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SETS OF p-UNIQUENESS ON NONCOMMUTATIVE LOCALLY COMPACT GROUPS

BY

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Abstract. We prove that a closed subgroup H of a locally compact group G is a set of p-uniqueness ($1 < p < \infty$) if and only if H is locally negligible. We also obtain the inverse projection theorem for sets of p-uniqueness.

1. Introduction. Let G be a locally compact group and $1 < p < \infty$. This paper is concerned with the study of the class of closed subsets of G of p-uniqueness.

We recall some definitions. Let $\mathcal{L}(L^p(G))$ be the Banach algebra of all bounded operators of $L^p(G)$, the L^p-space with respect to a left Haar measure. We denote by $PF_p(G)$ the norm closure in $\mathcal{L}(L^p(G))$ of $\{\lambda_G^p(f) \mid f \in \mathcal{L}^1(G)\}$ where $\lambda_G^p(f)$ is the operator $\varphi \mapsto \int_G \varphi(xy)\Delta_G(y)^{1/p} f(y)\,dy$. Then $PF_p(G)$ is a subalgebra of the Banach algebra $CV_p(G)$ of all convolution operators of $L^p(G)$.

We recall the definition of $CV_p(G)$: For φ a map of G into $\mathbb{C}$ we set ${}_a\varphi(x) = \varphi(ax)$ and $\varphi_a(x) = \varphi(xa)$ for $a, x \in G$. A bounded linear operator T of $L^p(G)$ is said to be a *convolution operator* (written $T \in CV_p(G)$) if $T({}_a\varphi) = {}_aT(\varphi)$ for every $a \in G$ and every $\varphi \in L^p(G)$.

Replacing above $f \in \mathcal{L}^1(G)$ by an arbitrary bounded measure μ, we obtain the convolution operator $\lambda_G^p(\mu)\varphi = \varphi * \Delta_G^{1/p'}\check{\mu}$. We have $\|\lambda_G^p(\mu)\|_p \le \|\mu\|$ and $\lambda_G^p(\delta_a) = \varphi_a \Delta_G(a)^{1/p}$. The algebra $PF_2(G)$ is the reduced C^*-algebra of the group G. Every element of $PF_p(G)$ is called a *p-pseudofunction* [11]. The norm closure in $\mathcal{L}(L^p(G))$ of $\{\rho_G^p(f) \mid f \in \mathcal{L}^1(G)\}$ where $\rho_G^p(f)$ is the operator $\varphi \mapsto f * \varphi$, is canonically isometric (as a Banach algebra) to $PF_p(G)$ via the map $T \mapsto \tau_p T \tau_p$ where $\tau_p\varphi(x) = \varphi(x^{-1})\Delta_G(x^{-1})^{1/p}$; this algebra leads therefore to similar results.

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A closed subset F of G is said to be a *set of p-uniqueness* if there are no non-zero p-pseudofunctions supported by F. A closed subset of $\mathbb{T}$ is a set of 2-uniqueness if and only if it is a set of uniqueness in the classical sense of the theory of trigonometric series [13, 16].

We prove that a closed subgroup H is a set of p-uniqueness if and only if H is locally negligible in G. This improves results of [1–5, 7, 10, 12, 15]. For H a closed subgroup and F a closed subset of H we show that if F a set p-uniqueness of H then F is a set of p-uniqueness of G; we also prove the converse if H is open in G. Finally, in analogy with the case of sets of synthesis, for H a closed normal subgroup of G and F a closed subset of G/H, we prove (the inverse projection theorem for sets of p-uniqueness) that F is a set of p-uniqueness of G/H if and only if $\omega^{-1}(F)$ is a set of p-uniqueness of G, where ω is the canonical map of G onto G/H. This generalizes similar results of [5] and [15].

