

ON SOME ARITHMETICAL FUNCTIONS AND
THE NUMBER OF PURE NUMBER FIELDS

BY

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Abstract. First, we define the Möbius and Liouville functions of order k over a number field F for a positive integer k . We give formulas for their partial sums. Moreover, we consider the number of k -free ideals of the integer ring of F .

Next, we investigate the number of pure number fields. In the previous paper, we considered lower and upper bounds of that number. However, the estimates were too coarse. One of the purposes of this paper is to improve the upper bound.

1. Introduction. For a large number $x > 0$, there are classical statements that

$$\sum_{n \leq x} \mu(n) = O(x \exp(-c\sqrt{\log x})) \quad \text{and} \quad \sum_{n \leq x} \lambda(n) = O(x \exp(-c\sqrt{\log x}))$$

where μ (resp. λ) is the Möbius (resp. Liouville) function. Here and throughout, c denotes a positive constant not necessarily the same at different occurrences. It is also well-known that the number of square-free positive integers not exceeding x is $(6/\pi^2)x + O(\sqrt{x})$ (see e.g. Montgomery and Vaughan [18, Theorem 2.2]). Our purpose is to generalize these classical results.

Let F be a number field of degree d , O_F the integer ring of F , and I_F the set of all non-zero ideals of O_F . Throughout this paper, $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \mathfrak{d}, \mathfrak{e}$ are non-zero ideals of O_F and \mathfrak{p} is a prime ideal of O_F . In the case $F \neq \mathbb{Q}$, an *arithmetical function* is a function from I_F to \mathbb{C} . For arithmetical functions f and g , we define their *Dirichlet convolution* $f * g$ as

$$f * g(\mathfrak{a}) = \sum_{\mathfrak{d}|\mathfrak{a}} f(\mathfrak{d})g\left(\frac{\mathfrak{a}}{\mathfrak{d}}\right).$$

This convolution is associative and commutative. The function δ such that $\delta(O_F) = 1$ and $\delta(\mathfrak{a}) = 0$ for $\mathfrak{a} \neq O_F$ is the identity element of this convolution. For an arithmetical function f , the inverse of f exists if and only if

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$f(O_F) \neq 0$. An arithmetical function f is called *multiplicative* if $f(O_F) = 1$ and $f(\mathfrak{a}\mathfrak{b}) = f(\mathfrak{a})f(\mathfrak{b})$ whenever \mathfrak{a} and \mathfrak{b} are coprime. The Dirichlet convolution of two multiplicative functions and the Dirichlet inverse of a multiplicative function are also multiplicative.

Let k be a positive integer. Now, we introduce the Möbius and Liouville functions of order k in F . The *Möbius function* $\mu_{k,F} : I_F \rightarrow \{0, \pm 1\}$ of order k is defined as follows. For a prime ideal \mathfrak{p} and a non-negative integer m , set

$$\mu_{k,F}(\mathfrak{p}^m) = \begin{cases} 1 & \text{if } m < k, \\ -1 & \text{if } m = k, \\ 0 & \text{if } k < m. \end{cases}$$

For $\mathfrak{a} = \mathfrak{p}_1^{e_1} \cdots \mathfrak{p}_g^{e_g} \in I_F$ where $\mathfrak{p}_1, \dots, \mathfrak{p}_g$ are distinct prime ideals and e_1, \dots, e_g are positive integers, we set

$$\mu_{k,F}(\mathfrak{a}) = \prod_{i=1}^g \mu_{k,F}(\mathfrak{p}_i^{e_i}).$$

The Möbius function of order k was introduced by Apostol [1]. In the case $F = \mathbb{Q}$, he obtained the asymptotic formula

$$\sum_{n \leq x} \mu_{k,\mathbb{Q}}(n) = A_k x + O(x^{1/k} \log x)$$

where A_k is a constant which depends on k , namely

$$A_k = \prod_p \left(1 - \frac{2}{p^k} + \frac{1}{p^k} \right).$$

Under the Riemann hypothesis, Suryanarayana [22] showed that the error term can be improved to $O(x^{4k/(4k^2+1)} \exp(A \log x / (\log \log x)))$. Bege [2, 3] introduced generalized Apostol's Möbius functions and showed some of their properties.

