infty$ we have*

$$\delta_p(M,N)\simeq S_p(t)M^{-p}N,$$

where $S_p(t)=\sum_{\pi\in\mathrm{NC}(p)}t^{|\pi|}$ is the Stirling polynomial of $\mathrm{NC}(p)$.

Proof. By the formula of Proposition 7.1, for $M=tN\to\infty$ we have

$$\delta_p(M,N)\simeq\sum_{\pi\triangleright\sigma}M^{|\pi|-p}N^{|\sigma|-p}.$$

We now use the standard fact that $\pi\triangleright\sigma$ implies $|\pi|+|\sigma|\le p+1$, with equality when $\pi,\sigma\in\mathrm{NC}(p)$ are inverse to each other, via Kreweras complementation. We obtain

$$\delta_p(M,N)\simeq\sum_{\pi\in\mathrm{NC}(p)}M^{|\pi|-p}N^{1-|\pi|},$$

which gives the formula in the statement (see [4]). ∎

Now back to our original question, concerning the case where $M,N\in\mathbb{N}$ are fixed, we can rewrite the formula of Proposition 7.1 in a more convenient way:

Proposition 7.3. *We have*

$$\delta_p(M,N)=\sum_{s=1}^M\sum_{t=1}^N\frac{M!}{(M-s)!}\cdot\frac{S_{ps}}{M^p}\cdot\frac{N!}{(N-t)!}\cdot\frac{S_{pt}}{N^p}\cdot P(\pi\triangleright\sigma\mid|\pi|=s,\,|\sigma|=t),$$

where $S_{ps}=\#\{\pi\in P(p)\mid|\pi|=s\}$ are the Stirling numbers of $P(p)$.

Proof. According to the formula of Proposition 7.1, we have

$$\delta_p(M,N)=\frac{1}{(MN)^p}\sum_{s=1}^M\sum_{t=1}^N\frac{M!}{(M-s)!}\cdot\frac{N!}{(N-t)!}\#\{\pi\triangleright\sigma\mid|\pi|=s,|\sigma|=t\}.$$

On the other hand, the probability in the statement is

$$P(\pi\triangleright\sigma\mid|\pi|=s,\,|\sigma|=t)=\frac{\#\{\pi\triangleright\sigma\mid|\pi|=s,\,|\sigma|=t\}}{S_{ps}S_{pt}}$$

By combining these two formulae, we obtain the result. ∎

Consider the probabilities which appear on the right in Proposition 7.3:

$$\varepsilon_p(s,t) = P(\pi \rhd \sigma \mid |\pi| = s,\ |\sigma| = t).$$

The corresponding contributions to $\delta_p(M,N)$ are

$$\delta_p^{st}(M,N) = \frac{M!}{(M-s)!} \cdot \frac{S_{ps}}{M^p} \cdot \frac{N!}{(N-t)!} \cdot \frac{S_{pt}}{N^p} \cdot \varepsilon_p(s,t).$$

The idea now will be to separate the contributions coming from indices $s=1$ or $t=1$. To be more precise, we can rewrite Proposition 7.3 as follows:

Theorem 7.4. *We have*

$$\delta_p(M,N) = \frac{1}{M^{p-1}} + \frac{1}{N^{p-1}} - \frac{1}{(MN)^{p-1}} + \sum_{s=2}^{M}\sum_{t=2}^{N} \delta_p^{st}(M,N),$$

where $\delta_p^{st}(M,N)$ are the contributions defined above.

Proof. According to Proposition 7.3,

$$\delta_p(M,N) = \sum_{s=1}^{M}\sum_{t=1}^{N} \delta_p^{st}(M,N).$$

Since $\varepsilon_p(1,t) = 1$, the contributions for $s=1$ are given by

$$\delta_p^{1t}(M,N) = M \cdot \frac{1}{M^p} \cdot \frac{N!}{(N-t)!} \cdot \frac{S_{pt}}{N^p} = \frac{1}{M^{p-1}} \cdot \frac{N!}{(N-t)!} \cdot \frac{S_{pt}}{N^p}.$$

Now by summing over $t \geq 1$, we obtain

$$\sum_{t=1}^{N} \delta_p^{1t}(M,N) = \frac{1}{M^{p-1}} \sum_{t=1}^{N} \frac{N!}{(N-t)!} \cdot \frac{S_{pt}}{N^p} = \frac{1}{M^{p-1}}.$$

Similarly,

$$\sum_{s=1}^{M} \delta_p^{s1}(M,N) = \frac{1}{N^{p-1}} \sum_{s=1}^{M} \frac{M!}{(M-s)!} \cdot \frac{S_{ps}}{M^p} = \frac{1}{N^{p-1}}.$$

Finally, for $s=1$ and $t=1$ the contribution is

$$\delta_p^{11}(M,N) = M \cdot \frac{1}{M^p} \cdot N \cdot \frac{1}{N^p} = \frac{1}{(MN)^{p-1}}.$$

By using the inclusion-exclusion principle, this gives the result. ∎

8. Moment estimates. In this section we estimate $\delta_p(M,N)$ by using the formula found in Theorem 7.4. In order to deal with the contributions of $s,t \geq 2$, we use the following fact:

Proposition 8.1. *The function constructed above,*

$$\varepsilon_p(s,t) = P(\pi \rhd \sigma \mid |\pi| = s,\ |\sigma| = t),$$

is decreasing in both $s \in \mathbb{N}$ and $t \in \mathbb{N}$.

