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Exponential decay and blow-up results for a nonlinear heat equation with a viscoelastic term and Robin conditions

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Abstract. We consider a nonlinear heat equation with a viscoelastic term and Robin conditions. First, we prove existence and uniqueness of a weak solution. Next, we prove that any weak solution with negative initial energy will blow up in finite time. Finally, we give a sufficient condition for the global existence and exponential decay of weak solutions. The main tools are the Faedo–Galerkin method and defining a modified energy functional together with the technique of Lyapunov functional.

1. Introduction. In this paper, we consider the nonlinear heat equation

$$u_t - \frac{\partial}{\partial x}[\mu_1(x,t)u_x] + \int_0^t g(t-s)\frac{\partial}{\partial x}[\mu_2(x,s)u_x(x,s)]\,ds = f(u) + f_1(x,t), \quad 0<x<1,\ t>0, \tag{1.1}$$

with the boundary conditions

$$u_x(0,t) - h_0u(0,t) = g_0(t), \quad u_x(1,t) + h_1u(1,t) = -g_1(t), \tag{1.2}$$

and the initial condition

$$u(x,0) = u_0(x), \tag{1.3}$$

where $h_0, h_1 \geq 0$ are real numbers with $h_0 + h_1 > 0$, and μ_1, μ_2, g, u_0, f, f_1, g_0, g_1 are given functions satisfying conditions to be specified later.

Equation (1.1) arises naturally in engineering and physical sciences. Problems of this type have been extensively studied, and several results concerning existence, nonexistence, regularity and asymptotic behavior have

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been established (see [1]–[8], [11]–[15] and references therein). Messaoudi [12] considered an initial boundary value problem related to the equation

$$u_t - \Delta u + \int_0^t g(t-s)\Delta u(x,s)\,ds = |u|^{p-2}u, \tag{1.4}$$

and proved, under suitable conditions on g and p, the blow-up of a weak solution with negative initial energy by the convexity method.

Li et al. [9] considered the semilinear heat equation with a memory term

$$u_t - \Delta u + \int_0^t g(t-s)\,\mathrm{div}[a(x)\nabla u(s)]\,ds = 0, \qquad x \in \Omega,\ t > 0, \tag{1.5}$$

and proved, under suitable conditions, that the energy functional decays to zero as time tends to infinity by the method of energy perturbation, in which the usual exponential and polynomial decay results are only special cases.

Recently, Han at el. [6], under suitable conditions on g, established a blow-up result for solutions with negative initial energy for the semilinear heat equation with a viscoelastic term

$$u_t - \Delta u + \int_0^t g(t-s)\Delta u(x,s)\,ds = 0, \qquad x \in \Omega,\ t > 0, \tag{1.6}$$

by defining a modified energy functional and using a concavity argument.

In the case $\mu_1(x,t) = \mu_2(x,t) = \mu(x,t)$, Problem (1.1)–(1.3) is studied in [15].

Motivated by the above mentioned works, we study the blow-up and exponential decay estimates for Problem (1.1)–(1.3). Our paper is organized as follows.

First, we present preliminaries in Section 2 and we prove the existence of a weak solution in Section 3. Next, in Sections 4 and 5, Problem (1.1)–(1.3) is considered with $g_0 = g_1 \equiv 0$ and $\mu_2(x,t) \equiv \mu_2(x)$. In the case of $f_1 \equiv 0$, if $u_0 \in H^1$ is such that $h_0\mu_1(0,0)u_0^2(0) + h_1\mu_1(1,0)u_0^2(1) + \int_0^1 \mu_1(x,0)u_{0x}^2(x)\,dx - 2\int_0^1 dx \int_0^{u_0(x)} f(z)\,dz < 0$, and if some auxiliary conditions are satisfied, then the weak solution u of Problem (1.1)–(1.3) blows up in finite time. For $\|f_1(t)\|$ small enough, if

$$h_0\mu_1(0,0)u_0^2(0) + h_1\mu_1(1,0)u_0^2(1) + \int_0^1 \mu_1(x,0)u_{0x}^2(x)\,dx - p\int_0^1 dx \int_0^{u_0(x)} f(z)\,dz > 0, \qquad p > 2,$$

and if the initial energy is also small enough, then the energy of the solution decays exponentially as $t \to \infty$. The main tools are the Faedo–Galerkin method and defining a modified energy functional together with the technique of Lyapunov functionals. Our results extend those in [15].

2. Preliminaries. Set $\Omega = (0,1)$ and $Q_T = \Omega \times (0,T)$. We omit the definitions of the usual function spaces $L^p = L^p(\Omega)$ and $H^m = H^m(\Omega)$. Let $\langle \cdot,\cdot\rangle$ be either the scalar product in L^2 or the dual pairing of a continuous linear functional and an element of a function space. The notation $\|\cdot\|$ stands for the L^2 norm, and $\|\cdot\|_X$ for the norm in the Banach space X. We denote by X' the dual space of X. We let $L^p(0,T;X)$, $1 \le p \le \infty$, be the Banach space of measurable functions $u:(0,T)\to X$ such that $\|u\|_{L^p(0,T;X)} < \infty$, with

$$\|u\|_{L^p(0,T;X)} = \begin{cases} \Big(\int_0^T \|u(t)\|_X^p\,dt\Big)^{1/p} < \infty & \text{if } 1 \le p < \infty,\\ \operatorname*{ess\,sup}_{0<t<T} \|u(t)\|_X & \text{if } p = \infty. \end{cases}$$

On H^1, we shall use the norms

$$\|v\|_{H^1} = (\|v\|^2 + \|v_x\|^2)^{1/2}, \qquad \|v\|_i = (v^2(i) + \|v_x\|^2)^{1/2}, \quad i = 0,1.$$

Let $\mu_i \in C^0(\overline{Q}_T)$ with $\mu_i(x,t) \ge \underline{\mu}_i > 0$ $(i = 1,2)$ for all $(x,t) \in \overline{Q}_T$. We consider two famillies of symmetric bilinear forms $\{a_i(t;\cdot,\cdot)\}_{0\le t\le T}$ on $H^1 \times H^1$ defined by

$$\begin{aligned} (2.1)\qquad & a_i(t;u,v) \\ &= \int_0^1 \mu_i(x,t)u_x(x)v_x(x)\,dx + h_0\mu_i(0,t)u(0)v(0) + h_1\mu_i(1,t)u(1)v(1) \\ &= \langle \mu_i(t)u_x, v_x\rangle + h_0\mu_i(0,t)u(0)v(0) + h_1\mu_i(1,t)u(1)v(1) \end{aligned}$$

for all $u,v \in H^1$ and $0 \le t \le T$.

Then we have the following lemmas. The proofs are straightforward, so we omit the details.

Lemma 2.1. *The embedding $H^1 \hookrightarrow C^0([0,1])$ is compact and*

$$(2.2)\qquad \begin{cases} \|v\|_{C^0([0,1])} \le \sqrt{2}\,\|v\|_{H^1}, & \forall v \in H^1,\\ \|v\|_{C^0([0,1])} \le \sqrt{2}\,\|v\|_i, & \forall v \in H^1,\ i = 0,1. \end{cases}$$

Lemma 2.2. *Let $\mu_i \in C^0(\overline{Q}_T)$ with $\mu_i(x,t) \ge \underline{\mu}_i > 0$ $(i = 1,2)$ for all $(x,t) \in \overline{Q}_T$. Then the symmetric bilinear forms $a_i(t;\cdot,\cdot)$ $(i = 1,2)$ are continuous on $H^1 \times H^1$ and coercive on H^1, i.e.,*

$$(2.3)\qquad \begin{aligned} &\text{(i)}\ |a_i(t;u,v)| \le a_{iT}\|u\|_{H^1}\|v\|_{H^1},\\ &\text{(ii)}\ a_i(t;v,v) \ge a_{0i}\|v\|_{H^1}^2, \end{aligned}$$

for all $u, v \in H^1$ and $0 \le t \le T$, where

$$a_{iT} = (1 + 2h_0 + 2h_1) \sup_{(x,t)\in\overline{Q}_T} \mu_i(x,t), \quad a_{0i} = \frac{1}{3}\underline{\mu}_i \min\{1, \max\{h_0, h_1\}\}.$$

REMARK 2.3. It follows from (2.2) that on H^1, $v \mapsto \|v\|_{H^1}$ and $v \mapsto \|v\|_i$ are equivalent norms satisfying

$$\frac{1}{\sqrt{3}}\|v\|_{H^1} \le \|v\|_i \le \sqrt{3}\,\|v\|_{H^1}, \qquad i = 0, 1.$$

3. The existence and uniqueness theorem. We make the following assumptions:

($\mathbf{A}_1$) $h_0, h_1 \ge 0$, $h_0 + h_1 > 0$;
($\mathbf{A}_2$) $g_0, g_1 \in H^1(0,T)$;
($\mathbf{A}_3$) $\mu_1 \in C^1([0,1]\times[0,T])$, $\mu_1(x,t) \ge \underline{\mu}_1 > 0$, $\forall (x,t) \in [0,1]\times[0,T]$;
($\mathbf{A}_4$) $\mu_2 \in C^1([0,1]\times[0,T])$, $\mu_2(x,t) \ge \underline{\mu}_2 > 0$, $\forall (x,t) \in [0,1]\times[0,T]$;
($\mathbf{A}_5$) $f \in C^0(\mathbb{R};\mathbb{R})$;
($\mathbf{A}_6$) $g \in H^1(0,T)$;
($\mathbf{A}_7$) $f_1 \in L^2(Q_T)$.

