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ON SOME GENERALIZATION OF SPECIAL AFFINE HYPERSPHERES

BY

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Abstract. We generalize the notion of special affine hypersphere. We study basic properties of such hyperspheres and give several examples. We also give a method of constructing higher-dimensional special hyperspheres using lower-dimensional examples.

1. Introduction. O. Baues and V. Cortés [BC] studied affine hypersurfaces equipped with an almost complex structure. They proved that every simply connected special Kähler manifold [F] can be realized in a canonical way as an improper affine hypersphere. In 2006 V. Cortés together with M.-A. Lawn and L. Schäfer [CLS] proved a similar result for special para-Kähler manifolds [CMMS]. Such hyperspheres were called by them *special affine hyperspheres*. In both cases an important role was played by the Kählerian (resp. para-Kählerian) 2-form ω.

The above results were for this author a motivation for a generalization of special affine hyperspheres. In the rest of the paper we refer to generalized special affine hyperspheres simply as *special hyperspheres*, whereas the spheres introduced in [BC] and [CLS] are called complex and para-complex special affine hyperspheres, respectively.

Recall [BC] that an improper affine hypersphere $f\colon M \to \mathbb{R}^{2n+1}$ is called a *complex special affine hypersphere* if on M there exists an *h-antisymmetric* almost complex structure J (i.e. $h(JX,Y) = -h(X,JY)$) such that the induced symplectic form ω is ∇-*parallel* (i.e. $\nabla\omega = 0$). Similarly [CLS] an improper affine hypersphere $f\colon M \to \mathbb{R}^{2n+1}$ is called a *para-complex special affine hypersphere* if on M there exists an h-antisymmetric almost para-complex structure J such that the induced symplectic form ω is ∇-parallel. In [BC] and [CLS] it was shown that an almost (para-)complex structure for a (para-)complex special affine hypersphere must be parallel relative to the Levi-Civita connection $\widehat{\nabla}$ of h. Thus in both cases we have in particular

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$\widehat{\nabla}\omega = 0$. This observation was the main motivation for the definition of a (generalized) special affine hypersphere.

An improper affine hypersphere $f\colon M \to \mathbb{R}^{2n+1}$ is called a (*generalized*) *special affine hypersphere* if there exists an almost symplectic structure ω on M such that $\nabla\omega = 0$ and $\widehat{\nabla}\omega = 0$, where ∇ is the Blaschke connection and $\widehat{\nabla}$ is the Levi-Civita connection of the Blaschke metric h. Every symplectic form satisfying the above conditions is called a *special symplectic form*.

In Section 2 we briefly recall the basic formulas of affine differential geometry. We also recall some basic definitions from symplectic geometry that we use.

Section 3 contains the main results of this paper. One of them gives equivalent conditions for an affine hypersphere to be special. In particular we prove that for a special hypersphere the eigenvalues of J_ω must be constant. We also prove that when the second fundamental form is positive definite, a special hypersphere must be a complex special hypersphere. In the 2-dimensional case improper affine hyperspheres are either special complex or special para-complex. We show that this is not true in higher dimensions: we give examples of special affine hyperspheres that are neither complex nor para-complex. Finally, we present a method of construction of higher-dimensional special hyperspheres using lower-dimensional examples.

Note that throughout this paper, the symbol ∇A^2, where A is a tensor of type $(1,1)$, should be understood as $\nabla(A^2)$ if not stated otherwise.

2. Preliminaries. We briefly recall the basic formulas of affine differential geometry. For more details, we refer to [NS]. Let $f\colon M \to \mathbb{R}^{n+1}$ be an orientable connected differentiable n-dimensional hypersurface immersed in the affine space $\mathbb{R}^{n+1}$ equipped with its usual flat connection D. Then for any transversal vector field ξ we have

$$\mathrm{D}_X f_*Y = f_*(\nabla_X Y) + h(X,Y)\xi, \tag{2.1}$$

$$\mathrm{D}_X \xi = -f_*(SX) + \tau(X)\xi, \tag{2.2}$$

where X, Y are vector fields tangent to M. It is known that ∇ is a torsion-free connection, h is a symmetric bilinear form on M, called the *second fundamental form*, S is a tensor of type $(1,1)$, called the *shape operator*, and τ is a 1-form, called the *transversal connection form*. The vector field ξ is called *equiaffine* if $\tau = 0$. When $d\tau = 0$ the vector field ξ will be called *locally equiaffine*.

When h is non-degenerate then h defines a pseudo-Riemannian metric on M. If h is non-degenerate, then we say that the hypersurface or the hypersurface immersion is *non-degenerate*. In this paper we always assume that f is non-degenerate. We have the following

THEOREM 2.1 ([NS], fundamental equations). *For any transversal vector field ξ, the induced connection ∇, the second fundamental form h, the shape operator S, and the 1-form τ satisfy the following equations:*

$$R(X,Y)Z = h(Y,Z)SX - h(X,Z)SY, \tag{2.3}$$

$$(\nabla_X h)(Y,Z) + \tau(X)h(Y,Z) = (\nabla_Y h)(X,Z) + \tau(Y)h(X,Z), \tag{2.4}$$

$$(\nabla_X S)(Y) - \tau(X)SY = (\nabla_Y S)(X) - \tau(Y)SX, \tag{2.5}$$

$$h(X,SY) - h(SX,Y) = 2d\tau(X,Y). \tag{2.6}$$

Equations (2.3)–(2.6) are called the *equations of Gauss*, *Codazzi for h*, *Codazzi for S*, and *Ricci*, respectively.

Let $f\colon M \to \mathbb{R}^{n+1}$ be an affine hypersurface with a transversal vector field ξ. On M we define a tensor field C of type $(0,3)$ by the formula

$$C(X,Y,Z) := (\nabla_X h)(Y,Z) + \tau(X)h(Y,Z)$$

for all $X,Y,Z \in \mathcal{X}(M)$. The tensor field C is called the *cubic form*. It is easy to verify that C is symmetric in all three variables.

On the space $\mathbb{R}^{n+1}$ we have the standard volume form determined by the determinant $\det[\cdot]$. If $f\colon M \to \mathbb{R}^{n+1}$ is an affine hypersurface with a transversal vector field ξ, then on M we define the induced volume form by

$$\theta(X_1,\ldots,X_n) := \det[f_*X_1,\ldots,f_*X_n,\xi]$$

for every $X_1,\ldots,X_n \in \mathcal{X}(M)$. The following theorem holds:

THEOREM 2.2 ([NS]). *If $f\colon M \to \mathbb{R}^{n+1}$ is an affine hypersurface with a transversal vector field ξ then*

$$\nabla_X\theta = \tau(X)\theta \quad \text{for every } X \in TM.$$

In particular, the following conditions are equivalent:

(1) $\nabla\theta = 0$,
(2) $\tau = 0$ (*that is, ξ is equiaffine*).

