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FUNCTIONAL ANALYSIS

On some geometric properties of Banach spaces of continuous functions on separable compact lines

by

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Summary. We study properties of Banach spaces $C(L)$ of all continuous scalar (real or complex) functions on compact lines L. First we show that if L is a separable compact line, then for every closed linear subspace X of $C(L)$ with separable dual the quotient space $C(L)/X$ possesses a sequence of continuous linear functionals separating its points. Next we show that for any compact line L the space $C(L)$ contains no subspace isomorphic to a $C(K)$ space where K is a separable nonmetrizable scattered compact Hausdorff space with countable height.

1. Introduction. A totally (= linearly) ordered set equipped with the order topology which is a compact space is called a *compact line*. We investigate Banach spaces $C(L)$ of all continuous scalar functions on compact lines L. Properties of those spaces have been studied by many authors. Locally uniformly convex and Kadec renorming of $C(L)$ was investigated in [7] and [8]. Properties of real continuous functions on scattered compact lines were studied in [11]. Correa and Tausk [2] showed that for any compact line L each subspace of $C(L)$ isomorphic to c_0 is complemented.

The present paper is devoted to showing that for every separable compact line L and for every closed linear subspace X of $C(L)$ with separable dual the quotient space $C(L)/X$ possesses a sequence of continuous linear functionals separating its points and in consequence:

(1) $C(L)/X$ has no subspace isomorphic to a nonseparable WCG Banach space,

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(2) every weakly compact subset of $C(L)/X$ is separable,
(3) $(C(L)/X)^*$ is separable in the $*$weak topology,
(4) for any compact line M the space $C(M)$ contains no subspace isomorphic to a $C(K)$ space where K is a separable nonmetrizable scattered compact Hausdorff space with countable height.

The assumption of separability of L cannot be omitted in our theorem. For every nonseparable compact line L and for every closed separable linear subspace X of $C(L)$ the quotient space $C(L)/X$ does not possess a sequence of continuous linear functionals separating points. This follows from the fact that every Radon measure on any compact line has separable support. If a compact Hausdorff space K is separable, then the evaluation functionals at points of a countable dense subset of K form a countable set of continuous linear functionals separating the points of $C(K)$. However it is not true in general that a quotient space $C(K)/X$ has the same property. For instance, if $K = \mathbb{L}$, the two-arrows space, then K is a separable compact line, $C(K)$ is isometrically isomorphic to the Banach space $D(0,1)$, and $D(0,1)/C([0,1])$ does not possess a sequence of continuous linear functionals separating its points. The latter is a consequence of the fact that $D(0,1)/C([0,1])$ is isomorphic to $c_0([0,1])$ (see [3, Example 2]). Recall that the Banach space $D(0,1)$ consists of all scalar functions on $[0,1]$ that are right continuous at each point of $[0,1)$ and left continuous at 1 with a left-hand limit at each point of $(0,1]$, and it is equipped with the supremum norm. The present author [10] studied the properties of the c_0-sum $(\bigoplus_{n=2}^{\infty} C(\mathbb{L}^n))_{c_0}$. Our main result for $C(\mathbb{L})$ may be deduced from [10, Thm. 3.1].

The paper is divided into three sections. Properties of compact lines and continuous functions on these spaces are gathered in the first part of the second section. Properties of countable intersections of hyperplanes in Banach spaces are investigated in the second part of that section. The main results of the paper are presented in the third section.

2. Preliminaries. The Banach space of all continuous scalar (real or complex) functions on a compact Hausdorff space K with the supremum norm is denoted by $C(K)$. The topological dual of a Banach space X is denoted by X^*. For a subset Y of a Banach space X, let $Y^\perp$ denote the subspace $\{x^* \in X^* : x^*(Y) = 0\}$ of X^*. The linear hull, the closed hull in the norm topology and the closed hull in the $*$weak topology of a subset A of a Banach space X are denoted by $\mathrm{lin}(A)$, $\overline{A}$ and $\overline{A}^{*w}$, respectively. For a nonempty set Γ, the Banach space (under the supremum norm) of all scalar functions f on Γ such that for every $\eta > 0$ the set $\{\gamma \in \Gamma : |f(\gamma)| > \eta\}$ is finite is denoted by $c_0(\Gamma)$. The Banach space of all scalar bounded sequences equipped with the supremum norm is denoted by l_∞. The cardinality of a set A is denoted by $\mathrm{card}(A)$.