The weak formulation of Problem (1.1)–(1.3) can be given in the following manner: Find $u \in L^\infty(0,T;H^1)$ with $u' \in L^2(0,T;L^2)$ such that $u(t)$ satisfies the following variational problem:

$$\begin{aligned}(3.1)\qquad \langle u'(t), v\rangle + a_1(t; u(t), v) - \int_0^t g(t-s) a_2(s; u(s), v)\, ds \\ = \langle f(u(t)), v\rangle + \langle f_1(t), v\rangle - \tilde{g}_0(t)v(0) - \tilde{g}_1(t)v(1)\end{aligned}$$

for all $v \in H^1$ and a.e. $t \in (0,T)$, and the initial condition

$$(3.2)\qquad u(0) = u_0,$$

where

$$\tilde{g}_i(t) = \mu_1(i,t)g_i(t) - \int_0^t g(t-s)\mu_2(i,s)g_i(s)\, ds, \qquad i = 0, 1.$$

Note that, with $f \in C(\mathbb{R};\mathbb{R})$, if we set

$$\Phi(r) = \begin{cases} \sup_{|u|\le r} |f(u)|, & r > 0, \\ |f(0)|, & r = 0, \end{cases}$$

then $\Phi \in C(\mathbb{R}_+;\mathbb{R}_+)$ is nondecreasing and

$$|f(u)| \le \Phi(|u|), \qquad \forall u \in \mathbb{R} \qquad \text{(see [15, Appendix 1])}.$$

We have the following theorem.

THEOREM 3.1. *Let* $(\mathbf{A}_1)$–$(\mathbf{A}_7)$ *hold and* $u_0 \in H^1$.

(i) *If* $\int_0^\infty \frac{dy}{1+y+\Phi^2(\sqrt{y})} = \infty$, *then Problem* (1.1)–(1.3) *has a weak solution* u *satisfying*

$$u \in L^\infty(0,T;H^1), \qquad u' \in L^2(0,T;L^2).$$

(ii) *If* $\int_0^\infty \frac{dy}{1+y+\Phi^2(\sqrt{y})} < \infty$, *then Problem* (1.1)–(1.3) *has a weak solution* u *satisfying*

$$u \in L^\infty(0,T_*;H^1), \qquad u' \in L^2(0,T_*;L^2),$$

with a certain T_* *small enough.*

Furthermore, if in addition

$(\bar{\mathbf{A}}_5)$ $\forall M > 0, \exists C_M > 0 : |f(y) - f(z)| \le C_M|y - z|, \forall y, z \in [-M, M]$,

then the solution is unique.

Proof. The proof consists of four steps:

STEP 1. *The Faedo–Galerkin approximation* (introduced by Lions [10]). Let $\{w_j\}$ be a denumerable base of H^1. We find an approximate solution of Problem (1.1)–(1.3) in the form

$$u_m(t) = \sum_{j=1}^{m} c_{mj}(t) w_j, \tag{3.3}$$

where the coefficient functions c_{mj}, $1 \le j \le m$, satisfy the system of ordinary differential equations

$$\begin{cases} \langle u_m'(t), w_j\rangle + a_1(t; u_m(t), w_j) - \int_0^t g(t-s) a_2(s; u_m(s), w_j)\, ds \\ \quad = \langle f(u_m(t)), w_j\rangle + \langle f_1(t), w_j\rangle - \tilde{g}_0(t) w_j(0) - \tilde{g}_1(t) w_j(1), \\ \hfill 1 \le j \le m, \\ u_m(0) = u_{0m}, \end{cases} \tag{3.4}$$

where

$$u_{0m} = \sum_{j=1}^{m} \alpha_{mj} w_j \to u_0 \quad \text{strongly in } H^1. \tag{3.5}$$

It is clear that, for each m, there exists a solution $u_m(t)$ of the form (3.3) which satisfies (3.4) and (3.5) almost everywhere on $0 \le t \le T_m$, for some $0 < T_m \le T$.

In what follows, we present a brief proof that a solution of (3.4)–(3.5) of the form (3.3) exists. The system (3.4) can be written as follows:

$$\begin{cases} A c_m'(t) + A_1(t) c_m(t) - \int_0^t g(t-s) A_2(s) c_m(s)\, ds = F(t, c_m(t)), \\ c_m(0) = \alpha_m, \end{cases} \tag{3.6}$$

where

$$(3.7)\quad \begin{cases} c_m(t) = (c_{m1}(t), \dots, c_{mm}(t))^T, \quad \alpha_m = (\alpha_{m1}, \dots, \alpha_{mm})^T, \\ A = (a_{ij}), \quad a_{ij} = \langle w_i, w_j \rangle, \quad 1 \le i, j \le m; \\ A_s(t) = (a_{ij}^{(s)}(t)), \quad a_{ij}^{(s)}(t) = a_s(t; w_i, w_j), \\ \qquad\qquad\qquad\qquad\qquad 1 \le i, j \le m,\ s = 1, 2; \\ F(t, c_m(t)) = (F_1(t, c_m(t)), \dots, F_m(t, c_m(t)))^T, \\ F_j(t, c_m(t)) = \langle f(u_m(t)), w_j \rangle + \langle f_1(t), w_j \rangle \\ \qquad\qquad -\tilde{g}_0(t) w_j(0) - \tilde{g}_1(t) w_j(1), \quad 1 \le j \le m. \end{cases}$$

Since A is invertible, (3.6) leads to

$$c_m(t) = \alpha_m - \int_0^t A^{-1} A_1(s) c_m(s)\, ds + \int_0^t ds \int_0^s g(s-r) A^{-1} A_2(r) c_m(r)\, dr + \int_0^t A^{-1} F(s, c_m(s))\, ds,$$

or

$$(3.8)\quad c_m(t) = \alpha_m + \int_0^t G(t,s) c_m(s)\, ds + \int_0^t A^{-1} F(s, c_m(s))\, ds \equiv (F c_m)(t),$$

where $G(t,s) = A^{-1}(\tilde{g}(t-s) A_2(s) - A_1(s))$ and $\tilde{g}(t) = \int_0^t g(s)\, ds$.

By the Schauder theorem, (3.8) has a solution $c_m(t)$ in a certain closed ball of the Banach space $C([0, T_m]; \mathbb{R}^m)$ with $T_m \in (0, T]$. Therefore, there exists $u_m(t)$ of the form (3.3) satisfying (3.4) and (3.5) on $0 \le t \le T_m$.

Step 2. *A priori estimates.* Multiplying the jth equation of (3.4) by $c'_{mj}(t)$ and summing over j, after some rearrangements we get

$$(3.9)\quad \|u'_m(t)\|^2 + a_1(t; u_m(t), u'_m(t)) - \int_0^t g(t-s) a_2(s; u_m(s), u'_m(t))\, ds = \langle f(u_m(t)), u'_m(t) \rangle + \langle f_1(t), u'_m(t) \rangle - \tilde{g}_0(t) u'_m(0,t) - \tilde{g}_1(t) u'_m(1,t).$$

Firstly, by direct calculation, we have

$$(3.10)\quad \frac{d}{dt} a_1(t; u_m(t), u_m(t)) = 2a_1(t; u_m(t), u'_m(t)) + \langle \mu'_1(t) u_{mx}(t), u_{mx}(t) \rangle + h_0 \mu'_1(0,t) u_m^2(0,t) + h_1 \mu'_1(1,t) u_m^2(1,t)$$

and

$$\frac{d}{dt}\int_0^t g(t-s)a_2(s;u_m(s),u_m(t))\,ds = g(0)a_2(t;u_m(t),u_m(t)) + \int_0^t g'(t-s)a_2(s;u_m(s),u_m(t))\,ds + \int_0^t g(t-s)a_2(s;u_m(s),u'_m(t))\,ds. \tag{3.11}$$

Hence, (3.9) can be rewritten as follows:

$$\begin{aligned} &2\|u'_m(t)\|^2 + \frac{d}{dt}a_1(t;u_m(t),u_m(t)) - 2\frac{d}{dt}\int_0^t g(t-s)a_2(s;u_m(s),u_m(t))\,ds \\ &= \langle \mu'_1(t)u_{mx}(t),u_{mx}(t)\rangle + h_0\mu'_1(0,t)u_m^2(0,t) + h_1\mu'_1(1,t)u_m^2(1,t) \\ &\quad - 2g(0)a_2(t;u_m(t),u_m(t)) - 2\int_0^t g'(t-s)a_2(s;u_m(s),u_m(t))\,ds \\ &\quad + 2\langle f(u_m(t)),u'_m(t)\rangle + 2\langle f_1(t),u'_m(t)\rangle - 2\tilde{g}_0(t)u'_m(0,t) - 2\tilde{g}_1(t)u'_m(1,t). \end{aligned} \tag{3.12}$$