A Blaschke hypersurface f is called an *affine hypersphere* if $S = \lambda I$, where $\lambda = \text{const}$. If $\lambda = 0$, then f is called an *improper affine hypersphere*, and if $\lambda \neq 0$, it is a *proper affine hypersphere.*

Let (M,g) be a pseudo-Riemannian manifold, and ∇ an affine connection on M. We can define another affine connection ∇^* by the formula

$$Xg(Y,Z) = g(\nabla_X Y,Z) + g(Y,\nabla^*_X Z).$$

We call ∇^* the *conjugate connection* of ∇ with respect to g.

Let ω be a non-degenerate 2-form on a manifold M. The form ω is called an *almost symplectic structure.* It is easy to see that if a manifold M admits some almost symplectic structure then M is orientable and of even dimension. The structure ω is called a *symplectic structure* if it is almost symplectic and additionally satisfies $d\omega = 0$. The pair (M, ω) is called an (*almost*) *symplectic manifold* if ω is an (almost) symplectic structure on M.

Recall [AP] that an affine connection ∇ on an almost symplectic manifold (M, ω) is called an *almost symplectic connection* if $\nabla\omega = 0$. An affine connection ∇ on an almost symplectic manifold (M, ω) is called a *symplectic connection* if it is almost symplectic and torsion-free.

If A is a tensor of type $(1, 1)$ on M, and ∇ is a connection on M, we define a new tensor of type $(1, 2)$ on M by the formula

$$d^{\nabla}A(X, Y) := (\nabla_X A)Y - (\nabla_Y A)X. \tag{2.7}$$

Moreover, if A is non-singular, we can define a new connection ∇^A on M as follows:

$$\nabla^A_X Y := A^{-1}\nabla_X(AY) = \nabla_X Y + A^{-1}(\nabla_X A)Y. \tag{2.8}$$

Let T^A and R^A be the torsion and curvature tensors of ∇^A, respectively. The following relations between the torsion and curvature tensors for ∇ and ∇^A are well known.

THEOREM 2.3. *For any non-singular tensor A of type $(1, 1)$ on (M, ∇) we have*

$$T^A(X, Y) = A^{-1}(d^{\nabla}A(X, Y)) + T(X, Y), \tag{2.9}$$

$$R^A(X, Y)Z = A^{-1}(R(X, Y)AZ). \tag{2.10}$$

Let (M, g) be a Riemannian manifold with a positive definite metric g and let $\mathcal{D}$ be a C^∞ distribution on M. We say that $\mathcal{D}$ is *parallel* if for any vector fields $X \in \mathcal{X}(M)$ and $Y \in \mathcal{D}$ the field $\widehat{\nabla}_X Y$ is in $\mathcal{D}$, where $\widehat{\nabla}$ is the Levi-Civita connection of g. If on M there exists a parallel distribution $\mathcal{D}$ of dimension $0 < \dim \mathcal{D} < \dim M$, then M is called *reducible*. If such a distribution does not exist, we call M *irreducible*.

From the de Rham decomposition theorem [KN] we know that a complete and simply connected Riemannian manifold (with positive definite metric) is isometric to a product of Riemannian manifolds $M_0 \times M_1 \times \cdots \times M_k$, where M_0 is a Euclidean space and $M_1, \ldots, M_k$ are complete simply connected and irreducible Riemannian manifolds. Moreover, when the manifold is neither simply connected nor complete, the theorem is still true locally. That is, every point $x \in M$ has a neighborhood isometrically equivalent to a product of a Euclidean space and irreducible Riemannian manifolds.

When the manifold M is irreducible, we have the following theorem:

THEOREM 2.4 ([Bi]). *Let (M, g) be an irreducible Riemannian manifold with positive definite metric g and let A be a parallel, relative to the Levi-Civita connection, tensor of type $(1, 1)$ on M. Then either $A = \alpha I$ or $A = \alpha I + \beta J$, where $\alpha, \beta \in \mathbb{R}$, $\beta \neq 0$ and J is a complex structure on M (dependent on A) such that $(J, g, \widehat{\nabla})$ is a Kählerian structure on M.*

The next theorem shows that a parallel tensor of type $(1,1)$ preserves the de Rham decomposition.

THEOREM 2.5 ([Bi]). *Let M be a complete simply connected Riemannian manifold and let $M_0 \times M_1 \times \cdots \times M_k$ be the de Rham decomposition for M. Let A be a $(1,1)$-tensor parallel relative to the Levi-Civita connection. Then $A = A_0 \times A_1 \times \cdots \times A_k$ where A_j is a parallel tensor of type $(1,1)$ on M_j for $j = 0, \dots, k$.*

REMARK 2.6. When the manifold is neither simply connected nor complete, the above theorem is still true locally.

Finally, we recall one more well known theorem related to eigenvalues and generalized eigenspaces of a $(1,1)$-tensor.

THEOREM 2.7. *Let A be a non-singular tensor of type $(1,1)$ on a manifold M. If ∇ is a torsion-free connection on M and $\nabla A = 0$, then*

(1) *A has constant eigenvalues,*
(2) *the generalized eigenspaces are ∇-parallel involutive distributions on M.*

3. Special affine hyperspheres. Let (M, ω) be an almost symplectic manifold and let h be a pseudo-Riemannian metric on M. On the manifold M there exists exactly one tensor J_ω of type $(1,1)$ such that $\omega(X, J_\omega Y) = h(X,Y)$. Since ω is non-degenerate, it is clear that J_ω is non-singular. Moreover, J_ω is h-antisymmetric (i.e. $h(X, J_\omega Y) = -h(J_\omega X, Y)$). Thus, there is one-to-one correspondence between almost symplectic structures on M and non-degenerate, h-antisymmetric $(1,1)$-tensors on M.

An improper affine hypersphere $f \colon M \to \mathbb{R}^{2n+1}$ is called a *special affine hypersphere* if there exists an almost symplectic structure ω on M such that $\nabla\omega = 0$ and $\widehat{\nabla}\omega = 0$, where ∇ is the Blaschke connection and $\widehat{\nabla}$ is the Levi-Civita connection of the Blaschke metric h. Every symplectic form satisfying the above conditions is called a *special symplectic form.*

First we recall two basic results from [S].