For $k \geq 2$, the *Liouville function* $\lambda_{k,F} : I_F \rightarrow \{0, \pm 1\}$ of order k is defined by

$$\lambda_{k,F}(\mathfrak{a}) = \sum_{\mathfrak{d}^k | \mathfrak{a}} \mu_{1,F} \left(\frac{\mathfrak{a}}{\mathfrak{d}^k} \right).$$

For a prime ideal \mathfrak{p} and a non-negative integer m , we see that

$$\lambda_{k,F}(\mathfrak{p}^m) = \begin{cases} 1 & \text{if } m \equiv 0 \pmod k, \\ -1 & \text{if } m \equiv 1 \pmod k, \\ 0 & \text{otherwise.} \end{cases}$$

In addition, $\lambda_{k,F}$ is multiplicative.

For simplicity, we denote $\mu_{k,F}$ (resp. $\lambda_{k,F}$) by μ_k (resp. λ_k) since we fix the number field F . Note that μ_1 (resp. λ_2) is the ordinary Möbius (resp. Liouville) function. For a large real number $x > 0$, we set

$$M_k(x) := \sum_{N(\mathfrak{a}) \leq x} \mu_k(\mathfrak{a}), \quad L_k(x) := \sum_{N(\mathfrak{a}) \leq x} \lambda_k(\mathfrak{a})$$

where $N(\mathfrak{a})$ is the *norm* of \mathfrak{a} , i.e., $N(\mathfrak{a}) = [O_F : \mathfrak{a}]$.

In Section 2 we show the following.

THEOREM 1.1. *For an integer $k \geq 2$, we have*

$$M_k(x) = \frac{c_F}{\zeta_F(k)} x \sum_{\mathfrak{a} \in I_F} \frac{\mu_1(\mathfrak{a}) J_1(\mathfrak{a})}{J_k(\mathfrak{a}) N(\mathfrak{a})} + O(x^{1/k} \log x)$$

where $J_k(\mathfrak{a}) = N(\mathfrak{a})^k \prod_{\mathfrak{p}|\mathfrak{a}} (1 - N(\mathfrak{p})^{-k})$, $\zeta_F(s)$ is the Dedekind zeta function of F , and c_F is the residue of $\zeta_F(s)$ at $s = 1$.

In Section 3 we show the following.

THEOREM 1.2. *Let k be a positive integer. There exist positive constants c and c' such that*

$$L_k(x) \ll \zeta_F(k(1 - c/\sqrt{\log x})) x \exp(-c' \sqrt{\log x}).$$

Since $1 < k(1 - c/\sqrt{\log x})$ for $k \geq 2$ and large $x > 0$, we see that $\zeta_F(k(1 - c/\sqrt{\log x})) = O(1)$. Therefore, we can roughly write $L_k(x) = O(x \exp(-c\sqrt{\log x}))$. The author and Minamide [12] showed that $M_1(x) = O(x \exp(-c\sqrt{\log x}))$ and $L_2(x) = O(x \exp(-c\sqrt{\log x}))$ for an arbitrary number field.

In addition, we consider the number of k -free ideals of O_F . In the case $F = \mathbb{Q}$, for a positive integer n , if there is no prime number l such that $l^k | n$, then n is said to be *k -free*. For an ideal $\mathfrak{a} \in I_F$, we say \mathfrak{a} is *k -free* if there is no prime ideal \mathfrak{p} such that $\mathfrak{p}^k | \mathfrak{a}$. For $k \geq 2$, we define the arithmetical function q_k as

$$q_k(\mathfrak{a}) = \begin{cases} 1 & \text{if } \mathfrak{a} \text{ is } k\text{-free,} \\ 0 & \text{otherwise.} \end{cases}$$

Note that $q_k = |\mu_{k-1}|$ and it is the characteristic function of the set of all k -free ideals of O_F . Let

$$Q_k(x) := \sum_{N(\mathfrak{a}) \leq x} q_k(\mathfrak{a}).$$

In Section 4 we show the following.

