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Normal forms for germs of vector fields with quadratic leading part. The remaining cases

by

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Abstract. We complete the classification of germs of plane vector fields with quadratic leading part initiated in Stróżyna (2015). There, two cases were completely analyzed, a simplest one and a most complex one. Here we study the remaining cases. In the proofs we use a new method introduced in Stróżyna & Żołądek (2015) concerning the Bogdanov–Takens singularity.

1. Introduction. The problem of classification of germs of vector fields in $(\mathbb{C}^n, 0)$, or in $(\mathbb{R}^n, 0)$, is natural and important (see [A]). In fact, two classification problems are considered: the usual one, when one applies local changes of coordinates, and the orbital one, when one additionally applies a reparametrization of time (i.e. when one is interested in classification of local phase portraits).

A standard approach to this problem is the following. Usually the vector fields under consideration are of the form

$$V(x) = V_0(x) + \cdots$$

where V_0 is a polynomial quasi-homogeneous vector field (with respect to some grading in the space $\mathbb{C}[[x]]$ of formal power series). The changes of variables $x = h(y)$ are generated by formal vector fields $Z(x)$, $h = \exp Z$; these Z's are subject to the same quasi-homogeneous grading. The linear (in Z) part of the transformed vector field is

$$\mathrm{ad}_V Z = [V, Z] = \mathrm{ad}_{V_0} Z + \cdots .$$

The operator $Z \mapsto \mathrm{ad}_{V_0} Z$ is the so-called first level homological operator. The first level normal form is defined by the choice of a space complemen-

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tary to the image of ad_{V_0}. The latter task is divided into finite-dimensional algebraic problems, by restricting the operator ad_{V_0} to spaces of polynomial vector fields of fixed quasi-homogeneous degree. For more details of this approach we refer the reader to [AFGG], [AGG], [BSa], [BaSl], [CWW], [KOW], [LS], [WCW], [Z].

G. Belitskiĭ [Be] proposed to choose the complementary subspace to ad_{V_0} to be the kernel of the conjugate operator $\mathrm{ad}^*_{V_0}$ with respect to some Hermitian products in the spaces of quasi-homogeneous vector fields of a given degree. In particular, if the vector field V_0 is linear, $V_0(x) = Ax$, then $(\mathrm{ad}_{Ax})^* = \mathrm{ad}_{A^*x}$. This approach was exploited by E. Lombardi and L. Stolovitch [LS]. Unfortunately, direct calculation of $\mathrm{Im}\,\mathrm{ad}_{V_0}$ and $\ker \mathrm{ad}^*_{V_0}$ is not easy, even in the case of a linear vector field V_0 [1].

In this paper we consider complex plane vector fields with zero linear part,

$$\dot{x} = \alpha x^2 + \beta xy + \gamma y^2 + \cdots, \qquad \dot{y} = \delta x^2 + \zeta xy + \eta y^2 + \cdots, \tag{1.1}$$

and classify them with respect to application of formal diffeomorphisms. This classification essentially depends on the classification of homogeneous quadratic parts

$$\boldsymbol{V}_0 = (\alpha x^2 + \beta xy + \gamma y^2)\partial_x + (\delta x^2 + \zeta xy + \eta y^2)\partial_y \tag{1.2}$$

with respect to linear changes of the coordinates. This classification was performed in [S, Section 2.2] (see also Section 2 below). From that classification one can see that the reduction process of the higher order terms in (1.1) is done recurrently with respect to some definite quasi-homogeneous grading (in the space of power series in two variables). In most cases that grading is standard, defined by the standard *Euler vector field*

$$\boldsymbol{E} = x\partial_x + y\partial_y, \tag{1.3}$$

but it can be nonstandard. In this paper we deal only with the cases when the standard grading is in use (as in [BSa]). Moreover, there are cases when the gradation is standard, but the critical point $(0,0)$ of system (1.1) is nonisolated (an infinite codimension phenomenon); we do not consider such singularities.

[1] In [LS] the example of $V_0 = Ax$ with a nilpotent matrix A is considered, $V_0 = x_2\partial_{x_1} + x_3\partial_{x_2}$; then $A^*x = x_1\partial_{x_2} + x_2\partial_{x_3}$. The authors claim that the first level normal form (an analogue of the 2-dimensional Takens normal form) is $(x_2+P_1)\partial_{x_1} + (x_3+x_2P_1+x_1P_2)\partial_{x_2} + (x_3P_1 + x_2P_2 + x_1P_3)\partial_{x_3}$, where $P_j = P_j(x_1,u)$ are formal power series in the first integrals x_1 and $u = x_2^2 - 2x_1x_3$ of A^*x (see [LS, Eq. (2)]). But also the vector fields $u^k\partial_{x_3}$ commute with A^*x; so in a correct normal form one should replace $x_1P_3(x_1,u)$ with $P_3(x_1,u)$ (see also [SZ1]).

The division of $\boldsymbol{V}_0$'s into different cases is determined by forms of the so-called *Principal First Integral* (PFI) of $\boldsymbol{V}_0$. Generally, this first integral is

$$F = x^a y^b (y-x)^c, \tag{1.4}$$

but in some limit cases logarithmic summands can appear. Important are also so-called *Inverse Integrating Multipliers* (IIMs), which should be polynomial.

In [S] the case with polynomial principal first integral, i.e., with $a = p$, $b = q$, $c = r$ relatively prime positive integers, was completely studied. There the polynomial inverse integrating multipliers are of the form

$$M_m = xy(y-x)F^m, \qquad m = 1, 2, \ldots;$$

they correspond to the first integrals F^{-m}. Also in [S] the 'road map' to treat other cases was sketched. Here we complete that task.

Let us say a few words about the method used in our reduction; it was invented by the author with H. Żołądek [SZ2] (and used in [S]). We want to reduce a vector field of the form $\boldsymbol{V}_0 + \boldsymbol{W}$ (as in (1.1)) to some normal form by application of a diffeomorphism $\exp \boldsymbol{Z}$ generated by a vector field $\boldsymbol{Z}$. Recall that the linear (in $\boldsymbol{Z}$) part of the action of $\exp \boldsymbol{Z}$ on $\boldsymbol{V}_0$ equals the commutator $-\mathrm{ad}_{\boldsymbol{V}_0}\boldsymbol{Z}$ plus higher order terms. We divide the perturbations $\boldsymbol{W}$ into two parts: 'transversal' to $\boldsymbol{V}_0$ and 'tangential' to $\boldsymbol{V}_0$; also the vector fields $\boldsymbol{Z}$ are subject to such division. We measure the part transversal to $\boldsymbol{V}_0$ by the bi-vector fields $\boldsymbol{V}_0 \wedge \boldsymbol{W} = h(x,y) \cdot \partial_x \wedge \partial_y$, i.e. by one function h. The part tangential to $\boldsymbol{V}_0$ is of the form $g(x,y)\boldsymbol{V}_0$, hence it is also measured by one function g. The homological operator $\mathrm{ad}_{\boldsymbol{V}_0}$ is split into two '1-dimensional' homological operators:

$$f \mapsto C(\boldsymbol{V}_0)f := \boldsymbol{V}_0(f), \qquad f \mapsto D(\boldsymbol{V}_0)f := \boldsymbol{V}_0(f) - \operatorname{div} \boldsymbol{V}_0 \cdot f. \tag{1.5}$$

In this way we realize the so-called first level reduction and obtain a first level normal form. In the second level reduction we use the homological operators $C(\boldsymbol{V}_0 + \boldsymbol{V}_1)$ and $D(\boldsymbol{V}_0 + \boldsymbol{V}_1)$ associated with two lowest degree terms from the first level normal form. In most complex cases we need four such steps.

An additional complication arises when $\boldsymbol{V}_0$ has nontrivial singular locus and can be represented as

$$\boldsymbol{V}_0 = G\boldsymbol{X} \tag{1.6}$$

for a noninvertible factor G, called the *Critical Factor* (CF). (It is also one of the elements which play a role in the classification of $\boldsymbol{V}_0$'s.) Then the corresponding homological operators (1.5) (and homological equations) are suitably modified.

Our method works well in the more general situation, when the 2-dimensional vector field V_0 is quasi-homogeneous (in [SZ2, Section 4.1] we sketched that situation). One of the advantages of the method is a clear distinction between the orbital and nonorbital normal forms. Unfortunately, we do not see any possibility to extend our approach beyond two dimensions.

The plan of the paper is the following. In Section 2 we present the algebraic apparatus used in the proofs. In Section 3 we discuss the classification of the leading homogeneous quadratic vector fields. In Sections 4–7 we study each of the cases determined in Section 3 separately. The list of the resulting normal forms (including those from [S]) is given in the last section; there we also compare our results with those of V. Basov and collaborators, and of L. Detchenya and A. Sadovskiĭ.

2. Homological equations. We briefly recall basic homological operators and their properties. For more details we refer the reader to [SZ2] and [S].

Recall that we study germs in $(\mathbb{C}^2, 0)$ of holomorphic vector fields of the form

$$\boldsymbol{V} = \boldsymbol{V}_0 + \cdots \tag{2.1}$$

where the dots denote higher order terms with respect to the standard grading, i.e., corresponding to systems (1.1) with homogeneous quadratic leading part $\boldsymbol{V}_0$.

2.1. Koszul complexes and homological operators. We deal with vector fields of the form (2.1). With $\boldsymbol{V}$ (and $\boldsymbol{V}_0$) we associate some linear operators. Let

$$\mathcal{F} = \mathbb{C}[[x, y]], \qquad \mathcal{Z} = \{\boldsymbol{Z} = z_1(x, y)\partial_x + z_2(x, y)\partial_y : z_i \in \mathbb{C}[[x, y]]\}$$

be the spaces of formal power series and formal vector fields. We denote by $\mathcal{F}_d$ and $\mathcal{Z}_d$ the spaces of functions and vector fields of degree d, where we set $\deg \partial_x = \deg \partial_y = -1$. We note the following identity:

$$[\boldsymbol{E}, \boldsymbol{Z}] = \deg \boldsymbol{Z} \cdot \boldsymbol{Z} \tag{2.2}$$

for a homogeneous vector field $\boldsymbol{Z}$.

We set

$$\begin{aligned} \mathrm{ad}_{\boldsymbol{V}} \boldsymbol{Z} &= [\boldsymbol{V}, \boldsymbol{Z}], \\ A(\boldsymbol{V})f &= f \cdot \boldsymbol{V}, \\ B(\boldsymbol{V})\boldsymbol{Z} &= \boldsymbol{V} \wedge \boldsymbol{Z}/\partial_x \wedge \partial_y, \\ C(\boldsymbol{V})f &= \boldsymbol{V}(f) = \partial f/\partial \boldsymbol{V}, \\ D(\boldsymbol{V})f &= \boldsymbol{V}(f) - \mathrm{div}(\boldsymbol{V}) \cdot f. \end{aligned} \tag{2.3}$$

The operators $C(\boldsymbol{V})$, $\mathrm{ad}_{\boldsymbol{V}}$ and $D(\boldsymbol{V})$ are called the *homological operators.* It turns out that the following diagram, with rows given by so-called *Koszul* *complexes*,

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{F} & \xrightarrow{A(\boldsymbol{V})} & \mathcal{Z} & \xrightarrow{B(\boldsymbol{V})} & \mathcal{F} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle C(\boldsymbol{V})} & & \downarrow{\scriptstyle \mathrm{ad}_{\boldsymbol{V}}} & & \downarrow{\scriptstyle D(\boldsymbol{V})} & & \\
0 & \longrightarrow & \mathcal{F} & \xrightarrow{A(\boldsymbol{V})} & \mathcal{Z} & \xrightarrow{B(\boldsymbol{V})} & \mathcal{F} & \longrightarrow & 0
\end{array}
\tag{2.4}
$$

is commutative.

It is easy to see that $\ker C(\boldsymbol{V})$ consists of all *First Integrals* (FIs) of $\boldsymbol{V}$, and $\ker D(\boldsymbol{V})$ consists of all *Inverse Integrating Multipliers* (IIMs) of $\boldsymbol{V}$, i.e., of functions M such that $\operatorname{div} M^{-1}\boldsymbol{V} \equiv 0$.

The further analysis depends on whether the origin is an isolated singularity of $\boldsymbol{V}_0$ or not.

2.2. The case with isolated singularity. If the origin is an isolated critical point of $\boldsymbol{V}_0$ then the above Koszul complexes are exact and we can split the reduction problem into the analysis of the operators $C(\boldsymbol{V})$ and $D(\boldsymbol{V})$. For this we have to resolve the singularity of the homogeneous vector field $\boldsymbol{V}_0$.

Recall that this resolution is a holomorphic map $\pi : (\mathbb{S}, E) \to (\mathbb{C}^2, 0)$ which is one-to-one outside the exceptional divisor $E \simeq \mathbb{CP}^1 = \pi^{-1}(0)$. We get a holomorphic foliation in the complex surface $\mathbb{S}$ such that in the general (nondicritical) case the divisor E is invariant with three singular points (counted with multiplicities); the so-called dicritical case corresponds to the situation $\boldsymbol{V}_0 = G\boldsymbol{E}$ which is beyond our analysis (see Remark 1 below). The above singular points on E correspond to invariant lines of $\boldsymbol{V}_0$.

In our analysis of homological operators we use the blowing-up coordinates:

$$(x, u) = (x, y/x).$$

These are coordinates in one chart of the surface $\mathbb{S}$ (and u is a coordinate along E); in another chart the coordinates are $(y, v) = (y, x/y)$.

Firstly, we recall that a homogeneous polynomial $f(x, y)$ of degree d takes the form

$$f = x^d \tilde{f}(u) \tag{2.5}$$

for a polynomial $\tilde{f}$. We also have

$$\boldsymbol{V}_0 = xa(u)\partial_u - x^2 b(u)\partial_x, \qquad \operatorname{div} \boldsymbol{V}_0 = xc(u) \tag{2.6}$$

for some polynomials a, b, c.

The homological equations

$$C(\boldsymbol{V}_0)f = g, \qquad D(\boldsymbol{V}_0)f = g \tag{2.7}$$

(for $f \in \mathcal{F}_d$ with given $g = x^{d+1}\tilde{g}(u) \in \mathcal{F}_{d+1}$) take the form

$$\begin{aligned} a(u)\frac{d\tilde{f}}{du} &= db(u)\tilde{f} + \tilde{g}, \\ a(u)\frac{d\tilde{f}}{du} &= [db(u) - c(u)]\tilde{f} + \tilde{g}. \end{aligned} \tag{2.8}$$

Generally, $a(u) = \mathrm{const} \cdot u(u-1)$ and the solutions to the latter equations are of the form

$$\begin{aligned} \tilde{f}(u) &= \mathrm{const} \cdot u^{\alpha}(u-1)^{\beta} \int^{u} \tau^{-\alpha-1}(\tau-1)^{-\beta-1}\tilde{g}(\tau)\, d\tau, \\ \tilde{f}(u) &= \mathrm{const} \cdot u^{\gamma}(u-1)^{\delta} \int^{u} \tau^{-\gamma-1}(\tau-1)^{-\delta-1}\tilde{g}(\tau)\, d\tau \end{aligned} \tag{2.9}$$

(for some exponents $\alpha, \beta, \gamma, \delta$). The integrals in (2.9) define *Schwarz–Christoffel functions* (SC functions), or incomplete Schwarz–Christoffel integrals.

Recall that the solutions to (2.7) should be polynomial; otherwise, the corresponding polynomial g lies outside $\operatorname{Im} C(\boldsymbol{V}_0)$ (or $\operatorname{Im} D(\boldsymbol{V}_0)$). Therefore we should find obstructions to (2.9) being polynomials. These obstructions are the *periods* of the SC functions defined as follows.

If $\alpha, \beta \notin \mathbb{Z}$ (respectively, $\gamma, \delta \notin \mathbb{Z}$) then the periods are defined to be the following complete SC integrals:

$$\begin{aligned} \Omega_C(g) &= \mathrm{P.V.} \int_0^1 \omega_C(g), \quad \omega_C = u^{-\alpha-1}(u-1)^{-\beta-1}\tilde{g}(u)\, du, \\ \Omega_D(g) &= \mathrm{P.V.} \int_0^1 \omega_D(g), \quad \omega_D = u^{-\gamma-1}(u-1)^{-\delta-1}\tilde{g}(u)\, du. \end{aligned} \tag{2.10}$$

Often $\alpha, \ldots, \delta > 0$ and the above integrals diverge, so one should take some regularization; this is indicated by the principal value symbol P.V. A natural regularization uses an analytic continuation of the Euler Beta function, as a function of parameters. Anyway, we get the following explicit description of the images of our homological operators in the cases when they are of codimension 1 in $\mathcal{F}_{d+1}$:

$$\operatorname{Im} C(\boldsymbol{V}_0) = \{\Omega_C = 0\}, \quad \operatorname{Im} D(\boldsymbol{V}_0) = \{\Omega_D = 0\}. \tag{2.11}$$

In special situations the periods are defined in special ways. Sometimes some of these images are of codimension 2 and they are defined by vanishing of two periods, e.g., residues of $\omega_{C,D}$ at $u = 0$ and $u = 1$. We will encounter such cases.

We finish this subsection with the following identity to be used later: we have $\boldsymbol{V}_0 \wedge \boldsymbol{E} = sxy(y-x)\partial_x \wedge \partial_y$, $s = a + b + c$, for $\boldsymbol{V}_0$ with the first integral (1.4) (see also (3.1) below), i.e.,

$$B(\boldsymbol{V}_0)\boldsymbol{E} = sxy(y-x). \tag{2.12}$$

Usually the orbital normal form is $\boldsymbol{V}_0+\varPhi(x,y)\boldsymbol{E}$ and the whole normal form is $(1+\varPsi(x,y))\cdot(\boldsymbol{V}_0+\varPhi\boldsymbol{E})$ for a special choice of the formal series $\varPhi$ and $\varPsi$. Therefore (2.12) indicates that the right-hand side of the second equation of (2.7) should equal $su(u-1)\tilde{\varPhi}(u)$. There are no such restrictions for the right-hand side of the first equation of (2.7).