Next, integrating (3.12), we get

$$\begin{aligned} &2\int_0^t \|u'_m(s)\|^2\,ds + a_1(t;u_m(t),u_m(t)) \\ &= a_1(0;u_{0m},u_{0m}) - 2g(0)\int_0^t a_2(s;u_m(s),u_m(s))\,ds \\ &\quad + \int_0^t [\langle \mu'_1(s)u_{mx}(s),u_{mx}(s)\rangle + h_0\mu'_1(0,s)u_m^2(0,s) + h_1\mu'_1(1,s)u_m^2(1,s)]\,ds \\ &\quad + 2\int_0^t g(t-s)a_2(s;u_m(s),u_m(t))\,ds \\ &\quad - 2\int_0^t\int_0^s g'(s-\tau)a_2(\tau;u_m(\tau),u_m(s))\,d\tau\,ds + 2\int_0^t \langle f(u_m(s)),u'_m(s)\rangle\,ds \\ &\quad + 2\int_0^t \langle f_1(s),u'_m(s)\rangle\,ds - 2\int_0^t \tilde{g}_0(s)u'_m(0,s)\,ds - 2\int_0^t \tilde{g}_1(s)u'_m(1,s)\,ds. \end{aligned} \tag{3.13}$$

By $(\mathbf{A}_1)$–$(\mathbf{A}_7)$, and using (2.2), (2.3) and the inequality

$$2ab \le \frac{1}{\beta}a^2 + \beta b^2, \qquad \forall a,b\in\mathbb{R},\ \forall\beta>0, \tag{3.14}$$

we estimate the terms on both sides of (3.13) as follows:

(3.15) $$a_1(t;u_m(t),u_m(t)) \geq a_{01}\|u_m(t)\|_{H^1}^2;$$

(3.16) $$-2g(0)\int_0^t a_2(s;u_m(s),u_m(s))\,ds \leq 2|g(0)|a_{2T}\int_0^t \|u_m(s)\|_{H^1}^2\,ds;$$

(3.17) $$\int_0^t [\langle \mu_1'(s)u_{mx}(s),u_{mx}(s)\rangle + h_0\mu_1'(0,s)u_m^2(0,s) + h_1\mu_1'(1,s)u_m^2(1,s)]\,ds \leq \tilde{a}_{1T}\int_0^t \|u_m(s)\|_{H^1}^2\,ds,$$

where $\tilde{a}_{1T} = 2h_0 + 2h_1 + \sup_{(x,t)\in\overline{Q}_T}|\mu_1'(x,t)|$;

(3.18) $$\begin{aligned} 2\int_0^t g(t-s)a_2(s;u_m(s),u_m(t))\,ds &\leq 2\|g\|_{L^\infty(0,T)}a_{2T}\|u_m(t)\|_{H^1}\int_0^t \|u_m(s)\|_{H^1}\,ds \\ &\leq \frac{1}{\beta}\Big(\|g\|_{L^\infty(0,T)}a_{2T}\int_0^t \|u_m(s)\|_{H^1}\,ds\Big)^2 + \beta\|u_m(t)\|_{H^1}^2 \\ &\leq \frac{1}{\beta}\|g\|_{L^\infty(0,T)}^2 T a_{2T}^2\int_0^t \|u_m(s)\|_{H^1}^2\,ds + \beta\|u_m(t)\|_{H^1}^2; \end{aligned}$$

(3.19) $$\begin{aligned} -2\int_0^t\int_0^s g'(s-\tau)a_2(\tau;u_m(\tau),u_m(s))\,d\tau\,ds &\leq a_{2T}\int_0^t \|u_m(s)\|_{H^1}\,ds\int_0^s |g'(s-\tau)|\,\|u_m(\tau)\|_{H^1}\,d\tau \\ &\leq a_{2T}\sqrt{T}\,\|g'\|_{L^2(0,T)}\int_0^t \|u_m(s)\|_{H^1}^2\,ds; \end{aligned}$$

(3.20) $$\begin{aligned} 2\int_0^t \langle f(u_m(s)),u_m'(s)\rangle\,ds &\leq 2\int_0^t \|f(u_m(s))\|^2\,ds + \frac{1}{2}\int_0^t \|u_m'(s)\|^2\,ds \\ &\leq 2\int_0^t \Phi^2(\sqrt{2}\,\|u_m(s)\|_{H^1})\,ds + \frac{1}{2}\int_0^t \|u_m'(s)\|^2\,ds; \end{aligned}$$

and

(3.21) $$2\int_0^t \langle f_1(s),u_m'(s)\rangle\,ds \leq 2\|f_1\|_{L^2(Q_T)}^2 + \frac{1}{2}\int_0^t \|u_m'(s)\|^2\,ds.$$

By using integration by parts, it follows that

$$(3.22)\qquad -2\int_0^t \tilde{g}_0(s)u_m'(0,s)\,ds$$
$$= -2\tilde{g}_0(t)u_m(0,t) + 2\tilde{g}_0(0)u_{0m}(0) + 2\int_0^t \tilde{g}_0'(s)u_m(0,s)\,ds$$
$$\le 2|\tilde{g}_0(0)||u_{0m}(0)| + \frac{2}{\beta}\|\tilde{g}_0\|^2_{L^\infty(0,T)} + \beta\|u_m(t)\|^2_{H^1} + 2\|\tilde{g}_0'\|^2_{L^2(0,T)}$$
$$+ \int_0^t \|u_m(s)\|^2_{H^1}\,ds.$$

Similarly,

$$(3.23)\qquad -2\int_0^t \tilde{g}_1(s)u_m'(1,s)\,ds$$
$$\le 2|\tilde{g}_1(0)|\,|u_{0m}(1)| + \frac{2}{\beta}\|\tilde{g}_1\|^2_{L^\infty(0,T)} + \beta\|u_m(t)\|^2_{H^1} + 2\|\tilde{g}_1'\|^2_{L^2(0,T)}$$
$$+ \int_0^t \|u_m(s)\|^2_{H^1}\,ds.$$

As $u_{0m} \to u_0$ strongly in H^1, we have

$$(3.24)\qquad a_1(0;u_{0m},u_{0m}) + 2|\tilde{g}_0(0)|\,|u_{0m}(0)| + 2|\tilde{g}_1(0)|\,|u_{0m}(1)| \le \tilde{C}_0,\ \forall m \in \mathbb{N},$$

where $\tilde{C}_0$ always indicates a bound depending on u_0.

By choosing $\beta = a_{01}/6$, it follows from (3.13) and (3.15)–(3.24) that

$$(3.25)\qquad S_m(t) \le C_T^{(1)} + \int_0^t \omega(S_m(s))\,ds,$$

where

$$(3.26)\qquad \begin{cases} S_m(t) = \int_0^t \|u_m'(s)\|^2\,ds + \frac{a_{01}}{2}\|u_m(t)\|^2_{H^1}, \\ C_T^{(1)} = \tilde{C}_0 + 2\|f_1\|^2_{L^2(Q_T)} + \frac{12}{a_{01}}(\|\tilde{g}_0\|^2_{L^\infty(0,T)} + \|\tilde{g}_1\|^2_{L^\infty(0,T)}) \\ \qquad + 2(\|\tilde{g}_0'\|^2_{L^2(0,T)} + \|\tilde{g}_1'\|^2_{L^2(0,T)}), \\ C_T^{(2)} = \frac{2}{a_{01}}\Big(2 + \tilde{a}_{1T} + 2|g(0)|a_{2T} + \frac{6}{a_{01}}\|g\|^2_{L^\infty(0,T)}Ta_{2T}^2 \\ \qquad + a_{2T}\sqrt{T}\,\|g'\|_{L^2(0,T)}\Big), \end{cases}$$

where the function $\omega(S) = 1 + C_T^{(2)}S + 2\Phi^2\big(\frac{2}{\sqrt{a_{01}}}\sqrt{S}\big)$ is continuous and nondecreasing on $0 \le S < \infty$.

We will use the following lemma (see I. Bihari [3]).