Let L be a compact line. For every $v \in L \setminus \{\min(L)\}$, $w \in L \setminus \{\max(L)\}$ and $s, t \in L$ with $s < t$ the sets $\{u \in L : u < v\} = [\min(L), v)$, $\{u \in L : u > w\} = (w, \max(L)]$ and $(s,t) = \{u \in L : s < u < t\}$ are called *open intervals*. The family of open intervals forms a base of the topology of L. Every nonempty closed subset of L has the minimum and maximum, which are members of the set (see [6] or [4, p. 53]). The closure of a subset A of L is denoted by $\overline{A}$. For every $u \in L$, we denote

$$u_- = \max \overline{\{v \in K : v < u\}} \quad \text{and} \quad u_+ = \min \overline{\{v \in K : v > u\}}.$$

We set $\min(L)_- = \min(L)$ and $\max(L)_+ = \max(L)$. Note that if for some $t \in L$ we have $t_- \neq t$, then the interval (t_-, t) is an empty set and $(t_-)_+ = t$. Similarly if $t_+ \neq t$, then $(t, t_+) = \emptyset$ and $(t_+)_- = t$.

Proposition 2.1. *Let L be a compact line.*

(a) *For every continuous function $f : L \to \mathbb{C}$ and $\varepsilon > 0$ the set*

$$\{t \in L : |f(t_+) - f(t)| > \varepsilon\}$$

is finite.

(b) *If the set $\{t \in L : t_+ \neq t\}$ is uncountable, then L is nonmetrizable.*

(c) *If there exists a subset A of L such that $\mathrm{card}(\overline{A}) > \mathrm{card}(A)$, then*
   (1) *there exists a nondecreasing continuous surjection $f : L \to [0,1]$,*
   (2) *there exists a countable subset B of L with $\overline{B}$ uncountable.*

Parts (a) and (b) are well known. Part (c) follows from [11, Thm. 1] and considerations in that paper. We present a short proof of (c) for completeness.

Proof of (c). It is clear that A is infinite. Suppose that $\mathrm{card}(A) = \gamma$. Let

$$\Pi = \{k/2^n : 0 \le k \le 2^n,\ k, n \in \mathbb{N} \cup \{0\}\}.$$

First we show that for every $a \in \Pi$ there exists $t_a \in \overline{A}$ such that $t_a < t_b$ if $a < b$. Let

$$D = \{s \in \overline{A} : \mathrm{card}([\min(L), s) \cap \overline{A}) \le \gamma\},$$
$$E = \{s \in \overline{A} : \mathrm{card}((s, \max(L)] \cap \overline{A}) \le \gamma\}.$$

Let $v = \max \overline{D}$ and $w = \min \overline{E}$. It is clear that $v, w \in \overline{A}$. By the Hessenberg theorem ($\gamma \cdot \gamma = \gamma$) we have

$$\mathrm{card}\Big(\bigcup_{s \in A,\, s<v} [\min(L), s) \cap \overline{A}\Big) \le \gamma, \qquad \mathrm{card}\Big(\bigcup_{s \in A,\, s>w} (s, \max(L)] \cap \overline{A}\Big) \le \gamma.$$

If $\max \overline{\{s \in A : s < v\}} < v$, then $v \in D$. If $\max \overline{\{s \in A : s < v\}} = v$, then $[\min(L), v) \cap \overline{A} = \bigcup_{s \in A,\, s<v} [\min(L), s) \cap \overline{A}$ and also $v \in D$. If

$\min\overline{\{s \in A : s > w\}} > w$, then $w \in E$. If $\min\overline{\{s \in A : s > w\}} = w$, then $(w, \max(L)] \cap \overline{A} = \bigcup_{s\in A,\, s>w}(s, \max(L)] \cap \overline{A}$ and also $w \in E$. Since $\operatorname{card}(\overline{A}) > \gamma$, for every $u \in (v, w) \cap \overline{A}$ we have $\operatorname{card}((v, u) \cap \overline{A}) > \gamma$ and $\operatorname{card}((u, w) \cap \overline{A}) > \gamma$. We set $t_0 = v$ and $t_1 = w$. Let $t_{1/2}$ be any element of $(v, w) \cap \overline{A}$. Continuing the procedure we find a set $\{t_a : a \in \Pi\}$ with the desired property. Let

$$f(x) = \begin{cases} \inf\{a \in \Pi : x \in [\min(L), t_a)\} & \text{if } x < t_1, \\ 1 & \text{if } x \geq t_1. \end{cases}$$