2. Subgroups of p-uniqueness. We denote by $(u,T)\mapsto u\cdot T$ the canonical structure of left $\mathcal{L}(L^p(G))$-module on the Figà-Talamanca–Herz algebra $A_p(G)$. Let H be an arbitrary closed subgroup of G. We recall (see [9]) that there exists a contractive linear map $\mathcal{P}$ of $\mathcal{L}(L^p(G))$ into $\mathcal{L}(L^p(H))$ such that

$$\mathcal{P}(u\cdot T)=\mathrm{Res}_H\, u\cdot \mathcal{P}(T)$$

for $u\in A_p(G)$ and $T\in \mathcal{L}(L^p(G))$ and with $\mathcal{P}(i(S))=S$ for every $S\in CV_p(H)$, where i denotes the canonical Banach algebra isometry of $CV_p(H)$ into $CV_p(G)$.

THEOREM 2.1. *Let G be a locally compact group, H a closed subgroup of G and $1<p<\infty$. Then H is a set of p-uniqueness in G if and only if H is locally negligible in G.*

Proof. Let m_G be a left Haar measure of G and m_H a left Haar measure of H.

Observe first that H is locally negligible in G if and only if H is non-open in G. Indeed, suppose that H is not locally negligible. Then there is a compact subset K of G such that $m_G(K\cap H)>0$. According to Steinhaus $(K\cap H)(K\cap H)^{-1}$ is a neighborhood of e in G, and therefore H is open in G. Conversely, suppose H is locally negligible and open in G. There is a compact neighborhood V of e in G with $V\subset H$. We have $m_G(V)>0$, and consequently H is not locally negligible.

We show now that if H is a set of p-uniqueness in G then H is locally negligible in G.

Assume that H is not locally negligible. Let T be a p-pseudofunction on H with $T\neq 0$. There is a sequence $(f_n)\subset C_{00}(H)$ with $\|\lambda_H^p(f_n)-T\|\to 0$ and therefore $\|i(\lambda_H^p(f_n))-i(T)\|\to 0$. The subgroup H being open, there is $c > 0$ such that $\mathrm{Res}_H\, m_G = c m_H$, hence $i(\lambda_H^p(f_n)) = \lambda_G^p((c^{-1}f_n)')$ where for $f : H \to \mathbb{C}$ we define $f'(x) = f(x)$ if $x \in H$ and $f'(x) = 0$ otherwise. We find that $i(T)$ is a p-pseudofunction of G supported by H, and therefore $T = 0$, a contradiction.

Conversely, suppose that H is locally negligible. We will prove that H is a set of p-uniqueness.

Let T be a p-pseudofunction of G supported by H. There is a sequence $(f_n) \subset C_{00}(G)$ such that $\|\lambda_G^p(f_n) - T\|_p \to 0$. Therefore

$$\|\mathcal{P}(\lambda_G^p(f_n)) - \mathcal{P}(T)\|_p \to 0.$$

According to N. Lohoué [14, Théorème 5, p. 190] and [8, Theorem 10, p. 129] there is $S \in CV_p(H)$ with $i(S) = T$, and so

$$\|\mathcal{P}(\lambda_G^p(f_n)) - S\|_p \to 0.$$

It suffices to show that $\mathcal{P}(\lambda_G^p(f)) = 0$ for every $f \in C_{00}(G)$.

Let f be a complex valued continuous function on G with compact support such that $f \neq 0$.

Suppose at first $\mathrm{supp}\, f \subset H$. We have $f = 1_H f$, and consequently $\lambda_G^p(f) = \lambda_G^p(f(1_H m_G))$. The subgroup H being locally negligible, we have $1_H m_G = 0$; this implies $\lambda_G^p(f) = 0$, and therefore $\mathcal{P}(\lambda_G^p(f)) = 0$.