LEMMA 3.1 ([S]). *Let ω and h be an almost symplectic form and a pseudo-Riemannian metric on M, respectively. If ∇ is a connection on M, then*

$$(\nabla_X\omega)(Y, J_\omega Z) = (\nabla_X h)(Y,Z) - \omega(Y, (\nabla_X J_\omega)(Z)) \tag{3.1}$$

for all $X, Y, Z \in \mathcal{X}(M)$. In particular, if $\widehat{\nabla}$ is the Levi-Civita connection for h, then the above formula can be rewritten as follows:

$$(\widehat{\nabla}_X\omega)(Y, J_\omega Z) = -\omega(Y, (\widehat{\nabla}_X J_\omega)(Z)), \tag{3.2}$$

thus $\widehat{\nabla}\omega = 0$ if and only if $\widehat{\nabla} J_\omega = 0$.

THEOREM 3.2 ([S]). *Let $f\colon M \to \mathbb{R}^{2n+1}$ be a non-degenerate affine hypersurface with a transversal vector field ξ and an almost symplectic form ω. The following conditions are equivalent:*

(1) $\nabla\omega = 0$;
(2) $d^\nabla J_\omega(X,Z) = \tau(Z)J_\omega X - \tau(X)J_\omega Z$;
(3) ∇ *and* ∇^J_ω *are* h*-conjugate;*
(4) $T^J_\omega(X,Z) = \tau(Z)X - \tau(X)Z$.

Now we give equivalent conditions for an improper affine hypersphere to be special.

THEOREM 3.3. *Let $f\colon M \to \mathbb{R}^{2n+1}$ be an improper affine hypersphere. Then the following conditions are equivalent:*

(1) f *is a special affine hypersphere;*
(2) *there exists a non-singular* h*-antisymmetric* $(1,1)$*-tensor* J *on* M *such that* $\widehat{\nabla}J = 0$ *and* $d^\nabla J = 0$;
(3) *there exists an almost symplectic structure* ω *on* M *such that* $\nabla\omega = 0$ *and* $\nabla J^2_\omega = 0$;
(4) *there exists a non-singular* h*-antisymmetric* $(1,1)$*-tensor* J *on* M *such that* $\nabla J^2 = 0$ *and* $d^\nabla J = 0$.

Proof. (1)⇒(2). If $f\colon M \to \mathbb{R}^{2n+1}$ is a special affine hypersphere then on M we have a tensor J_ω given by $\omega(X, J_\omega Y) = h(X,Y)$ for all $X, Y \in TM$, where h is the Blaschke metric. Of course J_ω is non-singular and h-antisymmetric. Moreover, as $\widehat{\nabla}\omega = 0$, we also have $\widehat{\nabla}J_\omega = 0$ (Lemma 3.1). Now it remains to note that Theorem 3.2 and $\tau = 0$ imply that $d^\nabla J_\omega = 0$.

(2)⇒(3). Define on M a tensor ω of type $(0,2)$ by

$$\omega(X,Y) := h(X, J^{-1}Y)$$

for all $X, Y \in TM$. Since J is h-antisymmetric and non-singular, ω is an almost symplectic form on M. From the definition of ω it easily follows that $J_\omega = J$. Now Theorem 3.2 and the condition $d^\nabla J = 0$ imply that $\nabla\omega = 0$. It is enough to show that $\nabla J^2_\omega = 0$. For this purpose, denote by ∇^* the connection h-conjugate to ∇. Then for all $X, Y, Z \in \mathcal{X}(M)$ we have

$$X(h(Y,Z)) = h(\nabla_X Y, Z) + h(Y, \nabla^*_X Z).$$

From the above we obtain

$$\begin{aligned} X(h(J_\omega Y, Z)) &= h(\nabla_X J_\omega Y, Z) + h(J_\omega Y, \nabla^*_X Z) \\ &= h(\nabla_X J_\omega Y, Z) - h(Y, J_\omega(\nabla^*_X Z)), \end{aligned} \tag{3.3}$$

$$\begin{aligned} X(h(Y, J_\omega Z)) &= h(\nabla_X Y, J_\omega Z) + h(Y, \nabla^*_X J_\omega Z) \\ &= -h(J_\omega(\nabla_X Y), Z) + h(Y, \nabla^*_X J_\omega Z). \end{aligned} \tag{3.4}$$

Adding (3.3) and (3.4) we get

$$0 = h((\nabla_X J_\omega)Y, Z) + h(Y, (\nabla^*_X J_\omega)Z). \tag{3.5}$$

Since ξ is equiaffine, $\widehat{\nabla}$ can be expressed as follows:

$$\widehat{\nabla}_X Y = \tfrac{1}{2}\nabla_X Y + \tfrac{1}{2}\nabla^*_X Y.$$

The above and (3.5) imply

$$\begin{aligned} h((\widehat{\nabla}_X J_\omega)Y, Z) &= \tfrac{1}{2}h((\nabla_X J_\omega)Y, Z) + \tfrac{1}{2}h((\nabla^*_X J_\omega)Y, Z) \\ &= \tfrac{1}{2}h((\nabla_X J_\omega)Y, Z) - \tfrac{1}{2}h(Y, (\nabla_X J_\omega)Z). \end{aligned} \tag{3.6}$$

Now using Lemma 3.1 and the relation $\nabla\omega = 0$ we get

$$\begin{aligned} (\nabla_X h)(Y, Z) = \omega(Y, (\nabla_X J_\omega)Z) &= h\big(Y, J_\omega^{-1}((\nabla_X J_\omega)Z)\big) \\ &= h\big(Z, J_\omega^{-1}((\nabla_X J_\omega)Y)\big), \end{aligned}$$

since $\nabla_X h$ is symmetric. Replacing Y by $J_\omega Y$ in the last equality we obtain

$$h(Y, (\nabla_X J_\omega)Z) = -h(J_\omega^{-1}\big((\nabla_X J_\omega)J_\omega Y), Z\big).$$

Using the above and (3.6) we get

$$\begin{aligned} h((\widehat{\nabla}_X J_\omega)Y, Z) &= \tfrac{1}{2}h\big(\nabla_X J_\omega Y - J_\omega(\nabla_X Y) + J_\omega^{-1}\nabla_X J_\omega^2 Y - \nabla_X J_\omega Y, Z\big) \\ &= \tfrac{1}{2}h(J_\omega^{-1}((\nabla_X J_\omega^2)Y), Z). \end{aligned}$$

Since h is non-degenerate, we have

$$(\widehat{\nabla}_X J_\omega)Y = \tfrac{1}{2}J_\omega^{-1}((\nabla_X J_\omega^2)Y). \tag{3.7}$$

By the assumption $\widehat{\nabla}J_\omega = 0$ the last equality thus implies $\nabla J_\omega^2 = 0$.