LEMMA 3.2. *Let $x : [0,T] \to \mathbb{R}_+$ be a continuous function satisfying*

$$x(t) \le M + \int_0^t k(s)\omega(x(s))\,ds, \qquad t \in [0,T], \tag{3.27}$$

where $M \ge 0$, $k : [0,T] \to \mathbb{R}_+$ is continuous and $\omega : \mathbb{R}_+ \to (0,\infty)$ is continuous and nondecreasing. Set

$$\Psi(u) = \int_0^u \frac{dy}{\omega(y)}, \qquad u \ge 0.$$

(i) *If $\int_0^\infty \frac{dy}{\omega(y)} = \infty$, then*

$$x(t) \le \Psi^{-1}\Big(\Psi(M) + \int_0^t k(s)\,ds\Big), \qquad \forall t \in [0,T]. \tag{3.28}$$

(ii) *If $\int_0^\infty \frac{dy}{\omega(y)} < \infty$, then there exists $T_* \in (0,T]$ such that*

$$x(t) \le \Psi^{-1}\Big(\Psi(M) + \int_0^t k(s)\,ds\Big), \qquad \forall t \in [0,T^*], \tag{3.29}$$

where

$$\int_0^{T_*} k(s)\,ds \le \int_0^\infty \frac{dy}{\omega(y)}. \ \blacksquare \tag{3.30}$$

By the convergence of $\int_0^\infty \frac{dy}{\omega(y)}$ and $\int_0^\infty \frac{dy}{1+y+\Phi^2(\sqrt{y})}$, applying Lemma 3.2 with $x(t) = S_m(t)$, $M = C_T^{(1)}$, $k(s) = 1$, $\omega(S) = 1 + C_T^{(2)} S + 2\Phi^2\big(\frac{2}{\sqrt{a_{01}}}\sqrt{S}\big)$, we deduce from (3.28) or (3.29) that

$$S_m(t) \le \Psi^{-1}(\Psi(C_T^{(1)}) + t) \le C_T, \qquad \forall m \in \mathbb{N}, \tag{3.31}$$

for all $t \in [0,T]$ or all $t \in [0,T^*]$, with T^* chosen as in (3.30), where C_T is a constant independent of m. This allows one to take the constant $T_m = T_*$ or $T_m = T$ for all $m \in \mathbb{N}$.

In what follows, we will write T_* for both T or T_*.

STEP 3. *The limiting process.* By (3.31), we deduce that there exists a subsequence of $\{u_m\}$, still denoted by $\{u_m\}$, such that

$$\begin{cases} u_m \to u & \text{in } L^\infty(0,T_*;H^1) \text{ weakly}^*, \\ u'_m \to u' & \text{in } L^2(Q_{T_*}) \text{ weakly}. \end{cases} \tag{3.32}$$

By the compactness lemma [10, p. 57], (3.32) implies the existence of a subsequence, still denoted by $\{u_m\}$, such that

$$u_m \to u \quad \text{strongly in } L^2(Q_{T_*}) \text{ and a.e. in } Q_{T_*}. \tag{3.33}$$

Remark 3.1. As $f \in C^0(\mathbb{R};\mathbb{R})$ is not supposed to be monotone, in order to do Step 3, we need to use the compactness lemma of Lions [10, p. 57] as above. This leads to the requirement $u_0 \in H^1$.

By the continuity of f, we have

$$f(u_m(x,t)) \to f(u(x,t)) \quad \text{a.e. in } Q_{T_*}.$$

Using $(\mathbf{A}_5)$ and (3.31), we get

$$|f(u_m(x,t))| \le \sup_{|z|\le\sqrt{2}\,\|u_m\|_{L^\infty(0,T^*;H^1)}} |f(z)| \le \sup_{|z|\le C_T} |f(z)| \le \bar{C}_T \tag{3.34}$$

a.e. in Q_{T_*}, where $\bar{C}_T$ is a constant independent of m.

It follows from the dominated convergence theorem that

$$f(u_m) \to f(u) \quad \text{strongly in } L^2(Q_{T_*}). \tag{3.35}$$

Passing to the limit in (3.4), by (3.32), (3.33) and (3.35), we see that u satisfies the equation

$$\begin{cases} \langle u'(t),v\rangle + a_1(t;u(t),v) - \int_0^t g(t-s)a_2(s;u(s),v)\,ds \\ \quad = \langle f(u(t)),v\rangle + \langle f_1(t),v\rangle - \tilde{g}_0(t)v(0) - \tilde{g}_1(t)v(1), \quad \forall v \in H^1, \\ u(0) = u_0. \end{cases} \tag{3.36}$$

Step 4. *Uniqueness of the solution.* Let u_1 and u_2 be weak solutions of (1.1)–(1.3) such that

$$u_i \in L^\infty(0,T_*;H^1), \quad u_i' \in L^2(0,T_*;L^2), \quad i = 1,2. \tag{3.37}$$

Then $u = u_1 - u_2$ satisfies

$$\begin{cases} \langle u'(t),v\rangle + a_1(t;u(t),v) - \int_0^t g(t-s)a_2(s;u(s),v)\,ds \\ \quad = \langle f(u_1(t)) - f(u_2(t)),v\rangle, \quad \forall v \in H^1, \\ u(0) = 0. \end{cases} \tag{3.38}$$

Taking $v = u$ in $(3.38)_1$ and integrating with respect to t, we obtain

$$\begin{aligned} \|u(t)\|^2 + 2\int_0^t a_1(s;u(s),u(s))\,ds \\ = 2\int_0^t\int_0^s g(s-\tau)a_2(\tau;u(\tau),u(s))\,d\tau\,ds \\ + 2\int_0^t \langle f(u_1(t)) - f(u_2(t)),u(s)\rangle\,ds. \end{aligned} \tag{3.39}$$

Set $\varrho(t) = \|u(t)\|^2 + \int_0^t \|u(s)\|_{H^1}^2\,ds$. As in Step 1, we can easily estimate all terms on the right hand side of (3.39) and obtain

$$\varrho(t) \le \frac{\frac{1}{\beta}a_{2T}^2 T\|g\|_{L^\infty(0,T)}^2 + C_M}{\min\{1, 2a_{01}\} - \beta} \int_0^t \varrho(s)\,ds, \tag{3.40}$$

where $M = \sqrt{2}\max_{i=1,2} \|u_i\|_{L^\infty(0,T^*;H^1)}$ and $0 < \beta < \min\{1, 2a_{01}\}$.

By Gronwall's lemma, (3.40) leads to $\varrho \equiv 0$, i.e., $u_1 = u_2$. Theorem 3.1 is proved. ■

4. Blow-up of solutions. Here, we consider Problem (1.1)–(1.3) with $g_0 = g_1 \equiv 0$, $f_1 \equiv 0$ and $\mu_2(x,t) \equiv \mu_2(x)$:

$$\begin{cases} u_t - \dfrac{\partial}{\partial x}[\mu_1(x,t)u_x] + \displaystyle\int_0^t g(t-s)\frac{\partial}{\partial x}[\mu_2(x)u_x(x,s)]\,ds = f(u), \\ \hfill 0 < x < 1,\ t > 0, \\ u_x(0,t) - h_0 u(0,t) = u_x(1,t) + h_1 u(1,t) = 0, \\ u(x,0) = u_0(x). \end{cases} \tag{4.1}$$

We make the following assumptions:

($\mathbf{A}_3'$) $\mu_1 \in C^1([0,1]\times\mathbb{R}_+)$, $\mu_1(x,t) \ge \underline{\mu}_1 > 0$, $\frac{\partial \mu_1}{\partial t}(x,t) \le 0$ for all $(x,t) \in [0,1]\times\mathbb{R}_+$;

($\mathbf{A}_4'$) $\mu_2 \in C^1([0,1])$, $\mu_2(x) \ge \underline{\mu}_2 > 0$ for all $x \in [0,1]$;

($\mathbf{A}_5'$) $f \in C(\mathbb{R};\mathbb{R})$ and there exist constants $p > 2$ and $\gamma > 0$ satisfying

(i) $uf(u) \ge \gamma|u|^p$, $\forall u \in \mathbb{R}$,

(ii) $uf(u) \ge pF(u) \equiv p\int_0^u f(z)\,dz$, $\forall u \in \mathbb{R}$;

($\mathbf{A}_6'$) $g \in C^1(\mathbb{R}_+;\mathbb{R}_+)$ and $0 < g(t) \le g(0)$, $g'(t) \le 0$, for all $t \ge 0$.

Remark 4.1. Below we present two examples of functions f satisfying ($\mathbf{A}_5'$). In the second example, f is more general than the function given in [12], [13].

Example 1. Set

$$f(u) = \gamma|u|^{p-2}u + \sum_{i=1}^N \alpha_i|u|^{q_i-2}u, \tag{4.2}$$

where $\gamma > 0$, $p > 2$, $\alpha_i \ge 0$, $q_i > 2$ are constants, with $2 < p \le q_i$, $i = 1,\dots,N$. It is obvious that ($\mathbf{A}_5'$) holds, since

$$uf(u) = \gamma|u|^p + \sum_{i=1}^N \alpha_i|u|^{q_i} \ge \gamma|u|^p,$$

$$p\int_0^u f(z)\,dz = \gamma|u|^p + \sum_{i=1}^N \alpha_i\frac{p}{q_i}|u|^{q_i} \le \gamma|u|^p + \sum_{i=1}^N \alpha_i|u|^{q_i} = uf(u).$$

EXAMPLE 2. Another example is

$$f(u) = |u|^{p-2}u\ln^k(e+u^2), \tag{4.3}$$

where $k > 1$ and $p > 2$ are constants. Because $\ln(e+u^2) \geq \ln e = 1$, we have

$$uf(u) = |u|^p \ln^k(e+u^2) \geq |u|^p = \gamma|u|^p, \quad \forall u \in \mathbb{R}, \quad \text{with } \gamma = 1.$$

Integration by parts gives

$$\int_0^u f(z)\,dz = \frac{1}{p}uf(u) - \frac{2k}{p}\int_0^u \ln^{k-1}(e+z^2)\frac{|z|^p z}{e+z^2}\,dz. \tag{4.4}$$

Note that $\int_0^u \ln^{k-1}(e+z^2)\frac{|z|^p z}{e+z^2}\,dz \geq 0$ for all $u \in \mathbb{R}$, so

$$\int_0^u f(z)\,dz \leq \frac{1}{p}uf(u), \qquad \forall u \in \mathbb{R}.$$

Hence, $(\mathbf{A}_5')$ holds.