It remains to consider the case of $\mathrm{supp}\, f \not\subset H$. We have

$$m_G(\mathrm{supp}\, f \cap H) = (1_H m_G)(\mathrm{supp}\, f) = 0.$$

Let $\varepsilon > 0$. There is an open subset U_1 of G with $\mathrm{supp}\, f \cap H \subset U_1$ and

$$m_G(U_1) < \varepsilon/\|f\|_\infty.$$

The relation

$$(\mathrm{supp}\, f \setminus U_1) \cap H = \emptyset$$

implies the existence of an open subset U_2 of G with $\mathrm{supp}\, f \setminus U_1 \subset U_2$ and $U_2 \cap H = \emptyset$. Consequently,

$$\mathrm{supp}\, f \subset U_1 \cup U_2.$$

We can therefore find $\tau_1, \tau_2 \in C_{00}(G)$ with the following properties:

- $\tau_1, \tau_2 \geq 0$,
- $\mathrm{supp}\, \tau_1 \subset U_1$, $\mathrm{supp}\, \tau_2 \subset U_2$,
- $\tau_1 + \tau_2 \leq 1_G$, and
- $\tau_1(x) + \tau_2(x) = 1$ for every $x \in \mathrm{supp}\, f$.

Hence $f = \tau_1 f + \tau_2 f$ and

$$\mathcal{P}(\lambda_G^p(f)) = \mathcal{P}(\lambda_G^p(\tau_1 f)) + \mathcal{P}(\lambda_G^p(\tau_2 f)).$$

We have

$$\|\mathcal{P}(\lambda_G^p(\tau_1 f))\|_p \leq \|\lambda_G^p(\tau_1 f)\|_p \leq \int_G \tau_1(x)|f(x)|\, dx \leq \int_{U_1} |f(x)|\, dx < \varepsilon.$$

We now choose $u \in A_{p'} \cap C_{00}(G)$ with $u(x) = 1$ for every $x \in \operatorname{supp} \tau_2$ and $\operatorname{supp} u \subset U_2$. Setting $v = \tilde{u}$ (where $\tilde{u}(x) = \overline{u(x^{-1})}$), we have $\tilde{v}\tau_2 = \tau_2$, and therefore

$$\lambda_G^p(\tau_2 f) = v\lambda_G^p(\tau_2 f).$$

This implies

$$\mathcal{P}(\lambda_G^p(\tau_2 f)) = (\operatorname{Res}_H v) \cdot \mathcal{P}(\lambda_G^p(\tau_2 f));$$

but $\operatorname{Res}_H v = 0$, so $\mathcal{P}(\lambda_G^p(\tau_2 f)) = 0$. We finally get $\mathcal{P}(\lambda_G^p(f)) = \mathcal{P}(\lambda_G^p(\tau_1 f))$, and therefore

$$\|\mathcal{P}(\lambda_G^p(f))\|_p < \varepsilon. \ \blacksquare$$

COROLLARY 2.2. *Let G be a locally compact group and H a locally negligible closed subgroup. Then*

$$i(CV_p(H)) \cap PF_p(G) = \{0\}$$

for every $1 < p < \infty$.

Ghahramani and Lau [10] proved that for G non-discrete the Banach algebra $PF_p(G)$ has no unit for $1 < p < \infty$. For $p = 2$ the result is due to Akemann and Walter [1] (they showed that for G non-discrete, $C^*(G)$ has no unit). Kaniuth [12] showed that $L^1(H) \cap C^*(G) = \{0\}$ for every closed normal subgroup H of G non-open in G.

Corollary 2.2 was obtained in [7] for a class of locally negligible closed subgroups including the neutral and the amenable subgroups. For $p = 2$ and G locally compact and second countable, Theorem 2.1 is due to Shulman, Todorov and Turowska [15].

3. Injection property for sets of uniqueness

THEOREM 3.1. *Let G be a locally compact group, H a closed subgroup of G, F a closed subset of H and* $1 < p < \infty$.