THEOREM 1.3. *Let $k \geq 2$ and $d = [F : \mathbb{Q}] > 1$. Then*

$$Q_k(x) = \frac{c_F}{\zeta_F(k)}x + \begin{cases} O(x^{1/2}) & \text{if } (d, k) = (2, 2), \\ O(x^{1/3} \log x) & \text{if } (d, k) = (2, 3), \\ O(x^{1/2} \log x) & \text{if } (d, k) = (3, 2), \\ O(x^{\frac{d-1}{d+1}}) & \text{otherwise.} \end{cases}$$

The number of k -free positive integers was considered e.g. in [7, 24]. In the case $F = \mathbb{Q}$ and $k \geq 2$, Walfisz [24] showed

$$Q_k(x) = \frac{x}{\zeta(k)} + O\left(\sqrt[k]{x} \exp\left(-c \frac{(\log x)^{3/5}}{k^{85}(\log \log x)^{1/5}}\right)\right).$$

It is conjectured that $Q_k(x) = x/\zeta(k) + O(x^{1/(2k)+\varepsilon})$.

We also investigate the number of pure number fields. Let p be a prime number, and F a number field of degree p . If $F = \mathbb{Q}(\sqrt[p]{n})$ for some p -free integer $n > 1$, we shall call F a *pure number field of degree p* . Note that we have $[\mathbb{Q}(\sqrt[p]{n}) : \mathbb{Q}] = p$ for a p -free integer $n > 1$ [16, VII, Theorem 9.1]. Pure number fields have been studied by several authors. For example, Nakano [20, 21] investigated the ideal class groups of pure number fields. The present author [11] considered the distribution of pure number fields of odd prime degree by elementary methods. Let

$$P_p(x) = \#\{\text{pure number fields } F \text{ of degree } p \text{ with } |d(F)| \leq x\}.$$

Here $d(F)$ is the discriminant of the number field F . In [11], the author proved

$$B_p x^{1/(p-1)} \leq P_p(x) \leq A_p x \quad \text{for some } A_p, B_p$$

by using the Cohen–Robinson theorem on k -free numbers in arithmetic progressions [8]. However, this estimate is very rough and the constants A_p and B_p are essentially meaningless. In this note we improve the upper bound of this estimate. In Section 5, we show the following.

THEOREM 1.4. *Let p be an odd prime number, and ε a positive number. Then*

$$P_p(x) \leq C_{p,\varepsilon} x^{1/(p-1)} (\log x)^{p-2}$$

where

$$C_{p,\varepsilon} = \frac{1}{\zeta(2)^{p-1}} \left(\frac{1}{p^{1/(p-1)}(p-1)} \right)^{p-2} + \varepsilon,$$

and $\zeta(s)$ is the Riemann zeta function.

The distributions of arithmetic objects like number fields are quite difficult. Let n be a positive integer. Set

$$N_n(x) = \#\{\text{number fields } F \text{ of degree } n \text{ with } |d(F)| \leq x\}.$$

An old conjecture asserts that $N_n(x) \sim c_n x$ for some constant c_n . This conjecture has been proved for $n = 2, 3, 4$ and 5 . (See [9, 5, 6] and the references given there.)

2. Proof of Theorem 1.1. Our proof follows closely the arguments given in [1]. First, we introduce some lemmas needed later. We denote by $[x]_F$ the number of non-zero ideals of O_F whose norm is less than or equal to x .

