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A NOTE ON THE EXPONENTIAL DIOPHANTINE EQUATION $(am^2+1)^x+(bm^2-1)^y=(cm)^z$

BY

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Abstract. Let a, b, c, m be positive integers such that $a+b=c^2$, $2 \nmid c$, $m>1$ and $m \equiv \pm 1 \pmod{c}$. We prove that if $a \equiv 4$ or $5 \pmod{8}$, $((a+1)/c)=-1$ and $m>6c^2 \log c$, where $((a+1)/c)$ is the Jacobi symbol, then the equation $(am^2+1)^x+(bm^2-1)^y=(cm)^z$ only has the positive integer solution $(x,y,z)=(1,1,2)$.

1. Introduction. Let $\mathbb{Z}, \mathbb{N}$ be the sets of all integers and positive integers respectively. In recent years, many papers investigated the solutions of ternary pure exponential diophantine equations (see [LTZ]–[WWZ]). Let a, b, c, m be positive integers such that

$$a+b=c^2, \quad 2 \nmid c, \quad m>1, \quad m \equiv \pm 1 \pmod{c}. \tag{1.1}$$

In this paper, we deal with the equation

$$(am^2+1)^x+(bm^2-1)^y=(cm)^z, \quad x,y,z \in \mathbb{N}. \tag{1.2}$$

Recently, J. P. Wang, T. T. Wang and W. P. Zhang [WWZ] proved that if $(a,b,c)=(4,5,3)$, then (1.2) has only the solution $(x,y,z)=(1,1,2)$. This improved an early result of N. Terai [T1]. We prove a general result as follows:

THEOREM 1.1. *If* $a \equiv 4$ *or* $5 \pmod{8}$, $((a+1)/c)=-1$ *and* $m>6c^2 \log c$, *where* $(*/*)$ *is the Jacobi symbol, then* (1.2) *has only the solution* $(x,y,z)=(1,1,2)$.

Obviously, since $(5/3)=-1$, the above theorem incorporates the result of [WWZ] for $m>59$.

2. Preliminaries. For any nonnegative integer r, let F_r and L_r denote the rth Fibonacci number and Lucas number respectively.

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Lemma 2.1 (see [BMS]). *The equation*

$$F_r = X^n, \quad r, X, n \in \mathbb{N},\ X > 1,\ n > 1,$$

has only the solutions $(r, X, n) = (6, 2, 3)$ *and* $(12, 12, 2)$.

For any positive integer D, let $h(-4D)$ denote the class number of positive binary quadratic primitive forms of discriminant $-4D$.

Lemma 2.2 (see [H, Theorems 11.4.3, 12.10.1 and 12.14.3]). $h(-4D) < 4\pi^{-1}\sqrt{D}\log(2e\sqrt{D})$.

Let D_1, D_2, k be positive integers such that $\min\{D_1, D_2\} > 1$, $\gcd(D_1, D_2) = 1$, $2 \nmid k$ and $\gcd(D_1D_2, k) = 1$.

Lemma 2.3 (see [Le, Theorems 1 and 2]). *If the equation*

$$D_1X^2 + D_2Y^2 = k^Z, \quad X, Y, Z \in \mathbb{Z},\ \gcd(X, Y) = 1,\ Z > 0, \tag{2.1}$$

has solutions (X, Y, Z), *then every solution* (X, Y, Z) *of* (2.1) *can be expressed as*

$$Z = Z_1 t, \quad t \in \mathbb{N},\ 2 \nmid t. \tag{2.2}$$

Observe that $X\sqrt{D_1} + Y\sqrt{-D_2} = \lambda_1(X_1\sqrt{D_1} + \lambda_2 Y_1\sqrt{-D_2})^t$, $\lambda_1, \lambda_2 \in \{-1, 1\}$, where X_1, Y_1, Z_1 are positive integers satisfying $D_1X_1^2 + D_2Y_1^2 = k^{Z_1}$, $\gcd(X_1, Y_1) = 1$ and $h(-4D_1D_2) \equiv 0 \pmod{2Z_1}$.