(3)⇒(4). It is enough to set $J := J_\omega$. Then $\nabla J^2 = 0$ follows directly from the assumption, and $d^\nabla J = 0$ is an immediate consequence of Theorem 3.2 and the fact that ξ is equiaffine.

(4)⇒(1). In a similar way to the proof of (2)⇒(3) one can define an almost symplectic structure ω by

$$\omega(X, Y) = h(X, J^{-1}Y).$$

Then $J_\omega = J$. Now in the same way as in (2)⇒(3) one may show that $\nabla\omega = 0$. It is enough to prove that $\widehat{\nabla}\omega = 0$. For this purpose note that since $\nabla\omega = 0$ and ξ is equiaffine, we have (3.7). Now, from $\nabla J^2 = 0$ we get $\widehat{\nabla}J = 0$, which by Lemma 3.1 is equivalent to $\widehat{\nabla}\omega = 0$. ∎

From Theorem 3.3 we easily get the following corollary.

COROLLARY 3.4. *If $f\colon M \to \mathbb{R}^{2n+1}$ is a special affine hypersphere, then for every $x \in M$ there exists a neighborhood U and a coordinate system on U, flat relative to the Blaschke connection ∇, such that both ω and J_ω^2 have constant coordinates in this system.*

Proof. Since f is an improper affine hypersphere, ∇ is flat. Thus for every point of M there exists a local flat coordinate system (relative to ∇). Now using Theorem 3.3 we get $\nabla\omega = 0$ and $\nabla J_\omega^2 = 0$, that is, both ω and J_ω^2 have constant coordinates in this local coordinate system. ∎

We also have

COROLLARY 3.5. *If $f\colon M \to \mathbb{R}^{2n+1}$ is a special affine hypersphere with a special symplectic form ω, then the tensor J_ω has constant eigenvalues.*

Proof. From Theorem 3.3 we have $\widehat{\nabla} J_\omega = 0$. Now, since $\widehat{\nabla}$ is torsion-free, the proof is an immediate consequence of the well known fact that if a $(1,1)$-tensor is parallel relative to a connection, then its eigenvalues must be constant. ∎

In [BC] one may find a remark that 2-dimensional improper affine spheres with positive definite Blaschke metric are complex special affine spheres. Below, for completeness, we state this result precisely and present a proof. We also take into account the case when the Blaschke metric is not positive definite.

THEOREM 3.6. *If $f\colon M \to \mathbb{R}^3$ is an improper affine sphere then f is special. Moreover, if h is definite (positive or negative), then f is a complex special affine sphere. When h is not definite, f is a para-complex special affine sphere.*

Proof. To show that f is special it is enough to set $\omega := \omega_h$ where ω_h is the volume form induced by the Blaschke metric h. Indeed, since $\dim M = 2$, ω is a symplectic structure on M. Moreover, since f is a Blaschke immersion, we also have

$$\nabla\omega = \nabla\omega_h = 0.$$

The condition $\widehat{\nabla}\omega = 0$ easily follows from the well known fact that the metric volume form ω_h is parallel relative to the Levi-Civita connection $\widehat{\nabla}$ for h.

To prove the second part of the theorem note that by Theorem 3.3 there exists an h-antisymmetric tensor J of type $(1,1)$ on M such that $\nabla J^2 = 0$. Now let $x \in M$ and let $\{e_1, e_2\}$ be an h-orthonormal basis for T_xM. Then there exist $\alpha, \alpha', \beta, \beta' \in \mathbb{R}$ such that

$$Je_1 = \alpha e_1 + \beta e_2 \quad \text{and} \quad Je_2 = \alpha' e_1 + \beta' e_2.$$

Since $h(Je_1, e_1) = 0$ and $h(Je_2, e_2) = 0$, we have $\alpha = 0$ and $\beta' = 0$. It follows that $J^2 e_1 = \beta\alpha' e_1$ and $J^2 e_2 = \beta\alpha' e_2$. In consequence, for every $W \in T_xM$ we have $J^2 W = \gamma W$, where γ is some real number. In general there exists a non-vanishing function $\Gamma \in C^\infty(M)$ such that

$$J^2 X = \Gamma X$$

for every $X \in \mathcal{X}(M)$. Since $\nabla J^2 = 0$, it follows that $\varGamma \equiv \text{const}$. Now we can define a new tensor J' of type $(1,1)$ on M by the formula

$$J' := \frac{1}{\sqrt{|\varGamma|}} J.$$

Then J' is an almost complex structure (when h is definite) or an almost para-complex structure (when h is indefinite), which is h-antisymmetric. Moreover, $\nabla \omega' = 0$ for the symplectic form ω' induced by J'. Finally, f is a complex (respectively, a para-complex) special affine sphere. ∎

The above theorem is not true in higher dimensions, as shown by the following example.

Example 3.7. Let $F\colon (0,\infty)^4 \to \mathbb{R}$ be given by

$$F(x,y,z,t) = \frac{4}{5 \cdot 2^{1/4}} (x^2 + y^2 + z^2 + t)^{5/4}.$$

Then

$$f\colon (0,\infty)^4 \ni (x,y,z,t) \mapsto (x,y,z,t,F(x,y,z,t)) \in \mathbb{R}^5$$

with the transversal vector field

$$\xi\colon (0,\infty)^4 \ni (x,y,z,t) \mapsto (0,0,0,0,1) \in \mathbb{R}^5$$

is an improper affine hypersphere which is not a special affine hypersphere.

Indeed, by straightforward computations we see that in the canonical basis $\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}, \frac{\partial}{\partial t}\}$ the second fundamental form h reads

$$h = \frac{1}{2} \frac{2^{3/4}}{(x^2 + y^2 + z^2 + t)^{3/4}} \cdot \begin{bmatrix} A_1 & xy & xz & \frac{1}{2}x \\ xy & A_2 & yz & \frac{1}{2}y \\ xz & yz & A_3 & \frac{1}{2}z \\ \frac{1}{2}x & \frac{1}{2}y & \frac{1}{2}z & \frac{1}{4} \end{bmatrix},$$

where

$$A_1 = 3x^2 + 2y^2 + 2z^2 + 2t,$$
$$A_2 = 3y^2 + 2x^2 + 2z^2 + 2t,$$
$$A_3 = 3z^2 + 2x^2 + 2y^2 + 2t.$$

Now it is easy to compute that $\det h = 1$, thus $\omega_h = \theta$, and so f is an improper affine hypersphere.