By the Urysohn Lemma (see [4, p. 43]) the function f is continuous. For every $a \in \Pi$, we have $t_a \in \bigcap_{b\in\Pi,\, b>a}[\min(L), t_b)$ and $t_a \notin \bigcup_{b\in\Pi,\, b\leq a}[\min(L), t_b)$. Hence $f(t_a) = a$ for every $a \in \Pi$. Since L is a compact space, $f(L) = [0, 1]$. It is clear that $\overline{\{t_a : a \in \Pi\}}$ has cardinality continuum. ∎

The *support* of a Radon measure μ on a compact Hausdorff space K is the smallest closed subset M of K such that $|\mu|(K \setminus M) = 0$ where $|\mu|$ is the variation of μ.

PROPOSITION 2.2. *Let L be a compact line.*

(a) *Every Radon measure on L has separable support.*
(b) *If X is a closed linear subspace of $C(L)$ which is isomorphic to a subspace of l_∞, then there exists a separable and closed subset M of L such that the restriction operator $R : C(L) \to C(M)$ given by the formula*

$$R(f) = f|_M \quad \textit{for every } f \in C(L)$$

is an isomorphism on X.

Part (a) is known but for completeness we present its proof.

Proof. (a) It is enough to show that any probability Radon measure μ on L with $\mu(\{t\}) = 0$ for every $t \in L$ has separable support. Suppose that μ has these properties. Let $f : L \to [0, 1]$ be given by

$$f(t) = \mu([\min(L), t]).$$

It is clear that f is a nondecreasing function and $\{0, 1\} \subset f(L)$.

Let $a \in (0, 1)$. Suppose that $a \notin f(L)$. Let $A = f^{-1}([0, a))$ and $B = f^{-1}((a, 1])$. Then $A \cup B = L$. Let $v = \max\overline{A}$. If $v \in A$, then $A = [\min(L), v]$. If $v \notin A$, then $v \in B$ and $A = [\min(L), v)$. Indeed, otherwise there exists $t \in B$ such that $t < v$; since $v \in \overline{A}$, the set $(t, v) \cap A$ is nonempty; since f is nondecreasing, $f(t) < a$, a contradiction.

This shows that either $A = [\min(L), v]$ and $B = (v, \max(L)]$, or $A = [\min(L), v)$ and $B = [v, \max(L)]$. Consider the first case. For every compact subset K of B we have $K \subset [\min(K), \max(L)] \subset B$ and $\mu(K) \leq 1 - a$. Since μ is a Radon measure, we have

$$\mu(B) = \sup\{\mu(K) : K \subset B,\ K \text{ compact}\} \leq 1 - a.$$

Hence

$$1 = \mu(L) = \mu(A) + \mu(B) < a + 1 - a = 1,$$

a contradiction. This shows that $f(L) = [0,1]$. Considerations in the second case are similar.

For every $q \in \mathbb{Q} \cap [0,1]$, let $t_q \in L$ be such that $f(t_q) = q$. Let $K = \overline{\{t_q : q \in \mathbb{Q} \cap [0,1]\}}$. The set $L \setminus K$ is a union of open intervals with endpoints in K. Let $(t_1, t_2) \subset L \setminus K$ be an interval such that $t_1, t_2 \in K$. If $\mu((t_1, t_2)) > \varepsilon > 0$, then $(f(t_1), f(t_1) + \varepsilon) \not\subset f(L)$. This shows that $\mu((t_1, t_2)) = 0$. Since μ is a Radon measure, $\mu(L \setminus K) = 0$.

Part (b) is a straightforward consequence of (a). ∎

The most important example of a separable nonmetrizable compact line is the two-arrows space $\mathbb{L} = \{(x,0) : 0 < x \le 1\} \cup \{(x,1) : 0 \le x < 1\}$. The order on $\mathbb{L}$ is defined by $(x,p) < (y,r)$ if either $x < y$, or $x = y$ and $p < r$. Ostaszewski [12] showed that a totally ordered set K equipped with the order topology is compact and separable if and only if K is order-isomorphic to a subset L of $\mathbb{L}$ such that $\{s \in [0,1] : (s,0) \in L\}$ is a closed subset of $[0,1]$, and $(s,0) \in L$ whenever $(s,1) \in L$.

We will need the following well known facts on separable compact lines.