LEMMA 2.1 (see Berndt [4], Murty and Van Order [19] and other papers cited there). *For a large real number $x > 0$, we have*

$$[x]_F = c_F x + O(x^{\frac{d-1}{d+1}}).$$

LEMMA 2.2. *For a positive integer k , we have*

$$|\mu_k(\mathfrak{a})| = \sum_{\mathfrak{d}^{k+1}|\mathfrak{a}} \mu_1(\mathfrak{d}) \quad \text{and} \quad \mu_k(\mathfrak{a}) = \sum_{\mathfrak{d}^k|\mathfrak{a}} \mu_{k-1}\left(\frac{\mathfrak{a}}{\mathfrak{d}^k}\right) \mu_{k-1}\left(\frac{\mathfrak{a}}{\mathfrak{d}}\right)$$

for any $\mathfrak{a} \in I_F$.

Proof. For a prime ideal \mathfrak{p} and a positive integer m , we see that

$$q_{k+1}(\mathfrak{a}) = |\mu_k(\mathfrak{p}^m)| = \sum_{\mathfrak{d}^{k+1}|\mathfrak{p}^m} \mu_1(\mathfrak{d}).$$

Since these are multiplicative functions, the first assertion holds. We can check that $\mu_k(\mathfrak{p}^m) = \sum_{\mathfrak{d}^k|\mathfrak{p}^m} \mu_{k-1}(\mathfrak{p}^m/\mathfrak{d}^k) \mu_{k-1}(\mathfrak{p}^m/\mathfrak{d})$ for every prime ideal \mathfrak{p} and integer $m > 0$. Since both sides are multiplicative, we have the second assertion. ■

The next lemma plays an important role in proving Theorem 1.1. Recall that $N(\mathfrak{a})$ denotes the norm of \mathfrak{a} . Let $\sigma_s(\mathfrak{a}) = \sum_{\mathfrak{d}|\mathfrak{a}} N(\mathfrak{d})^s$ for $s \in \mathbb{C}$.

LEMMA 2.3. *Let α be a constant such that $(d-1)/(d+1) \leq \alpha < 1$. For $k \geq 2$ and $\mathfrak{a} \in I_F$, set*

$$G_{\mathfrak{a}}(x) = \sum_{N(\mathfrak{b}) \leq x} \mu_{k-1}(\mathfrak{b}) \mu_{k-1}(\mathfrak{a}^{k-1} \mathfrak{b}).$$

Then

$$G_{\mathfrak{a}}(x) = \frac{c_F \mu_1(\mathfrak{a}) J_1(\mathfrak{a}) N(\mathfrak{a})^{k-1}}{\zeta_F(k) J_k(\mathfrak{a})} x + O(x^{1/k} \sigma_{-\alpha}(\mathfrak{a}))$$

unless $k\alpha = 1$, where $J_k(\mathfrak{a}) = N(\mathfrak{a})^k \prod_{\mathfrak{p}|\mathfrak{a}} (1 - N(\mathfrak{p})^{-k})$.

Proof. If $(\mathfrak{a}, \mathfrak{b}) \neq 1$, then $\mu_{k-1}(\mathfrak{a}^{k-1} \mathfrak{b}) = 0$. If $(\mathfrak{a}, \mathfrak{b}) = 1$, we see that

$$\mu_{k-1}(\mathfrak{b}) \mu_{k-1}(\mathfrak{a}^{k-1} \mathfrak{b}) = |\mu_{k-1}(\mathfrak{b})| \mu_1(\mathfrak{a})$$

because $\mu_k(\mathfrak{a}^k) = \mu_1(\mathfrak{a})$. Hence,

$$G_{\mathfrak{a}}(x) = \mu_1(\mathfrak{a}) \sum_{\substack{N(\mathfrak{b}) \leq x \\ (\mathfrak{a}, \mathfrak{b})=1}} |\mu_{k-1}(\mathfrak{b})|.$$