We will show that f is not a special affine hypersphere even locally, that is, there is no open subset $U \subset (0,\infty)^4$ such that $f|_U$ is special. Suppose that f is special on some open subset $U \subset (0,\infty)^4$. Then there exists an almost symplectic structure ω on U such that

$$\omega = \begin{bmatrix} 0 & \omega_{12} & \omega_{13} & \omega_{14} \\ -\omega_{12} & 0 & \omega_{23} & \omega_{24} \\ -\omega_{13} & -\omega_{23} & 0 & \omega_{34} \\ -\omega_{14} & -\omega_{24} & -\omega_{34} & 0 \end{bmatrix}$$

and $\omega_{ij} \in \mathbb{R}$ since $\nabla\omega = 0$ and the basis $\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}, \frac{\partial}{\partial t}\}$ is flat relative to ∇. Since ω is non-degenerate, we also have

$$\det\omega = (\omega_{12}\omega_{34} + \omega_{23}\omega_{14} - \omega_{13}\omega_{24})^2 \neq 0. \tag{3.8}$$

Moreover, $\nabla J_\omega^2 = 0$ implies that the matrix $J_\omega^2 = (\omega^{-1} \cdot h)^2$ has constant elements. By straightforward computations we check that the $(1,4)$ and $(2,4)$ components of the tensor J_ω^2 are

$$(J_\omega^2)_{14} = -\frac{\sqrt{2}}{4} \cdot \frac{(2x(\omega_{24}^2 + \omega_{34}^2) - 2y\omega_{14}\omega_{24} - 2z\omega_{14}\omega_{34} + \omega_{12}\omega_{24} + \omega_{13}\omega_{34})}{\det\omega\sqrt{x^2+y^2+z^2+t}},$$

$$(J_\omega^2)_{24} = \frac{\sqrt{2}}{4} \cdot \frac{(2y(\omega_{14}^2 + \omega_{34}^2) - 2x\omega_{14}\omega_{24} - 2z\omega_{34}\omega_{24} - \omega_{14}\omega_{12} + \omega_{34}\omega_{23})}{\det\omega\sqrt{x^2+y^2+z^2+t}}.$$

From the above formulas we obtain $\omega_{24} = \omega_{34} = 0$ and $\omega_{14} = 0$, respectively. In particular, $\det\omega = 0$, which contradicts (3.8).

When h is positive definite, we have the following theorems:

THEOREM 3.8. *Let $f\colon M \to \mathbb{R}^{2n+1}$ be a special affine hypersphere with a positive definite second fundamental form h. If (M,h) is irreducible, then f is a complex special affine hypersphere.*

Proof. Since f is special, there exists a symplectic structure ω on M such that $\widehat{\nabla}\omega = 0$ and $\nabla\omega = 0$. In particular $\widehat{\nabla}J_\omega = 0$. Now, using Theorem 2.4 we deduce that the tensor J_ω equals $\alpha I + \beta J$, $\alpha, \beta \in \mathbb{R}$, and $(J, g, \widehat{\nabla})$ is a Kählerian structure on M. Since J_ω is h-antisymmetric, we get $\alpha = 0$, so $J_\omega = \beta J$. The tensor J induces in a natural manner a symplectic form $\omega'(X,Y) = h(X,JY)$. In particular

$$\omega'(X,JY) = -h(X,Y) = -\omega(X,J_\omega Y) = -\beta\omega(X,JY)$$

for $X, Y \in TM$. The last equality implies that $\omega' = -\beta\omega$, thus $\nabla\omega' = 0$. ∎

THEOREM 3.9. *Let $f\colon M \to \mathbb{R}^{2n+1}$ be a special affine hypersphere with a positive definite second fundamental form h. Then for every point $x \in M$ there exists some neighborhood U such that $f|_U$ is a complex special affine hypersphere.*

Proof. By de Rham's theorem (local version), every point of the manifold M has a neighborhood isometrically equivalent to the product of a Euclidean space and irreducible Riemannian manifolds. Let $x \in M$ and $U =$

$U_0 \times \cdots \times U_k$ be the de Rham decomposition in some neighborhood U of x. We also have a decomposition of the metric h as a product of Riemannian metrics: $h = h_0 \oplus \cdots \oplus h_k$. Since f is special, there exists a symplectic structure ω on M such that $\widehat{\nabla}\omega = 0$ and $\nabla\omega = 0$. In particular we have $\widehat{\nabla} J_\omega = 0$. Now, Theorem 2.4, Remark 2.6 and Theorem 2.5 imply that the tensor J_ω can be expressed as a product

$$J_\omega = \beta_0 J_0 \times \cdots \times \beta_k J_k,$$

where $\beta_0, \ldots, \beta_i \in \mathbb{R}$ and (h_i, J_i) is a Kählerian structure on U_i for $i = 0, \ldots, k$. Without loss of generality we may assume that the U_i are ordered in such a way that $\beta_0^2 \leq \cdots \leq \beta_k^2$. Merging submanifolds U_i with equal β_i^2 we get $U = V_1 \times \cdots \times V_s$, $J_\omega = \bar{J}_1 \times \cdots \times \bar{J}_s$ and $h = \bar{h}_1 \oplus \cdots \oplus \bar{h}_s$. Furthermore $\bar{J}_1^2 = -\alpha_i^2$ for $i = 1, \ldots, s$ and $0 < \alpha_1^2 < \alpha_2^2 < \cdots < \alpha_s^2$.

Now we can define a new tensor J on U by the formula

$$J := \frac{1}{\alpha_1}\bar{J}_1 \times \cdots \times \frac{1}{\alpha_s}\bar{J}_s.$$

Directly from the definition it follows that $J^2 = -I$. Moreover, J is h-antisymmetric since $\bar{J}_i$ is $\bar{h}_i$-antisymmetric for $i = 1, \ldots, s$. By Theorem 3.2 we see that $d^\nabla J = 0$ if and only if $\nabla\omega' = 0$, where ω' is the symplectic structure induced by the almost complex structure J. Therefore, in order to show that $f|_U$ is a complex special affine hypersphere it is enough to prove that $d^\nabla J = 0$. Since $-\alpha_1^2, \ldots, -\alpha_s^2$ are the eigenvalues of J_ω^2 and all these eigenvalues are different, by Theorem 2.7 (since $\nabla J_\omega^2 = 0$) the spaces TV_i are ∇-parallel involutive distributions.