Lemma 2.4 (see [BS, Theorem 2]). *If* (X, Y, Z) *is a solution of* (2.1) *with* $Y = \pm 1$, *then* $t = 1$ *and* $Z = Z_1$ *in* (2.2), *except for the following cases:*

(i) $D_1X_1^2 = \frac{1}{4}(k^{Z_1} \pm 1)$, $D_2 = \frac{1}{4}(3k^{Z_1} \mp 1)$, $(X, Z) = (\pm X_1(D_1X_1^2 - 3D_2), 3Z_1)$,

(ii) $D_1X_1^2 = \frac{1}{4}F_{3s+3\epsilon}$, $D_2 = \frac{1}{4}L_{3s}$, $k^{Z_1} = F_{3s+\epsilon}$, $(X, Z) = (\pm X_1(D_1^2X_1^4 - 10D_1D_2X_1^2 + 5D_2^2), 5Z_1)$, *where* $\epsilon \in \{1, -1\}$ *and* s *is a positive integer.*

3. Further lemmas concerning the solutions to (1.2).

Let a, b, c, m be positive integers satisfying (1.1), and let (x, y, z) be a solution of (1.2).

Lemma 3.1. *If* $((a+1)/c) = -1$, *then* $2 \nmid xy$.

Proof. Since $m \equiv \pm 1 \pmod{c}$, we have

$$m^2 \equiv 1 \pmod{c}. \tag{3.1}$$

Further, by (3.1), we get

$$am^2 + 1 \equiv a + 1 \pmod{c}. \tag{3.2}$$

Since $((a+1)/c) = -1$, we have $\gcd(a+1, c) = 1$ and $\gcd(am^2+1, c) = 1$ by (3.2). Therefore, we see from (1.2) that

$$\gcd(am^2+1, cm) = \gcd(bm^2-1, cm) = \gcd(am^2+1, bm^2-1) = 1. \tag{3.3}$$

Since $(cm)^z = (am^2+1)^x + (bm^2-1)^y \geq (am^2+1) + (bm^2-1) = (a+b)m^2 = (cm)^2$, we have $z \geq 2$. Hence, by (1.2), we get $0 \equiv (cm)^z \equiv (am^2+1)^x + (bm^2-1)^y \equiv 1 + (-1)^y \pmod{m^2}$.

Since $m > 1$, we obtain

$$2 \nmid y. \tag{3.4}$$

If $2 \mid x$, then from (1.1), (1.2), (3.1) and (3.4) we get $1 = (-(bm^2-1)^y/c) = (-(bm^2-1)/c) = ((-b+1)/c) = ((a+1)/c) = -1$, a contradiction. So we have $2 \nmid x$. ∎

Lemma 3.2. *If* $((a+1)/c) = -1$ *and* $(x,y,z) \neq (1,1,2)$, *then* $2 \nmid m$.

Proof. As $(x,y,z) \neq (1,1,2)$, we have $z \geq 3$ and $0 \equiv (cm)^z \equiv (am^2+1)^x + (bm^2-1)^y \equiv (axm^2+1) + (bym^2-1) \equiv (ax+by)m^2 \pmod{m^3}$, whence

$$ax + by \equiv 0 \pmod{m}. \tag{3.5}$$

By Lemma 3.1, we have $x \equiv y \equiv 1 \pmod 2$. Since $a + b = c^2$ and $2 \nmid c$, we get $ax + by \equiv a + b \equiv c^2 \equiv 1 \pmod 2$. Therefore, we see from (3.5) that $2 \nmid m$. ∎

Lemma 3.3. *Under the assumption of Lemma* 3.2, *if* $a \equiv 0$ *or* $1 \pmod 4$, *then* $2 \mid z$.

Proof. By Lemma 3.2, we have $2 \nmid m$. This implies that

$$(am^2+1, bm^2-1) = \begin{cases} (1,\ 0) \pmod 2 & \text{if } 2 \mid a, \\ (0,\ 1) \pmod 2 & \text{if } 2 \nmid a. \end{cases} \tag{3.6}$$

We now assume that $2 \nmid z$. Then since $2 \nmid xy$, by (1.2) and (3.6), we have

$$1 = \begin{cases} \left(\dfrac{(bm^2-1)cm}{am^2+1}\right) & \text{if } 2 \mid a, \\ \left(\dfrac{(am^2+1)cm}{bm^2-1}\right) & \text{if } 2 \nmid a. \end{cases} \tag{3.7}$$