2.3. The case with a critical factor. Assume that

$$\boldsymbol{V}_0 = G\boldsymbol{X}$$

where the *Critical Factor* (CF) $G(x,y)$ is linear or quadratic and $\boldsymbol{X}$ has an isolated singularity, i.e., the form (1.6).

We consider the diagram (2.4) restricted to suitable homogeneous components $\mathcal{F}_d$ and $\mathcal{Z}_d$ consisting of homogeneous polynomials and vector fields.

Then $\operatorname{Im} B(\boldsymbol{V}_0) = G\mathcal{F}_k$, $\ker B(\boldsymbol{V}_0) = \mathcal{F}_l\boldsymbol{X}$ and $\operatorname{Im} A(\boldsymbol{V}_0) = G\mathcal{F}_m\boldsymbol{X}$ (for suitable k,l,m). Of course, the corresponding Koszul complexes are not exact and the homological operators (and homological equations) must be modified.

The operator $D(\boldsymbol{V}_0)$ remains unchanged, but we must be careful when choosing the right-hand side of the homological equation $D(\boldsymbol{V}_0)f = g$: it is desirable that g be divisible by G. But we encounter a situation when $\operatorname{Im} D(\boldsymbol{V}_0) = \operatorname{Im} B(\boldsymbol{V}_0)$ and then the normal forms are derived in another (more direct) way (see Section 4.11).

To reduce the orbital factor we should use vector fields $f\boldsymbol{X}$, from $\ker B(\boldsymbol{V}_0)$, and apply them to $\boldsymbol{V}_0 = G\boldsymbol{X}$. We have

$$[G\boldsymbol{X}, f\boldsymbol{X}] = C'(\boldsymbol{V}_0)f\cdot\boldsymbol{X}$$

where

$$C'(\boldsymbol{V}_0)f = G\boldsymbol{X}(f) - \boldsymbol{X}(G)f, \tag{2.13}$$

Thus $C'(\boldsymbol{V}_0)$ is our *modified homological operator*. It is easy to see that

$$\ker C'(\boldsymbol{V}_0) = \{f = hG : \boldsymbol{X}(h) = 0\}. \tag{2.14}$$

When we expect a normal form like $(1+\varPsi)(\boldsymbol{V}_0+\varPhi\boldsymbol{E})$ then the right-hand side of the homological equation

$$C'(\boldsymbol{V}_0)f = g \tag{2.15}$$

should be $g = hG$ for a homogeneous polynomial $h(x,y)$.

The analogues of (2.8)–(2.11) are

$$a(u)\frac{d\tilde{f}}{du} = [db(u)-e(u)]\tilde{f} + \tilde{g}, \tag{2.16}$$

$$\tilde{f}(u) = \mathrm{const}\cdot u^{\mu}(u-1)^{\nu}\int^{u}\tau^{-\mu-1}(\tau-1)^{-\nu-1}\tilde{g}(\tau)\,d\tau, \tag{2.17}$$

$$\varOmega_{C'}(g) = \mathrm{P.V.}\int_0^1 \omega_C(g), \qquad \omega_C = u^{-\mu-1}(u-1)^{-\nu-1}\tilde{g}(u)\,du, \tag{2.18}$$

$$\operatorname{Im} C'(\boldsymbol{V}_0) = \{\varOmega_{C'} = 0\}, \tag{2.19}$$

for a suitable polynomial $e(u)$ and exponents μ, ν.

It occurs, but only once in Section 4.11, that $\operatorname{Im} C'(\boldsymbol{V}_0) = G\mathcal{F}_d$ and the direct method is applicable (in that case also $\operatorname{Im} D(\boldsymbol{V}_0) = \operatorname{Im} B(\boldsymbol{V}_0)$, see Remark 4).

3. Classification of the homogeneous quadratic vector fields from the further reduction perspective. Recall that we study germs $\boldsymbol{V} = \boldsymbol{V}_0 + \cdots$ as in (2.1). The vector field $\boldsymbol{V}_0$ can have critical locus of three sorts: the origin, a straight line, two straight lines and a double straight line. In [S] the first case is denoted as crit $= 0$, the second one as crit $= 1$, and the third and fourth as crit $= 2$. They correspond to the degrees of the Critical Factor (CF) G such that $\boldsymbol{V}_0 = G\boldsymbol{X}$ (as in (1.6)) and crit $= \deg G$.

Next, $\boldsymbol{V}_0$ has invariant lines (and some of them can be critical). Their number (denoted lin in [S]) can be 3, 2 or 1.

If lin $= 3$ then the invariant lines can be taken to be $x = 0$, $y = 0$ and $y = x$. Moreover $\boldsymbol{V}_0$ has a Darboux First Integral (FI) of the form (1.4), i.e., $x^a y^b (y-x)^c$, and

$$\boldsymbol{V}_0 = x[(b+c)y - bx]\partial_x + y[(a+c)x - ay]\partial_y. \tag{3.1}$$

Note that one of the exponents a, b, c can be zero; then the corresponding line is critical. For example, if $F = x^a y^b (y-x)^0$ then $\boldsymbol{V}_0 = (y-x)(bx\partial_x - ay\partial_y)$ and we have the critical factor $G = y - x$.

Since a power of a first integral is also a first integral, we will define something like a *Principal First Integral* (PFI).

Below we use the following notation:

$$p, q, r \in \mathbb{Z}_{>0} = \{1, 2, \ldots\}, \qquad \mathbb{Z}_{\geq k} = \{i \in \mathbb{Z} : i \geq k\}, \qquad s = a + b + c. \tag{3.2}$$

In the *'Generic Nonresonant' case* we have

$$a + b + c = 1, \qquad a \notin \mathbb{Q}, \qquad bc \neq 0, \tag{3.3}$$

in the function (1.4). Here there are no polynomial FIs and no polynomial Inverse Integrating Multipliers (IIMs).

In the *'Polynomial PFI' case* we have

$$a = p, \qquad b = q, \qquad c = r, \qquad \gcd(p, q, r) = 1. \tag{3.4}$$

Here $F_m = F^m = (x^p y^q (y-x)^r)^m$, are polynomial FIs $m = 1, 2, \ldots$, and $M_m = xy(y-x)F^m$, $m = 1, 2, \ldots$, are polynomial IIMs. (In Sections 7–8 it is called *Subcase 1.*)

In the *'Rational PFI without Polynomial IIMs' case* we have either

$$(3.5)\qquad \begin{array}{l} a=p,\quad b=q,\quad -c=p,\quad s\neq 0,\quad \gcd(p,s)=1\quad (\textit{Subcase 1}),\quad \text{or}\\ -a=p,\quad b=q,\quad c=r,\quad s\neq 0,\quad \gcd(p,s)=1\quad (\textit{Subcase 2}). \end{array}$$

Here there are no polynomial FIs and no polynomial IIMs. The division into two subcases is dictated by uniformity of the final normal form for $\boldsymbol{V}$ (see Remark 1 below).

Next, we have the situations with rational (but not polynomial) PFIs and polynomial IIMs.

In the *'Rational PFI with 1-Factor IIM' case* we have either

$$(3.6)\qquad \begin{array}{l} a=b=1,\quad -c=r\geq 3,\quad r\neq 4\quad (\textit{Subcase 1}),\quad \text{or}\\ a=b=1,\quad c=-4\quad (\textit{Subcase 2}). \end{array}$$

Here $F=xy/(y-x)^r$ and $M=(y-x)^{r+1}$. In the two subcases the final normal forms are different.

In the *'Rational PFI with 2-Factor IIM' case* we have

$$(3.7)\qquad \begin{array}{l} a=1,\quad -b=q,\quad -c=r,\quad 1\leq r\leq q>1\quad (\textit{Subcase 1}),\quad \text{or}\\ a=1,\quad b=c=1\quad (\textit{Subcase 2}),\quad \text{or}\\ a=1,\quad b=0,\quad -c=r\geq 2\quad (\textit{Subcase 3}). \end{array}$$

Here $F=x/y^q(y-x)^r$ and $M=y^{q+1}(y-x)^{r+1}$ and in Subcase 3 there is a CF $G=y$.

Now we consider the situations with nontrivial critical factor. In the *'Nonresonant with CF' case* we have

$$(3.8)\qquad a\notin\mathbb{Q},\qquad a+b=1,\qquad c=0.$$

Here the CF equals $G=y-x$ and there are no polynomial FIs and polynomial IIMs.

In the *'Polynomial PFI with Linear CF' case* we have

$$(3.9)\qquad a=p,\quad b=q,\quad c=0,\quad \gcd(p,q)=1.$$

Here the CF equals $G=y-x$, we have FIs $F_m=F^m=(x^py^q)^m$ and IIMs $M_m=xy(y-x)F^m$, $m=1,2,\ldots$. (In Sections 7–8 it is called *Subcase 2*.)

In the *'Rational PFI without Polynomial IIMs and with CF' case* we have

$$(3.10)\qquad a=p,\quad -b=q,\quad c=0,\quad 1<p<q,\quad \gcd(p,q)=1.$$

Here $F=x^q/y^p$ and $G=y-x$.

Finally, we have the *'Quadratic CF' case* with

$$(3.11)\qquad a=c=0,\qquad b=1.$$

Here $F=y$, $F_m=y^m$ are polynomial FIs, $M_m=xy^{m+1}(y-x)$, $m=1,2,\ldots,$

are polynomial IIMs, and $G = x(y-x)$ (2). (In Sections 7–8 it is called *Subcase 3*.)

There remain cases with a multiple invariant line; recall that the multiplicity of an invariant line $y = u_0 x$ equals the order of the zero $u = u_0$ in $\dot{u}$ (for $u = y/x$).

In the *'Double Invariant Line' case* we have

$$\begin{aligned} &\boldsymbol{V}_0 = xy\partial_y + (ax+y)\boldsymbol{E}, && \text{and either} \\ &a \neq 0, 1, 1/2, 1/3, \ldots && (\textit{Subcase 1}), \quad \text{or} \\ &a = 1/n, \quad n = 1, 2, \ldots && (\textit{Subcase 2}). \end{aligned} \tag{3.12}$$

Here $F = ay - (1+a)\ln x + y/x$ is a FI, and $M = x^2$ is a polynomial IIM.

In the *'Triple Invariant Line' case* we have

$$\boldsymbol{V}_0 = x^2\partial_y + (ax+y)\boldsymbol{E}. \tag{3.13}$$

Here $F = ay/x + y^2/2x^2 - \ln x$ with the corresponding IIM $M = x^3$.

Finally, in the *'Double Invariant Line with CF' case* we have either

$$\begin{aligned} &\boldsymbol{V}_0 = (bx+y)(x\partial_y + a\boldsymbol{E}), && a \neq 0 \quad (\textit{Subcase 1}), \quad \text{or} \\ &\boldsymbol{V}_0 = x(x\partial_y + a\boldsymbol{E}), && a \neq 0 \quad (\textit{Subcase 2}). \end{aligned} \tag{3.14}$$

Here $F = (bx+y)^0(ay/x - \ln x)$ or $F = x^0(ay/x - \ln x)$ and $M = x^2(bx+y)$ or $M = x^3$.

Remark 1. L. Detchenya and A. Sadovskiĭ [DS] present the following classification of homogeneous quadratic vector fields:

$$\begin{aligned} &x\boldsymbol{E} \text{ (DS1)}, \quad x[bx+(c-1)y]\partial_x + y[(b-1)x+cy]\partial_y \text{ (DS2)}, \\ &(bx+y)\boldsymbol{E} - xy\partial_y \text{ (DS3)}, \quad x(x\partial_x - 2y\partial_y) \text{ (DS4)}, \quad x\boldsymbol{E} - xy\partial_y \text{ (DS5)}, \\ &x\boldsymbol{E} - x^2\partial_y \text{ (DS6)}, \quad -x^2\partial_y \text{ (DS7)}, \quad \pm x^2\partial_y + y\boldsymbol{E} \text{ (DS8)}, \\ &x(y\partial_x - x\partial_y) + (bx+cy)\boldsymbol{E} \text{ (DS9)}. \end{aligned}$$

Cases DS1, DS4 and DS7 are beyond our analysis (either the grading is nonstandard or the singularity is of infinite codimension). In case DS2 there are three real invariant lines ($x = 0$, $y = 0$ and $y = x$) and in case DS9 there is one real invariant line $x = 0$ and two nonreal ones $y = \pm ix$ (the authors consider real vector fields). In cases DS3 and DS5 we have the double

(2) In [S] other cases with crit $= \deg G = 2$ were considered: $x^2\partial_x$, $xy\partial_x$ and $y^2\partial_x$. In all of them either the Newton diagram of $\boldsymbol{V}$ contains an edge leading to a nonstandard grading or $\boldsymbol{V}$ has nonisolated critical locus.

The same remark applies to the case $y(\alpha x\partial_x + \beta y\partial_y)$.

This is in analogy with the cases when the leading part is linear. In the elementary cases (at least one nonzero eigenvalue) there exist formal separatrices, which can be taken as $x = 0$ and $y = 0$. Then the Newton diagram of the whole vector field has only one vertex and one uses the standard grading. In the nilpotent case $y\partial_x + \cdots$ a quasi-homogeneous grading is applied (see [SZ2]).

invariant line $x=0$ and in cases DS6 and DS8 we have the triple invariant line $x=0$. In the above list one cannot find the cases with linear critical factor and with quadratic critical factor.

Another list of homogeneous quadratic vector fields was produced by V. Basov and E. Fedorova [BaF2] (it extends a list from [BaS1]):

$$x^2\partial_x+y^2\partial_y\ (\mathrm{CF}_1^0),\quad y^2\partial_x+x^2\partial_y\ (\mathrm{CF}_2^0),\quad x(x^2+ay)\partial_x+y^2\partial_y\ (\mathrm{CF}_3^0),$$
$$(ax^2+y^2)\partial_x+y^2\partial_y\ (\mathrm{CF}_4^0),\quad (ax^2\pm y^2)\partial_x+xy\partial_y\ (\mathrm{CF}_5^0),$$
$$(ax^2+y^2)\partial_x+x^2\partial_y\ (\mathrm{CF}_6^0),\quad y(ax+y)\partial_x+x^2\partial_y\ (\mathrm{CF}_7^0),$$
$$y(ax+y)\partial_x+y(x+by)\partial_y\ (\mathrm{CF}_8^0),\quad (ax^2\pm y^2)\partial_x+y(x+by)\partial_y\ (\mathrm{CF}_9^0),$$
$$(x^2/2+axy-y^2)\partial_x+xy\partial_y\ (\mathrm{CF}_{10}^0),\quad x(ax\partial_x+y\partial_y)\ (\mathrm{CF}_1^1),$$
$$x(\pm y\partial_x+x\partial_y)\ (\mathrm{CF}_2^1),\quad x[(ax+y)\partial_x+y\partial_y]\ (\mathrm{CF}_3^1),$$
$$x[x\partial_x+(x+y)\partial_y]\ (\mathrm{CF}_4^1),\quad x[(ax+y)\partial_x+x\partial_y]\ (\mathrm{CF}_5^1),\quad x^2\partial_x\ (\mathrm{DCF}_1^2),$$
$$xy\partial_x\ (\mathrm{DCF}_2^2),\quad y^2\partial_x\ (\mathrm{DCF}_3^2),\quad x(x+y)\partial_x\ (\mathrm{DCF}_4^2),\quad (x^2+y^2)\partial_x\ (\mathrm{DCF}_5^2),$$
$$x^2(\partial_x+\partial_y)\ (\mathrm{CF}_1^2),\quad (x^2\pm y^2)(\partial_x+\partial_y)\ (\mathrm{CF}_2^2),\quad (x-y)^2(\partial_x+\partial_y)\ (\mathrm{CF}_3^2),$$
$$xy(\partial_x+\partial_y)\ (\mathrm{CF}_4^2),\quad x(x-y)(\partial_x+\partial_y)\ (\mathrm{ACF}_2^2).$$

Here CF (respectively DCF or ACF) means canonical form (respectively degenerate canonical form or additional canonical form), and the upper index equals crit (see [DS]). Some cases in the latter list are irrelevant from our point of view, and others can be grouped into classes from our list.

The 'Generic Nonresonant' case is the simplest one and was solved in [S, Theorem 2] (see also Section 4.2). The 'Polynomial PFI' case is the most complicated. Its classification consists of nine subcases and is given in [S, Theorem 1] (see also Section 8).

The further plan of work on the above cases is the following. In Section 4 we consider all the cases without polynomial FIs and polynomial IIMs of degree ≥ 4. (The IIMs with degree <4 do not influence the normal forms because they do not belong to $B(\boldsymbol{V}_0)\mathcal{F}_d$, $d\geq 2$; see also Section 4.1). They include: the 'Generic Nonresonant' case, the 'Rational PFI without Polynomial IIMs' case, the 'Nonresonant with CF' case, the 'Rational PFI without Polynomial IIMs and with CF' case, the 'Double Invariant Line' case, the 'Triple Invariant Line' case and the 'Double Invariant Line with CF' case. They need only one level of analysis, i.e., using the homological operators $D(\boldsymbol{V}_0)$ and $C(\boldsymbol{V}_0)$, or $D(\boldsymbol{V}_0)$ and $C'(\boldsymbol{V}_0)$ in the cases with CF. They are considered in separate subsections.

In Section 5 we consider the 'Rational PFI with 1-Factor IIM' case. There we will need three levels of analysis associated with the operators $D(\boldsymbol{V})$ and $C(\boldsymbol{V})$ for different $\boldsymbol{V}$'s.

In Section 6 we handle the ‘Rational PFI with 2-Factor IIM’ case. Again, three levels of analysis are needed.