Now for any $X \in \mathcal{X}(V_i)$ and $Y \in \mathcal{X}(V_j)$ we get

$$\begin{aligned} d^\nabla J(X,Y) &= \nabla_X JY - \nabla_Y JX - J(\nabla_X Y) + J(\nabla_Y X) \\ &= \frac{1}{\alpha_j}\nabla_X \bar{J}_j Y - \frac{1}{\alpha_i}\nabla_Y \bar{J}_i X - \frac{1}{\alpha_j}\bar{J}_j(\nabla_X Y) + \frac{1}{\alpha_i}\bar{J}_i(\nabla_Y X) \\ &= \frac{1}{\alpha_j}(\nabla_X J_\omega)Y - \frac{1}{\alpha_i}(\nabla_Y J_\omega)X. \end{aligned}$$

If $i = j$ then the above implies

$$d^\nabla J(X,Y) = \frac{1}{\alpha_i} d^\nabla J_\omega(X,Y) = 0.$$

If $i \neq j$ then it is enough to show that $(\nabla_X J_\omega)Y = 0$. For this purpose, note that if $\mathfrak{B}_i$ and $\mathfrak{B}_j$ are the bases related to local coordinate systems on V_i and V_j respectively, then for $X \in \mathfrak{B}_i$ and $Y \in \mathfrak{B}_j$ we have $\nabla_X Y = \nabla_Y X = 0$ since $\nabla_X Y \in TV_j$, $\nabla_Y X \in TV_i$ and $[X, Y] = 0$. The above implies that

$$(\nabla_X J_\omega)Y = \nabla_X J_\omega Y - J_\omega(\nabla_X Y) = \nabla_X J_\omega Y = \nabla_X \bar{J}_j Y.$$

Now, if $\bar{J}_j Y = \sum a_p Y_p$ where $Y_p \in \mathfrak{B}_j$ and $a_p \in C^\infty(V_j)$ for $p = 1, \ldots, \dim V_j$, then the above equality takes the form

$$(\nabla_X J_\omega)Y = \nabla_X \bar{J}_j Y = \sum \underbrace{X(a_p)}_{0} Y_p + \sum a_p \underbrace{\nabla_X Y_p}_{0} = 0.$$

Summarizing, we have $d^\nabla J(X,Y) = 0$ for every $X, Y \in X(U)$. Thus $f|_U$ is a complex special affine hypersphere. ∎

Theorems 3.6, 3.8 and 3.9 imply that special affine hyperspheres that are neither complex nor para-complex must be of dimension at least four, and their Blaschke metric must be indefinite. The following examples show that such hyperspheres really exist.

EXAMPLE 3.10. Let $F\colon \mathbb{R}^4 \to \mathbb{R}$ be given by

$$F(x,y,z,t) = \tfrac{1}{2}x^2 + \tfrac{1}{2}y^2 + \tfrac{1}{2}z^2 - \tfrac{1}{2}t^2.$$

Then

$$f\colon \mathbb{R}^4 \ni (x,y,z,t) \mapsto (x,y,z,t,F(x,y,z,t)) \in \mathbb{R}^5$$

with the transversal vector field

$$\xi\colon \mathbb{R}^4 \ni (x,y,z,t) \mapsto (0,0,0,0,1) \in \mathbb{R}^5$$

is a special affine hypersphere which is neither complex nor para-complex.

It is easy to verify that in the canonical basis $\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}, \frac{\partial}{\partial t}\}$ the second fundamental form h is

$$h = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}.$$

The above implies that $\omega_h = \theta$, thus f is an improper affine hypersphere. Now take any non-degenerate ω of the form

$$\omega = \begin{bmatrix} 0 & \omega_{12} & \omega_{13} & \omega_{14} \\ -\omega_{12} & 0 & \omega_{23} & \omega_{24} \\ -\omega_{13} & -\omega_{23} & 0 & \omega_{34} \\ -\omega_{14} & -\omega_{24} & -\omega_{34} & 0 \end{bmatrix},$$

where $\omega_{ij} \in \mathbb{R}$. Then $J_\omega^2 = (\omega^{-1} \cdot h)^2$ has constant components. By Theorem 3.3, f is special. By straightforward computations we obtain the diagonal entries of the matrix J_ω^2:

$$(J_\omega^2)_{11} = \frac{1}{\det\omega}(-\omega_{34}^2 - \omega_{24}^2 + \omega_{23}^2), \quad (J_\omega^2)_{33} = \frac{1}{\det\omega}(-\omega_{24}^2 - \omega_{14}^2 + \omega_{12}^2),$$

$$(J_\omega^2)_{22} = \frac{1}{\det\omega}(-\omega_{34}^2 - \omega_{14}^2 + \omega_{13}^2), \quad (J_\omega^2)_{44} = \frac{1}{\det\omega}(\omega_{23}^2 + \omega_{13}^2 + \omega_{12}^2).$$

Since ω is non-degenerate, we have

$$(J_\omega^2)_{44} - (J_\omega^2)_{11} - (J_\omega^2)_{22} - (J_\omega^2)_{33} = \frac{2}{\det\omega}(\omega_{34}^2 + \omega_{24}^2 + \omega_{14}^2) > 0.$$

Therefore, since $(J_\omega^2)_{44} > 0$, the diagonal entries of the matrix J_ω^2 cannot be equal. That is, f is neither a complex nor a para-complex special affine hypersphere.

EXAMPLE 3.11. Let $U = (0,\infty)^2 \times \mathbb{R}^4$ and let $F\colon U \to \mathbb{R}$ be given by

$$F(x,y,z,t,u,v) = \left(\tfrac{4}{9}x^2 + y\right)^{3/2} - \tfrac{1}{2}z^2 - \tfrac{1}{2}t^2 + \tfrac{1}{2}u^2 - \tfrac{1}{2}v^2.$$

Then

$$f\colon U \ni (x,y,z,t,u,v) \mapsto (x,y,z,t,u,v,F(x,y,z,t,u,v)) \in \mathbb{R}^7$$

with the transversal vector field

$$\xi\colon U \ni (x,y,z,t,u,v) \mapsto (0,0,0,0,0,0,1) \in \mathbb{R}^7$$

is a special affine hypersphere which is neither complex nor para-complex.

In the canonical basis $\left\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}, \frac{\partial}{\partial t}, \frac{\partial}{\partial u}, \frac{\partial}{\partial v}\right\}$ the second fundamental form h is

$$h = \begin{bmatrix} \frac{4}{9}\cdot\frac{(8x^2+9y)}{\sqrt{4x^2+9y}} & \frac{2x}{\sqrt{4x^2+9y}} & 0 & 0 & 0 & 0 \\ \frac{2x}{\sqrt{4x^2+9y}} & \frac{9}{4\sqrt{4x^2+9y}} & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix}.$$