PROPOSITION 2.3. *Let L be a separable compact line.*

(a) *L is first countable.*

(b) *If X is a subset of $C(L)$ such that the set*

$$U = \{t \in L : \text{there exists } f \in X \text{ such that } f(t_+) \ne f(t)\}$$

is countable, then X is separable.

(c) *The linear operator $J : C(L) \to c_0(L)$ given by the formula*

$$J(f)(t) = f(t_+) - f(t) \quad \text{for every } t \in L$$

is a well defined continuous operator with separable kernel.

(d) *If L is uncountable, then there exists a nondecreasing continuous surjection $f : L \to [0,1]$.*

Proof. Let Q_0 be a countably dense subset of L such that

$$\{\min(L), \max(L)\} \subset Q_0.$$

Let $Q = \bigcup_{t \in Q_0} \{t_-, t, t_+\}$.

(a) Let $t \in L$. If $t_- = t = t_+$, then $\{(s,u) : s,u \in Q,\ s < t < u\}$ is a base of open neighbourhoods of t. If $t_- = t < t_+$, then $\{(s,t] : s \in Q,\ s < t\}$ is a base of open neighbourhoods of t. If $t_- < t = t_+$, then $\{[t,s) : s \in Q,\ s > t\}$ is a base of open neighbourhoods of t. If $t_- < t < t_+$, then $\{\{t\}\}$ is a base of open neighbourhoods of t.

(b) Since L is a compact Hausdorff space, for every $s,u \in Q$ with $s < u$ there exists $f_{s,u} \in C(L)$ such that $f_{s,u}(L) \subset [0,1]$, $f_{s,u}([\min(L), s]) = \{0\}$

and $f_{s,u}([u,\max(L)]) = \{1\}$. For every $s \in U$, let $g_s = \chi_{[s_+,\max(L)]}$. It is clear that $[s_+,\max(L)]$ is open and closed. Hence g_s is continuous for every $s \in U$. Let

$$F = \{f_{s,u} : s,u \in Q,\ s < u\} \cup \{g_s : s \in U\}.$$

Let $\mathcal{R}$ be an equivalence relation on L such that $v\mathcal{R}w$ if $f(v) = f(w)$ for every $f \in F$. Let $L_0 = L/\mathcal{R}$ with the quotient topology. It is clear that L_0 is Hausdorff, compact and metrizable. Let $\mathfrak{q} : L \to L_0$ be the quotient map.

Suppose that $f \in X$ and there exist $v, w \in L$ such that $v < w$ and $v\mathcal{R}w$ and $f(v) \neq f(w)$.

If the interval (v, w) contains at least two members of Q, then there exist $s, u \in Q$ such that $s, u \in (v, w)$, $s < u$, $f_{s,u}(v) = 0$ and $f_{s,u}(w) = 1$. Hence v and w are not $\mathcal{R}$-related. If (v, w) contains exactly one point $s \in Q$, then $w = s_+ \in Q$, $f_{s,w}(v) = 0$ and $f_{s,w}(w) = 1$. If $(v, w) = \emptyset$, then $w = v_+$, $v \in U$, $g_v(v) = 0$ and $g_v(w) = 1$.

Thus in each case above we have arrived at a contradiction. This shows that there exists $h \in C(L_0)$ such that $f = h \circ \mathfrak{q}$. Since L_0 is compact and metrizable, X is a subset of the separable subspace $\{h \circ \mathfrak{q} : h \in C(L_0)\}$ of $C(L)$.

(c) By Proposition 2.1(a) the operator J is well defined. It is easy to check that J is continuous. The separability of $\ker J$ easily follows from (b).

(d) This part follows from Proposition 2.1(c). ∎

We will need the following well known facts.

Proposition 2.4.

(a) *For a closed linear subspace Y of a Banach space X the following conditions are equivalent:*

  (1) *there exists a sequence $(x_n^*) \subset X^*$ such that $Y = \bigcap_{n=1}^{\infty} \ker x_n^*$,*
  (2) *there exists a sequence $(x_n^*) \subset (X/Y)^* = Y^\perp$ separating the points of X/Y.*

(b) *If X is a separable Banach space, then for every closed linear subspace Y of X there exists a sequence $(x_n^*) \subset X^*$ such that $Y = \bigcap_{n=1}^{\infty} \ker x_n^*$.*

(b′) *If Y is a closed linear subspace of a Banach space X such that X/Y is separable, then there exists a sequence $(x_n^*) \subset X^*$ such that $Y = \bigcap_{n=1}^{\infty} \ker x_n^*$.*