Section 7 is devoted to three cases with polynomial PFI. But the ‘Polynomial PFI’ case was considered in [S]. So we concentrate on the ‘Polynomial PFI with Linear CF’ case and the ‘Quadratic CF’ case where we also have nonisolated critical locus, of dimension 1 and 2 respectively. We will follow the four steps of analysis from [S]; of course, we will use the operators $D(\boldsymbol{V})$ and $C'(\boldsymbol{V})$. In [S, Remark 2] it is claimed that the classification is the same as in the ‘Polynomial PFI’ case with trivial modifications; we substantiate this statement.

4. The cases with injective homological operators. In this section we completely analyze the following cases:

- the ‘Generic Nonresonant’ case,
- the ‘Rational PFI without Polynomial IIMs’ case,
- the ‘Double Invariant Line’ case,
- the ‘Triple Invariant Line’ case,
- the ‘Nonresonant with CF’ case,
- the ‘Rational PFI without Polynomial IIMs and with CF’ case,
- the ‘Double Invariant Line with CF’ case.

4.1. The first level homological operators. Define the following *first level homological operators*:

$$C_d(\boldsymbol{V}_0), D_d(\boldsymbol{V}_0), C'_d(\boldsymbol{V}_0) : \mathcal{F}_d \to \mathcal{F}_{d+1},$$

i.e., the restrictions of $C(\boldsymbol{V})$, $D(\boldsymbol{V})$ and $C'(\boldsymbol{V}_0)$ to the $d+1$-dimensional spaces of homogeneous polynomials.

LEMMA 1. *The operators $D_d(\boldsymbol{V}_0)$ for $d \geq 4$ and $C_d(\boldsymbol{V}_0)$ for $d > 0$ (or $C'_d(\boldsymbol{V}_0)$ for $d > 0$, when* crit > 0) *have trivial kernels in all the above cases.*

Proof. The kernel of $C_d(\boldsymbol{V}_0)$ consists of homogeneous polynomial first integrals. Any such integral should be a function of the principal first integral F. By definition in all the above cases there are no polynomial first integrals.

Analogously the kernel of $C'_d(\boldsymbol{V}_0)$ consists of functions $f = hG$ such that h is a polynomial homogeneous first integral of X. Such an h is a function of F.

Finally, the kernel of $D_d(\boldsymbol{V}_0)$ consists of homogeneous polynomial inverse integrating multipliers. If $\boldsymbol{X}_F = F'_y\partial_x - F'_x\partial_y$ is a Hamiltonian vector field with Hamilton function F, then $\boldsymbol{V}_0 = M\boldsymbol{X}_F$. In all cases where such a polynomial IIM exists, its degree is ≤ 3. ∎

The statement about D_d implies the following for the problem of orbital normal form. When we apply the operator $\mathrm{ad}_{\boldsymbol{V}_0}$ to $\boldsymbol{Z}$ with $\deg \boldsymbol{Z} \geq 1$ (with quadratic and higher order components), it results in the application of the operator $D(\boldsymbol{V}_0)$ to the series $B(\boldsymbol{V}_0)\boldsymbol{Z}$ which contains terms of degree ≥ 4. Here is the reason why, in Lemma 1, we consider only the operators D_d with $d \geq 4$.

Therefore, in applications, we should choose a 1-dimensional subspaces $\mathcal{N}(D_d) \subset \mathcal{F}_{d+1}$, $d \geq 4$, complementary to $\operatorname{Im} D_d(\boldsymbol{V}_0)$, and identify its elements with $B(\boldsymbol{V}_0)\boldsymbol{W}_{d-2}$ for some vector field $\boldsymbol{W}_{d-2} \subset \mathcal{Z}_{d-2}$. These $\boldsymbol{W}_j$'s (which are usually of the form $a_{j-1}x^{j-1}\boldsymbol{E}$) will contribute to the *formal orbital normal form*, which will equal $\boldsymbol{V}_0 + \sum \boldsymbol{W}_{d-2}$.

The second statement of the lemma implies that $\operatorname{Im} C_d(\boldsymbol{V}_0)$ is of codimension 1 (respectively, $\operatorname{Im} C'_d(\boldsymbol{V}_0)$ is also of codimension 1). If we choose a 1-dimensional subspace $\mathcal{N}(C_d) \subset \mathcal{F}_{d+1}$ then its elements g_{d+1} (which are usually of the form $b_{d+1}x^{d+1}$) will correspond to summands in the *formal orbital factor*. The generators of $\mathcal{N}(C'_d)$ are chosen in the form $g_{d'+1}G$, $d' = d - \deg G$.

The *complete normal form* will be

$$\Big(1 + \sum g_d\Big)\Big(\boldsymbol{V}_0 + \sum \boldsymbol{W}_d\Big),$$

in almost all situations. The spaces $\mathcal{N}(D_d)$, $\mathcal{N}(C_d)$ and $\mathcal{N}(C'_d)$ will be specified in each of the above cases separately.

4.2. The 'General Nonresonant' case. Recall that this case was already solved in [S]. It also the simplest one; so it is reasonable to present it here, as a kind of instructive example.

For $\boldsymbol{V}_0$ as in (3.1) we have (2.9)–(2.10) with

$$\begin{aligned} &\alpha = db/s, \quad \beta = dc/s, \quad \gamma = (d-3)b/s + 1, \quad \delta = (d-3)c/s + 1, \\ &\alpha + \beta = d - da/s, \quad \gamma + \delta = d - 1 - (d-3)a/s, \end{aligned} \tag{4.1}$$

where $s = a + b + c = 1$ and $a = a/s$ is irrational (see (3.3)). We choose the following generators of $\mathcal{N}(C_d)$ and $\mathcal{N}(D_d)$:

$$g^C = x^{d+1}, \quad g^D = x^{d-1}y(y-x) = x^{d+1}u(u-1), \tag{4.2}$$

and we will demonstrate that they lie outside the images of the operators C_d and D_d respectively. Here $g^D = B(\boldsymbol{V}_0) \cdot x^{d-2}\boldsymbol{E}$ (see (2.12) for $s = 1$) and corresponds to the term $x^{d-2}\boldsymbol{E}$ in the orbital normal form.

The integrands in (2.10) are

$$\omega_C(g^C) = \frac{du}{u^{\alpha+1}(u-1)^{\beta+1}}, \quad \omega_D(g^D) = \frac{du}{u^{\gamma}(u-1)^{\delta}}. \tag{4.3}$$

Since $a/s \notin \mathbb{Q}$, the numbers $\alpha + \beta$, $\gamma + \delta$ are not integers. Therefore the periods $\Omega_C(g^C)$ and $\Omega_D(g^D)$, given in (2.10), are computed via the Euler

Beta function and are nonzero, e.g.,

$$\Omega_C(g^C) = \text{const} \cdot \frac{\Gamma(-\alpha)\Gamma(-\beta)}{\Gamma(-\alpha-\beta)} \neq 0 \ (^3). \tag{4.4}$$

It follows that:

- *The complete formal normal form in the 'Generic Nonresonant' case is*

$$(1+\psi(x))(\boldsymbol{V}_0 + \varphi(x)\boldsymbol{E}), \tag{4.5}$$

where $\varphi(x) = a_2x^2 + \cdots$ *and* $\psi(x) = b_1x + \cdots$ *are formal power series and where* $\boldsymbol{V}_0 + \varphi(x)\boldsymbol{E}$ *is the formal orbital normal form.*

This is the normal form from [S, Theorem 2].

4.3. The 'Rational PFI without Polynomial IIMs' case. Subcase 1. Recall that the PFI equals

$$F = \frac{x^py^q}{(y-x)^r}, \qquad \gcd(p,s) = 1,$$

$s = p+q-r \neq 0$ (otherwise y/x is also a first integral) and there are no polynomial IIMs (i.e., when $(p,q,r) \neq (1,1,r)$, $r \geq 3$, and $(p,q,r) \neq (p,q,1)$; cf. (3.5)).

Here we have formulas (2.9)–(2.10) with

$$\alpha = dq/s, \quad \beta = -dr/s, \quad \gamma = (d-3)q/s+1, \quad \delta = -(d-3)r/s+1,$$
$$\alpha+\beta = d - dp/s, \quad \gamma+\delta = d-1-(d-3)p/s.$$

Since $\gcd(p,s) = 1$, the number $\alpha+\beta$ (respectively, $\gamma+\delta$) is an integer only when d/s (respectively, $(d-3)/s$) is.

We choose the generators of $\mathcal{N}(C_d)$ and $\mathcal{N}(D_d)$ as in (4.2). If d/s is not an integer (respectively $(d-3)/s \notin \mathbb{Z}$) then an argument with the Beta function (from the previous case) works.

Suppose $d = ms$ with $m \in \mathbb{Z}_{>0}$ (thus $s > 0$). Then $\alpha = mq \in \mathbb{Z}_+$ and $\beta = -mr \in \mathbb{Z}_{<0}$. The function in the first equation of (2.9) equals

$$\tilde{f}^C = -\frac{1}{s}\frac{u^{mq}}{(u-1)^{mr}} \int^u \frac{(\tau-1)^{mr-1}}{\tau^{mq+1}}\, d\tau. \tag{4.6}$$

If $mr-1 \geq mq$ then the obstruction to $\tilde{f}$ being a polynomial is

$$\Omega_C(g^C) = \operatorname{Res}_{u=0} \omega_C(g^C) \neq 0. \tag{4.7}$$

Otherwise, the SC integral in (4.6) is a rational function $\int_\infty^u \omega_C(g^C)$ which

(3) Recall that the Euler Gamma function $\Gamma(z)$ does not vanish and has poles at $z = 0, -1, -2, \ldots$.

should vanish at $u=0$. Therefore the corresponding period is

$$\Omega_C(g^C) = \int_{\infty}^{0} \omega_C(g^C), \tag{4.8}$$

which is obviously nonzero. If $d=-ms$ with $m \in \mathbb{Z}_{<0}$ then after swapping q with r the argument is the same.

If $d-3=ms$ then the second equation of (2.9) gives

$$\tilde{f}^D = -\frac{1}{s}\frac{u^{mq+1}}{(u-1)^{mr-1}} \int^{u} \frac{(\tau-1)^{mr-1}}{\tau^{mq+1}}\, d\tau. \tag{4.9}$$

The analysis is the same as in the case of the function (4.6).

Therefore:

• *The normal form in Subcase 1 of the 'Rational PFI without Polynomial IIMs' case is the same as in* (4.5).

4.4. The 'Rational PFI without Polynomial IIMs' case. Subcase 2. Here

$$F = \frac{y^q(y-x)^r}{x^p}, \qquad \gcd(p,s)=1,$$

$s=q+r-p \neq 0$, and there are no polynomial IIMs (cf. (3.5)). Now we have

$$\alpha = dq/s, \quad \beta = dr/s, \quad \gamma = (d-3)q/s+1, \quad \delta = (d-3)r/s+1,$$
$$\alpha+\beta = d+dp/s, \quad \gamma+\delta = d-1+(d-3)p/s.$$

Again, the corresponding generators are as in (4.2). If d/s is not an integer (respectively $(d-3)/s \notin \mathbb{Z}$), then an argument with the Beta function is used. If $d/s<0$ is an integer (respectively, $(d-3)/s<0$ is an integer) then the corresponding period $\Omega_{C,D}$ is an integral along the interval $[0,1]$ of the form $\omega_{C,D}$ with positive exponents; so, it is nonzero.

Suppose $d=ms$ with $m \in \mathbb{Z}_+$. Then $\alpha = mq \in \mathbb{Z}_{>0}$ and $\beta = mr \in \mathbb{Z}_{>0}$. The function in the first equation of (2.9) equals

$$\tilde{f}^C = -\frac{1}{s} u^{mq}(u-1)^{mr} \int_{u_0}^{u} \frac{d\tau}{\tau^{mq+1}(\tau-1)^{mr+1}} \tag{4.10}$$

for some u_0. By considering $u_0=\infty$ and the behavior as $u\to\infty$ one checks that $\tilde{f}^C$ cannot be a polynomial. (In another approach one calculates two periods: either $\operatorname{Res}_{u=0}\omega_C$ or $\operatorname{Res}_{u=1}\omega_C$).

The analysis when $(d-3)/s = m \in \mathbb{Z}_{>0}$ is the same and this completes the proof that:

• *The normal form in Subcase 2 of the 'Rational PFI without Polynomial IIMs' case is as in* (4.5).

Remark 2. The reason why we have divided the situation with rational PFI without polynomial IIMs into two subcases is that the final normal form is the same.

4.5. The 'Double Invariant Line' case. Subcase 1. We have

$$F = ay - (1+a)\ln x + y/x, \qquad \boldsymbol{V}_0 = xy\partial_y + (ax+y)\boldsymbol{E},$$

where in Subcase 1 we assume that

$$a \neq 0, 1, 1/2, 1/3, \ldots$$

(see (3.12)).

Note that $\operatorname{div} \boldsymbol{V}_0 = x + 3(ax+y)$. In the blowing-up coordinates we get the system $\dot{x} = x^2(a+u)$, $\dot{u} = xu$.

Assume first that the generators of $\mathcal{N}(C_d)$ and $\mathcal{N}(D_d)$ are

$$g^C = x^{d+1}, \qquad g^D = \boldsymbol{V}_0 \wedge x^{d-2}\boldsymbol{E}/\partial_x \wedge \partial_y = -x^d y.$$

Then the homological equations are

$$u(\tilde{f}^C)' + d(a+u)\tilde{f}^C = 1, \tag{4.11}$$

$$u(\tilde{f}^D)' + (\gamma + (d-3)u)\tilde{f}^D = -u, \tag{4.12}$$

where

$$\gamma = (d+3)a - 1.$$

By comparing the highest order terms in (4.11) we find that it cannot have polynomial solutions. Also, the only possible solution to (4.12) is the constant $\tilde{f}^D(u) \equiv 1/(3-d)$, provided $\gamma = 0$.

The constant γ vanishes when $a = 1/(d-3)$, where $d \geq 5$. Therefore:

• *In Subcase 1 of the 'Double Invariant Line' case the normal form is as in* (4.5).

4.6. The 'Double Invariant Line' case. Subcase 2. The first integral and the quadratic vector field are as in the previous case, but with

$$a = 1/n, \qquad n \in \mathbb{Z}_{>0}.$$

We choose the generator of $\mathcal{N}(D_d)$, $d = n+3$, to be $g^D = \boldsymbol{V}_0 \wedge x^{d-3}\partial_y/\partial_x \wedge \partial_y = x^d(ax+y)$; for other degrees this generator is standard. Therefore:

• *In Subcase 2 of the 'Double Invariant Line' case the normal form for* $a = 1/n$ *is*

$$(1+\psi(x))(\boldsymbol{V}_0 + a_{n+1}x^{n+2}\partial_y + \varphi(x)\boldsymbol{E}) \tag{4.13}$$

where the index sets for the series $\varphi = \sum_{i\in\mathcal{I}(\varphi)} a_i x^i$ *and* $\psi = \sum_{i\in\mathcal{I}(\psi)} b_i x^i$ *are*

$$\mathcal{I}(\varphi) = \mathbb{Z}_{\geq 2} \setminus \{n+1\}, \qquad \mathcal{I}(\psi) = \mathbb{Z}_{\geq 1}.$$

REMARK 3. It seems possible to apply a linear change of the coordinates x, y (with a change of F and $\boldsymbol{V}_0$) so that the normal form would be as in (4.5) for any value of the 'modulus' a. But formulas become very complicated and we have decided to stick to simple $\boldsymbol{V}_0$ with a slightly nonstandard orbital normal form $\boldsymbol{V}_0 + \varphi(x)\boldsymbol{E} + a_{n+1}x^{n+2}\partial_y$.

4.7. The 'Triple Invariant Line' case. We have

$$F = a\frac{y}{x} + \frac{y^2}{2x^2} - \ln x, \qquad \boldsymbol{V}_0 = x^2\partial_y + (ax+y)\boldsymbol{E},$$

with $\operatorname{div}\boldsymbol{V}_0 = 3(ax+y)$ (see (3.13)). Hence $\dot{x} = x^2(a+u)$, $\dot{u} = 1$. The generators

$$g^C = x^{d+1} \quad \text{and} \quad g^D = \boldsymbol{V}_0 \wedge x^{d-2}\boldsymbol{E}/\partial_x \wedge \partial_y = x^{d+1}$$

lead to the homological equations

$$(\tilde{f}^C)' + d(a+u)\tilde{f}^C = 1, \qquad (\tilde{f}^D)' + (d-3)(a+u)\tilde{f}^D = 1.$$

By looking at $\deg\tilde{f}$ we find that they do not have polynomial solutions.

Therefore:

- *In the 'Triple Invariant Line' case the normal form is as in* (4.5).

4.8. The 'Nonresonant with CF' case. The principal first integral is

$$F = (y-x)^0 x^a y^b,$$

$s = a+b = 1$ and $a = a/s \notin \mathbb{Q}$ (see (3.8)). The zero exponent of $y-x$ means that the line $G := y-x = 0$ is critical, thus

$$\boldsymbol{V}_0 = (y-x)(bx\partial_x - ay\partial_y) = G\boldsymbol{X}. \tag{4.14}$$

In the x, u coordinates we have $\dot{x} = bx^2(u-1)$, $\dot{u} = -su(u-1)$ with $s = a+b$.

The operator $D_d(\boldsymbol{V}_0)$ is defined as above. But, instead of $C_d(\boldsymbol{V}_0)$, we should consider the operator $C_d'(\boldsymbol{V}_0) = G\partial/\partial\boldsymbol{X} - \boldsymbol{X}(G)$; its image kills terms $g\boldsymbol{X}$. (Also in the remaining cases from this section we consider the operator C_d'.)