Further, since $(am^2+1) + (bm^2-1) = (cm)^2$, we have

$$1 = \begin{cases} \left(\dfrac{(bm^2-1)}{am^2+1}\right) = \left(\dfrac{(cm)^2}{am^2+1}\right) & \text{if } 2 \mid a, \\ \left(\dfrac{(am^2+1)}{bm^2-1}\right) = \left(\dfrac{(cm)^2}{bm^2-1}\right) & \text{if } 2 \nmid a. \end{cases} \tag{3.8}$$

Substituting (3.8) into (3.7), we get

$$1 = \begin{cases} \left(\dfrac{c}{am^2+1}\right)\left(\dfrac{m}{am^2+1}\right) & \text{if } 2 \mid a, \\ \left(\dfrac{c}{bm^2-1}\right)\left(\dfrac{m}{bm^2-1}\right) & \text{if } 2 \nmid a. \end{cases} \tag{3.9}$$

If $a \equiv 0 \pmod 4$, then from (3.1) and (3.9) we obtain $am^2+1 \equiv 1 \pmod 4$ and

$$1 = \left(\frac{am^2+1}{c}\right)\left(\frac{am^2+1}{m}\right) = \left(\frac{a+1}{c}\right)\left(\frac{1}{m}\right) = -1,$$

a contradiction.

Similarly, if $a \equiv 1 \pmod 4$, then $b \equiv 0 \pmod 4$, $bm^2 - 1 \equiv 3 \pmod 4$ and

$$\begin{aligned}1 &= \left(\frac{c}{bm^2-1}\right)\left(\frac{m}{bm^2-1}\right)\\ &= (-1)^{(c-1)/2}\left(\frac{bm^2-1}{c}\right)(-1)^{(m-1)/2}\left(\frac{bm^2-1}{m}\right)\\ &= (-1)^{(c-1)/2}\left(\frac{b-1}{c}\right)(-1)^{(m-1)/2}\left(\frac{-1}{m}\right)\\ &= (-1)^{(c-1)/2}\left(\frac{-(a+1)}{c}\right) = \left(\frac{a+1}{c}\right) = -1,\end{aligned}$$

a contradiction. Thus, $2 \mid z$. ■

Lemma 3.4. *Under the assumption of Lemma* 3.2, *if* $a \equiv 4$ *or* $5 \pmod 8$, *then*

$$1 = \begin{cases} y & \text{if } a \equiv 4 \pmod 8,\\ x & \text{if } a \equiv 5 \pmod 8.\end{cases}$$

Proof. By Lemmas 3.1–3.3, we have $2 \nmid xy$, $2 \mid z$ and $2 \nmid m$. If $a \equiv 4 \pmod 8$, then $b \equiv 5 \pmod 8$, so $am^2 + 1 \equiv 5 \pmod 8$ and $bm^2 - 1 \equiv b-1 \equiv a \equiv 4 \pmod 8$. By (1.2), we get $1 \equiv (cm)^z \equiv (am^2+1)^x+(bm^2-1)^y \equiv 5^x + 4^y \equiv 5 + 4^y \pmod 8$, whence $y = 1$.

Similarly, if $a \equiv 5 \pmod 8$, then $am^2 + 1 \equiv 6 \pmod 8$, $bm^2 - 1 \equiv 3 \pmod 8$ and $1 \equiv (cm)^z \equiv 6^x + 3 \pmod 8$, whence $x = 1$. ■

Lemma 3.5. *Under the assumption of Lemma* 3.4, *if* $m > 6c^2 \log c$, *then* $z > m^4/4c^2$.

Proof. Notice that $b \neq 1$, $2 \nmid xy$ and $(x, y, z) \neq (1, 1, 2)$. If $z = 4$, then $(cm)^4 = (am^2 + 1)^x + (bm^2 - 1)^y > \min\{(am^2 + 1)^3, (bm^2 - 1)^3\} > m^6$, whence $c^4 > m^2 > 36c^4(\log c)^2$, a contradiction. So $z > 4$. Further, since $2 \mid z$ by Lemma 3.3, we get

$$z \geq 6. \tag{3.10}$$

When $a \equiv 4 \pmod 8$, by Lemma 3.4, we have $y = 1$. Hence, by (1.2) and (3.10), we get

$$\begin{aligned}0 &\equiv (cm)^z \equiv (am^2+1)^x + (bm^2-1)^y \\ &\equiv \left(1 + \binom{x}{1} am^2 + \binom{x}{2} a^2m^4\right) + (-1+bm^2) \\ &\equiv (ax+b)m^2 + \binom{x}{2} a^2m^4 \pmod{m^6},\end{aligned}$$