(c) *If for a Banach space X there exists a sequence $(x_n^*) \subset X^*$ separating the points of X, then there exists an injective continuous linear operator $S : X \to l_\infty$.*

(c′) *If there exists a separable closed linear subspace Y of a Banach space X such that $Y = \bigcap_{n=1}^{\infty} \ker x_n^*$ for some sequence $(x_n^*) \subset X^*$, then there exists an injective continuous linear operator $S : X \to l_\infty$.*

(d) *If K is a nonseparable compact Hausdorff space such that every Radon measure on K has separable support, then for any sequence $(x_n^*) \subset C(K)^*$ the subspace $Y = \bigcap_{n=1}^\infty \ker x_n^*$ of $C(K)$ is nonseparable.*

(e) *If for a Banach space X there exists a sequence $(x_n^*) \subset X^*$ separating the points of X and $P : X \to X$ is a continuous projection, then $P(X) = \bigcap_{n=1}^\infty \ker(I - P)^*(x_n^*)$.*

Proof. Parts (a), (b), (b′), (d) and (e) are obvious. Part (c) easily follows from (c′).

(c′) Let $(x_n^*) \subset X^*$ be a sequence such that $Y = \bigcap_{n=1}^\infty \ker x_n^*$ and $\|x_n^*\| = 1$ for every n. Since Y is separable, there exists a sequence $(y_n^*) \subset Y^*$ separating the points of Y and such that $\|y_n^*\| = 1$ for every n. For every n, let $\tilde{y}_n^* \in X^*$ be an extension of y_n^* with $\|\tilde{y}_n^*\| = \|y_n^*\|$. It is easy to check that the linear operator $S : X \to l_\infty$ given by the formula $S(x) = (z_n^*(x))$ is continuous and injective where $z_{2n-1}^* = x_n^*$ and $z_{2n}^* = \tilde{y}_n^*$ for every n. ∎

The Johnson–Lindenstrauss space JL_2 (see [9]) is an example of a Banach space X, not isomorphic to a subspace of l_∞, which possesses a sequence $(x_n^*) \subset X^*$ separating the points of X. A Banach space X is called *weakly compactly generated* (WCG) if there exists a weakly compact subset A of X such that the linear hull of A is dense in X in the norm topology. We refer to [15] for more information about this class. For any nonempty set Γ the space $c_0(\Gamma)$ is a WCG Banach space. The spaces $C(\mathbb{L})$ and l_∞ are not WCG. We will need the following property of WCG Banach spaces.

PROPOSITION 2.5. *If X is a WCG Banach space and Z is a Banach space, and $S : X \to Z$ is a continuous linear operator such that $S(X)$ and* $\ker(S)$ *are separable subspaces of Z and X, respectively, then X is separable.*

Proof. Let A be a weakly compact subset of X whose linear hull is dense in X. Let B be a countable and dense subset of $S(A)$. Let C be a countable subset of A such that $S(C) = B$. Let D be a countable and dense subset of $\ker(S)$. Let $x \in A$. We find a sequence $(v_n) \subset C$ such that $(S(v_n))$ converges to $S(x)$ in the norm topology. By the Eberlein–Šmulian theorem there exists a subsequence (v_{k_n}) of (v_n) that weakly converges to $y \in A$. It is clear that $x - y \in \ker(S)$. Then there exists a sequence (w_n) in D such that $x - y = \lim_{n\to\infty} w_n$. Therefore $(v_{k_n} + w_n)$ converges to x in the weak topology. Thus we have shown that the closed linear hull of $C+D$ contains A. Therefore A and X are separable in the norm topology. ∎

COROLLARY 2.6.

(a) *If X is a nonseparable WCG Banach space, then for any sequence $(x_n^*) \subset X^*$ the subspace $Y = \bigcap_{n=1}^\infty \ker x_n^*$ of X is nonseparable.*

(b) *If L is a nonseparable compact line, then for any sequence $(x_n^*) \subset C(L)^*$ the subspace $Y = \bigcap_{n=1}^\infty \ker x_n^*$ of $C(L)$ is nonseparable.*

Proof. (a) Every weakly compact subset of l_∞ is separable in the norm topology. Therefore (a) is a straightforward consequence of Propositions 2.4(c′) and 2.5.

Part (b) is a consequence of Propositions 2.2(a) and 2.4(d). ∎

The class of nonseparable compact Hausdorff spaces K such that every Radon measure on K has separable support is quite wide. It contains for example all nonseparable Rosenthal compacts (see [5]).