Now, on H^1, we consider the following symmetric bilinear forms $a(\cdot,\cdot)$, $b(\cdot,\cdot)$, $\frac{\partial a_1}{\partial t}(t;\cdot,\cdot)$:

$$a(u,v) = \int_0^1 u_x(x)v_x(x)\,dx + h_0u(0)v(0) + h_1u(1)v(1),$$

$$b(u,v) = \int_0^1 \mu_2(x)u_x(x)v_x(x)\,dx + h_0\mu_2(0)u(0)v(0) + h_1\mu_2(1)u(1)v(1),$$

$$\frac{\partial a_1}{\partial t}(t;u,v) = \int_0^1 \mu_1'(x,t)u_x(x)v_x(x)\,dx + h_0\mu_1'(0,t)u(0)v(0) + h_1\mu_1'(1,t)u(1)v(1), \qquad \forall u,v \in H^1.$$

It is easy to show that the forms $a(\cdot,\cdot)$, $b(\cdot,\cdot)$ are continuous on $H^1 \times H^1$ and coercive on H^1. On the other hand, the norm $v \mapsto \|v\|_{H^1}$ and the norms $v \mapsto \|v\|_a = \sqrt{a(v,v)}$ and $v \mapsto \|v\|_b = \sqrt{b(v,v)}$ are equivalent.

Furthermore, we have the following lemmas.

LEMMA 4.2. *There exist positive constants* $\underline{a}$, $\bar{a}$, $\underline{\mu}_1$, $\bar{\mu}_1$, $\underline{\mu}_2$, $\bar{\mu}_2$ *such that:*

(i) $a(v,v) \geq \underline{a}\|v\|_{H^1}^2$,
(ii) $|a(u,v)| \leq \bar{a}\|u\|_{H^1}\|v\|_{H^1}$,
(iii) $b(v,v) \geq \underline{\mu}_2\|v\|_a^2$,
(iv) $|b(u,v)| \leq \bar{\mu}_2\|u\|_a\|v\|_a$,
(v) $a_1(t;v,v) \geq \underline{\mu}_1\|v\|_a^2$,
(vi) $|a_1(t;u,v)| \leq \bar{\mu}_1\|u\|_a\|v\|_a$,
(vii) $\frac{\partial a_1}{\partial t}(t;v,v) \leq 0$,

for all u, $v \in H^1$ *and all* $t \geq 0$, *where*

$$\bar{\mu}_1 = \sup_{0 \leq x \leq 1} \mu_1(x,0), \qquad \bar{\mu}_2 = \max_{0 \leq x \leq 1} \mu_2(x),$$
$$\underline{a} = \tfrac{1}{3} \min\{1; \max\{h_0, h_1\}\}, \qquad \bar{a} = 1 + 2h_0 + 2h_1.$$

LEMMA 4.3. *On* H^1, *the norms* $v \mapsto \|v\|_a$ *and* $v \mapsto \|v\|_b$ *are equivalent and*

$$\sqrt{\underline{\mu}_2}\, \|v\|_a \leq \|v\|_b \leq \sqrt{\bar{\mu}_2}\, \|v\|_a, \qquad \forall v \in H^1.$$

Now we define the modified energy functional related to Problem (4.1):

$$E(t) = \frac{1}{2}(g \diamond u)(t) + \frac{1}{2} a_1(t; u(t), u(t)) - \frac{1}{2}\Big(\int_0^t g(s)\,ds\Big)\|u(t)\|_b^2 - \int_0^1 dx \int_0^{u(x,t)} f(z)\,dz, \tag{4.5}$$

where

$$(g \diamond u)(t) = \int_0^t g(t-s)\|u(s) - u(t)\|_b^2\,ds.$$

By multiplying $(4.1)_1$ by u', and integrating over $(0,1)$, we get

$$E'(t) = -\|u'(t)\|^2 + \frac{1}{2}\frac{\partial a_1}{\partial t}(t; u(t), u(t)) - \frac{1}{2} g(t)\|u(t)\|_b^2 + \frac{1}{2}(g' \diamond u)(t) \leq 0 \tag{4.6}$$

for any regular solutions. The same result can be established for weak solutions and for almost every t, by a density argument.

THEOREM 4.4. *Let the assumptions* $(\mathbf{A}_1)$, $(\mathbf{A}_3')$–$(\mathbf{A}_6')$, *and* $(\bar{\mathbf{A}}_5)$ *hold. If in addition*

$$\int_0^\infty g(s)\,ds < \frac{\underline{\mu}_1}{\bar{\mu}_2}\Big(1 - \frac{1}{(p-1)^2}\Big),$$

then, for any $u_0 \in H^1$ *such that* $E(0) < 0$, *the weak solution* u *of Problem* (4.1) *blows up in finite time.*

Proof. We define

$$H(t) = -E(t), \qquad t \geq 0. \tag{4.7}$$

Then it follows from (4.6) that $H'(t) \geq 0$ for all $t \geq 0$. This implies that

$$H(t) \geq H(0) = -E(0) > 0, \qquad \forall t \geq 0. \tag{4.8}$$

Set

$$L_1(t) = \tfrac{1}{2}\|u(t)\|^2. \tag{4.9}$$

By taking the time derivative of (4.9) and using (4.1), we obtain

$$\begin{aligned} L_1'(t) &= \langle u'(t), u(t)\rangle \\ &= \langle f(u(t)), u(t)\rangle - a_1(t; u(t), u(t)) + \int_0^t g(t-s) b(u(s), u(t))\,ds \\ &\geq \langle f(u(t)), u(t)\rangle - a_1(t; u(t), u(t)) + \Big(\int_0^t g(s)\,ds\Big)\|u(t)\|_b^2 \\ &\quad - \int_0^t g(t-s)|b(u(s)-u(t), u(t))|\,ds. \end{aligned} \tag{4.10}$$

By using the Schwarz inequality and Young inequality, we obtain

$$\begin{aligned} &\int_0^t g(t-s)\big|b\big(u(s)-u(t), u(t)\big)\big|\,ds \\ &\qquad \leq \int_0^t g(t-s)\|u(t)\|_b\|u(s)-u(t)\|_b\,ds \\ &\qquad \leq \frac{1}{2\delta_1}\|u(t)\|_b^2\int_0^t g(s)\,ds + \frac{\delta_1}{2}(g\diamond u)(t) \end{aligned} \tag{4.11}$$

for all $\delta_1 > 0$.

By the definition of $H(t)$,

$$\begin{aligned} &\int_0^1 dx \int_0^{u(x,t)} f(z)\,dz \\ &\quad = H(t) + \frac{1}{2}\Big[a_1(t; u(t), u(t)) - \Big(\int_0^t g(s)\,ds\Big)\|u(t)\|_b^2 + (g\diamond u)(t)\Big]. \end{aligned} \tag{4.12}$$

Since

$$\varPhi(x) = \frac{\mu_1}{\bar{\mu}_2}\Big(1 - \frac{1}{(x-1)^2}\Big)$$

is continuous and nondecreasing on $2 \leq x \leq p$, so that

$$0 = \varPhi(2) < \int_0^\infty g(s)\,ds < \frac{\mu_1}{\bar{\mu}_2}\Big(1 - \frac{1}{(p-1)^2}\Big) = \varPhi(p),$$

it follows that there exists a unique constant $\hat{p} \in (2, p)$ such that

$$\int_0^\infty g(s)\,ds = \varPhi(\hat{p}). \tag{4.13}$$

Set $\delta_2 = \hat{p}/p$ and $\delta_1 = p\delta_2 = \hat{p}$, $0 < \delta_2 < 1$. We deduce from (4.8) and (4.10)–(4.13) that

$$\begin{aligned}(4.14)\quad L_1'(t) &\geq (1-\delta_2)\langle f(u(t)), u(t)\rangle \\ &\quad + \delta_2 p\Big[\int_0^1 dx \int_0^{u(x,t)} f(z)\,dz\Big] - a_1(t; u(t), u(t)) \\ &\quad + \Big(\int_0^t g(s)\,ds\Big)\|u(t)\|_b^2 - \frac{1}{2\delta_1}\|u(t)\|_b^2 \int_0^t g(s)\,ds - \frac{\delta_1}{2}(g \diamond u)(t) \\ &\geq (1-\hat{p}/p)\langle f(u(t)), u(t)\rangle \\ &\quad + \frac{1}{2\hat{p}}(\hat{p}-1)^2\Big[\frac{\mu_1}{\overline{\mu_2}}\Big(1 - \frac{1}{(\hat{p}-1)^2}\Big) - \int_0^\infty g(s)\,ds\Big]\|u(t)\|_b^2 \\ &= (1-\hat{p}/p)\langle f(u(t)), u(t)\rangle \geq (1-\hat{p}/p)\gamma\|u(t)\|_{L^p}^p.\end{aligned}$$