whence

$$(ax+b) + \binom{x}{2} a^2m^2 \equiv 0 \pmod{m^4}. \tag{3.11}$$

We see from (3.11) that $ax+b \equiv 0 \pmod{m^2}$ and

$$ax+b = fm^2, \qquad f \in \mathbb{N}. \tag{3.12}$$

Substituting (3.12) into (3.11), we get

$$f + \binom{x}{2} a^2 \equiv 0 \pmod{m^2}.$$

Since $ax \equiv -b \pmod{m^2}$, we have

$$2\binom{x}{2} a^2 \equiv a^2x^2 - a^2x \equiv b^2 + ab \equiv b(a+b) \equiv bc^2 \pmod{m^2}.$$

and

$$2f + 2\binom{x}{2} a^2 \equiv 2f + bc^2 \equiv 0 \pmod{m^2}. \tag{3.13}$$

Further, since $2f + bc^2 > 0$, we see from (3.13) that $2f + bc^2 \geq m^2$ and

$$2f \geq m^2 - bc^2 > m^2 - c^4 > \frac{m^2}{2}, \tag{3.14}$$

since $m^2 > 36c^4$. By (3.12) and (3.14), we have

$$ax+b > \frac{m^4}{4}. \tag{3.15}$$

On the other hand, by (1.2), we have $(cm)^z > (am^2+1)^x$, so

$$z > x\left(\frac{\log(am^2+1)}{\log(cm)}\right) > x. \tag{3.16}$$

Therefore, by (3.15) and (3.16),

$$z > x > \frac{ax+b}{a+b} = \frac{ax+b}{c^2} > \frac{m^4}{4c^2}. \tag{3.17}$$

Similarly, when $a \equiv 5 \pmod 8$, we have $x = 1$ and

$$\begin{aligned}0 &\equiv (cm)^z \equiv (am^2+1) + (bm^2-1)^y \\ &\equiv (am^2+1) + \left(-\binom{y}{2}b^2m^4 + \binom{y}{1}bm^2 - 1\right) \\ &\equiv (a+by)m^2 - \binom{y}{2}b^2m^4 \pmod{m^6},\end{aligned}$$

whence

$$(a+by) - \binom{y}{2}b^2m^2 \equiv 0 \pmod{m^4}. \tag{3.18}$$

Therefore $a + by \equiv 0 \pmod{m^2}$ and

$$a + by = gm^2, \ g \in \mathbb{N}. \tag{3.19}$$

Substitute (3.19) into (3.18) to get

$$g - \binom{y}{2}b^2 \equiv 0 \pmod{m^2}. \tag{3.20}$$

Further, since $by \equiv -a \pmod{m^2}$, we have

$$2\binom{y}{2}b^2 \equiv ac^2 \pmod{m^2}. \tag{3.21}$$

Hence, by (3.20) and (3.21),

$$2g - ac^2 \equiv 0 \pmod{m^2}. \tag{3.22}$$

Furthermore, since $2 \nmid ac^2$, we have $2g \neq ac^2$. Since $m > 6c^2 \log c$, if $2g - ac^2$ is negative, from (3.22) we get $0 < ac^2 - m^2 < 0$, a contradiction. So

$$2g \geq m^2 + ac^2. \tag{3.23}$$

Therefore, by (3.19) and (3.23),

$$a + by \geq \left(\frac{m^2+ac^2}{2}\right)m^2 > \frac{m^4}{2}. \tag{3.24}$$

On the other hand, by (1.2), we have $(cm)^z > (bm^2-1)^y$, so $z > y$. Thus, by (3.24),

$$z > y > \frac{a+by}{a+b} = \frac{a+by}{c^2} > \frac{m^4}{2c^2}. \tag{3.25}$$

By (3.17) and (3.25), the lemma is proved. ∎

Lemma 3.6. *Under the assumption of Lemma* 3.4, *we have*

$$z < \frac{2}{\pi}\sqrt{(am^2+1)(bm^2-1)}\log(2e\sqrt{(am^2+1)(bm^2-1)}). \tag{3.26}$$