3. Main results. Now we are ready to state our main result.

THEOREM 3.1. *Let L be a separable compact line. For a closed linear subspace X of $C(L)$ such that X^* is separable there exists a sequence $(x_n^*) \subset C(L)^*$ such that $X = \bigcap_{n=1}^{\infty} \ker x_n^*$.*

Proof. Let Q_0 be a countably dense subset of L with $\{\min(L), \max(L)\} \subset Q_0$. Let $Q_1 = \bigcup_{t\in Q_0} \{t_-, t, t_+\}$. Since X is a separable subspace of $C(L)$, there exists a countable subset Q_2 of L such that $Q_1 \subset Q_2$ and $J(X) \subset \overline{\text{lin}}\{e_t : t \in Q_2\}$ where J is the operator defined in Proposition 2.3 and e_t is the tth unit vector in $c_0(L)$. For every $t \in L \setminus Q_2$ we have the equality

$$\delta_{t_+} + X^\perp = \delta_t + X^\perp \in C(L)^*/X^\perp$$

where δ_u is the Dirac measure at u. The quotient space $C(L)^*/X^\perp$ is isometric to X^*. Hence it is separable. By the Bartle and Graves theorem (see [1, Thm. 4]) there exists a continuous map $S : C(L)^*/X^\perp \to C(L)^*$ such that $\mathfrak{q} \circ S(x) = x$ and $\|S(x)\| \le 2\|x\|$ for every $x \in C(L)^*/X^\perp$, where $\mathfrak{q} : C(L)^* \to C(L)^*/X^\perp$ is the quotient map. Hence the subset $\{S(\delta_t + X^\perp) : t \in L\}$ of $C(L)^*$ is separable in the norm topology.

For every $a, b \in Q_2$ with $a < b$, let $T_{a,b}$ be a countable subset of $[a, b]$ such that

$$\{S(\delta_t + X^\perp) : t \in [a, b]\} \subset \overline{\{S(\delta_t + X^\perp) : t \in T_{a,b}\}}.$$

Since $C(L)^*/X^\perp$ is separable, such sets exist. Let

$$T = \bigcup_{a,b\in Q_2,\, a<b} T_{a,b}.$$

For every $\sigma > 0$, let A_σ be the subset of all $t \in \{s \in L : s = s_+\}$ such that there exists $u_t \in Q_2$ such that $u_t > t$ and

$$\|S(\delta_t + X^\perp) - S(\delta_s + X^\perp)\| > \sigma$$

for every $s \in (t, u_t)$.

Suppose that A_σ is uncountable. Then there exists $u \in Q_2$ and an uncountable subset C of A_σ such that $u_t = u$ for every $t \in C$. It is clear that for every $s, t \in C$ with $s \ne t$ we have

$$\|S(\delta_t + X^\perp) - S(\delta_s + X^\perp)\| > \sigma.$$

Since $\{S(\delta_t + X^\perp) : t \in L\}$ is a separable subset of the metric space $C(L)^*$, this is impossible. Therefore A_σ is countable for every $\sigma > 0$. Hence also $\bigcup_{\sigma>0} A_\sigma$ is countable.

For every $\sigma > 0$, let B_σ be the subset of all $t \in \{s \in L : s = s_-\}$ such that there exists $v_t \in Q_2$ such that $v_t < t$ and

$$\|S(\delta_t + X^\perp) - S(\delta_s + X^\perp)\| > \sigma$$

for every $s \in (v_t, t)$. It is clear that also $\bigcup_{\sigma>0} B_\sigma$ is countable. Let $Q_3 = Q_2 \cup (\bigcup_{\sigma>0} B_\sigma) \cup \{t \in L : t_+ \in \bigcup_{\sigma>0} A_\sigma\}$. Note that

$$(*) \qquad \begin{aligned} \{t \in L : t \neq t_+\} &\subset \{t \in L : t = t_-, t \neq t_+\} \cup Q_1 \\ &\subset \{t \in L \setminus Q_3 : t = t_-,\ t \neq t_+,\ (t_+)_+ = t_+\} \cup Q_3 \cup \{t \in L : t_+ \in Q_1\}. \end{aligned}$$