On the other hand, $\|v\| \leq \|v\|_{L^p}$ for all $v \in L^p$. Hence

$$(4.15)\qquad \sqrt{2}\,L_1^{p/2}(t) = \|u(t)\|^p \leq \|u(t)\|_{L^p}^p.$$

We deduce from (4.14) and (4.15) that

$$(4.16)\qquad L_1'(t) \geq (1-\hat{p}/p)\gamma\sqrt{2}\,L_1^{p/2}(t) \equiv \gamma_1 L_1^{p/2}(t).$$

A direct integration of (4.16) then yields

$$(4.17)\qquad L_1^{p/2-1}(t) \geq \frac{1}{L_1^{1-p/2}(0) - (p/2-1)\gamma_1 t}.$$

Therefore $L_1(t)$ blows up in time $T^* \leq \frac{2}{(p-2)\gamma_1} L_1^{1-p/2}(0)$. Theorem 4.1 is proved. ∎

5. Exponential decay of solution. This section investigates the decay of the solution of (1.1)–(1.3) corresponding to $g_0 = g_1 \equiv 0$ and $\mu_2(x,t) \equiv \mu_2(x)$.

We prove that if $a_1(0; u_0, u_0) - p\int_0^1 dx \int_0^{u_0(x)} f(z)\,dz > 0$ and if the initial energy $\|f_1(t)\|$ is small enough, then the energy of the solution decays exponentially as $t \to \infty$. For this purpose, we make the following assumptions:

($\mathbf{A}_5''$) $f \in C(\mathbb{R}, \mathbb{R})$ and there exist constants $\bar{d}_2 > 0$, $d_2 > p > 2$, $q_i > 2$, with $2 < p \leq q_i$, $i = 1, \ldots, N$, satisfying

- (i) $f(0) = 0$, $uf(u) > 0$, $\forall u \in \mathbb{R}$, $u \neq 0$,
- (ii) $uf(u) \leq d_2 F(u) \equiv d_2 \int_0^u f(z)\,dz$, $\forall u \in \mathbb{R}$,
- (iii) $F(u) = \int_0^u f(z)\,dz \leq \bar{d}_2(|u|^p + \sum_{i=1}^N |u|^{q_i})$, $\forall u \in \mathbb{R}$;

$(\mathbf{A}_6'')$ $g \in C^1(\mathbb{R}_+;\mathbb{R}_+)$ and

(i) $0 < g(t) \le g(0)$, $g'(t) \le 0$, $\forall t \ge 0$;
$L \equiv \underline{\mu}_1 - \bar{\mu}_2 \int_0^\infty g(s)\,ds > 0$ for $\bar{\mu}_2 = \max_{0\le x\le 1} \mu_2(x)$;
(ii) $g'(t) \le -\xi_1 g(t)$, $\forall t \ge 0$, $\xi_1 > 0$;

$(\mathbf{A}_7'')$ $f_1 \in L^2(\mathbb{R}_+;L^2)$, and there exist constants $C_0, \gamma_0 > 0$ such that

$$\|f_1(t)\| \le C_0 e^{-\gamma_0 t}, \qquad \forall t \ge 0.$$

REMARK 5.1. We will show that the examples of f in Section 4 also satisfy $(\mathbf{A}_5'')$.

1. Consider

$$f(u) = \gamma|u|^{p-2}u + \sum_{i=1}^N \alpha_i |u|^{q_i-2}u, \tag{5.1}$$

where $\gamma > 0$, $\alpha_i \ge 0$, p, q_i are constants with $2 < p \le q_i$, $i = 1,\dots,N$. It is obvious that $(\mathbf{A}_5'')$ holds, because of

$$uf(u) = \gamma|u|^p + \sum_{i=1}^N \alpha_i|u|^{q_i} \ge \gamma|u|^p > 0, \qquad \forall u \in \mathbb{R},\ u \ne 0;$$

$$\int_0^u f(z)\,dz = \frac{\gamma}{p}|u|^p + \sum_{i=1}^N \alpha_i \frac{1}{q_i}|u|^{q_i} \le \bar{d}_2\Big(|u|^p + \sum_{i=1}^N |u|^{q_i}\Big);$$

$$\begin{aligned} uf(u) &= \gamma|u|^p + \sum_{i=1}^N \alpha_i|u|^{q_i} \le d_2\Big(\frac{\gamma}{p}|u|^p + \sum_{i=1}^N \alpha_i\frac{1}{q_i}|u|^{q_i}\Big) \\ &= d_2 \int_0^u f(z)\,dz, \end{aligned}$$

with $\bar{d}_2 = \max\{\gamma/p, \alpha_1/q_1, \dots, \alpha_N/q_N\}$, $d_2 = 1 + \max\{p, q_1, \dots, q_N\} > p$.

2. Consider

$$f(u) = |u|^{p-2}u \ln^k(e + u^2), \tag{5.2}$$

where $k > 1$ and $p > 2k$ are constants. Obviously, $(\mathbf{A}_5'', \mathrm{i})$ holds.

As for $(\mathbf{A}_5'', \mathrm{iii})$, we have $uf(u) = |u|^p \ln^k(e + u^2)$, so

$$\int_0^u f(z)\,dz = \frac{1}{p}uf(u) - \frac{2k}{p}\int_0^u \ln^{k-1}(e + z^2)\frac{|z|^p z}{e + z^2}\,dz. \tag{5.3}$$

As $\int_0^u \ln^{k-1}(e + z^2)\frac{|z|^p z}{e+z^2}\,dz \ge 0$ for all $u \in \mathbb{R}$ by the inequality $\ln(1 + x) \le x$

for all $x \geq 0$, this yields

$$\int_0^u f(z)\,dz \leq \frac{1}{p} u f(u) = \frac{1}{p}|u|^p \ln^k(e+u^2)$$
$$= \frac{1}{p}|u|^p \Big[1+\ln\Big(1+\frac{u^2}{e}\Big)\Big]^k \leq \frac{1}{p}|u|^p \Big(1+\frac{u^2}{e}\Big)^k.$$

Choosing an integer $N \geq k$ and applying the Newton formula, we get

$$\frac{1}{p}|u|^p\Big(1+\frac{u^2}{e}\Big)^k \leq \frac{1}{p}|u|^p\Big(1+\frac{u^2}{e}\Big)^N = \frac{1}{p}|u|^p\Big[1+\sum_{i=1}^N C_N^i\Big(\frac{u^2}{e}\Big)^i\Big]$$
$$= \bar{d}_2\Big(|u|^p + \sum_{i=1}^N |u|^{q_i}\Big), \qquad \forall u \in \mathbb{R},$$

where $\bar{d}_2 = \frac{1}{p}\max\{1; \max_{1\leq i\leq N} C_N^i/e^i\}$ and $q_i = p+2i$. Thus, $(\mathbf{A}_5'', \mathrm{iii})$ holds.

For $(\mathbf{A}_5'', \mathrm{ii})$, note that

$$G(u) = \int_0^u \ln^{k-1}(e+z^2)\frac{|z|^p z}{e+z^2}\,dz = G(-u) \geq 0, \qquad \forall u \in \mathbb{R}.$$

For $u \geq 0$,

$$G(u) = \int_0^u \ln^{k-1}(e+z^2)\frac{|z|^p z}{e+z^2}\,dz \leq \ln^{k-1}(e+u^2)\int_0^u \frac{|z|^p z}{e+z^2}\,dz$$
$$\leq \ln^{k-1}(e+u^2)\int_0^u z^{p-1}\,dz = \frac{1}{p}u^p \ln^{k-1}(e+u^2)$$
$$\leq \frac{1}{p}u^p\ln^k(e+u^2) = \frac{1}{p}|u|^p\ln^k(e+u^2).$$

For $u \leq 0$,

$$G(u) = G(-u) \leq \frac{1}{p}|-u|^p\ln^k(e+(-u)^2) = \frac{1}{p}|u|^p\ln^k(e+u^2).$$

Therefore

$$G(u) = \int_0^u \ln^{k-1}(e+z^2)\frac{|z|^p z}{e+z^2}\,dz$$
$$\leq \frac{1}{p}|u|^p\ln^k(e+u^2) = \frac{1}{p}u f(u), \qquad \forall u \in \mathbb{R}.$$