We choose the following generators of the cokernels of $C_d'(\boldsymbol{V}_0)$ and $D_d(\boldsymbol{V}_0)$:

$$g^{C'} = x^d(y-x) = x^d\boldsymbol{V}_0/\boldsymbol{X}, \qquad g^D = x^{d-2}y(y-x); \tag{4.15}$$

note that $g^{C'}$ is divisible by the CF $G = y-x$. We get the following homological equations:

$$-su(u-1)^2(\tilde{f}^{C'})' + [db(u-1)^2 + (au+b)(u-1)]\tilde{f}^{C'} = u-1, \tag{4.16}$$

and the second equation of (2.8), with the solutions

$$\begin{aligned}\tilde{f}^{C'} &= -\frac{1}{s}u^{\mu}(u-1)\int^{u}\frac{d\tau}{\tau^{\mu+1}(\tau-1)^2},\\ \tilde{f}^{D} &= -\frac{1}{s}u^{\gamma}(u-1)\int^{u}\frac{d\tau}{\tau^{\gamma}(\tau-1)^2},\end{aligned} \tag{4.17}$$

where

$$\gamma = (d-3)(1-a)+1, \qquad \mu = (d-1)(1-a).$$

Due to the nonresonance condition $a \notin \mathbb{Q}$, μ and γ are not rational, and hence the integrands have nonzero residues at $\tau = 1$.

• *In the 'Nonresonant with CF' case the normal form is as in* (4.5).

4.9. The 'Rational PFI without Polynomial IIMs and with CF' case. We have the rational PFI

$$F = (y-x)^0\frac{x^p}{y^q}$$

with $1 < p < q$, $\gcd(p,q) = 1$, and the critical line $G = y - x = 0$ (see (3.10)). So

$$\boldsymbol{V}_0 = G\boldsymbol{X} = (y-x)(-qx\partial_x - py\partial_y).$$

We proceed as in the previous case with $a = p$, $b = -q$ and $s = p - q < 0$. With $g^{C'}$ and g^D as in (4.15) we get formulas (4.17) with

$$\mu = -(d-1)q/s > 0, \qquad \gamma = -(d-3)q/s + 1 > 0.$$

Therefore the forms $\omega_{C,D}$ have nonzero residues at $u = 1$.

• *In the 'Rational PFI without Polynomial IIMs and with CF' case the normal form is as in* (4.5).

4.10. The 'Double Invariant Line with CF' case. Subcase 1. We have

$$F = (bx+y)^0\left(a\frac{y}{x} - \ln x\right), \qquad a \neq 0$$

(see (3.14)), i.e.,

$$\boldsymbol{V}_0 = (bx+y)(x\partial_y + a\boldsymbol{E}) = G\boldsymbol{X}$$

with $G = bx + y$, $\boldsymbol{X}(G) = x + aG$ and $\operatorname{div}\boldsymbol{V}_0 = x + 3aG$. In the x, u coordinates we get $\dot{x} = ax^2(b+u)$, $\dot{u} = x(b+u)$. We choose

$$g^{C'} = x^d \cdot \boldsymbol{V}_0/\boldsymbol{X} = x^{d+1}(b+u), \qquad g^D = \boldsymbol{V}_0 \wedge x^{d-2}\boldsymbol{E}/\partial_x \wedge \partial_y = -x^{d+1}(b+u).$$

Then the homological equations become

$$\begin{aligned}(b+u)(\tilde{f}^{C'})' + [(d-1)a(b+u) - 1]\tilde{f}^{C'} &= b+u,\\ (b+u)(\tilde{f}^{D})' + [(d-1)a(b+u) - 1]\tilde{f}^{D} &= -(b+u).\end{aligned}$$

The degree of $\tilde{f}^{C',D}$ cannot be positive. But the valuation of both sides in each equation at $u = -b$ shows that they do not have constant solutions either.

• *In Subcase 1 of the 'Double Invariant Line' case the normal form is as in* (4.5).

4.11. The 'Double Invariant Line with CF' case. Subcase 2. Here

$$F = x^0\left(a\frac{y}{x} - \ln x\right), \qquad \boldsymbol{V}_0 = x(x\partial_y + a\boldsymbol{E}) = G\boldsymbol{X},$$

with $a \neq 0$ (see (3.14)).

One can see that the homological equations become

$$(\tilde{f}^{C'})' + (d-1)a\tilde{f}^{C'} = \tilde{g}^{C'}, \qquad (\tilde{f}^{D})' + (d-1)a\tilde{f}^{D} = \tilde{g}^{D}.$$

They have polynomial solutions of the same degree as the degrees of the right hand sides. Therefore, if we do not want to have solutions of degree $\leq d$, we should take polynomials $\tilde{g}^{C'}$ and $\tilde{g}^{D}$ of degree $d+1$.

But such choices are not compatible with the choice of $\tilde{g}^{C'}$ as $h\cdot\boldsymbol{V}_0/\boldsymbol{X} = hx$ with $h(x,y) \in \mathcal{F}_d$ and the choice of $\tilde{g}^{D}$ as $B(\boldsymbol{V}_0)\boldsymbol{Z} = xh$, $\boldsymbol{Z} \in \mathcal{Z}_{d-2}$, $h \in \mathcal{F}_d$.

We apply a direct calculation of the operator $\mathrm{ad}_{\boldsymbol{V}_0}$. With $\boldsymbol{Z} = A(x,y)\partial_x + B(x,y)\partial_y \in \mathcal{Z}_d$, $d \geq 1$, we get

$$[\boldsymbol{V}_0, \boldsymbol{Z}] = x[adA + xA'_y]\partial_x + [x(a(d+1)B + xB'_y - ayA)]\partial_y.$$

We see that, after projection onto the vector fields divisible by x and subsequent division by x, the resulting operator takes a triangular form with nonzero terms (ad and $a(d+1)$) on the diagonal; it is one-to-one.

Therefore:

• *In Subcase 2 of the 'Double Invariant Line with CF' case the unique formal normal form is*

$$\boldsymbol{V}_0 + \varphi(y)\partial_x + \psi(y)\partial_y, \qquad \varphi = a_3y^3 + \cdots, \qquad \psi = b_3y^3 + \cdots. \tag{4.18}$$

Remark 4. A natural question arises about the orbital normal form. To be precise, one asks which terms from the series $\varphi(y)$ and $\psi(y)$ can be deleted by multiplying $\boldsymbol{V}$ of the form (4.18) by a formal power series.

The answer depends on the first nonzero term in the perturbation $\varphi\partial_x + \psi\partial_y$. Let

$$\varphi(y) = a_my^m + \cdots, \qquad \psi(y) = b_ny^n + \cdots, \qquad a_mb_n \neq 0.$$

We can assume that $\min(m,n) < \infty$, as the opposite case is of infinite codimension.

If $m \leq n$ then the formal orbital normal form is

$$V_0 + a_my^m\partial_x + \psi(y)\partial_y; \tag{4.19}$$

if $n < m$ then the formal orbital normal form is $V_0 + \varphi(y)\partial_x + b_ny^n\partial_y$.

This follows from the possibility of multiplication of $\boldsymbol{V} = \boldsymbol{V}_0 + \cdots$ by $1 + c_1 y + c_2 y^2 + \cdots$, combined with the above reduction.

REMARK 5. In this case we observe the following unusual phenomenon. Since the critical locus of $\boldsymbol{V}_0$ is 1-dimensional, the operator $B(\boldsymbol{V}_0)$ is not surjective: $\operatorname{Im} B(\boldsymbol{V}_0)$ is of codimension 1. It turns out that this image coincides with $\operatorname{Im} D(\boldsymbol{V}_0)$. Also $\operatorname{Im} C'(\boldsymbol{V}_0)$ coincides with $G \cdot \mathbb{C}[[x, y]]$.

5. The 'Rational PFI with 1-Factor IIM' case. The principal first integral is

$$F = \frac{xy}{(y-x)^r}, \qquad r \geq 3;$$

thus we have the vector field (3.1) with $a = b = 1$, $c = -r$. When $r \neq 4$ we have Subcase 1, otherwise we have Subcase 2 (see (3.6)).

5.1. Discussion. The principal difference between this case and the cases considered in the previous section is that one of the homological operators has nontrivial kernel. Namely, we have the polynomial IIM $(y-x)^{r+1}$ generating the kernel of $D_{r+1}(\boldsymbol{V}_0)$. This fact has some implications.

Firstly, the cokernel of $D_{r+1}(\boldsymbol{V}_0)$ is 2-dimensional. Secondly, the generator of its kernel can be used in further reductions, which can be realized in two ways.

On the one hand, we can cancel some term from the orbital normal form (which was not killed using $D(\boldsymbol{V}_0)$). For this one uses a so-called second level homological operator defined by means of $D(\boldsymbol{V}_0 + \boldsymbol{V}_1)$, where $\boldsymbol{V}_1$ is a lowest degree term of the first level normal form (in fact, $\boldsymbol{V}_1$ is from the first level orbital normal form). Sometimes, for special $\boldsymbol{V}_1$, the second level homological operators still have nontrivial kernels and a third level analysis is needed.

But our IIM is also of the form $B(\boldsymbol{V}_0)\boldsymbol{T}$, for a concrete vector field $\boldsymbol{T}$, and one can use $\operatorname{ad}_{\boldsymbol{V}_0} \boldsymbol{T}$, which turns out to be proportional to $\boldsymbol{V}_0$. In this way we can kill some term from the orbital factor.

There arises a problem of priority. For us, it is the orbital normal form that we put first. Only when nothing remains to be killed on the orbital side, we reduce a term from the orbital factor.

The same remarks apply in the next section (about the 'Rational PFI with 2-Factor IIM' case). In the following subsections we present the first, second and third level analysis of the homological operators.

5.2. The first level analysis

LEMMA 2. *We have* $\ker C_d(\boldsymbol{V}_0) = 0$ *for any* d, $\ker D_d(\boldsymbol{V}_0) = 0$ *if* $d \neq r+1$, *and* $\ker D_{r+1}(\boldsymbol{V}_0) = \mathbb{C} \cdot M$, *where*

$$M = (2-r)(y-x)^{r+1}. \tag{5.1}$$

Proof. The first statement is obvious. Next, $\boldsymbol{X}_F = (y-x)^{-r-1}\boldsymbol{V}_0$ with

$$\boldsymbol{V}_0 = x((1-r)y - x)\partial_x + y((1-r)x - y)\partial_y. \ \blacksquare$$

LEMMA 3. *The IIM* (5.1) *equals* $B(\boldsymbol{V}_0)\boldsymbol{T}$ *with*

$$\begin{aligned}(5.2)\quad \boldsymbol{T} &= F^{-1}\boldsymbol{E} + \frac{x+y}{(y-x)^2}F^{-1}\boldsymbol{V}_0 \\ &= (y-x)^{r-2}\{((2-r)y - (2+r)x)\partial_x + ((2-r)x - (2+r)y)\partial_y)\}.\end{aligned}$$

Moreover,

$$\operatorname{ad}_{\boldsymbol{V}_0}\boldsymbol{T} = 2(4-r)(y-x)^{r-2}\boldsymbol{V}_0 \tag{5.3}$$

(*which is nonzero for* $r \neq 4$).

Proof. The fact that $\boldsymbol{V}_0 \wedge F^{-1}\boldsymbol{E} = M \cdot \partial_x \wedge \partial_y$ follows from (2.12) with $s = 2-r$; of course, $B(\boldsymbol{V}_0)\cdot h\boldsymbol{V}_0 = 0$. But the vector field $F^{-1}\boldsymbol{E}$ has poles along the lines $x = 0$ and $y = 0$. We remove these poles by adding a term proportional to $\boldsymbol{V}_0$.

The derivation of (5.3) uses (2.2) and the equalities $\boldsymbol{V}_0((x+y)/(y-x)^2) = (x^2+y^2+2(3-r)xy)/(y-x)^3$ and $\boldsymbol{V}_0(F^{-1}) = 0$. ■

REMARK 6. We observed the similar vanishing of $\operatorname{ad}_{\boldsymbol{V}_0}\boldsymbol{T}$ for one value of the exponent r, as above, in the generalized node case of the Bogdanov–Takens singularity in [SZ2, Theorem II and Lemma 3.16]. Compare also Lemma 6 below.

In the first level analysis of the homological operators we use only the operators associated with $\boldsymbol{V} = \boldsymbol{V}_0$. Firstly we specify the subspaces $\mathcal{N}(C_d)$ and $\mathcal{N}(D_d)$ complementary to $\operatorname{Im} C_d$ and $\operatorname{Im} D_d$. Recall that $\dim \mathcal{N}(C_d) = 1$ for any d, and $\dim \mathcal{N}(D_d)$ is 1 for $d \neq r+1$ and 2 otherwise.

With $a = b = 1$, $c = -r$ and $s = 2-r$ we have

$$\alpha = -d\frac{1}{r-2}, \quad \beta = d\frac{r}{r-2}, \quad \gamma = -(d-3)\frac{1}{r-2} + 1,$$
$$\delta = (d-3)\frac{r}{r-2} + 1,$$
$$\alpha + \beta = d + d\frac{1}{r-2}, \quad \gamma + \delta = d - 1 + (d-3)\frac{1}{r-2}.$$

(see (4.1)). We set

$$\begin{aligned} & g^C = x^{d+1}, \\ (5.4)\quad & g^D = x^{d-1}y(y-x) \quad \text{if } d \neq r+1, \\ & g_0^D = y^{r+1}((1-r)x - y), \quad g_1^D = x^{r+1}((1-r)y - x) \quad \text{if } d = r+1,\end{aligned}$$

as potential bases for $\mathcal{N}(C_d)$ and $\mathcal{N}(D_d)$. Note that

$$g_0^D = B(\boldsymbol{V}_0)y^r\partial_x, \quad g_1^D = B(\boldsymbol{V}_0)x^r\partial_y.$$

We get the form ω_C as in (4.3); for $d/(r-2)$ noninteger, its period $\Omega_C(g^C)$ equals const $\cdot B(\alpha,\beta)\neq 0$. If $d/(r-2)=m\in\mathbb{Z}$ then we get the function

$$\tilde f^C=\frac{1}{2-r}\,\frac{(u-1)^{mr}}{u^m}\int^u\frac{\tau^{m-1}}{(\tau-1)^{mr+1}}\,d\tau$$

(cf. (4.6)). The residue of $\omega_C(g^C)$ at $u=1$ vanishes; therefore the correct period is $\Omega_C(g^C)=\int_\infty^0\omega_C(g^C)\neq 0$ (cf. (4.8)).

If $(d-3)/(r-2)\notin\mathbb{Z}$ then $\Omega_D(g^D)\neq 0$. If $d-3=m(r-2)$, $m\in\mathbb{Z}$, then we arrive at the function

$$\tilde f^D=\frac{1}{2-r}\,\frac{(u-1)^{mr+1}}{u^{m-1}}\int^u\frac{\tau^{m-2}}{(\tau-1)^{mr+2}}\,d\tau.$$

For $m>1$ the above argument as for $\tilde f^C$ works, the unique period is $\Omega_D(g^D)=\int_\infty^0\omega_D(g^D)\neq 0$.

For $m=1$, i.e., $d=r+1$, we have two generators, g_0^D and g_1^D, and we define two periods $\Omega_D^{0,1}(g_j^D)=\mathrm{Res}_{u=0,1}\,\omega_D(g_j^D)$, $j=0,1$. We have

$$\omega_D(g_0^D)=\frac{(1-r)u-1}{u(u-1)^{r+2}}\,du,\qquad \omega_D(g_1^D)=\frac{u^r(u+r-1)}{(u-1)^{r+2}}\,du,$$

and we define the *period matrix*

$$\begin{pmatrix}\Omega_D^0(g_0^D) & \Omega_D^0(g_1^D)\\ \Omega_D^1(g_0^D) & \Omega_D^1(g_1^D)\end{pmatrix}. \tag{5.5}$$

Since $\Omega_D^0(g_0^D)=(-1)^{r-1}\neq 0$, $\Omega_D^1(g_1^D)=1\neq 0$ and $\Omega_D^0(g_1^D)=0$, this matrix has triangular form, with nonzero entries on the diagonal, and hence is nondegenerate.

But this is not the end of the first level analysis. We have not yet used the kernel of $D_{r+1}(\boldsymbol{V}_0)$ generated by $(y-x)^{r+1}$, via the vector field $\boldsymbol{T}$ from Lemma 3. If the orbital normal form differs from $\boldsymbol{V}_0$ then we can use this $\boldsymbol{T}$ to cancel higher order terms. But when the orbital normal form is $\boldsymbol{V}_0$, then we get the term

$$(4-r)(y-x)^{r-2}\boldsymbol{V}_0\neq 0$$

for $r\neq 4$; for $r=4$ we get nothing. It turns out that the function $g=(y-x)^{r-2}$ lies outside $\mathrm{Im}\,C_d(\boldsymbol{V}_0)$ for $d=r-3$.

Indeed, the corresponding period $\Omega_C(g)=\mathrm{P.V.}\int_0^1 u^{-\alpha-1}(u-1)^{-\beta-1}\,du$, with $\alpha=-(r-3)/(r-2)$, $\beta=r(r-3)/(r-2)$, $\alpha+\beta\notin\mathbb{Z}$, is nonzero.

Therefore:

• *The first level normal forms in the 'Rational PFI with 1-Factor IIM' case are*

$$\begin{aligned}&(1+\psi(x))(\boldsymbol{V}_0+\boldsymbol{U}+\varphi(x)\boldsymbol{E}),\\ &\boldsymbol{U}=ay^r\partial_x+bx^r\partial_y,\quad \mathcal{I}(\varphi)=\mathbb{Z}_{\geq 2},\quad \mathcal{I}(\psi)=\mathbb{Z}_{\geq 1};\end{aligned} \tag{5.6}$$

or

$$(5.7)\qquad (1+\psi(x))\boldsymbol{V}_0,\quad \mathcal{I}(\psi)=\begin{cases}\mathbb{Z}_{\geq 1}\setminus\{r-2\} & \text{(Subcase 1)},\\ \mathbb{Z}_{\geq 1} & \text{(Subcase 2)}\end{cases}$$

(*the latter form is complete*).