Proof. The problem being symmetric in s,t, it is enough to prove that $\varepsilon_p(s,t)$ is decreasing in t. By splitting the problem over the partitions π satisfying $|\pi|=s$, it is enough to prove that for any partition $\pi\in P(p)$, the following quantity is decreasing in t:

$$\varepsilon_\pi(t)=P(\pi\triangleright\sigma\mid|\sigma|=t).$$

In order to do so, recall from Proposition 7.1 that $\pi\triangleright\sigma$ is equivalent to

$$|\beta\cap\gamma|=|(\beta-1)\cap\gamma|,\qquad\forall\beta\in\pi,\ \forall\gamma\in\sigma.$$

Now observe that when merging two blocks of σ, say $(\gamma_1,\gamma_2)\to\gamma$, the condition is satisfied for γ, simply by summing the equalities for γ_1,γ_2. We deduce that the probability $\varepsilon_\pi(t)$ gets bigger when decreasing $t=|\sigma|$, as desired. ∎

Let us now combine Theorem 7.4 with Proposition 8.1. We obtain:

PROPOSITION 8.2. *We have*

$$\delta_p(M,N)\le 1-\left(1-\frac{1}{M^{p-1}}\right)\left(1-\frac{1}{N^{p-1}}\right)(1-\varepsilon_p(2,2))$$

for any $M,N\ge 2$.

Proof. The formula of Theorem 7.4 can be written as follows:

$$\begin{aligned}\delta_p(M,N)&=\frac{1}{M^{p-1}}+\frac{1}{N^{p-1}}-\frac{1}{(MN)^{p-1}}+\sum_{s=2}^M\sum_{t=2}^N\delta_p^{st}(M,N)\\&=1-\left(1-\frac{1}{M^{p-1}}\right)\left(1-\frac{1}{N^{p-1}}\right)+\sum_{s=2}^M\sum_{t=2}^N\delta_p^{st}(M,N).\end{aligned}$$

According to Proposition 8.1, for any $s,t\ge 2$ we have

$$\delta_p^{st}(M,N)\le\frac{M!}{(M-s)!}\cdot\frac{S_{ps}}{M^p}\cdot\frac{N!}{(N-t)!}\cdot\frac{S_{pt}}{N^p}\cdot\varepsilon_p(2,2).$$

By summing over all $s,t\ge 2$, and using the inclusion-exclusion principle, as in the proof of Theorem 7.4, we obtain

$$\begin{aligned}\sum_{s=2}^M\sum_{t=2}^N\delta_p^{st}(M,N)&\le\sum_{s=2}^M\sum_{t=2}^N\frac{M!}{(M-s)!}\cdot\frac{S_{ps}}{M^p}\cdot\frac{N!}{(N-t)!}\cdot\frac{S_{pt}}{N^p}\cdot\varepsilon_p(2,2)\\&=\left(1-\frac{1}{M^{p-1}}-\frac{1}{N^{p-1}}+\frac{1}{(MN)^{p-1}}\right)\varepsilon_p(2,2)\\&=\left(1-\frac{1}{M^{p-1}}\right)\left(1-\frac{1}{N^{p-1}}\right)\varepsilon_p(2,2).\end{aligned}$$

This gives the formula in the statement. ∎

On the other hand, by using the results of Section 6, we obtain:

Proposition 8.3. *We have*

$$\varepsilon_p(2,N)\simeq\frac{1}{2\cdot N!}\sqrt{\frac{N^N}{(\pi p)^{N-1}}}\qquad as\ p\to\infty.$$

Proof. As $p\to\infty$ we have

$$\begin{aligned}\delta_p^{st}(M,N)&=\frac{M!}{(M-s)!}\cdot\frac{S_{ps}}{M^p}\cdot\frac{N!}{(N-t)!}\cdot\frac{S_{pt}}{N^p}\cdot\varepsilon_p(s,t)\\&\simeq\frac{M!}{(M-s)!}\cdot\frac{s^p}{M^p}\cdot\frac{N!}{(N-t)!}\cdot\frac{t^p}{N^p}\cdot\varepsilon_p(s,t)\\&=\frac{M!}{(M-s)!}\cdot\frac{N!}{(N-t)!}\left(\frac{st}{MN}\right)^p\varepsilon_p(s,t).\end{aligned}$$

Here we have used the estimate $S_{ps}\simeq s^p$, which follows from the fact that choosing a partition $\pi\in P(p)$ with $\leq s$ blocks amounts to assigning a number $1,\ldots,s$ to any of the points $1,\ldots,p$, and the assignments which lead to $|\pi|<s$ can be neglected.