Hence $\det h = -1$, and thus $\omega_h = \theta$. That is, f is an improper affine hypersphere. If f is a complex (respectively, a para-complex) special hypersphere, then there must exists an antisymmetric matrix ω with constant elements such that $J_\omega^2 = (\omega^{-1}\cdot h)^2 = -I$ (respectively, $J_\omega^2 = (\omega^{-1}\cdot h)^2 = I$). Denote $\alpha := \omega^{-1}$. Then α is a non-degenerate antisymmetric matrix with constant elements:

$$\alpha = \begin{bmatrix} 0 & \alpha_{12} & \alpha_{13} & \alpha_{14} & \alpha_{15} & \alpha_{16} \\ -\alpha_{12} & 0 & \alpha_{23} & \alpha_{24} & \alpha_{25} & \alpha_{26} \\ -\alpha_{13} & -\alpha_{23} & 0 & \alpha_{34} & \alpha_{35} & \alpha_{36} \\ -\alpha_{14} & -\alpha_{24} & -\alpha_{34} & 0 & \alpha_{45} & \alpha_{46} \\ -\alpha_{15} & -\alpha_{25} & -\alpha_{35} & -\alpha_{45} & 0 & \alpha_{56} \\ -\alpha_{16} & -\alpha_{26} & -\alpha_{36} & -\alpha_{46} & -\alpha_{56} & 0 \end{bmatrix}.$$

From the form of h it follows that the (i,j) entry of $J_\omega^2 = (\alpha \cdot h)^2$ can be expressed as

$$\frac{A_{ij}x^2 + B_{ij}x + C_{ij}y + D_{ij}}{\sqrt{4x^2+9y}} + E_{ij},$$

where $A_{ij}, B_{ij}, C_{ij}, D_{ij}, E_{ij} \in \mathbb{R}$. Therefore, J_ω^2 has constant elements if and only if $A_{ij} = B_{ij} = C_{ij} = D_{ij} = 0$ for $i,j = 1,\dots,6$.

By straightforward computations we get

$$C_{33} = 4\alpha_{13}^2, \quad C_{44} = 4\alpha_{14}^2, \quad C_{55} = -4\alpha_{15}^2, \quad C_{66} = 4\alpha_{16}^2$$

and

$$D_{33} = \tfrac{9}{4}\alpha_{23}^2, \quad D_{44} = \tfrac{9}{4}\alpha_{24}^2, \quad D_{55} = -\tfrac{9}{4}\alpha_{25}^2, \quad D_{66} = \tfrac{9}{4}\alpha_{26}^2.$$

The above implies that

$$\alpha_{13} = \alpha_{14} = \alpha_{15} = \alpha_{16} = \alpha_{23} = \alpha_{24} = \alpha_{25} = \alpha_{26} = 0.$$

Now we get

$$(J_\omega^2)_{11} = -\alpha_{12}^2 \le 0 \quad \text{and} \quad (J_\omega^2)_{55} = \alpha_{35}^2 + \alpha_{56}^2 + \alpha_{45}^2 \ge 0.$$

The last two inequalities imply that neither $J_\omega^2 = -I$ nor $J_\omega^2 = I$. That is, f is neither a complex nor a para-complex special affine hypersphere. It remains to show that f is indeed special. For this purpose, take

$$\omega = \begin{bmatrix} 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

Then

$$J_\omega^2 = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

which means that f is a special affine hypersphere.

The next example shows that there are special affine hyperspheres that are both complex and para-complex.

EXAMPLE 3.12. Let $F_1\colon \mathbb{R}^4 \to \mathbb{R}$ and $F_2\colon \mathbb{R}^4 \to \mathbb{R}$ be given by

$$F_1(x,y,z,t) = xz + yt, \tag{3.9}$$

$$F_2(x,y,z,t) = tx + zy - tz^2 + t^2z + \tfrac{1}{3}t^3 - \tfrac{1}{3}z^3. \tag{3.10}$$

Then the function

$$f_i\colon \mathbb{R}^4 \ni (x,y,z,t) \mapsto (x,y,z,t,F_i(x,y,z,t)) \in \mathbb{R},$$

where $i = 1,2$, is a complex special affine hypersphere as well as a para-complex special affine hypersphere.

Indeed, in the canonical basis $\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}, \frac{\partial}{\partial t}\}$ we have the following formulas for the second fundamental forms of f_1 and f_2:

$$h_1 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \quad h_2 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & -2t-2z & -2z+2t \\ 1 & 0 & -2z+2t & 2z+2t \end{bmatrix}.$$

It is easy to see that $\det h_1 = \det h_2 = 1$, thus f_1 and f_2 are improper affine hyperspheres. Let

$$\omega_A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}, \quad \omega_B = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix},$$

$$\omega_C = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}.$$

Straightforward computations give $(\omega_A^{-1} \cdot h_1)^2 = -I$ and $(\omega_B^{-1} \cdot h_1)^2 = I$, that is, f_1 is both a complex and a para-complex special affine hypersphere. In a similar way we get $(\omega_B^{-1} \cdot h_2)^2 = -I$ and $(\omega_C^{-1} \cdot h_2)^2 = I$, so f_2 is also both a complex and a para-complex special affine hypersphere.

Finally, we give a method that allows us to construct higher-dimensional examples of special affine hyperspheres using lower-dimensional special affine hyperspheres:

THEOREM 3.13. *Let U_1 and U_2 be open subsets in $\mathbb{R}^{2n}$ and $\mathbb{R}^{2m}$ respectively. Let*

$$f_1\colon U_1 \ni (x_1,\ldots,x_{2n}) \mapsto (x_1,\ldots,x_{2n},F_1(x_1,\ldots,x_{2n})) \in \mathbb{R}^{2n+1}$$

with $\xi_1 = (0, 0, \ldots, 0, 1) \in \mathbb{R}^{2n+1}$ *and*

$$f_2 \colon U_2 \ni (y_1, \ldots, y_{2m}) \mapsto (y_1, \ldots, y_{2m}, F_2(y_1, \ldots, y_{2m})) \in \mathbb{R}^{2m+1}$$

with $\xi_2 = (0, \ldots, 0, 1) \in \mathbb{R}^{2m+1}$ *be two improper affine hyperspheres. If* f_1 *and* f_2 *are special, then also*

$$\begin{aligned} f \colon U_1 \times U_2 \ni (x_1, \ldots, x_{2n}, y_1, \ldots, y_{2m}) \mapsto \\ (x_1, \ldots, x_{2n}, y_1, \ldots, y_{2m}, F_1(x_1, \ldots, x_{2n}) + F_2(y_1, \ldots, y_{2m})) \in \mathbb{R}^{2n+2m+1} \end{aligned}$$

with $\xi = (0, \ldots, 0, 1) \in \mathbb{R}^{2n+2m+1}$ *is a special affine hypersphere. In particular* $h = h_1 \oplus h_2$ *where* h, h_1, h_2 *are the Blaschke metrics for* f, f_1, f_2*, respectively. Moreover, if* ω_1*,* ω_2 *are special symplectic forms for* f_1, f_2 *respectively, then* $\omega = \omega_1 \oplus \omega_2$ *is a special symplectic form for* f*. Further,* $J_\omega = J_{\omega_1} \oplus J_{\omega_2}$*. If* f_1 *and* f_2 *are both complex [para-complex] special affine hyperspheres, then* f *is also a complex [para-complex] special affine hypersphere.*