5.3. The second level analysis. In this section we study the homological operators associated with vector fields of the form

$$\boldsymbol{V}=\boldsymbol{V}_0+\boldsymbol{V}_1,$$

where $\boldsymbol{V}_1$ is a homogeneous vector field of lowest degree $\deg \boldsymbol{V}_1 > 1$ which was not reduced in the first level analysis. We have to consider several possibilities:

$$(5.8)\qquad \boldsymbol{V}_1 = ax^k\boldsymbol{E} \quad\text{or}\quad \boldsymbol{V}_1 = ay^r\partial_x + bx^r\partial_y$$

(from the orbital normal form) or

$$(5.9)\qquad \boldsymbol{V}_1 = cx^l\boldsymbol{V}_0$$

(associated with the orbital factor). Of course, it is possible that there appear terms (5.8) and (5.9) simultaneously, of the same degree or of different degrees. However, we prefer to consider actions of the homological operators associated with them separately, with the orbital normal form priority (see Section 5.1). Therefore, the case (5.8) is used when terms of the form (5.9) are present, even of degree smaller than the degree of (5.8). The terms (5.9) are used when the orbital normal form is $\boldsymbol{V}_0$.

Our analysis is essentially reduced to the operator $D(\boldsymbol{V})$ acting on functions of the form $\xi M + f$, where $\xi\in\mathbb{C}$ and $M=(y-x)^{r+1}$ is the generator of $\ker D_{r+1}(\boldsymbol{V}_0)$, and followed by projection onto a space of homogeneous functions. Therefore we get the following *second level homological operator*:

$$(5.10)\qquad \widetilde{D}(\boldsymbol{V}):\mathbb{C}\oplus\mathcal{F}_d\to\mathcal{F}_{d+1},\qquad (\xi,f)\mapsto \xi D(\boldsymbol{V}_1)M + D(\boldsymbol{V}_0)f,$$

where $d=k+r=\deg \boldsymbol{V}_1 + r$. This operator acts between spaces of the same dimension.

Lemma 4. *We have:*

(a) $D(x^k\boldsymbol{E})M = (r-1-k)x^kM$,
(b) $D(ay^r\partial_x + bx^r\partial_y)(y-x)^{r+1} = (r+1)(bx^r - ay^r)(y-x)^r$.

5.3.1. *The case* $\boldsymbol{V}_1 = ax^k\boldsymbol{E}$. We assume that

$$(5.11)\qquad k\neq r-1.$$

We have $d=k+r>r+1$, as $k=\deg \boldsymbol{V}_1 > 1$. Therefore $\operatorname{Im} D_d(\boldsymbol{V}_0)$ is of codimension 1 (by Lemma 2). Therefore, in order to demonstrate the surjectivity of the operator $\widetilde{D}(\boldsymbol{V})$ from (5.10), it is sufficient to show that $D(\boldsymbol{V}_1)M = \mathrm{const}\cdot x^k(y-x)^{r+1}$ (by Lemma 4(a)) lies outside $\operatorname{Im} D_d(\boldsymbol{V}_0)$.

For this we consider the period $\Omega_D(g)$ for $g = x^k(y-x)^{r+1}$. We have

$$\omega_D(g) = u^\vartheta (u-1)^\theta \, du \tag{5.12}$$

with $\vartheta = \frac{k+r-3}{r-2} - 2 > 0$ and $\theta = r - 1 - (k+r-3)\frac{r}{r-2}$. If $\frac{d-3}{r-2} = \frac{k+r-3}{r-2}$ is not an integer then $\Omega_D(g) = \mathrm{const} \cdot B(\vartheta+1, \theta+1) \neq 0$. Otherwise, either $\Omega_D(g) = \mathrm{Res}_{u=1}\, \omega_D(g) \neq 0$ or $\Omega_D(g) = \int_\infty^0 \omega_D(g) \neq 0$.

• *The complete normal form in the 'Rational PFI with 1-Factor IIM' case with* $\boldsymbol{V}_1 = ax^k\boldsymbol{E}$ *is either*

$$\begin{aligned}&(1+\psi(x))(\boldsymbol{V}_0 + \boldsymbol{V}_1 + bx^r\partial_y + \varphi(x)\boldsymbol{E}),\\ &k < r-1, \quad \mathcal{I}(\psi) = \mathbb{Z}_{\geq 1}, \quad \mathcal{I}(\varphi) = \mathbb{Z}_{>k} \setminus \{r-1, k+r-2\},\end{aligned} \tag{5.13}$$

or

$$\begin{aligned}&(1+\psi(x))(\boldsymbol{V}_0 + \boldsymbol{V}_1 + \varphi(x)\boldsymbol{E}),\\ &k > r-1, \quad \mathcal{I}(\psi) = \mathbb{Z}_{\geq 1}, \quad \mathcal{I}(\varphi) = \mathbb{Z}_{>k} \setminus \{k+r-2\}.\end{aligned} \tag{5.14}$$

5.3.2. *The case* $\boldsymbol{V}_1 = ay^r\partial_x + bx^r\partial_y$. We assume $(a,b) \neq (0,0)$. Here $d = 2r-1 \neq r-1$. As in the previous case, by considering a suitable period $\Omega_D(g)$ for $g = (bx^r - ay^r)(y-x)^r$ (see Lemma 4(b)) we have

$$\tilde{f}^D = \mathrm{const} \cdot \frac{(u-1)^{2r+1}}{u} \int^u \frac{(a - b\tau^r)}{(\tau-1)^{r+2}}\, d\tau.$$

Note that the form $\omega_D(g) = (a - bu^r)(u-1)^{-r-2}\, du$ has trivial residues at $u = 0, 1$. So, the only period is

$$\Omega_D(g) = \int_\infty^0 \omega_D(g) = \frac{1}{r+1}(a + (-1)^{r+1}b).$$

If $a + (-1)^{r+1}b \neq 0$ then we are done. Otherwise, we can use the IIM M either to cancel another term from the orbital normal form (provided it is $\neq \boldsymbol{V}_0 + \boldsymbol{V}_1$), or to improve the orbital factor.

• *The second level normal form in the 'Rational PFI with 1-Factor IIM' case with* $\boldsymbol{V}_1 = ay^r\partial_x + bx^r\partial_y$ *is either*

$$\begin{aligned}&(1+\psi(x))(\boldsymbol{V}_0 + \boldsymbol{V}_1 + \varphi(x)\boldsymbol{E}),\\ &a + (-1)^{r+1}b \neq 0, \quad \mathcal{I}(\varphi) = \mathbb{Z}_{\geq r} \setminus \{2r-3\}, \quad \mathcal{I}(\psi) = \mathbb{Z}_{\geq 1}\end{aligned} \tag{5.15}$$

(*this form is complete*), *or*

$$\begin{aligned}&(1+\psi(x))(\boldsymbol{V}_0 + \boldsymbol{V}_1 + \varphi(x)\boldsymbol{E}),\\ &\boldsymbol{V}_1 = a(y^r\partial_x + (-1)^r x^r\partial_y), \quad \mathcal{I}(\varphi) = \mathbb{Z}_{\geq r}, \quad \mathcal{I}(\psi) = \mathbb{Z}_{\geq 1},\end{aligned} \tag{5.16}$$

or

$$\begin{aligned}&(1+\psi(x))(\boldsymbol{V}_0 + \boldsymbol{V}_1),\\ &\boldsymbol{V}_1 = a(y^r\partial_x + (-1)^r x^r\partial_y), \quad \mathcal{I}(\psi) = \mathbb{Z}_{\geq 1} \setminus \{r-2\} \quad \text{(Subcase 1)}\end{aligned} \tag{5.17}$$

(*this form is complete*), *or*

$$(5.18)\qquad \begin{aligned}&(1+\psi(x))(\boldsymbol{V}_0+\boldsymbol{V}_1),\\ &\boldsymbol{V}_1 = a(y^4\partial_x + x^4\partial_y), \quad \mathcal{I}(\psi)=\mathbb{Z}_{\geq 1} \quad \text{(Subcase 2)}.\end{aligned}$$

5.3.3. *The case* $\boldsymbol{V}_1 = cx^l\boldsymbol{V}_0$. Here the orbital normal form is $\boldsymbol{V}_0$. So M is used in killing only one term from the series $\psi(x)$. The result is in (5.7).

5.4. Third level. We consider homological operators associated with the vector fields $\boldsymbol{V} = \boldsymbol{V}_0 + \boldsymbol{V}_1 + \boldsymbol{V}_2$ such that $\boldsymbol{V}_0 + \boldsymbol{V}_1$ has nontrivial IIM, i.e.,

$$(5.19)\qquad \boldsymbol{V}_1 = a(y^r\partial_x + (-1)^r x^r\partial_y).$$

In fact we are left with two subcases.

5.4.1. *The case* $\boldsymbol{V}_2 = cx^l\boldsymbol{E}$. Of course, $c\neq 0$ and $l\geq r$. As before, we use $D(\boldsymbol{V}_2)M \notin \operatorname{Im} D(\boldsymbol{V}_0)$ to kill the term $x^{l+r}\boldsymbol{E}$ from the orbital normal form.

• *The complete normal form in the 'Rational PFI with 1-Factor IIM' case with* $\boldsymbol{V}_1 = a(y^r\partial_x + (-1)^r x^r\partial_y)$ *and* $\boldsymbol{V}_2 = cx^l\boldsymbol{E}$ *is*

$$(5.20)\qquad \begin{aligned}&(1+\psi(x))(\boldsymbol{V}_0+\boldsymbol{V}_1+\boldsymbol{V}_2+\varphi(x)\boldsymbol{E}),\\ &\mathcal{I}(\varphi)=\mathbb{Z}_{>l}\setminus\{l+r-2\}, \quad \mathcal{I}(\psi)=\mathbb{Z}_{\geq 1}.\end{aligned}$$

5.4.2. *The case* $\boldsymbol{V}_2 = cx^j\boldsymbol{V}_0$. We assume additionally that

$$(5.21)\qquad r=4,$$

i.e., we are in Subcase 2 (see (5.18)), and we admit $c=0$. Thus the orbital normal form is $\boldsymbol{V}_0 + a(y^4\partial_x + x^4\partial_y)$. Moreover, $\operatorname{ad}_{\boldsymbol{V}_0}\boldsymbol{T} = 0$, where $\boldsymbol{T} = -2(y-x)^2\,\{(3x+y)\partial_x + (x+3y)\partial_y\}$ (see (5.2)). But we have $\operatorname{ad}_{\boldsymbol{V}_2}\boldsymbol{T} = 2jx^{j-1}(y-x)^2(3x+2y)\boldsymbol{V}_0$ and the factor $g = x^{j-1}(y-x)^2\,(3x+2y)$ does not lie in $\operatorname{Im} C_{j+1}(\boldsymbol{V}_0)$. Indeed, here

$$\omega_C(g) = (3+u)^{(j-1)/2}(u-1)^{-2j-1}\,du.$$

If j is even then $\operatorname{Res}_{u=1}\omega_C(g)\neq 0$. If $j=2m-1$ is odd then we compute $\Omega_C(g) = \int_{-\infty}^0 \omega_C(g)$; the substitution $u=\tau/(\tau-1)$ leads to a Beta type integral with $\Omega_C(g) = -\frac{11}{4}B(m,3m)\neq 0$.

• *The complete normal form in Subcase 2 of the 'Rational PFI with 1-Factor IIM' case with* $\boldsymbol{V}_1 = a(y^4\partial_x + x^4\partial_y)$ *and* $\boldsymbol{V}_2 = cx^j\boldsymbol{V}_0$ *is either*

$$(5.22)\qquad (1+cx^j+\psi(x))(\boldsymbol{V}_0+\boldsymbol{V}_1), \quad \mathcal{I}(\psi)=\mathbb{Z}_{>j}\setminus\{j-2\},$$

or

$$(5.23)\qquad \boldsymbol{V}_0+\boldsymbol{V}_1.$$

6. The 'Rational PFI with 2-factor IIM' case. We have

$$F = \frac{x}{y^q(y-x)^r},$$

i.e., the vector field (2.1) with $a = 1$, $b = -q$, $c = -r$ and $s = 1-q-r = -|s|$ (see (3.7)). When $1 \le r \le q > 1$ we have Subcase 1, when $q = r = 1$ we have Subcase 2, and when $q = 0$ and $r \ge 2$ we have Subcase 3. In the latter subcase we have the critical line $y = 0$.

Analysis of these cases is essentially the same as in the previous section, and the discussion from Section 5.1 is applicable. One difference is in Subcase 3, where one has to use the operator $C'(\boldsymbol{V}_0)$ (not $C(\boldsymbol{V}_0)$).

6.1. The first level analysis

Lemma 5. *We have* $\ker C_d(\boldsymbol{V}_0) = 0$ *for any* d (*respectively,* $\ker C'_d(\boldsymbol{V}_0) = 0$ *for any* d) *and* $\ker D_d(\boldsymbol{V}_0) = 0$ *if* $d \ne q+r+2$ *and* $d \ge 4$, *while* $\ker D_{q+r+2}(\boldsymbol{V}_0) = \mathbb{C}\cdot M$, *where*

$$M = sy^{q+1}(y-x)^{r+1}, \qquad s = 1-q-r. \tag{6.1}$$

Lemma 6. *The IIM* (6.1) *equals* $B(\boldsymbol{V}_0)\boldsymbol{T}$ *for*

$$\boldsymbol{T} = F^{-1}\boldsymbol{E} - ((y-x)F)^{-1}\boldsymbol{V}_0 = y^q(y-x)^{r-1}\{((1-q)x+sy)\partial_x - ry\partial_y\}.$$

Moreover,

$$\mathrm{ad}_{\boldsymbol{V}_0}\boldsymbol{T} = (1-q)y^q(y-x)^{r-1}\boldsymbol{V}_0 \tag{6.2}$$

(*which is nonzero for* $q \ne 1$).

Proof. The property $B(\boldsymbol{V}_0)\boldsymbol{T} = M$ follows from (2.12). The term $((y-x)F)^{-1}\boldsymbol{V}_0$ is extracted from $F^{-1}\boldsymbol{V}_0$ in order to remove the pole along the line $x = 0$. ∎

Let us specify the subspaces $\mathcal{N}(C_d)$, $\mathcal{N}(C'_d)$ and $\mathcal{N}(D_d)$. By Lemma 5, $\dim\mathcal{N}(C_d) = \dim\mathcal{N}(C'_d) = 1$ for any d, while $\dim\mathcal{N}(D_d) = 1$ for $d \ne q+r+2$ and $\dim\mathcal{N}(D_{q+r+2}) = 2$.

We have

$$\alpha = dq/|s|, \quad \beta = dr/|s|, \quad \gamma = (d-3)q/|s|+1, \quad \delta = (d-3)r/|s|+1,$$
$$\alpha+\beta = d+d/|s|, \quad \gamma+\delta = d-1+(d-3)/|s|,$$

in (2.10). Moreover, in Subcase 3 we have $\boldsymbol{V}_0 = G\boldsymbol{X}$ with $G = y$ and $\boldsymbol{X}(G) = (1-r)x - y$.

We choose

$$g^C = x^{d+1}, \qquad g^{C'} = x^d y, \tag{6.3}$$

$$g^D = x^{d-1}y(y-x) \qquad (d \ne q+r+2), \tag{6.4}$$

$$g_0^D = g^D, \qquad g_1^D = y^{q+r+2}((r-1)x+y) \qquad (d = q+r+2), \tag{6.5}$$

i.e., $g_1^D = \boldsymbol{V}_0 \wedge y^{q+r+1}\partial_x/\partial_x \wedge \partial_y$. The solutions to the homological equations are

$$\tilde{f}^C = \frac{1}{|s|} u^\alpha (u-1)^\beta \int^u \frac{d\tau}{\tau^{\alpha+1}(\tau-1)^{\beta+1}}, \tag{6.6}$$

$$\tilde{f}^D = \frac{1}{|s|} u^\gamma (u-1)^\delta \int^u \tau^{-\gamma}(\tau-1)^{-\delta}\, d\tau, \qquad d \neq q+r+2, \tag{6.7}$$

$$\tilde{f}^{C'} = \frac{1}{r-1} u(u-1)^\nu \int^u \tau^{-1}(\tau-1)^{-\nu-1}\, d\tau, \quad \nu = (d-1)r/(r-1). \tag{6.8}$$

Equations (6.6) and (6.7) are treated as in the previous section. We have either $\alpha+\beta \notin \mathbb{Z}$ for $d/|s| \notin \mathbb{Z}$ (respectively, $\gamma+\delta \notin \mathbb{Z}$ for $d/|s| \notin \mathbb{Z}$) and $\Omega_{C,D}$ is expressed via nonzero Beta functions, or $\Omega_{C,D} = \operatorname{Res}_{u=0} \omega_{C,D} \neq 0$ (because $\alpha, \beta, \gamma, \delta > 0$). In the case of (6.8) the residue of the form $\omega_{C'} = u^{-1}(u-1)^{-\nu-1}\, du$ at $u = 0$ is nonzero.

If $d = q+r+2$ then $\gamma = q+1$, $\delta = r+1$ and we have two forms

$$\omega_D(g_0^D) = \frac{du}{u^{q+1}(u-1)^{r+1}}, \qquad \omega_D(g_1^D) = \frac{u^r(u+r-1)\, du}{(u-1)^{r+2}}$$

(associated with the choice (6.5)). If we define the periods as $\Omega_D^0 = \operatorname{Res}_{u=0} \omega_D$ and $\Omega_D^1 = \operatorname{Res}_{u=1} \omega_D$ then the corresponding period matrix becomes triangular with nonzero determinant (see Section 5.2). This proves that the functions (6.5) generate $\mathcal{N}(D_{q+r+2})$ indeed.

Finally, when the orbital normal form is $\boldsymbol{V}_0$ and $q \neq 1$, then the IIM $M = \mathrm{const} \cdot y^{q+1}(y-x)^{r+1}$ from the kernel of $D_{q+r+2}(\boldsymbol{V}_0)$ can be used, via the vector field $\boldsymbol{T}$ from Lemma 6, to cancel some term in the orbital factor.