In particular, for $s=M=2$ we obtain

$$\delta_p^{2t}(2,N)\simeq 2\cdot\frac{N!}{(N-t)!}\left(\frac{t}{N}\right)^p\varepsilon_p(2,t).$$

By combining this estimate with Theorem 7.4 for $M=2$, we obtain

$$\begin{aligned}\delta_p(2,N)&=\frac{1}{2^{p-1}}+\frac{1}{N^{p-1}}-\frac{1}{(2N)^{p-1}}+\sum_{t=2}^N\delta_p^{2t}(2,N)\\&\simeq\frac{1}{2^{p-1}}+2\sum_{t=2}^N\frac{N!}{(N-t)!}\left(\frac{t}{N}\right)^p\varepsilon_p(2,t).\end{aligned}$$

With this formula in hand, we can proceed by recurrence on $N\geq 2$. Since the quantity in the statement converges as $p\to\infty$ to 0 much more slowly than the various powers α^N, with $\alpha\in(0,1)$, only the last term will matter, and our estimate simply reads

$$\delta_p(2,N)\simeq 2\cdot N!\varepsilon_p(2,N).$$

Now by using the $M=2$ estimate from Theorem 6.4, we obtain

$$\varepsilon_p(2,N)\simeq\frac{1}{2\cdot N!}\cdot\delta_p(2,N)\simeq\frac{1}{2\cdot N!}\sqrt{\frac{N^N}{(\pi p)^{N-1}}}.\ \blacksquare$$

We can now prove our result:

Theorem 8.4. *We have* $\lim_{p\to\infty}\delta_p(M,N)=0$ *for any* $M,N\geq 2$.

Proof. By combining Propositions 8.2 and 8.3, we obtain

$$\delta_p(M,N)\leq 1-\left(1-\frac{1}{M^{p-1}}\right)\left(1-\frac{1}{N^{p-1}}\right)\big(1-\varepsilon_p(2,2)\big).$$

Since the product on the right converges to $1 \times 1 \times 1 = 1$, this gives the result. ∎

9. Poisson laws. We recall that the free analogue of the Poisson law of parameter $t > 0$, in the sense of the Bercovici–Pata bijection [9], is the Marchenko–Pastur law of parameter t, also called the free Poisson law of parameter t. We denote this measure by π_t. See [16], [17], [22].

We have the following result, summarizing our findings:

THEOREM 9.1. *Given two finite abelian groups G, H with $|G| = M$ and $|H| = N$, let χ be the main character of the quantum group associated to $\mathcal{F}_{G\times H}$. Then:*

(1) $\mu = \mathrm{law}\big(\frac{\chi}{MN}\big)$ *is supported on* $[0,1]$.
(2) *The measure μ has no atom at* 1.
(3) *If $M = tN \to \infty$ we have*

$$\mathrm{law}\left(\frac{\chi}{N}\right) = \left(1 - \frac{1}{M}\right)\delta_0 + \frac{1}{M}\pi_t \quad \textit{in moments.}$$

Proof. Here (1) is trivial, (2) is new, and (3) is known from [4] in the case of the generic fibers. To be more precise, the proof goes as follows:

(1) follows from the fact that χ is by definition the main character for a certain quantum group $\mathcal{G} \subset S_{MN}^+$, and is therefore a sum of MN projections.

(2) follows from Theorem 8.4 and from the fact that an atom at 1 would make the moments converge to a nonzero quantity.

(3) According to our various normalizations, we have

$$\int_{\mathcal{G}}^{r} \left(\frac{\chi}{N}\right)^p = \frac{c_p^r(M,N)}{N^p} = \frac{(MN)^{p-1} d_p^r(M,N)}{N^p} = \frac{M^{p-1}}{N} d_p^r(M,N).$$

By using Proposition 7.2 we obtain, as $M = tN \to \infty$,

$$\int_{\mathcal{G}} \left(\frac{\chi}{N}\right)^p \simeq \frac{M^{p-1}}{N}\delta_p(M,N) \simeq \frac{M^{p-1}}{N} S_p(t) M^{-p} N = \frac{1}{M} S_p(t).$$

Since $S_p(t)$ is the pth moment of π_t, this gives the result. ∎

10. Concluding remarks. There are several questions related to the above results. First, we do not know how to improve Theorem 8.4 with a precise estimate, as in Theorem 6.4.

There are also some interesting questions related to [1], [21]. The main problem here, well-known and open, is that of understanding how a general deformed Fourier matrix $\mathcal{F}_K$ can be defined directly in terms of the finite abelian group K.

In connection with [7], observe that the representations there are also of the form $\pi : C(S_{\dim B}^+) \to C(U_B, \mathcal{L}(B))$ for a certain finite-dimensional

C^*-algebra B. In the present paper this algebra is a commutative one, $B = C(G\times H)$. We believe that the unification with [7] is an important question, which could lead to a substantial "boost" in the understanding and use of the integration formula of [6], [24].

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