Proof. By Lemma 3.4, if $a \equiv 4 \pmod{8}$, then $y = 1$ and the equation

$$(am^2+1)X^2 + (bm^2-1)Y^2 = (cm)^Z, \quad X, Y, Z \in \mathbb{Z},\ \gcd(X, Y) = 1,\ Z > 0, \tag{3.27}$$

has the solution

$$(X, Y, Z) = ((am^2+1)^{(x-1)/2}, 1, z) \tag{3.28}$$

with $Y = 1$. Therefore, by (3.3), applying Lemma 2.4 to (3.27) and (3.28), we have

$$z = Z_1, \tag{3.29}$$

except for the following cases:

$$bm^2 - 1 = \tfrac{1}{4}(3(cm)^{Z_1} \mp 1), \qquad z = 3Z_1 \tag{3.30}$$

or

$$(cm)^{Z_1} = F_{3s+\epsilon}, \qquad z = 5Z_1. \tag{3.31}$$

When (3.29) holds, by Lemma 2.3, we have $h(-4(am^2+1)(bm^2-1)) \equiv 0 \pmod{2z}$, so

$$z \leq \tfrac{1}{2}h(-4(am^2+1)(bm^2-1)). \tag{3.32}$$

Therefore, applying Lemma 2.2 to (3.32), we obtain (3.26).

When (3.30) holds, as $(cm)^{Z_1} = (am^2+1)X_1^2 + (bm^2-1)Y_1^2 \geq (am^2+1) + (bm^2-1) = (cm)^2$, we have $Z_1 \geq 2$. Hence, the first equality of (3.30) yields $4 \equiv \pm 1 \pmod{m^2}$. But, since $m > 1$, this is impossible.

When (3.31) holds, since $2 \nmid cm$, applying Lemma 2.1 to the first equality of (3.31), we get $Z_1 = 1$. Therefore, the second equality of (3.31) gives $z = 5$ and (3.26) holds. Thus, z satisfies (3.26) if $a \equiv 4 \pmod{8}$.

Similarly, if $a \equiv 5 \pmod{8}$, then $x = 1$ and the equation

$$(bm^2-1)X^2 + (am^2+1)Y^2 = (cm)^Z, \quad X, Y, Z \in \mathbb{Z},\ \gcd(X, Y) = 1,\ Z > 0,$$

has the solution

$$(X, Y, Z) = ((bm^2-1)^{(y-1)/2}, 1, z)$$

with $Y = 1$. Therefore, using the same method as above, we can prove that z satisfies (3.26) if $a \equiv 5 \pmod{8}$. ■

4. Proof of Theorem 1.1

Proof. We now assume that (x, y, z) is a solution of (1.2) with $(x, y, z) \neq (1, 1, 2)$. By Lemmas 3.5 and 3.6,

$$\frac{m^4}{4c^2} < z < \frac{2}{\pi}\sqrt{(am^2+1)(bm^2-1)}\log(2e\sqrt{(am^2+1)(bm^2-1)}). \tag{4.1}$$

Since $(am^2+1)+(bm^2-1)=(cm)^2$, we have $(am^2+1)(bm^2-1)<(cm)^4$. Hence, by (4.1),

$$m^2 < \frac{8}{\pi}c^4\log(2ec^2m^2). \tag{4.2}$$

Let

$$f(t)=t^2-\frac{8}{\pi}c^4\log(2ec^2t^2), \qquad t>1. \tag{4.3}$$

Since $f'(t)=2t-16c^4/(\pi t)$, we conclude that if $t>\sqrt{8/\pi}\,c^2$, then $f'(t)>0$, so $f(t)$ is increasing. Further, since $c\ge 3$, we have

$$\begin{aligned}
&f(6c^2\log c)\\
&\quad = 36c^4(\log c)^2-\frac{8}{\pi}c^4(\log 2+1+2\log c+2\log 6+4\log c+2\log\log c)\\
&\quad = c^4(\log c)^2\left(36-\frac{8}{\pi\log c}\left(\frac{\log 2+1+2\log 6}{\log c}+6+\frac{2\log\log c}{\log c}\right)\right)\\
&\quad > c^4(\log c)^2\left(36-\frac{8}{\pi\log 3}\left(\frac{6}{\log 3}+8\right)\right) > c^4(\log c)^2\left(36-\frac{105}{\pi\log 3}\right)>0.
\end{aligned}$$

This implies that $f(t)>0$ if $t\ge 6c^2\log c$. Therefore, we see from (4.3) that if $m>6c^2\log c$, then (4.2) is false. Thus, the theorem is proved. ∎

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