By Lemma 2.2 and the formula

$$\sum_{\substack{N(\mathfrak{c}) \leq x \\ (\mathfrak{c}, \mathfrak{a})=1}} 1 = \sum_{\mathfrak{c}|\mathfrak{a}} \mu_1(\mathfrak{c}) \left[\frac{x}{N(\mathfrak{c})} \right]_F,$$

we have

$$\begin{aligned} G_{\mathfrak{a}}(x) &= \mu_1(\mathfrak{a}) \sum_{\substack{N(\mathfrak{b}) \leq x \\ (\mathfrak{a}, \mathfrak{b})=1}} \sum_{\mathfrak{d}^k | \mathfrak{b}} \mu_1(\mathfrak{d}) = \mu_1(\mathfrak{a}) \sum_{\substack{N(\mathfrak{d}^k) \leq x \\ (\mathfrak{a}, \mathfrak{d})=1}} \mu_1(\mathfrak{d}) \sum_{\substack{N(\mathfrak{c}) \leq x/N(\mathfrak{d}^k) \\ (\mathfrak{c}, \mathfrak{a})=1}} 1 \\ &= \mu_1(\mathfrak{a}) \sum_{\substack{N(\mathfrak{d}^k) \leq x \\ (\mathfrak{a}, \mathfrak{d})=1}} \mu_1(\mathfrak{d}) \sum_{\mathfrak{c}|\mathfrak{a}} \mu_1(\mathfrak{c}) \left[\frac{x}{N(\mathfrak{c}\mathfrak{d}^k)} \right]_F. \end{aligned}$$

By the estimate $[x]_F = c_F x + O(x^\alpha)$ (Lemma 2.1), the above expression is

$$(2.1) \quad c_F x \mu_1(\mathfrak{a}) \sum_{\mathfrak{c}|\mathfrak{a}} \frac{\mu_1(\mathfrak{c})}{N(\mathfrak{c})} \sum_{\substack{N(\mathfrak{d}^k) \leq x \\ (\mathfrak{a}, \mathfrak{d})=1}} \frac{\mu_1(\mathfrak{d})}{N(\mathfrak{d}^k)} + O\left(\sum_{\mathfrak{c}|\mathfrak{a}} \sum_{\substack{N(\mathfrak{d}^k) \leq x \\ (\mathfrak{a}, \mathfrak{d})=1}} \frac{x^\alpha}{N(\mathfrak{c}\mathfrak{d}^k)^\alpha} \right).$$

Since

$$\begin{aligned} \sum_{\substack{N(\mathfrak{d}^k) \leq x \\ (\mathfrak{a}, \mathfrak{d})=1}} \frac{\mu_1(\mathfrak{d})}{N(\mathfrak{d}^k)} &= \sum_{(\mathfrak{a}, \mathfrak{d})=1} \frac{\mu_1(\mathfrak{d})}{N(\mathfrak{d}^k)} + O\left(\sum_{N(\mathfrak{d}) > \sqrt[k]{x}} \frac{1}{N(\mathfrak{d})^k} \right) \\ &= \frac{1}{\zeta_F(k)} \prod_{\mathfrak{p}|\mathfrak{a}} (1 - N(\mathfrak{p})^{-k})^{-1} + O(x^{(1-k)/k}) \\ &= \frac{N(\mathfrak{a})^k}{\zeta_F(k) J_k(\mathfrak{a})} + O(x^{(1-k)/k}) \end{aligned}$$

and $\sum_{\mathfrak{c}|\mathfrak{a}} \mu_1(\mathfrak{c})/N(\mathfrak{c}) = J_1(\mathfrak{a})/N(\mathfrak{a})$, the first term in (2.1) is

$$c_F x \frac{\mu_1(\mathfrak{a}) J_1(\mathfrak{a}) N(\mathfrak{a})^{k-1}}{\zeta_F(k) J_k(\mathfrak{a})} + O(x^{1/k}).$$