It follows from (5.3) that

$$\int_0^u f(z)\,dz \geq \frac{1}{p}u f(u) - \frac{2k}{p^2}u f(u) = \frac{p-2k}{p^2}u f(u),$$

or

$$uf(u) \leq \frac{p^2}{p-2k}\int_0^u f(z)\,dz \equiv d_2\int_0^u f(z)\,dz, \qquad \forall u \in \mathbb{R}, \tag{5.4}$$

Consequently, $(\mathbf{A}_5'', \mathrm{ii})$ is true with

$$d_2 = \frac{p^2}{p-2k} = p + 2k + \frac{4k^2}{p-2k} > p + 2k > p.$$

Now, we define the following Lyapunov functional:

$$L(t) = E(t) + \frac{\delta}{2}\|u(t)\|^2 \equiv E(t) + \delta L_1(t), \tag{5.5}$$

where

$$\begin{aligned} E(t) &= \frac{1}{2}(g \diamond u)(t) + \frac{1}{2}a_1(t; u(t), u(t)) \\ &\quad - \frac{1}{2}\Big(\int_0^t g(s)\,ds\Big)\|u(t)\|_b^2 - \int_0^1 dx \int_0^{u(x,t)} f(z)\,dz \\ &= \frac{1}{2}(g \diamond u)(t) \\ &\quad + \Big(\frac{1}{2} - \frac{1}{p}\Big)\Big(a_1(t; u(t), u(t)) - \|u(t)\|_b^2 \int_0^t g(s)\,ds\Big) + \frac{1}{p}I(t), \end{aligned}$$

$$I(t) = I(u(t)) = a_1(t; u(t), u(t)) - \|u(t)\|_b^2 \int_0^t g(s)\,ds - p\int_0^1 dx \int_0^{u(x,t)} f(z)\,dz.$$

Then we have the following lemmas.

Lemma 5.2. *Assume that* $(\mathbf{A}_1)$, $(\mathbf{A}_3')$, $(\mathbf{A}_4')$, $(\mathbf{A}_5'')$–$(\mathbf{A}_7'')$, *and* $(\bar{\mathbf{A}}_5)$ *hold. Then*

$$\begin{aligned} E'(t) &\leq -(1 - \varepsilon_1/2)\|u'(t)\|^2 - \frac{1}{2}g(t)\|u(t)\|_b^2 \\ &\quad - \frac{1}{2}\xi_1(g \diamond u)(t) + \frac{1}{2\varepsilon_1}\|f_1(t)\|^2, \qquad \forall \varepsilon_1 > 0. \end{aligned} \tag{5.6}$$

Proof. Multiplying $(1.1)_1$ by $u'(x,t)$ and integrating over $[0,1]$, we get

$$\begin{aligned} E'(t) &= -\|u'(t)\|^2 + \frac{1}{2}\frac{\partial a_1}{\partial t}(t; u(t), u(t)) \\ &\quad - \frac{1}{2}g(t)\|u(t)\|_b^2 + \frac{1}{2}(g' \diamond u)(t) + \langle f_1(t), u'(t)\rangle \end{aligned} \tag{5.7}$$

for any regular solution u. We can extend (5.7) to weak solutions by using density arguments.

On the other hand, we have

$$\langle f_1(t), u'(t)\rangle \le \frac{\varepsilon_1}{2}\|u'(t)\|^2 + \frac{1}{2\varepsilon_1}\|f_1(t)\|^2, \tag{5.8}$$

$$\frac{1}{2}(g' \diamond u)(t) \le -\frac{1}{2}\xi_1 (g \diamond u)(t), \tag{5.9}$$

$$\frac{1}{2}\frac{\partial a_1}{\partial t}(t; u(t), u(t)) \le 0. \tag{5.10}$$

From (5.7)–(5.10), we get (5.6), and Lemma 5.2 is proved. ∎

Lemma 5.3. *Assume that* $(\mathbf{A}_1)$, $(\mathbf{A}_3')$, $(\mathbf{A}_4')$, $(\mathbf{A}_5'')$–$(\mathbf{A}_7'')$, *and* $(\bar{\mathbf{A}}_5)$ *hold. Suppose* $I(0) > 0$ *and*

$$\eta_* = L - p\bar{d}_2\Big(\tilde{D}_p^p R_*^{p-2} + \sum_{i=1}^{N} \tilde{D}_{q_i}^{q_i} R_*^{q_i-2}\Big) > \Big(1 - \frac{p}{d_2}\Big)\overline{\mu}_1 > 0, \tag{5.11}$$

where

$$R_* = \sqrt{\frac{2p}{(p-2)L}\Big(E(0) + \frac{1}{2}\int_0^\infty \|f_1(t)\|^2\, dt\Big)}, \qquad \tilde{D}_p = \sup_{0\neq v \in H^1} \frac{\|v\|_{L^p}}{\|v\|_a},\ p > 2.$$

Then $I(t) > 0$ *for all* $t \ge 0$.

Proof. By the continuity of $I(t)$ and $I(0) > 0$, there exists $T_1 > 0$ such that

$$I(t) = I(u(t)) \ge 0, \qquad \forall t \in [0, T_1]. \tag{5.12}$$

This gives

$$\begin{aligned} E(t) &\ge \Big(\frac{1}{2} - \frac{1}{p}\Big)\Big(a_1(t; u(t), u(t)) - \|u(t)\|_b^2 \int_0^t g(s)\, ds\Big) \\ &\ge \Big(\frac{1}{2} - \frac{1}{p}\Big)\Big(\underline{\mu}_1 - \bar{\mu}_2 \int_0^\infty g(s)\, ds\Big)\|u(t)\|_a^2 \\ &= \frac{(p-2)L}{2p}\|u(t)\|_a^2, \qquad \forall t \in [0, T_1]. \end{aligned}$$

Hence

$$\|u(t)\|_a^2 \le \frac{2p}{(p-2)L} E(t), \qquad \forall t \in [0, T_1]. \tag{5.13}$$

From (5.6) with $\varepsilon_1 = 1$ and (5.13), we deduce that

$$\begin{aligned} \|u(t)\|_a^2 &\le \frac{2p}{(p-2)L} E(t) \\ &\le \frac{2p}{(p-2)L}\Big[E(0) + \frac{1}{2}\int_0^\infty \|f_1(t)\|^2\, dt\Big] \equiv R_*^2, \qquad \forall t \in [0, T_1]. \end{aligned} \tag{5.14}$$

We note that

$$\begin{aligned}E(0) &= \frac{1}{2}a_1(0;u_0,u_0) - \int_0^1 dx \int_0^{u_0(x)} f(z)\,dz \\ &= \frac{1}{2}\Big(a_1(0;u_0,u_0) - 2\int_0^1 dx \int_0^{u_0(x)} f(z)\,dz\Big) \\ &\geq \frac{1}{2}\Big(a_1(0;u_0,u_0) - p\int_0^1 dx \int_0^{u_0(x)} f(z)\,dz\Big) = \frac{1}{2}I(0) > 0.\end{aligned}$$

By $(\mathbf{A}_5'', \text{iii})$, we get

$$\begin{aligned}p\int_0^1 dx \int_0^{u(x,t)} f(z)\,dz &\leq p\bar{d}_2\Big(\|u(t)\|_{L^p}^p + \sum_{i=1}^N \|u(t)\|_{L^{q_i}}^{q_i}\Big) \\ &\leq p\bar{d}_2\Big(\tilde{D}_p^p\|u(t)\|_a^{p-2} + \sum_{i=1}^N \tilde{D}_{q_i}^{q_i}\|u(t)\|_a^{q_i-2}\Big)\|u(t)\|_a^2 \\ &\leq p\bar{d}_2\Big(\tilde{D}_p^p R_*^{p-2} + \sum_{i=1}^N \tilde{D}_{q_i}^{q_i} R_*^{q_i-2}\Big)\|u(t)\|_a^2, \qquad \forall t \in [0,T_1].\end{aligned}$$

Therefore

$$\begin{aligned}I(t) &= a_1(t;u(t),u(t)) - \|u(t)\|_b^2 \int_0^t g(s)\,ds \\ &\quad - p\int_0^1 dx \int_0^{u(x,t)} f(z)\,dz \\ &\geq \Big[L - p\bar{d}_2\Big(\tilde{D}_p^p R_*^{p-2} + \sum_{i=1}^N \tilde{D}_{q_i}^{q_i} R_*^{q_i-2}\Big)\Big]\|u(t)\|_a^2 \\ &\equiv \eta_*\|u(t)\|_a^2 > 0, \qquad \forall t \in [0,T_1].\end{aligned}$$

Now, we set $T_\infty = \sup\{T > 0 : I(t) > 0 \text{ for all } t \in [0,T]\}$. If $T_\infty < \infty$ then, by the continuity of $I(t)$, we have $I(T_\infty) \geq 0$. By the same arguments as above, we can deduce that there exists $\bar{T}_\infty > T_\infty$ such that $I(t) > 0$ for all $t \in [0,\bar{T}_\infty]$. Hence $I(t) > 0$ for all $t \geq 0$. Lemma 5.3 is proved. ∎