For this it is enough to show that the function $g = y^q(y-x)^{r-1}$ from (6.2) does not lie in $\operatorname{Im} C_d(\boldsymbol{V}_0)$, $d = q+r-2$, if $q \neq 1$; respectively, $y(y-x)^{r-1} \notin \operatorname{Im} C'_d(\boldsymbol{V}_0)$, $d = r-1$, if $q = 1$.

Hence one considers the corresponding period $\Omega_C(g)$; respectively, $\Omega_{C'}(g) = \operatorname{Res}_{u=0} u^{-1}(u-1)^{r-\nu-1}\, du$, $\nu = (d-1)r/(r-1)$, if $q = 1$.

• *The first level normal forms in the 'Rational PFI with 2-Factor IIM' case are*

$$\begin{aligned} &(1+\psi(x))(\boldsymbol{V}_0 + \boldsymbol{U} + \varphi(x)\boldsymbol{E}),\\ &\boldsymbol{U} = ax^{q+r}\boldsymbol{E} + by^{q+r+1}\partial_x, \quad \mathcal{I}(\varphi) = \mathbb{Z}_{\geq 2} \setminus \{q+r\}, \quad \mathcal{I}(\psi) = \mathbb{Z}_{\geq 1}, \end{aligned} \tag{6.9}$$

or

$$\begin{aligned} &(1+\psi(x))\boldsymbol{V}_0,\\ &\mathcal{I}(\psi) = \begin{cases} \mathbb{Z}_{\geq 1} \setminus \{q+r-1\} & \text{(Subcases 1 and 3)},\\ \mathbb{Z}_{\geq 1} & \text{(Subcase 2)}. \end{cases} \end{aligned} \tag{6.10}$$

6.2. The second level. Here we study the homological operators associated with vector fields of the form $\boldsymbol{V} = \boldsymbol{V}_0 + \boldsymbol{V}_1$ where $\boldsymbol{V}_1$ is the lowest degree part of the orbital normal form. Thus $\boldsymbol{V}_1 = ax^k\boldsymbol{E}$ or $\boldsymbol{V}_1 = ax^{q+r}\boldsymbol{E} + by^{q+r+1}\partial_x$. As in the previous section, there are no cancellations when $\boldsymbol{V}_1 = cx^k\boldsymbol{V}_0$.

We consider the operator $\widetilde{D}(\boldsymbol{V})$ defined in (5.10) with $d = \deg \boldsymbol{V}_1 + q + r + 2$ and $M = y^{q+1}(y-x)^{r+1}$.

Lemma 7. *We have:*

(a) $D(x^k\boldsymbol{E})M = (q+r-k)x^kM$,
(b) $D(y^{q+r+1}\partial_x)M = -(r+1)y^{2q+r+2}(y-x)^r$.

6.2.1. *The case* $\boldsymbol{V}_1 = ax^k\boldsymbol{E}$. We assume

$$k \neq q+r. \tag{6.11}$$

We have $d = k+(q+r+2) > q+r+2$ and hence $\operatorname{Im} D_d(\boldsymbol{V}_0)$ is of codimension 1 (Lemma 5). Therefore it is enough to show that $g = x^k y^{q+1}(y-x)^{r+1} = x^{k+q+r+2}u^{q+1}(u-1)^{r+1}$ (Lemma 7(a)) lies outside $\operatorname{Im} D_d(\boldsymbol{V}_0)$, i.e., $\Omega_D(g) \neq 0$. The corresponding form equals $\omega_D(g) = u^{\vartheta}(u-1)^{\theta}\,du$ with $\vartheta = -(d-3)q/|s| + q - 1$, $\theta = -(d-3)r/|s| + r - 1$.

We have either $\Omega_D = \text{const} \cdot B(\vartheta+1, \theta+1) \neq 0$ when $\vartheta + \theta \notin \mathbb{Z}$ ($(d-3)/|s|) \notin \mathbb{Z}$), or $d - 3 = m|s|$ and $\Omega_D = \operatorname{Res}_{u=0}\omega_D \neq 0$ (and also $\operatorname{Res}_{u=1}\omega_D \neq 0$).

• *The complete formal normal form in the 'Rational PFI with 2-Factor IIM' case with* $\boldsymbol{V}_1 = ax^k\boldsymbol{E}$ *is either*

$$\begin{aligned} &(1+\psi(x))(\boldsymbol{V}_0 + \boldsymbol{V}_1 + \boldsymbol{U} + \varphi(x)\boldsymbol{E}),\\ &\boldsymbol{U} = ax^{q+r}\boldsymbol{E} + by^{q+r+1}\partial_x, \quad k < q+r,\\ &\mathcal{I}(\varphi) = \mathbb{Z}_{\geq k} \setminus \{q+r, k+q+r-1\}, \quad \mathcal{I}(\psi) = \mathbb{Z}_{\geq 1}, \end{aligned} \tag{6.12}$$

or

$$\begin{aligned} &(1+\psi(x))(\boldsymbol{V}_0 + \boldsymbol{V}_1 + \varphi(x)\boldsymbol{E}),\\ &k > q+r, \quad \mathcal{I}(\varphi) = \mathbb{Z}_{\geq k} \setminus \{k+q+r-1\}, \quad \mathcal{I}(\psi) = \mathbb{Z}_{\geq 1}. \end{aligned} \tag{6.13}$$

6.2.2. *The case* $\boldsymbol{V}_1 = ax^{q+r}\boldsymbol{E} + by^{q+r+1}\partial_x$. We assume $(a,b) \neq (0,0)$. By Lemma 7 we have $D(\boldsymbol{V}_1)M = -b(r+1)y^{2q+r+2}(y-x)^r$. So we want to know whether $g = y^{2q+r+2}(y-x)^r \notin \operatorname{Im} D_d(\boldsymbol{V}_0)$ with $d = 2q+2r+1$. The solution of the corresponding homological equation is

$$\tilde{f}^D = \frac{1}{|s|}u^{2q+1}(u-1)^{2r+1}\int_{\infty}^{u} \frac{\tau^r}{(\tau-1)^{r+2}}\,d\tau.$$

It is a polynomial of degree d, which demonstrates that the operator $\widetilde{D}(\boldsymbol{V})$ (see (5.10)) has kernel generated by $M + \text{const} \cdot f^D$ and is not surjective. Therefore we gain nothing.

If the orbital normal form is richer than $\boldsymbol{V}_0 + \boldsymbol{V}_1$ then M is used in the third level reduction of the orbital normal form. Otherwise, we reduce the corresponding term x^{q+r-1} from the orbital factor $1+\psi(x)$. This holds when $q \neq 1$.

• *The second level normal form in the 'Rational PFI with 2-Factor IIM' case with* $\boldsymbol{V}_1 = ax^{q+r}\boldsymbol{E} + by^{q+r+1}\partial_x$ *is either*

$$(1+\psi(x))(\boldsymbol{V}_0 + \boldsymbol{V}_1 + \varphi(x)\boldsymbol{E}), \quad \mathcal{I}(\varphi) = \mathbb{Z}_{>q+r}, \quad \mathcal{I}(\psi) = \mathbb{Z}_{\geq 1}, \tag{6.14}$$

or

$$\begin{aligned} &(1+\psi(x))(\boldsymbol{V}_0 + \boldsymbol{V}_1), \\ &\mathcal{I}(\psi) = \begin{cases} \mathbb{Z}_{\geq 1} \setminus \{q+r-1\} & \text{(Subcases 1 and 3)}, \\ \mathbb{Z}_{\geq 1} & \text{(Subcase 2)}. \end{cases} \end{aligned} \tag{6.15}$$

6.3. Third level. We deal with $\boldsymbol{V} = \boldsymbol{V}_0 + \boldsymbol{V}_1 + \boldsymbol{V}_2$, where

$$\boldsymbol{V}_1 = ax^{q+r}\boldsymbol{E} + by^{q+r+1}\partial_x \tag{6.16}$$

and $\boldsymbol{V}_2 = cx^l\boldsymbol{E}$ or $\boldsymbol{V}_2 = cx^j\boldsymbol{V}_0$.

6.3.1. *The case* $\boldsymbol{V}_2 = cx^l\boldsymbol{E}$. Of course, $l > q + r$. We use $D(\boldsymbol{V}_2)M \notin \operatorname{Im} D(\boldsymbol{V}_0)$ to kill $x^{l+q+r-1}\boldsymbol{E}$.

• *The complete normal form in the 'Rational PFI with 2-Factor IIM' case with* $\boldsymbol{V}_1 = ax^{q+r}\boldsymbol{E} + by^{q+r+1}\partial_x$ *and* $\boldsymbol{V}_2 = cx^l\boldsymbol{E}$ *is*

$$\begin{aligned} &(1+\psi(x))(\boldsymbol{V}_0 + \boldsymbol{V}_1 + \boldsymbol{V}_2 + \varphi(x)\boldsymbol{E}), \\ &\mathcal{I}(\varphi) = \mathbb{Z}_{\geq l} \setminus \{l+q+r-1\}, \quad \mathcal{I}(\psi) = \mathbb{Z}_{\geq 1}. \end{aligned} \tag{6.17}$$

6.3.2. *The case* $\boldsymbol{V}_2 = cx^j\boldsymbol{V}_0$. This case is essential only when

$$q = r = 1, \tag{6.18}$$

i.e., $F = x/y(y-x)$ (Subcase 2). Recall that $\boldsymbol{T} = -y^2(\partial_x + \partial_y)$ and that $[\boldsymbol{V}_0, \boldsymbol{T}] = 0$ (Lemma 6). Therefore

$$[x^j\boldsymbol{V}_0, \boldsymbol{T}] = x^j[\boldsymbol{V}_0, \boldsymbol{T}] - \boldsymbol{T}(x^j)\boldsymbol{V}_0 = jx^{j-1}y^2\boldsymbol{V}_0 := jg(x,y)\boldsymbol{V}_0.$$

The polynomial g is not in $\operatorname{Im} C_d(\boldsymbol{V}_0)$ for $d = j > 1$.

Indeed, we have

$$\tilde{f}^C = u^j(u-1)^j \int^u \frac{d\tau}{\tau^{j-1}(\tau-1)^{j+1}},$$

where the integrand has nonzero residue at $u = 1$ for $j > 1$. Thus we can cancel the term x^{j+1} from $\psi(x)$.

If $j = 1$ then we cannot cancel x^2 from $\psi(x)$. But if ψ contains a higher order term ex^m then we can cancel the term x^{m+1}.

• *The complete normal form in Subcase 2 of the 'Rational PFI with 2-Factor IIM' case with* $\boldsymbol{V}_1 = ax^2\boldsymbol{E} + by^3\partial_x$ *and* $\boldsymbol{V}_2 = cx^j\boldsymbol{V}_0$ *is either*

$$(1+cx^j+\psi(x))(\boldsymbol{V}_0 + \boldsymbol{V}_1), \quad j > 1, \quad \mathcal{I}(\psi) = \mathbb{Z}_{>j+1}, \tag{6.19}$$

or

$$(1 + cx + ex^m + \psi(x))(\boldsymbol{V}_0 + \boldsymbol{V}_1), \qquad \mathcal{I}(\psi) = \mathbb{Z}_{>m+1}, \tag{6.20}$$

or

$$(1 + cx)(\boldsymbol{V}_0 + \boldsymbol{V}_1). \tag{6.21}$$

7. The case with polynomial PFI. This case is characterized by the polynomial form of the principal first integral:

$$F = x^p y^q (y - x)^r, \qquad \gcd(p, q, r) = 1,$$

with $p, q, r \in \mathbb{Z}_{\geq 0}$. The case with $p, q, r > 0$, i.e., the 'Polynomial PFI' case (see (3.4)), was completely studied in [S]; we will call it also *Subcase 1.* When $r = 0$ and $p, q > 0$ we have the 'Polynomial PFI with linear CF' case (see (3.9)); we will call it *Subcase 2.* When $p = r = 0$ and $q = 1$ we have 'Quadratic CF' case (see (3.11)), which we will call *Subcase 3.*

In [S, Remark 2] it was suggested that the same list of normal forms (as in the 'Polynomial PFI' case) holds in Subcases 2 and 3 (with obvious modifications). The aim of this section is to verify that statement.

Therefore we shall follow (rather quickly) the four levels of analysis from the 'Polynomial PFI' case and adapt them to the remaining types.

Recall that

$$\boldsymbol{V}_0 = \begin{cases} (y - x)(qx\partial_x - py\partial_y) = G\boldsymbol{X} & \text{(Subcase 2)}, \\ x(y - x)\partial_y = G\boldsymbol{X} & \text{(Subcase 3)}. \end{cases}$$

The operators $D_d(\boldsymbol{V}_0)$ are as in Sections 2–3. But, instead of $C_d(\boldsymbol{V}_0)$, we consider the operator $C'_d(\boldsymbol{V}_0) = G\partial/\partial\boldsymbol{X} - \boldsymbol{X}(G)$ (see (2.13)) with the homological equation $C'_d(\boldsymbol{V}_0)f = g$ resulting from $\operatorname{ad}_{G\boldsymbol{X}} f\boldsymbol{X} = g\boldsymbol{X}$.

Below $s = p + q + r$, i.e., $s = p + q$ (Subcase 2) or $s = 1$ (Subcase 3).

7.1. The first level analysis

LEMMA 8.

(a) *In Subcase 2,* $\ker C'_d(\boldsymbol{V}_0) = 0$ *for* $d \neq 1 \pmod{s}$ *and* $\ker C'_d(\boldsymbol{V}_0) = \mathbb{C} \cdot L_m$ *for* $d = ms + 1 \in s\mathbb{Z} + 1$ *with*

$$L_m = (y - x)F^m.$$

(b) *In Subcase 3,* $\ker C'_d(\boldsymbol{V}_0) = \mathbb{C} \cdot L_{d-2}$ *with* $L_m = x(y - x)y^m$.

(c) *We have* $\ker D_d(\boldsymbol{V}_0) = 0$ *if* $d \neq 3 \pmod{s}$ *and* $\ker D_d(\boldsymbol{V}_0) = \mathbb{C} \cdot M_m$ *if* $d = ms + 3 \in s\mathbb{Z} + 3$ *with*

$$M_m = sxy(y - x)F^m.$$

LEMMA 9. *We have* $M_m = B(\boldsymbol{V}_0)\boldsymbol{T}_m$, *where*

$$\boldsymbol{T}_m = F^m\boldsymbol{E} \quad \textit{and} \quad \operatorname{ad}_{\boldsymbol{V}_0} T_m = F^m\boldsymbol{V}_0.$$

The solutions to the homological equations $D_d(\boldsymbol{V}_0)f^D = g^D$ and $C'_d(\boldsymbol{V}_0)f^{C'} = g^{C'}$ are

$$(7.1)\qquad \tilde f^D = -\frac{1}{s}u^\gamma(u-1)^\delta \int^u \omega_D(g^D), \qquad \tilde f^{C'} = -\frac{1}{s}u^\mu(u-1)^\nu \int^u \omega_{C'}(g^{C'}),$$

where $\omega_D(g) = \tilde g(u)u^{-\gamma-1}(u-1)^{-\delta-1}\,du$, $\omega_{C'}(g) = \tilde g(u)u^{-\mu-1}(u-1)^{-\nu-1}\,du$ and

$$(7.2)\qquad \gamma = (d-2)q/s + 1, \qquad \delta = 1$$

(thus $\gamma = d-2$ for Subcase 3),

$$(7.3)\qquad \begin{aligned} &\mu = (d-1)q/s, \quad &&\nu = 1 \quad &&\text{(Subcase 2)},\\ &\mu = d-2, &&\nu = 1 &&\text{(Subcase 3)}. \end{aligned}$$

We choose generators of the subspaces $\mathcal{N}(D_d)$ as in [S]:

$$(7.4)\qquad \begin{aligned} &g^D = x^{d-2}y(y-x), \quad d \neq 3 \pmod s,\\ &g_0^D = g^D, \quad g_1^D = xy^{d-2}(y-x). \end{aligned}$$

For Subcase 2 we choose generators of $\mathcal{N}(C'_d)$ as

$$(7.5)\qquad \begin{aligned} &g^{C'} = x^d(y-x), \quad d \neq 1 \pmod s,\\ &g_0^{C'} = g^{C'}, \quad g_1^{C'} = y^d(y-x); \end{aligned}$$

for Subcase 3 we choose

$$(7.6)\qquad g_0^{C'} = x^d(y-x), \quad g_1^{C'} = xy^{d-1}(y-x).$$

The fact that, for $d \neq 3 \pmod s$, the function g^D is good follows from the nonvanishing of $\Omega_D = \operatorname{Res}_{u=1}\omega_D$ where $\omega_D(g^D) = u^{-\gamma}(u-1)^{-1}\,du$. For $d = ms+3$ the period matrix with the periods $\Omega_D^{0,1} = \operatorname{Res}_{u=0,1}\omega_D$ is triangular and nondegenerate.

In Subcase 2 with $d \neq 1 \pmod s$ we consider $\Omega_{C'} = \operatorname{Res}_{u=1}\omega_{C'}(g^{C'})$. When $d = ms+1$ we consider the period matrix with $\Omega_{C'}^{0,1} = \operatorname{Res}_{u=0,1}\omega_{C'}$.

In Subcase 3 we have the corresponding forms

$$\frac{du}{u^{d-1}(u-1)}, \qquad \frac{du}{u-1}$$

and the period matrix with $\Omega_{C'}^{0,1} = \operatorname{Res}_{u=0,1}\omega_{C'}$ is nondegenerate.

If the orbital normal form is simply $\boldsymbol{V}_0$ then we use the IIMs M_m from Lemma 9 to kill some additional terms from the orbital factor. To this end, one considers the equations $C'_d(\boldsymbol{V}_0)f^{C'} = g$ with $g = GF_m = x^{pm}y^{qm}(y-x)$ (Subcase 2) or $g = xy^{d-1}(y-x)$ (Subcase 3). In Subcase 2, $d = ms \neq 1 \pmod s$, and hence $\omega_{C'}(g) = u^\lambda(u-1)^{-1}\,du$ with $\lambda \notin \mathbb{Z}$ has nonzero residue at $u=1$. So, we kill the terms $x^{(p+q)m}$, $m = 1,2,\ldots,$ from the orbital factor.