For every $t \in \{s \in L : s = s_-,\ s \neq s_+,\ (s_+)_+ = s_+\} \setminus Q_3$ and $s, u \in L$ such that $s < t < t_+ < u$ we have

$$(**) \qquad \begin{aligned} S(\delta_t + X^\perp) &\in \overline{\{S(\delta_v + X^\perp) : v \in (s,t)\}}, \\ S(\delta_{t_+} + X^\perp) &\in \overline{\{S(\delta_v + X^\perp) : v \in (t_+,u)\}}. \end{aligned}$$

Let

$$U = \{S(\delta_t + X^\perp) - \delta_t : t \in T\}.$$

The set U is a countable subset of $X^\perp$. Suppose that $t \in \{s \in L : s = s_-,\ s \neq s_+,\ (s_+)_+ = s_+\} \setminus Q_3$. By $(**)$ and Proposition 2.3(a) there exist strictly increasing sequences $(a_n), (b_n) \subset Q_1$ and a sequence $(t_n) \subset L$ such that $a_n < b_n < a_{n+1}$ and $t_n \in T_{a_n,b_n}$ for every n, $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = t$, and

$$S(\delta_t + X^\perp) = S(\delta_{t_+} + X^\perp) = \lim_{n\to\infty} S(\delta_{t_n} + X^\perp)$$

where the limit is taken in the norm topology of $C(L)^*$. It is clear that (t_n) converges to t in L. Hence (δ_{t_n}) converges to δ_t in the $*$weak topology of $C(L)^*$. By $(**)$ and Proposition 2.3(a) there exist strictly decreasing sequences $(c_n), (d_n) \subset Q_1$ and a sequence $(s_n) \subset L$ such that $c_n > d_n > c_{n+1}$ and $s_n \in T_{d_n,c_n}$ for every n, $\lim_{n\to\infty} c_n = \lim_{n\to\infty} d_n = t_+$, and

$$S(\delta_t + X^\perp) = S(\delta_{t_+} + X^\perp) = \lim_{n\to\infty} S(\delta_{s_n} + X^\perp)$$

where the limit is taken in the norm topology of $C(L)^*$. It is clear that (s_n) converges to t_+ in L. Hence (δ_{s_n}) converges to δ_{t_+} in the $*$weak topology of $C(L)^*$. Thus we have shown that

$$S(\delta_t + X^\perp) - \delta_t \in \overline{U}^{*w} \quad \text{and} \quad S(\delta_{t_+} + X^\perp) - \delta_{t_+} \in \overline{U}^{*w}$$

for every $t \in \{s \in L : s = s_-,\ s \neq s_+,\ (s_+)_+ = s_+\} \setminus Q_3$. Therefore

$$\delta_t - \delta_{t_+} \in \overline{U}^{*w} - \overline{U}^{*w} \subset \overline{\mathrm{lin}(U)}^{*w}$$

for every $t \in \{s \in L : s = s_-,\ s \neq s_+,\ (s_+)_+ = s_+\} \setminus Q_3$.

Let $Y = \bigcap_{x^* \in U} \ker x^*$. Suppose that $f \in Y$. Then $y^*(f) = 0$ for every $y^* \in \overline{\operatorname{lin}(U)}^{*w}$. Hence $f(t) - f(t_+) = 0$ for every $t \in \{s \in L : s = s_-, s \neq s_+, (s_+)_+ = s_+\} \setminus Q_3$. By $(*)$ and Proposition 2.3(b), Y is a separable subspace of $C(L)$. An appeal to Proposition 2.4(b) completes the proof. ∎

Let us record the following consequences of Theorem 3.1 and Corollary 2.6.

Corollary 3.2. *Let L be a separable compact line. If X is a closed linear subspace of $C(L)$ such that X^* is separable, then*

(a) *$C(L)/X$ has no subspace isomorphic to a nonseparable WCG Banach space,*
(b) *every weakly compact subset of $C(L)/X$ is separable in the norm topology,*
(c) *$(C(L)/X)^*$ is separable in the $*$weak topology.*

Part (a) of our next result follows from the corollary above.