LEMMA 5.4. *Let $I(0) > 0$ and (5.11) hold. Set*

$$E_1(t) = (g \diamond u)(t) + \|u(t)\|_a^2 + I(t).$$

Then there exist positive constants β_1, β_2 such that

$$\beta_1 E_1(t) \leq L(t) \leq \beta_2 E_1(t), \qquad \forall t \geq 0. \tag{5.15}$$

Proof. It is not difficult to see that

$$L(t)=\frac{1}{2}(g\diamond u)(t)+\Big(\frac{1}{2}-\frac{1}{p}\Big)\Big(a_1(t;u(t),u(t))-\|u(t)\|_b^2\int_0^t g(s)\,ds\Big)$$
$$+\frac{1}{p}I(t)+\frac{\delta}{2}\|u(t)\|^2$$
$$\geq\frac{1}{2}(g\diamond u)(t)+\frac{(p-2)L}{2p}\|u(t)\|_a^2+\frac{1}{p}I(t)\geq\beta_1E_1(t),$$

where $\beta_1=\min\{\frac{(p-2)L}{2p},\frac{1}{p}\}$. Similarly

$$L(t)\leq\frac{1}{2}(g\diamond u)(t)+\Big(\frac{1}{2}-\frac{1}{p}\Big)\bar{\mu}_1\|u(t)\|_a^2+\frac{1}{p}I(t)+\frac{\delta}{2}\tilde{D}_2^2\|u(t)\|_a^2$$
$$=\frac{1}{2}(g\diamond u)(t)+\Big[\Big(\frac{1}{2}-\frac{1}{p}\Big)\bar{\mu}_1+\frac{\delta}{2}\tilde{D}_2^2\Big]\|u(t)\|_a^2+\frac{1}{p}I(t)\leq\beta_2E_1(t),$$

where $\beta_2=\max\{\frac{1}{2},(\frac{1}{2}-\frac{1}{p})\bar{\mu}_1+\frac{\delta}{2}\tilde{D}_2^2\}$. Lemma 5.4 is proved. ∎

LEMMA 5.5. *Suppose $I(0)>0$ and (5.11) holds. Then*

$$(5.16)\qquad L_1'(t)\leq\frac{1}{2\varepsilon_2}(g\diamond u)(t)-\frac{\varepsilon_3d_2}{p}I(t)+\frac{1}{2\varepsilon_2}\|f_1(t)\|^2$$
$$-\Big[(1-\varepsilon_3)\frac{d_2}{p}\eta_*-\Big(\frac{d_2}{p}-1\Big)\overline{\mu}_1-\frac{\varepsilon_2}{2}\tilde{D}_2^2\Big]\|u(t)\|_a^2$$
$$-\Big(\frac{d_2}{p}-1-\frac{\varepsilon_2}{2}\Big)\|u(t)\|_b^2\int_0^t g(s)\,ds$$

for all $\varepsilon_2>0$ and $\varepsilon_3\in(0,1)$.

Proof. By multiplying $(1.1)_1$ by $u(x,t)$ and integrating over $[0,1]$, we obtain

$$(5.17)\qquad L_1'(t)=-a_1(t;u(t),u(t))+\|u(t)\|_b^2\int_0^t g(s)\,ds$$
$$+\int_0^t g(t-s)b\big(u(s)-u(t),u(t)\big)\,ds+\langle f(u(t)),u(t)\rangle+\langle f_1(t),u(t)\rangle.$$

Note that

$$(5.18)\qquad\begin{cases}a_1(t;u(t),u(t))-\|u(t)\|_b^2\int_0^t g(s)\,ds\leq\overline{\mu}_1\|u(t)\|_a^2-\|u(t)\|_b^2\int_0^t g(s)\,ds,\\ \int_0^t g(t-s)b\big(u(s)-u(t),u(t)\big)\,ds\leq\frac{\varepsilon_2}{2}\|u(t)\|_b^2\int_0^t g(s)\,ds+\frac{1}{2\varepsilon_2}(g\diamond u)(t),\\ \langle f_1(t),u(t)\rangle\leq\frac{\varepsilon_2}{2}\tilde{D}_2^2\|u(t)\|_a^2+\frac{1}{2\varepsilon_2}\|f_1(t)\|^2,\qquad\forall\varepsilon_2>0,\end{cases}$$

and

$$(5.19)\qquad \langle f(u(t)),u(t)\rangle \le d_2\int_0^1 dx\int_0^{u(x,t)} f(z)\,dz$$
$$= \frac{d_2}{p}\Big[a_1(t;u(t),u(t)) - \|u(t)\|_b^2\int_0^t g(s)\,ds - I(t)\Big]$$
$$\le \frac{d_2}{p}\Big(a_1(t;u(t),u(t)) - \|u(t)\|_b^2\int_0^t g(s)\,ds\Big)$$
$$- \frac{\varepsilon_3 d_2}{p}I(t) - \frac{(1-\varepsilon_3)d_2}{p}\eta_*\|u(t)\|_a^2,\qquad \forall\varepsilon_3\in(0,1).$$

By using (5.17)–(5.19), we deduce Lemma 5.5. ∎

Finally, we have the following theorem.

THEOREM 5.6. *Assume that* $(\mathbf{A}_1)$, $(\mathbf{A}_3')$, $(\mathbf{A}_4')$, $(\mathbf{A}_5'')$–$(\mathbf{A}_7'')$ *and* $(\bar{\mathbf{A}}_5)$ *hold and* $u_0\in H^1$. *Let* $I(0)>0$ *and suppose the initial energy* $E(0)$ *satisfies* (5.11). *Then there exist positive constants* C, γ *such that*

$$(5.20)\qquad E_1(t)\le Ce^{-\gamma t},\qquad \forall t\ge 0.$$

Proof. It follows from (5.5), (5.6) and (5.16) that

$$(5.21)\qquad L'(t)\le -\Big(1-\frac{\varepsilon_1}{2}\Big)\|u'(t)\|^2 - \frac12\Big(\xi_1-\frac{\delta}{\varepsilon_2}\Big)(g\diamond u)(t) - \frac{\delta\varepsilon_3 d_2}{p}I(t)$$
$$-\delta\Big\{\frac{d_2}{p}\Big[\eta_*-\Big(1-\frac{p}{d_2}\Big)\overline{\mu}_1\Big] - \varepsilon_3\frac{d_2}{p}\eta_* - \frac{\varepsilon_2}{2}\tilde{D}_2^2\Big\}\|u(t)\|_a^2$$
$$-\delta\Big(\frac{d_2}{p}-1-\frac{\varepsilon_2}{2}\Big)\|u(t)\|_b^2\int_0^t g(s)\,ds + \rho(t)$$

for all $\delta,\varepsilon_1,\varepsilon_2>0$ and $0<\varepsilon_3<1$, where

$$(5.22)\qquad \rho(t)=\frac12\Big(\frac{1}{\varepsilon_1}+\frac{\delta}{\varepsilon_2}\Big)\|f_1(t)\|^2\le C_*e^{-2\gamma_0 t}.$$

As $\eta_*>(1-p/d_2)\overline{\mu}_1>0$, we can choose $\varepsilon_2>0$ and $\varepsilon_3\in(0,1)$ such that

$$(5.23)\qquad \begin{aligned} &\frac{d_2}{p}-1-\frac{\varepsilon_2}{2}>0,\\ &\sigma_1=\frac{d_2}{p}\Big[\eta_*-\Big(1-\frac{p}{d_2}\Big)\overline{\mu}_1\Big]-\varepsilon_3\frac{d_2}{p}\eta_*-\frac{\varepsilon_2}{2}\tilde{D}_2^2>0.\end{aligned}$$

We continue by choosing δ, ε_1 such that

$$(5.24)\qquad 1-\frac{\varepsilon_1}{2}>0,\qquad \sigma_2=\frac12\Big(\xi_1-\frac{\delta}{\varepsilon_2}\Big)>0.$$

Then, we deduce from (5.15) and (5.21)–(5.24) that there exists a constant $\gamma > 0$ such that

$$L'(t) \le -\gamma_1[(g \diamond u)(t) + \|u(t)\|_a^2 + I(t)] + C_* e^{-2\gamma_0 t} \le -\gamma L(t) + C_* e^{-2\gamma_0 t}, \tag{5.25}$$

where

$$\gamma_1 = \min\{\varepsilon_3 d_2 \delta/p, \delta\sigma_1, \sigma_2\} > 0, \qquad 0 < \gamma < \min\{\gamma_1, \gamma_1/\beta_2, 2\gamma_0\}.$$

By directly integrating (5.25), we deduce

$$\beta_1 E_1(t) \le L(t) \le \left(L(0) + \frac{C_*}{2\gamma_0 - \gamma}\right) e^{-\gamma t}, \qquad \forall t \ge 0.$$

This implies (5.20) and Theorem 5.1 is proved. ∎

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