In Subcase 3 we have $g = g_1^{C'}$ from (7.6); so, the term y^{d-1} from the orbital factor is killed.

• *The first level normal form in Subcase 2 is the same as in* [S, *Proposition* 1], *and in Subcase 3 we have either*

$$(1+\psi(x)+\chi(y))(\boldsymbol{V}_0+\varphi(x)\boldsymbol{E}+\phi(y)\boldsymbol{E}), \quad \mathcal{I}(\varphi)=\mathcal{I}(\phi)=\mathbb{Z}_{\geq 2},\quad \mathcal{I}(\psi)=\mathcal{I}(\chi)=\mathbb{Z}_{\geq 1}, \tag{7.7}$$

or

$$(1+\psi(x))\boldsymbol{V}_0,\quad \mathcal{I}(\psi)=\mathbb{Z}_{\geq 1}\ (^4). \tag{7.8}$$

7.2. The second level analysis. We study the operator $\widetilde{D}(\boldsymbol{V})$ from (5.10) with $\boldsymbol{V}=\boldsymbol{V}_0+\boldsymbol{V}_1$ and $\boldsymbol{V}_1=ax^k\boldsymbol{E}$ or $\boldsymbol{V}_1=(ax^k+by^k)\boldsymbol{E}$ and with the IIMs M_m from the kernel of $D(\boldsymbol{V}_0)$ (see Lemma 8).

But the operator $C'(\boldsymbol{V}_0)$ also has huge kernel generated by the $L_m=GF^m$'s from Lemma 8. Therefore we define the operators

$$\widetilde{C'}(\boldsymbol{V}):\mathbb{C}\cdot GF_m\oplus\mathcal{F}_d\to\mathcal{F}_{d+1},\quad (\xi GF^m,f)\mapsto \xi GC(\boldsymbol{V}_1)F^m+C'_d(\boldsymbol{V}_0)f, \tag{7.9}$$

where the argument corresponds to the vector field $\boldsymbol{Z}=\xi F^m(\boldsymbol{V}_0+\boldsymbol{V}_1)+f\boldsymbol{X}$.

LEMMA 10.

(a) $C((ax^k+by^k)\boldsymbol{E})F^m=ms(ax^k+by^k)F^m$,
(b) $D((ax^k+by^k)\boldsymbol{E})M_m=(ms+1-k)(ax^k+by^k)M_m$.

7.2.1. *The case* $\boldsymbol{V}_1=ax^k\boldsymbol{E}$. We assume Subcase 2 with

$$k\neq 1\ (\mathrm{mod}\ s). \tag{7.10}$$

By Lemma 10(b) we should check whether the polynomials $g^D=x^kM_m=x^{k+ms+3}u^{mq}$ lie outside $\operatorname{Im}D_d(\boldsymbol{V}_0)$ with $d=k+ms+2\neq 3\ (\mathrm{mod}\ s)$. Because $\Omega_D(g^D)=\operatorname{Res}_{u=1}\omega_D(g^D)\neq 0$, the answer is 'yes' and we can kill the terms $x^{k+ms}\boldsymbol{E}$ from the orbital normal form.

To prove the invertibility of the operator $\widetilde{C'}(\boldsymbol{V})$ we show that the polynomial $g^{C'}=x^kF_mG$ (Lemma 10(a)) lies outside $\operatorname{Im}C'_d(\boldsymbol{V}_0)$ with $d=k+ms\neq 1\ (\mathrm{mod}\ s)$. But now $\Omega_{C'}(g^{C'})=\operatorname{Res}_{u=1}\omega_{C'}(g^{C'})\neq 0$. We kill the terms x^{k+ms} from the orbital factor.

• *The complete normal form in Subcase 2 with* $\boldsymbol{V}_1=ax^k\boldsymbol{E}$ *is as in* [S, *Proposition* 2].

7.2.2. *The case* $\boldsymbol{V}_1=(ax^{ns+1}+by^{ns+1})\boldsymbol{E}$. Of course, $(a,b)\neq(0,0)$. Here the operator $\widetilde{D}(\boldsymbol{V})$ acts between $\mathbb{C}\oplus\mathcal{F}_d$ and $\mathcal{F}_{d+1}$ for $d=(m+n)s+3$. The space $\mathcal{F}_{d+1}/\operatorname{Im}D_d(\boldsymbol{V}_0)$ is generated by g_0^D and g_1^D from (7.4).

By Lemma 10(b), if $m=n$ then $\ker\widetilde{D}(\boldsymbol{V})=\mathbb{C}M_n\oplus\mathbb{C}M_{m+n}$ and nothing can be reduced from the orbital normal form (cf. [S, Lemma 15]). If this

(^{4}) The form (7.8) is slightly different from the one following from [S, Proposition 1]; the latter would be $(1+\chi(y))\boldsymbol{V}_0$.

normal form is richer than $\boldsymbol{V}_0+\boldsymbol{V}_1$ then these M_m's can be used in the further orbital normal form reduction. Otherwise we use them to kill some powers of x in the orbital factor, as in the case of orbital normal form $\boldsymbol{V}_0$ from the first level analysis.

Let $m\neq n$. Then we should compare three elements from $\mathcal{F}_{d+1}$: g_0^D, g_1^D and $g_2^D=(ax^{ns+1}+by^{ns+1})M_m$. Let $\Omega_D^{0,1}=\mathrm{Res}_{u=0,1}\,\omega_D$ be the corresponding periods.

If $a\neq 0$ then the period matrix $(\Omega_D^i(g_j^D))_{j=1,2}^{i=0,1}$ is nondegenerate; so, (g_1^D,g_2^D) is a basis of $\mathcal{N}(D_d)$ and $x^{(m+n)s+1}$ can be removed from $\varphi(x)$. If $a=0\neq b$ then the matrix $(\Omega_D^i(g_j^D))_{j=0,2}^{i=0,1}$ is nondegenerate and one removes $y^{(m+n)s+1}$ from $\phi(y)$.

Next, we consider the operator $\widetilde{C'}(\boldsymbol{V})$ defined in (7.9) with $d=(m+n)s+1$ (Subcase 2) and $d=m+n+1$ (Subcase 3). We have the periods $\Omega_{C'}^i(g_j^{C'})$, where $g_{0,1}^{C'}$ are given in (7.6) and $g_2^{C'}=(ax^{ns+1}+by^{ns+1})GF_m$.

If $a\neq 0$ then the period matrix $(\Omega_{C'}^i(g_j^{C'}))_{j=1,2}^{i=0,1}$ is nondegenerate; so, $(g_1^{C'},g_2^{C'})$ is a basis of $\mathcal{N}(C'_d)$ and $x^{(m+n)s+1}$ can be removed from $\psi(x)$. If $a=0\neq b$ then the matrix $(\Omega_{C'}^i(g_j^{C'}))_{j=0,2}^{i=0,1}$ is nondegenerate and one removes the term $y^{(m+n)s+1}$ from $\chi(y)$.

• *The second level normal form with* $\boldsymbol{V}_1=(ax^{ns+1}+by^{ns+1})\boldsymbol{E}$ *is as in* [S, *Proposition* 3] *for all cases but Subcase 3 when it is*

$$(1+\psi(x))(\boldsymbol{V}_0+\boldsymbol{V}_1),\quad \mathcal{I}(\psi)=\mathbb{Z}_{\geq 1}. \tag{7.11}$$

7.3. The third level. Here we use homological operators associated with $\boldsymbol{V}=\boldsymbol{V}_0+\boldsymbol{V}_1+\boldsymbol{V}_2$ where

$$\boldsymbol{V}_1=(ax^{ns+1}+by^{ns+1})\boldsymbol{E}, \tag{7.12}$$

i.e., when $\boldsymbol{V}_0+\boldsymbol{V}_1$ has the IIM M_n. For the third vector field we have the following possibilities: $\boldsymbol{V}_2=hx^k\boldsymbol{E}$, $k\neq 1 \pmod s$; $\boldsymbol{V}_2=hy^{ls+1}\boldsymbol{E}$, $l\neq 2n$, when $a\neq 0$; $\boldsymbol{V}_2=hx^{ls+1}\boldsymbol{E}$, $l\neq 2n$, when $a=0\neq b$; and $\boldsymbol{V}_2=(cx^{2n+1}+ey^{2n+1})\boldsymbol{E}$.

Using M_n we can reduce only one term, either from the second order orbital normal form or from the orbital factor.

7.3.1. *The case* $\boldsymbol{V}_2=hx^k\boldsymbol{E}$. Recall that

$$k\neq 1 \pmod s, \tag{7.13}$$

i.e. we have Subcase 2. We use $g=D(\boldsymbol{V}_2)M_n=h(ns-k)x^kM_n$ to reduce the term $x^{ns+k}\boldsymbol{E}$ from $\varphi(x)\boldsymbol{E}$. Here the corresponding period is $\mathrm{Res}_{u=1}\,\omega_D(g)$.

7.3.2. *The case* $\boldsymbol{V}_2=hy^{ls+1}\boldsymbol{E}$. Recall that

$$a\neq 0\quad\text{and}\quad l\neq n. \tag{7.14}$$

We use simultaneously M_l and M_n to reduce the cokernel of $D_d(\boldsymbol{V}_0)$, $d = (n+l)s+3$, i.e., the polynomials $g_0^D = (ax^{ns+1} + by^{ns+1})M_l$ and $g_1^D = y^{ls+1}M_n$. The corresponding period matrix is nondegenerate. We cancel the terms $x^{(n+l)s+1}\boldsymbol{E}$ and $y^{(n+l)s+1}\boldsymbol{E}$.

7.3.3. *The case* $\boldsymbol{V}_2 = hx^{ls+1}\boldsymbol{E}$. Recall that

$$a = 0 \neq b \quad \text{and} \quad l \neq n. \tag{7.15}$$

This case is analogous to the previous one, but with $g_0^D = by^{ns+1}M_l$ and $g_1^D = x^{ls+1}M_n$. The terms $x^{(n+l)s+1}\boldsymbol{E}$ and $y^{(n+l)s+1}\boldsymbol{E}$ are killed.

7.3.4. *The case* $\boldsymbol{V}_2 = (cx^{2n+1} + ey^{2n+1})\boldsymbol{E}$. Here nothing from the 2-dimensional space $\operatorname{coker} D_d(\boldsymbol{V}_0)$ was reduced in the second level analysis. Following [S] we introduce the determinant of the period matrix

$$\varDelta_n = \varDelta_n(a,b,c,d) = \det(\varOmega_D^i(g_j^D)), \tag{7.16}$$

where $g_0^D = (ax^{ns+1} + by^{ns+1})M_{2n}$ and $g_1^D = (cx^{2ns+1} + ey^{2ns+1})M_n$.

If $\varDelta_n \neq 0$ then the terms $x^{(n+l)s+1}\boldsymbol{E}$ and $y^{(n+l)s+1}\boldsymbol{E}$ are reduced.

7.4. The fourth level. Now $\boldsymbol{V} = \boldsymbol{V}_0 + \boldsymbol{V}_1 + \boldsymbol{V}_2 + \boldsymbol{V}_3$, where $\boldsymbol{V}_1 = (ax^{ns+1} + by^{ns+1})\boldsymbol{E}$, $\boldsymbol{V}_2 = (cx^{2n+1} + ey^{2n+1})\boldsymbol{E}$ and

$$\varDelta_n = 0. \tag{7.17}$$

The following subsections are devoted to different choices of $\boldsymbol{V}_3$.

7.4.1. *The case* $\boldsymbol{V}_3 = hx^k\boldsymbol{E}$. We have Subcase 2 and

$$k \neq 1 \pmod{s}. \tag{7.18}$$

We use $D_d(\boldsymbol{V}_3)M_n$, $d = k + ns + 2 \neq 3 \pmod{s}$, to kill the term $x^{k+ns}\boldsymbol{E}$.

7.4.2. *The case* $\boldsymbol{V}_3 = hy^{ls+1}\boldsymbol{E}$. We assume

$$a \neq 0. \tag{7.19}$$

We use $D(\boldsymbol{V}_3)M_n$ and $D(\boldsymbol{V}_1)M_l$ to reduce $x^{(n+l)s+1}\boldsymbol{E}$ and $y^{(n+l)s+1}\boldsymbol{E}$ (as in Section 7.3.2).

7.4.3. *The case* $\boldsymbol{V}_3 = hx^{ls+1}\boldsymbol{E}$. This holds when

$$a = 0 \neq b. \tag{7.20}$$

The same terms as in the previous case are reduced.

7.4.4. *The case* $\boldsymbol{V}_3 = 0$. Here $\boldsymbol{V}_1 = (ax^{ns+1} + by^{ns+1})\boldsymbol{E}$ and we have two situations: $\boldsymbol{V}_2 = 0$ or $\boldsymbol{V}_2 = (cx^{2n+1} + ey^{2n+1})\boldsymbol{E}$ with zero determinant $\varDelta_n(a,b,c,e)$ of the period matrix.

We use M_n, via $\boldsymbol{T}_n \in B(\boldsymbol{V}_0)^{-1}M_n$, to reduce x^{ns} from $\psi(x)$ (Subcase 2) or $y^{ns} = y^n$ from $\chi(y)$ (Subcase 3).

• *The list of normal forms in Subcase 2 is the same as in* [S, *Theorem* 1], *and in Subcase 3 one should replace*

– $\{1+\psi(x)+\psi(y)\}\boldsymbol{V}_0$, $\mathcal{I}(\psi)=\mathbb{Z}_{>0}\setminus I_0$, $\mathcal{I}(\chi)=I_1$ ([S, (33)]) *with* $\{1+\psi(x)\}\boldsymbol{V}_0$, $\mathcal{I}(\psi)=\mathbb{Z}_{>0}$;

– $\mathcal{I}(\psi)=\mathbb{Z}_{>0}\setminus(\mathbb{Z}_{>ns+1}\cap I_1)\setminus\{ns\}$, $\mathcal{I}(\chi)=\mathbb{Z}_{\le ns+1}\cap I_1$ ([S, (37)]) *with* $\mathcal{I}(\psi)=\mathbb{Z}_{\le n+1}$, $\mathcal{I}(\chi)=\mathbb{Z}_{>0}\setminus\{n\}$;

– $\mathcal{I}(\psi)=\mathbb{Z}_{>0}\setminus\{ns\}$, $\mathcal{I}(\chi)=\mathbb{Z}_{\le n+1}\cap I_1$ ([S, (38)]) *with* $\mathcal{I}(\psi)=\mathbb{Z}_{>0}$, $\mathcal{I}(\chi)=\mathbb{Z}_{>0}\cap\mathbb{Z}_{\le n+1}\setminus\{n\}$.

Here and below,

$$I_l=\{i\in\mathbb{Z}_{>0}: i=l \ (\mathrm{mod}\ s)\}\quad \text{for } l\in\mathbb{Z}. \tag{7.21}$$

8. The main theorem. *The list of complete and inequivalent normal forms for vector fields* (1.1) *with the leading parts* $\boldsymbol{V}_0$ *having the principal first integral from the list of types in Section* 3 *is the following.*

For the 'Generic Non-resonant' case, the 'Rational PFI without Polynomial IIMs' case, the 'Double Invariant Line' case (*Subcase 1*), *the 'Triple Invariant Line' case, the 'Nonresonant with CF' case, the 'Rational PFI without Polynomial IIMs and with CF' case, and the 'Double Invariant Line with CF' case* (*Subcase 1*) *we have the normal forms as in* (4.5), *i.e.*,

$$\{1+\psi(x)\}\{\boldsymbol{V}_0+\varphi(x)\boldsymbol{E}\},\quad \mathcal{I}(\varphi)=\mathbb{Z}_{>1},\quad \mathcal{I}(\psi)=\mathbb{Z}_{>0}; \tag{8.1}$$

for the 'Double Invariant Line' case (*Subcase 2*) *we have the normal forms as in* (4.13), *i.e.*,

$$\{1+\psi(x)\}\{\boldsymbol{V}_0+a_{n+1}x^{n+2}\partial_y+\varphi(x)\boldsymbol{E}\},\quad \mathcal{I}(\varphi)=\mathbb{Z}_{>1}\setminus\{n+1\}; \tag{8.2}$$

for the 'Double Invariant Line with CF' case (*Subcase 2*) *we have the normal forms as in* (4.18), *i.e.*,

$$\boldsymbol{V}_0+\varphi(y)\partial_x+\psi(y)\partial_y,\quad \mathcal{I}(\varphi)=\mathbb{Z}_{>2},\quad \mathcal{I}(\psi)=\mathbb{Z}_{>2}. \tag{8.3}$$

For the 'Rational PFI with 1-Factor IIM' case we have either

$$\begin{aligned}&\{1+\psi(x)\}\{\boldsymbol{V}_0+\boldsymbol{V}_1+bx^r\partial_y+\varphi(x)\boldsymbol{E}\},\\ &\boldsymbol{V}_1=ax^k\boldsymbol{E},\quad k<r-1,\\ &\mathcal{I}(\psi)=\mathbb{Z}_{\ge1},\quad \mathcal{I}(\varphi)=\mathbb{Z}_{>k}\setminus\{r-1,k+r-2\}\end{aligned} \tag{8.4}$$

(*the normal forms as in* (5.13)), *or*

$$\begin{aligned}&\{1+\psi(x)\}\{\boldsymbol{V}_0+\boldsymbol{V}_1+\varphi(x)\boldsymbol{E}\},\\ &\boldsymbol{V}_1=ax^k\boldsymbol{E},\quad k>r-1,\\ &\mathcal{I}(\psi)=\mathbb{Z}_{\ge1},\quad \mathcal{I}(\varphi)=\mathbb{Z}_{>k}\setminus\{k+r-2\}\end{aligned} \tag{8.5}$$