Proof. Let ∇_i and h_i be the Blaschke connection and the Blaschke metric for f_i, $i = 1, 2$. A transversal vector field ξ induces on $U_1 \times U_2$ a metric h and a connection ∇. If $\partial_1, \ldots, \partial_{2n}$ and $\tilde{\partial}_1, \ldots, \tilde{\partial}_{2m}$ are the canonical bases on U_1 and U_2 respectively, then $\partial_1, \ldots, \partial_{2n}, \tilde{\partial}_1, \ldots, \tilde{\partial}_{2m}$ is a canonical basis on $U_1 \times U_2$ (recall that $\partial_i = \frac{\partial}{\partial x_i}$ and $\tilde{\partial}_i = \frac{\partial}{\partial y_i}$). The form of f implies that

$$h(\partial_i, \partial_j) = \frac{\partial^2 F_1}{\partial x_i \partial x_j} = h_1(\partial_i, \partial_j), \qquad h(\tilde{\partial}_i, \tilde{\partial}_j) = \frac{\partial^2 F_2}{\partial y_i \partial y_j} = h_2(\tilde{\partial}_i, \tilde{\partial}_j)$$

and

$$h(\partial_i, \tilde{\partial}_j) = h(\tilde{\partial}_j, \partial_i) = 0,$$

thus we get

$$h = h_1 \oplus h_2.$$

Moreover,

$$\nabla_{\partial_i} \partial_j = \nabla_{\partial_i} \tilde{\partial}_j = \nabla_{\tilde{\partial}_i} \partial_j = \nabla_{\tilde{\partial}_i} \tilde{\partial}_j = 0,$$

since f is a graph-immersion. Note that $\theta(\partial_1, \ldots, \partial_{2n}, \tilde{\partial}_1, \ldots, \tilde{\partial}_{2m}) = 1$. Since f_1 and f_2 are improper affine hyperspheres we also have

$$\begin{aligned} \omega_h(\partial_1, \ldots, \partial_{2n}, \tilde{\partial}_1, \ldots, \tilde{\partial}_{2m}) &= \sqrt{|\det[h_1(\partial_i, \partial_j)] \cdot \det[h_2(\tilde{\partial}_i, \tilde{\partial}_j)]|} \\ &= \omega_{h_1}(\partial_1, \ldots, \partial_{2n}) \cdot \omega_{h_2}(\tilde{\partial}_1, \ldots, \tilde{\partial}_{2m}) = 1. \end{aligned}$$

This implies that f is an improper affine hypersphere.

Since f_i is special, Theorem 3.3 implies that on U_i there exists a non-singular h_i-antisymmetric $(1, 1)$-tensor J_i (and the corresponding form ω_i) such that $\nabla_i J_i^2 = 0$ and $d^{\nabla_i} J_i = 0$. On $U_1 \times U_2$ we can define a new tensor of type $(1, 1)$ by

$$J = J_1 \oplus J_2.$$

Of course J is non-singular since both J_1 and J_2 are non-singular. Moreover, $J(TU_1) \subset TU_1$ and $J(TU_2) \subset TU_2$. Now, it follows that

$$h(J\partial_i, \partial_j) = h_1(J_1\partial_i, \partial_j) = -h_1(\partial_i, J_1\partial_j) = -h(\partial_i, J\partial_j),$$
$$h(J\tilde{\partial}_i, \tilde{\partial}_j) = h_2(J_2\tilde{\partial}_i, \tilde{\partial}_j) = -h_2(\tilde{\partial}_i, J_2\tilde{\partial}_j) = -h(\tilde{\partial}_i, J\tilde{\partial}_j).$$

We also have

$$h(J\partial_i, \tilde{\partial}_j) = 0 = -h(\partial_i, J\tilde{\partial}_j),$$

thus J is h-antisymmetric.

Now let $X, Y \in \{\partial_1, \ldots, \partial_{2n}, \tilde{\partial}_1, \ldots, \tilde{\partial}_{2m}\}$. Then

$$(d^\nabla J)(X, Y) = (\nabla_X J)Y - (\nabla_Y J)X = \nabla_X JY - \nabla_Y JX.$$

In particular for $X, Y \in \{\partial_1, \ldots, \partial_{2n}\}$ we get

$$(d^\nabla J)(X, Y) = \nabla_X J_1 Y - \nabla_Y J_1 X = \nabla_{1X} J_1 Y - \nabla_{1Y} J_1 X$$
$$= (d^{\nabla_1} J_1)(X, Y) = 0.$$

Similarly, for $X, Y \in \{\tilde{\partial}_1, \ldots, \tilde{\partial}_{2m}\}$ we have

$$(d^\nabla J)(X, Y) = (d^{\nabla_2} J_2)(X, Y) = 0.$$

When $X \in \{\partial_1, \ldots, \partial_{2n}\}$ and $Y \in \{\tilde{\partial}_1, \ldots, \tilde{\partial}_{2m}\}$, or $Y \in \{\partial_1, \ldots, \partial_{2n}\}$ and $X \in \{\tilde{\partial}_1, \ldots, \tilde{\partial}_{2m}\}$, we obtain

$$(d^\nabla J)(X, Y) = 0.$$

Using the same methods we prove that

$$(\nabla_X J^2)Y = 0$$

for any $X, Y \in \{\partial_1, \ldots, \partial_{2n}, \tilde{\partial}_1, \ldots, \tilde{\partial}_{2m}\}$. Summarizing, J is an h-antisymmetric non-singular tensor of type $(1, 1)$ with $d^\nabla J = 0$ and $\nabla J^2 = 0$, that is, f is a special affine hypersphere. Directly from the definition of J we deduce that

$$\omega = \omega_1 \oplus \omega_2.$$

To prove the second part of the theorem first note that if f_1, f_2 are complex (respectively, para-complex) special affine hyperspheres, then J_1 and J_2 can be chosen in such a way that $J_1^2 = -I$ and $J_2^2 = -I$ (respectively, $J_1^2 = I$ and $J_2^2 = I$). Then $J^2 = J_1^2 \oplus J_2^2 = -I$ (respectively, $J^2 = J_1^2 \oplus J_2^2 = I$), so J is an almost complex (respectively, an almost para-complex) structure on $U_1 \times U_2$. That is, f is a complex (respectively, a para-complex) special affine hypersphere. ■

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