Next, we consider the O -term in (2.1). We see that

$$\begin{aligned} x^\alpha \sum_{\mathfrak{c}|\mathfrak{a}} \frac{1}{N(\mathfrak{c})^\alpha} \sum_{\substack{N(\mathfrak{d}) \leq \sqrt[k]{x} \\ (\mathfrak{a}, \mathfrak{d})=1}} \frac{1}{N(\mathfrak{d})^{k\alpha}} &< x^\alpha \sigma_{-\alpha}(\mathfrak{a}) \sum_{N(\mathfrak{d}) \leq \sqrt[k]{x}} \frac{1}{N(\mathfrak{d})^{k\alpha}} \\ &\ll \begin{cases} x^{1/k} \sigma_{-\alpha}(\mathfrak{a}) & \text{if } k\alpha \neq 1, \\ x^\alpha \sigma_{-\alpha}(\mathfrak{a}) \log x^{1/k} & \text{if } k\alpha = 1. \blacksquare \end{cases} \end{aligned}$$

By Lemma 2.2,

$$\begin{aligned} M_k(x) &= \sum_{N(\mathfrak{a}) \leq x} \sum_{\mathfrak{d}^k | \mathfrak{a}} \mu_{k-1}\left(\frac{\mathfrak{a}}{\mathfrak{d}^k}\right) \mu_{k-1}\left(\frac{\mathfrak{a}}{\mathfrak{d}}\right) \\ &= \sum_{N(\mathfrak{d})^k \leq x} \sum_{N(\mathfrak{b}) \leq x/N(\mathfrak{d})^k} \mu_{k-1}(\mathfrak{b}) \mu_{k-1}(\mathfrak{d}^{k-1}\mathfrak{b}) = \sum_{N(\mathfrak{d})^k \leq x} G_{\mathfrak{d}}\left(\frac{x}{N(\mathfrak{d})^k}\right). \end{aligned}$$

If $k\alpha \neq 1$, Lemma 2.3 implies

$$(2.2) \quad M_k(x) = \frac{c_F x}{\zeta_F(k)} \sum_{N(\mathfrak{a}) \leq \sqrt[k]{x}} \frac{\mu_1(\mathfrak{a}) J_1(\mathfrak{a})}{J_k(\mathfrak{a}) N(\mathfrak{a})} + O\left(x^{1/k} \sum_{N(\mathfrak{a}) \leq \sqrt[k]{x}} \frac{\sigma_{-\alpha}(\mathfrak{a})}{N(\mathfrak{a})}\right).$$

Since

$$\begin{aligned} \sum_{N(\mathfrak{a}) \leq \sqrt[k]{x}} \frac{\mu_1(\mathfrak{a}) J_1(\mathfrak{a})}{J_k(\mathfrak{a}) N(\mathfrak{a})} &= \sum_{\mathfrak{a} \in I_F} \frac{\mu_1(\mathfrak{a}) J_1(\mathfrak{a})}{J_k(\mathfrak{a}) N(\mathfrak{a})} - \sum_{N(\mathfrak{a}) > x^{1/k}} \frac{\mu_1(\mathfrak{a}) J_1(\mathfrak{a})}{J_k(\mathfrak{a}) N(\mathfrak{a})} \\ &= \sum_{\mathfrak{a} \in I_F} \frac{\mu_1(\mathfrak{a}) J_1(\mathfrak{a})}{J_k(\mathfrak{a}) N(\mathfrak{a})} + O\left(\sum_{N(\mathfrak{a}) > x^{1/k}} \frac{1}{N(\mathfrak{a})^k}\right), \end{aligned}$$

the first term in (2.2) is

$$\frac{c_F}{\zeta_F(k)} x \sum_{\mathfrak{a} \in I_F} \frac{\mu_1(\mathfrak{a}) J_1(\mathfrak{a})}{J_k(\mathfrak{a}) N(\mathfrak{a})} + O(x^{1/k}).$$

Since

$$\begin{aligned} \sum_{N(\mathfrak{a}) \leq \sqrt[k]{x}} \frac{\sigma_{-\alpha}(\mathfrak{a})}{N(\mathfrak{a})} &= \sum_{N(\mathfrak{a}) \leq \sqrt[k]{x}} N(\mathfrak{a})^{-1} \sum_{\mathfrak{d} | \mathfrak{a}} N(\mathfrak{d})^{-\alpha} \\ &= \sum_{N(\mathfrak{c}) \leq \sqrt[k]{x}} N(\mathfrak{c})^{-1} \sum_{N(\mathfrak{d}) \leq \sqrt[k]{x}/N(\mathfrak{c})} N(\mathfrak{d})^{-\alpha-1} \\ &\ll \sum_{N(\mathfrak{c}) \leq \sqrt[k]{x}} N(\mathfrak{c})^{-1} \ll \log x^{1/k}, \end{aligned}$$

the O -term in (2.2) is $\ll x^{1/k} \log x$. Therefore, Theorem 1.1 is proved.