(1) *If F is a set of p-uniqueness in H then it is so in G.*
(2) *The converse holds if and only if H is open in G.*

Proof. Suppose that F is a set of p-uniqueness in H. Let T be a p-pseudofunction of G supported by F. If H is not open in G then $T = 0$ by Theorem 2.1. Suppose that H is open in G. By [7] there is a projection $\mathcal{E}$ of $CV_p(G)$ onto $\{U \in CV_p(G) \mid \operatorname{supp} U \subset H\}$ such that $\mathcal{E}(\lambda_G^p(\mu)) = \lambda_G^p(1_H \mu)$ for every bounded measure μ. Let (f_n) be a sequence in $C_{00}(G)$ such that $\|\lambda_G^p(f_n) - T\| \to 0$. Then $\|\mathcal{E}(\lambda_G^p(f_n)) - \mathcal{E}(T)\| \to 0$. There is $S \in CV_p(H)$ with $i(S) = T$; consequently, $\mathcal{E}(T) = i(S)$. For every $n \in \mathbb{N}$ we have

$$\mathcal{E}(\lambda_G^p(f_n)) = \lambda_G^p(1_H f_n m_G);$$

choosing $c > 0$ such that $\operatorname{Res}_H m_G = c m_H$, we obtain

$$\mathcal{E}(\lambda_G^p(f_n)) = i(\lambda_H^p(\operatorname{Res}_H c f_n)).$$

We get $\|i(\lambda_H^p(\mathrm{Res}_H\, cf_n)) - i(S)\| \to 0$. This implies $\|\lambda_H^p(\mathrm{Res}_H\, cf_n) - S)\| \to 0$, and consequently $S \in PF_p(H)$. Taking in account that $\operatorname{supp} S \subset F$, we get $S = 0$ and finally $T = 0$.

Assume that H is open in G and that F is a set of p-uniqueness in G. Consider a p-pseudofunction T of H supported by F. As above, $i(T)$ is a p-pseudofunction of G, thus $i(T) = 0$ and therefore $T = 0$. Finally suppose H is non-open in G. Then H is a set of p-uniqueness in G but not in H. ∎

For G abelian, H a compact open infinite subgroup and $p = 2$, the above result is due to V. Tardivel [16]. For G an arbitrary locally compact group, H an open subgroup and $1 < p < \infty$, see Delaporte [5, Théorème 8.4].

COROLLARY 3.2. *Let G a locally compact group, H an open subgroup and $1 < p < \infty$. Then i is a Banach algebra isometry of $PF_p(H)$ onto*

$$\{T \in PF_p(G) \mid \operatorname{supp} T \subset H\}.$$

We easily obtain the following complement to Theorem 2.1.

COROLLARY 3.3. *Let G be a locally compact group, H a closed subgroup of G and $1 < p < \infty$. Then H is a set of p-uniqueness in G if and only if $\mathcal{P}(PF_p(G)) = 0$.*

4. The inverse projection theorem for sets of uniqueness. We denote by $\mathcal{M}_{00}^{\infty}(G)$ the set of all $f : G \to \mathbb{C}$ bounded, m_G-measurable and with compact support. In this section, G is a locally compact group, H a closed normal subgroup and ω the canonical map of G onto G/H.

DEFINITION 4.1. For $1 < p < \infty$, $r, s \in \mathcal{M}_{00}^{\infty}(G)$ and $T \in CV_p(G)$ we set

$$\langle f_{r,s}(T)[m], [n]\rangle = \int\limits_{G/H} \langle T[{}_y(m\circ\omega)(\Delta_{G/H}\circ\omega)^{1/p} r], [{}_y(n\circ\omega)(\Delta_{G/H}\circ\omega)^{1/p'} s]\rangle \, d\dot{y}$$

for every $m, n \in C_{00}(G/H)$.