(*as in* (5.14)), *or*

$$\begin{aligned}&\{1+\psi(x)\}\{\boldsymbol{V}_0+\boldsymbol{V}_1+\varphi(x)\boldsymbol{E}\},\\ &\boldsymbol{V}_1=ay^r\partial_x+bx^r\partial_y,\quad a+(-1)^{r+1}b\neq0,\\ &\mathcal{I}(\varphi)=\mathbb{Z}_{\ge r}\setminus\{2r-3\},\quad \mathcal{I}(\psi)=\mathbb{Z}_{\ge1}\end{aligned} \tag{8.6}$$

(*as in* (5.15)), *or*

$$\begin{aligned} &\{1+\psi(x)\}\{\boldsymbol{V}_0+\boldsymbol{V}_1+\boldsymbol{V}_2+\varphi(x)\boldsymbol{E}\},\\ &\boldsymbol{V}_1=a(y^r\partial_x+(-1)^r x^r\partial_y),\quad \boldsymbol{V}_2=cx^l\boldsymbol{E},\\ &\mathcal{I}(\varphi)=\mathbb{Z}_{>l}\setminus\{l+r-2\},\quad \mathcal{I}(\psi)=\mathbb{Z}_{\geq 1}\end{aligned} \tag{8.7}$$

(*as in* (5.20)), *or*

$$\begin{aligned} &\{1+\psi(x)\}\{\boldsymbol{V}_0+\boldsymbol{V}_1\}\\ &\boldsymbol{V}_1=a(y^r\partial_x+(-1)^r x^r\partial_y),\quad \mathcal{I}(\psi)=\mathbb{Z}_{\geq 1}\setminus\{r-2\}\quad \text{(Subcase 1)}\end{aligned} \tag{8.8}$$

(*as in* (5.17)), *or*

$$\begin{aligned} &\{1+cx^j+\psi(x)\}\{\boldsymbol{V}_0+\boldsymbol{V}_1\},\\ &\boldsymbol{V}_1=a(y^r\partial_x+(-1)^r x^r\partial_y),\quad r=4,\quad \mathcal{I}(\psi)=\mathbb{Z}_{>j}\setminus\{j-2\}\end{aligned} \tag{8.9}$$

(*as in* (5.22)), *or*

$$\{1+\psi(x)\}\boldsymbol{V}_0,\qquad \mathcal{I}(\psi)=\begin{cases}\mathbb{Z}_{\geq 1}\setminus\{r-2\} & \text{(Subcase 1)},\\ \mathbb{Z}_{\geq 1} & \text{(Subcase 2)}\end{cases} \tag{8.10}$$

(*as in* (5.7)), *or*

$$\boldsymbol{V}_0+\boldsymbol{V}_1,\qquad \boldsymbol{V}_1=a(y^r\partial_x+(-1)^r x^r\partial_y),\quad r=4 \tag{8.11}$$

(*as in* (5.23)).

For the 'Rational PFI with 2-Factor IIM' case we have either

$$\begin{aligned} &\{1+\psi(x)\}\{\boldsymbol{V}_0+\boldsymbol{V}_1+\boldsymbol{U}+\varphi(x)\boldsymbol{E}\},\\ &\boldsymbol{V}_1=ax^k\boldsymbol{E},\quad k<q+r,\quad \boldsymbol{U}=ax^{q+r}\boldsymbol{E}+by^{q+r+1}\partial_x,\\ &\mathcal{I}(\varphi)=\mathbb{Z}_{\geq k}\setminus\{q+r,k+q+r-1\},\quad \mathcal{I}(\psi)=\mathbb{Z}_{\geq 1}\end{aligned} \tag{8.12}$$

(*as in* (6.12)), *or*

$$\begin{aligned} &\{1+\psi(x)\}\{\boldsymbol{V}_0+\boldsymbol{V}_1+\varphi(x)\boldsymbol{E}\},\\ &\boldsymbol{V}_1=ax^k\boldsymbol{E},\qquad k>q+r,\\ &\mathcal{I}(\varphi)=\mathbb{Z}_{\geq k}\setminus\{k+q+r-1\},\quad \mathcal{I}(\psi)=\mathbb{Z}_{\geq 1}\end{aligned} \tag{8.13}$$

(*as in* (6.13)), *or*

$$\begin{aligned} &\{1+\psi(x)\}\{\boldsymbol{V}_0+\boldsymbol{V}_1+\boldsymbol{V}_2+\varphi(x)\boldsymbol{E}\},\\ &\boldsymbol{V}_1=ax^{q+r}\boldsymbol{E}+bx^{q+r+1}\partial_y,\quad \boldsymbol{V}_2=cx^l\boldsymbol{E},\\ &\mathcal{I}(\varphi)=\mathbb{Z}_{\geq l}\setminus\{l+q+r-1\},\quad \mathcal{I}(\psi)=\mathbb{Z}_{\geq 1}\end{aligned} \tag{8.14}$$

(*as in* (6.17)), *or*

$$\begin{aligned} &\{1+\psi(x)\}\{\boldsymbol{V}_0+\boldsymbol{V}_1\},\\ &\boldsymbol{V}_1=ax^{q+r}\boldsymbol{E}+bx^{q+r+1}\partial_y,\\ &\mathcal{I}(\psi)=\mathbb{Z}_{\geq 1}\setminus\{q+r-1\}\quad \text{(Subcases 1 and 3)}\end{aligned} \tag{8.15}$$

(*as in* (6.15)), *or*

(8.16) $$\{1 + cx^j + \psi(x)\}\{\boldsymbol{V}_0 + \boldsymbol{V}_1\},\quad \boldsymbol{V}_1 = ax^{q+r}\boldsymbol{E} + bx^{q+r+1}\partial_y, \quad \mathcal{I}(\psi) = \mathbb{Z}_{\geq j+2} \quad \text{(Subcase 2)}$$

(*as in* (6.19)), *or*

(8.17) $$\{1 + cx^j + ex^m + \psi(x)\}\{\boldsymbol{V}_0 + \boldsymbol{V}_1\},\quad \boldsymbol{V}_1 = ax^{q+r}\boldsymbol{E} + bx^{q+r+1}\partial_y, \quad \mathcal{I}(\psi) = \mathbb{Z}_{\geq m+2} \quad \text{(Subcase 2)}$$

(*as in* (6.20)), *or*

(8.18) $$\{1 + cx\}\{\boldsymbol{V}_0 + \boldsymbol{V}_1\} \quad \text{(Subcase 2)}$$

(*as in* (6.21)).

For the 'Polynomial PFI' case (*with three subcases*) *we have either*

(8.19) $$\{1 + \psi(x) + \chi(y)\}\boldsymbol{V}_0,\quad \mathcal{I}(\psi) = \mathbb{Z}_{>0} \setminus I_0, \quad \mathcal{I}(\chi) = I_1 \quad \text{(Subcases 1 and 2)},$$

or

(8.20) $$\{1 + \psi(x)\}\boldsymbol{V}_0, \quad \mathcal{I}(\psi) = \mathbb{Z}_{>0} \quad \text{(Subcase 3)},$$

or

(8.21) $$\begin{gathered}\{1 + \psi(x) + \chi(y)\}\{\boldsymbol{V}_0 + \boldsymbol{V}_1 + [\varphi(x) + \phi(y)]\boldsymbol{E}\},\\ \boldsymbol{V}_1 = ax^j\boldsymbol{E} \neq 0, \quad 2 \leq j \not\equiv 1 \pmod{s},\\ \mathcal{I}(\varphi) = \mathbb{Z}_{>j} \setminus I_j, \quad \mathcal{I}(\phi) = \mathbb{Z}_{>j} \cap I_1,\\ \mathcal{I}(\psi) = \mathbb{Z}_{>0} \setminus (\mathbb{Z}_{>j} \cap I_j), \quad \mathcal{I}(\chi) = I_1,\end{gathered}$$

or

(8.22)
$$\begin{gathered}\{1 + \psi(x) + \chi(y)\}\{\boldsymbol{V}_0 + \boldsymbol{V}_1 + \boldsymbol{V}_2 + [\varphi(x) + \phi(y)]\boldsymbol{E}\},\\ \boldsymbol{V}_1 = (ax^{ns+1} + by^{ns+1})\boldsymbol{E}, \quad \boldsymbol{V}_2 = (cx^{2ns+1} + ey^{2ns+1})\boldsymbol{E}, \quad a\Delta_n \neq 0,\\ \mathcal{I}(\varphi) = \mathbb{Z}_{>2ns+1} \setminus I_1, \quad \mathcal{I}(\phi) = (\mathbb{Z}_{>2ns+1} \cap I_1) \setminus \{3ns+1\},\\ \mathcal{I}(\psi) = \mathbb{Z}_{>0} \setminus (\mathbb{Z}_{>ns+1} \cap I_1), \quad \mathcal{I}(\chi) = I_1\end{gathered}$$

(*where* Δ_n *is defined in* (7.16)), *or*

(8.23) $$\begin{gathered}\{1 + \psi(x) + \chi(y)\}\{\boldsymbol{V}_0 + \boldsymbol{V}_1 + \boldsymbol{V}_2 + [\varphi(x) + \phi(y)]\boldsymbol{E}\},\\ \boldsymbol{V}_1 = by^{ns+1}\boldsymbol{E}, \quad \boldsymbol{V}_2 = (cx^{2ns+1} + ey^{ls+1})\boldsymbol{E}, \quad bc \neq 0,\\ \mathcal{I}(\varphi) = \mathbb{Z}_{>2ns+1} \setminus \{3ns+1\}, \quad \mathcal{I}(\phi) = \emptyset,\\ \mathcal{I}(\psi) = \mathbb{Z}_{>0}, \quad \mathcal{I}(\chi) = \mathbb{Z}_{\leq ns+1} \cap I_1,\end{gathered}$$

or

(8.24)

$$\begin{aligned}&\{1+\psi(x)+\chi(y)\}\{\boldsymbol{V}_0+\boldsymbol{V}_1+\boldsymbol{V}_2\},\\&\boldsymbol{V}_1=(ax^{ns+1}+by^{ns+1})\boldsymbol{E},\quad \boldsymbol{V}_2=(cx^{2ns+1}+ey^{2ns+1})\boldsymbol{E},\quad a\neq 0=\Delta_n,\\&\mathcal{I}(\psi)=\mathbb{Z}_{>0}\setminus(Z_{>ns+1}\cap I_1)\setminus\{ns\},\ \mathcal{I}(\chi)=I_1\quad\text{(Subcases 1 and 2)},\\&\mathcal{I}(\psi)=\mathbb{Z}_{\le n+1},\quad \mathcal{I}(\chi)=\mathbb{Z}_{>0}\setminus\{n\}\quad\text{(Subcase 3)},\end{aligned}$$

or

$$(8.25)\quad\begin{aligned}&\{1+\psi(x)+\chi(y)\}\{\boldsymbol{V}_0+\boldsymbol{V}_1+\boldsymbol{V}_2\},\\&\boldsymbol{V}_1=by^{ns+1}\boldsymbol{E}\neq 0,\quad \boldsymbol{V}_2=ey^{2ns+1}\boldsymbol{E},\\&\mathcal{I}(\psi)=\mathbb{Z}_{>0}\setminus\{ns\},\quad \mathcal{I}(\chi)=\mathbb{Z}_{\le ns+1}\cap I_1\quad\text{(Subcases 1 and 2)},\\&\mathcal{I}(\psi)=\mathbb{Z}_{>0},\quad \mathcal{I}(\chi)=\mathbb{Z}_{\le n+1}\setminus\{n\}\quad\text{(Subcase 3)},\end{aligned}$$

or

$$(8.26)\quad\begin{aligned}&\{1+\psi(x)+\chi(y)\}\{\boldsymbol{V}_0+\boldsymbol{V}_1+\boldsymbol{V}_2+\boldsymbol{V}_3+[\varphi(x)+\phi(y)]\boldsymbol{E}\},\\&\boldsymbol{V}_1=(ax^{ns+1}+by^{ns+1})\boldsymbol{E},\quad \boldsymbol{V}_3=hx^k\boldsymbol{E},\\&\boldsymbol{V}_2=(cx^{2ns+1}+ey^{2ns+1})\boldsymbol{E},\quad \boldsymbol{V}_2\equiv 0\text{ if }k<2ns+1,\\&ah\neq 0=\Delta_n,\quad ns+1<k\neq 1\ (\mathrm{mod}\ s),\\&\mathcal{I}(\varphi)=\mathbb{Z}_{>k}\setminus I_1\setminus\{k+ns\},\quad \mathcal{I}(\phi)=\mathbb{Z}_{>k}\cap I_1,\\&\mathcal{I}(\psi)=\mathbb{Z}_{>0}\setminus(\mathbb{Z}_{>ns+1}\cap I_1),\quad \mathcal{I}(\chi)=I_1,\end{aligned}$$

or

$$(8.27)\quad\begin{aligned}&\{1+\psi(x)+\chi(y)\}\{\boldsymbol{V}_0+\boldsymbol{V}_1+\boldsymbol{V}_2+\boldsymbol{V}_3+[\varphi(x)+\phi(y)]\boldsymbol{E}\},\\&\boldsymbol{V}_1=(ax^{ns+1}+by^{ns+1})\boldsymbol{E},\quad \boldsymbol{V}_3=hy^{ls+1}\boldsymbol{E},\\&\boldsymbol{V}_2=(cx^{2ns+1}+ey^{2ns+1})\boldsymbol{E},\quad \boldsymbol{V}_2\equiv 0\text{ if }l<2n,\\&ah\neq 0=\Delta_n,\quad n<l\neq 2n,\\&\mathcal{I}(\varphi)=\mathbb{Z}_{>ls+1}\setminus I_1,\quad \mathcal{I}(\phi)=(\mathbb{Z}_{>ls+1}\cap I_1)\setminus\{(l+n)s+1\},\\&\mathcal{I}(\psi)=\mathbb{Z}_{>0}\setminus(\mathbb{Z}_{>ns+1}\cap I_1),\quad \mathcal{I}(\chi)=I_1,\end{aligned}$$

or

$$(8.28)\quad\begin{aligned}&\{1+\psi(x)+\chi(y)\}\{\boldsymbol{V}_0+\boldsymbol{V}_1+\boldsymbol{V}_2+\boldsymbol{V}_3+[\varphi(x)+\phi(y)]\boldsymbol{E}\},\\&\boldsymbol{V}_1=by^{ns+1}\boldsymbol{E}\neq 0,\quad \boldsymbol{V}_3=hx^k\boldsymbol{E}\neq 0,\\&\boldsymbol{V}_2=ey^{2ns+1}\boldsymbol{E},\quad \boldsymbol{V}_2\equiv 0\text{ if }k<2ns+1,\\&ns+1<k\neq 2ns+1,\\&\mathcal{I}(\varphi)=\mathbb{Z}_{>k}\setminus\{(k+n)s+1\},\quad \mathcal{I}(\phi)=\emptyset,\\&\mathcal{I}(\psi)=\mathbb{Z}_{>0},\quad \mathcal{I}(\chi)=\mathbb{Z}_{\le ns+1}\cap I_1.\end{aligned}$$

Remark 7. The work of L. Detchenya and A. Sadovskiĭ [DS] is also devoted to the orbital normal forms of such vector fields. They consider the situations DS1–DS9 from Remark 1 and use only homogeneous grading (which is sometimes incorrect). Moreover, they use different notation and it seems that they restrict themselves to the first level analysis. It would be interesting, but quite laborious, to compare their results with our orbital normal forms (i.e., without the orbital factor $1+\psi(x)+\chi(y)$ or $1+\psi(x)$).

As already mentioned, V. Basov, A. Skitovich and E. Fedorova [Ba, BaS1, BaS2, BaF1, BaF2] have raised the problem of classification of germs of real plane vector fields with quadratic leading part. They used only homogeneous grading (as in [DS]). However, their normal forms are not explicit.

For example, in [BaS1] the authors consider the case of $\boldsymbol{V}_0 = x_1x_2(\partial_{x_1}+\partial_{x_2})$, i.e. Subcase 3 of the 'Polynomial PFI' case. Their analysis of homological equations leads to so-called resonant relations (for the coefficients of the transformed vector field $\sum_{j,k} y_1^j y_2^k [Y_1^{j,k}\partial_{y_1} + Y_2^{j,k}\partial_{y_2}]$) of the form

$$Y_1^{0,p} - Y_2^{0,p} = \tilde{c}, \qquad Y_1^{p,0} - Y_2^{p,0} = \tilde{c}, \qquad \sum_{s=0}^{p} a_s^p[(-1)^p Y_1^{s,p-s} + Y_2^{p-s,s}] = \tilde{c}$$

with $a_0^p = (p-1)(p-2)^2$, $a_1^p = (p-2)^2$, $a_s^p = (-1)^s \prod_{j=p-1}^{p-3}(p-j)/j$ and some definite (but not the same) constants $\tilde{c}$. In [BaS1, Theorem 3] it is stated that, in the normal form, all nonresonant coefficients $Y_i^{j,k}$ vanish and the resonant ones satisfy the above equations.

In Example 1 there (following Thm. 3), the normal forms are more explicit (but still not unique): either

$$V_0 + \sum_{p\ge 3}[Y_1^{q_p,p-q_p} y_1^{q_p} y_2^{p-q_p} + Y_1^{p,0} y_1^p]\partial_{y_1} + \sum_{p\ge 3} Y_1^{r_p,p-r_p} y_1^{r_p} y_2^{p-r_p}\partial_{y_2}$$

(where $q_p \in \{0,\dots,p-1\}$ and $r_p \in \{0,\dots,p\}$ are subject to the restriction $r_p = 0$ if $q_p > 1$), or

$$V_0 + \sum_{p\ge 3}[Y_1^{0,p} y_2^p + Y_1^{q_p,p-q_p} y_1^{q_p} y_2^{p-q_p} + Y_1^{p,0} y_1^p]\partial_{y_1}$$

(where $q_p \in \{1,\dots,p-1\}$).

Similar nonexplicit and nonunique statements are formulated in all other cases studied by V. Basov with coauthors.

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