Corollary 3.3. *Let L be a separable compact line.*

(a) *If a Banach space X contains a closed linear subspace Y such that Y^* is separable and X/Y contains a subspace isomorphic to a nonseparable WCG Banach space, then X is not isomorphic to a subspace of $C(L)$.*
(b) *If Y is a closed linear subspace of a Banach space X and $S : X \to C(L)$ is a continuous linear operator such that X/Y is a WCG Banach space and $\overline{S(Y)}^*$ is separable, then S has separable range.*

Proof. (b) Let $\mathfrak{q} : X \to X/Y$ be the quotient map. Let $T : X/Y \to C(L)/\overline{S(Y)}$ be the continuous linear operator given by the formula

$$T(x + Y) = S(x) + \overline{S(Y)}.$$

According to Corollary 3.2(b) the operator T has separable range. Therefore so does $T \circ \mathfrak{q}$, and in consequence $\overline{S(X)}/\overline{S(Y)}$ is a separable subset of $C(L)/\overline{S(Y)}$. ∎

A topological space K is *scattered* if every subset A of K has a relatively isolated point in A. The *derived set* of $A \subset K$ is the set $A^{(1)}$ of all accumulation points of A in K. For an ordinal number α we define the *αth derived set* by transfinite induction in the following way: $A^0 = A$, $A^{(\alpha+1)} = (A^{(\alpha)})^1$, and $A^{(\alpha)} = \bigcap_{\beta<\alpha} A^{(\beta)}$ if α is a limit ordinal.

The pair (α, m) is called the *characteristic system* of a scattered compact Hausdorff space K if $K^{(\alpha+1)} = \emptyset$ and $K^{(\alpha)}$ is a nonempty finite set with m elements (see [13]). The ordinal α is also called the *height* of K.

$C(K)$ spaces for scattered compact spaces K were investigated by many authors (see [15]). The Johnson–Lindenstrauss space JL_0 is an example of a Banach space which is isometrically isomorphic to a $C(K_1)$ space where K_1

is a separable nonmetrizable scattered compact space such that $K_1 \setminus K_1^{(1)}$ is countable, $K_1^{(1)} \setminus K_1^{(2)}$ is uncountable and $K_1^{(2)}$ is a singleton (see [14]). From Proposition 2.3(d) it follows that no compact line contains a subset homeomorphic to a separable nonmetrizable scattered compact space. On the other hand, if K is a closed subset of a compact line L, then $C(K)$ is isomorphic to a subspace of $C(L)$ (see [7, Lemma 3.4]). Our next result gathers together these two facts.

Corollary 3.4. *Let L be a compact line. If K is a separable scattered compact Hausdorff space with characteristic system (α, m) where α is a countable ordinal, and the space $C(K)$ is isomorphic to a subspace of $C(L)$, then K is countable.*

Proof. Since K is a separable compact Hausdorff space, the space $C(K)$ is isomorphic to a subspace of l_∞. According to Proposition 2.2(b) the space $C(K)$ is isomorphic to a subspace of $C(M)$ where M is a separable compact line.

Suppose that K is uncountable. Since K is separable, the set $K \setminus K^{(1)}$ is countable. Since α is a countable ordinal, there exists an ordinal $\beta < \alpha$ such that $K^{(\beta)} \setminus K^{(\beta+1)}$ is uncountable. Let γ be the smallest ordinal such that $K^{(\gamma)} \setminus K^{(\gamma+1)}$ is uncountable. Let $Y = \{f \in C(K) : f|_{K^{(\gamma)}} = 0\}$. It is easy to see that

$$\|f + Y\| = \sup\{|f(t)| : t \in K^{(\gamma)}\}$$

for every $f \in C(K)$, and the quotient space $C(K)/Y$ is isometrically isomorphic to $C(K^{(\gamma)})$. The set $K^{(\gamma)} \setminus K^{(\gamma+1)}$ is uncountable and each of its points is isolated in $K^{(\gamma)}$. Therefore $C(K^{(\gamma)})$ contains a subspace isomorphic to $c_0(K^{(\gamma)} \setminus K^{(\gamma+1)})$. Consequently, $C(K)/Y$ contains a subspace isomorphic to a nonseparable WCG Banach space. Let $T = (K \setminus K^{(\gamma)}) \cup \{\infty\}$. Let $\mathfrak{q} : K \to T$ be given by the formula

$$\mathfrak{q}(t) = \begin{cases} t & \text{if } t \in K \setminus K^{(\gamma)}, \\ \infty & \text{if } t \in K^{(\gamma)}. \end{cases}$$

We equip T with the strongest topology for which the map $\mathfrak{q}$ is continuous (the quotient topology). Since $K^{(\gamma)}$ is a closed subset of K, the space T is Hausdorff. Now it is clear that T is a compact countable scattered Hausdorff space. It is clear that Y is isomorphic to a closed subspace of $C(T)$ consisting of functions that vanish at ∞. Therefore Y^* is a separable space. But this contradicts Corollary 3.3(a). ∎

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