LEMMA 4.2. *For $1 < p < \infty$ and $r, s \in \mathcal{M}_{00}^{\infty}(G)$ we have*

$$f_{r,s}(PF_p(G)) \subset PF_p(G/H).$$

Proof. It is straightforward to verify that for $f \in C_{00}(G)$ we have

$$f_{r,s}(\lambda_G^p(f)) = \lambda_{G/H}^p\Big(T_H\big(f\overline{\tau_{p'} s} * (\tau_p r)^{\vee}\big)\Big)$$

where T_H is the Weil map of $C_{00}(G)$ onto $C_{00}(G/H)$ and where

$$(\tau_p r)(x) = r(x^{-1})\Delta_G(x^{-1})^{1/p}.$$

Let T be a p-pseudofunction on G. There is a sequence $(f_n) \subset C_{00}(G)$ such that $\|\lambda_G^p(f_n) - T\| \to 0$. Taking into account that for every n,

$$\|f_{r,s}(\lambda_G^p(f_n) - T)\| \le N_p(r)N_{p'}(s)\|\lambda_G^p(f_n) - T\|,$$

we get

$$\Big\|\lambda_{G/H}^p\Big(T_H\big(f_n\overline{\tau_{p'}s} * (\tau_p r)^{\vee}\big)\Big) - f_{r,s}(T)\Big\| \to 0.$$

Consequently, $f_{r,s}(T)$ is a p-pseudofunction on G/H. ■

THEOREM 4.3. *Let G be a locally compact group, H a closed normal subgroup and F a closed subset of G/H of p-uniqueness with $1 < p < \infty$. Then $\omega^{-1}(F)$ is a set of p-uniqueness in G.*

Proof. Let T be a p-pseudofunction of G supported by $\omega^{-1}(F)$. Consider arbitrary $r, s \in \mathcal{M}_{00}^{\infty}(G)$. From

$$\operatorname{supp} f_{r,s}(T) \subset \omega\big((\operatorname{supp} r)^{-1} \operatorname{supp} s\big) \cap \overline{\omega(\operatorname{supp} T)}$$

(see [6, Prop. 4]) we obtain $\operatorname{supp} f_{r,s}(T) \subset F$. Lemma 6 implies $f_{r,s}(T) = 0$. This relation being verified for $r, s \in \mathcal{M}_{00}^{\infty}(G)$, we finally get $T = 0$ (see [6, Prop. 5]). ■

We are going to prove the converse of Theorem 4.3.

DEFINITION 4.4. Let β be a Bruhat function associated to the subgroup H. For $1 < p < \infty$, $k, l \in C_{00}(G)$ and $T \in CV_p(G/H)$, we set

$$\langle \Omega_{k,l}(T)[\varphi], [\psi]\rangle = \int_G \big\langle T[T_H({}_{t^{-1}}k'\varphi)], [T_H({}_{t^{-1}}l'\psi)]\big\rangle\, dt$$

for every $\varphi, \psi \in C_{00}(G)$ where $k' = \check{k}\beta^{1/p'}$ and $l' = \check{l}\beta^{1/p}$.

THEOREM 4.5. *Let G be a locally compact group, H a closed normal subgroup, F a closed subset of G/H and $1 < p < \infty$. Suppose that $\omega^{-1}(F)$ is a set of p-uniqueness. Then F is a set of p-uniqueness.*

Proof. Let T be a p-pseudofunction of G/H supported by F. We claim that $T = 0$. According to [6, Cor. 18] it suffices to show that $\Omega_{k,l}(T) = 0$ for every $k, l \in C_{00}(G)$. Let $k, l \in C_{00}(G)$. Delaporte [5, Lem. 5.4] proved that the convolution operator $\Omega_{k,l}(T)$ is a p-pseudofunction of G. By [6, Prop. 16] the support of $\Omega_{k,l}(T)$ is contained in $\omega^{-1}(\operatorname{supp} T)$, and hence in $\omega^{-1}(F)$. This implies $\Omega_{k,l}(T) = 0$. ■

Theorems 4.3 and 4.5 are due to V. Tardivel [16] for $p = 2$ and for locally compact abelian second countable groups. Delaporte obtained them for amenable groups [5, Théorème 8.3].

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