3. Proof of Theorem 1.2. We show that the function q_k is the Dirichlet inverse of λ_k .

LEMMA 3.1. *For $k \geq 2$, we have $q_k * \lambda_k = \delta$ and*

$$\frac{\zeta_F(ks)}{\zeta_F(s)} = \sum_{\mathfrak{a} \in I_F} \frac{\lambda_k(\mathfrak{a})}{N(\mathfrak{a})^s}, \quad \text{Re } s > 1.$$

Proof. For a prime ideal \mathfrak{p} and a positive integer m , we see that

$$q_k * \lambda_k(\mathfrak{p}^m) = \begin{cases} 1 & \text{if } m = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Since q_k and λ_k are multiplicative, so is $q_k * \lambda_k$. Note that for arithmetical functions f and g we formally have

$$\left(\sum_{\mathfrak{a} \in I_F} \frac{f(\mathfrak{a})}{N(\mathfrak{a})^s} \right) \left(\sum_{\mathfrak{a} \in I_F} \frac{g(\mathfrak{a})}{N(\mathfrak{a})^s} \right) = \left(\sum_{\mathfrak{a} \in I_F} \frac{f * g(\mathfrak{a})}{N(\mathfrak{a})^s} \right).$$

Thus,

$$\left(\sum_{\mathfrak{a} \in I_F} \frac{\lambda_k(\mathfrak{a})}{N(\mathfrak{a})^s} \right) \left(\sum_{\mathfrak{a} \in I_F} \frac{q_k(\mathfrak{a})}{N(\mathfrak{a})^s} \right) = 1.$$

By the Euler product of the Dedekind zeta function,

$$\frac{\zeta_F(s)}{\zeta_F(ks)} = \prod_{\mathfrak{p}} \left(1 + \frac{1}{N(\mathfrak{p})^s} + \dots + \frac{1}{N(\mathfrak{p})^{(k-1)s}} \right) = \sum_{\mathfrak{a} \in I_F} \frac{q_k(\mathfrak{a})}{N(\mathfrak{a})^s}.$$

Therefore,

$$\sum_{\mathfrak{a} \in I_F} \frac{\lambda_k(\mathfrak{a})}{N(\mathfrak{a})^s} = \frac{\zeta_F(ks)}{\zeta_F(s)}. \blacksquare$$

According to the custom, we write $s = \sigma + it$ for a complex number s . To prove Theorem 1.2 we use the following lemma which is a special case of the classical Perron formula (see e.g. [23, p. 70]).

LEMMA 3.2. *Let*

$$f(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}, \quad B(\sigma) = \sum_{n=1}^{\infty} \frac{|a_n|}{n^\sigma},$$

$\psi(x) > 0$ a non-decreasing function such that $a_n \ll \psi(n)$, and σ_a the abscissa of absolute convergence of $f(s)$. Then, for $b > \sigma_a$,

$$\begin{aligned} \sum_{n \leq x} a_n &= \frac{1}{2\pi i} \int_{b-iT}^{b+iT} f(s) \frac{x^s}{s} ds + O\left(\frac{\psi(2x)x \log x}{T}\right) \\ &\quad + O\left(\frac{x^b B(b)}{T}\right) + O\left(\psi(N) \min\left[\frac{x}{T|x-N|}, 1\right]\right) \end{aligned}$$

where N is the integer nearest to x , and T is a sufficiently large positive number.

In the case $F = \mathbb{Q}$, when applying Lemma 3.2 to $f(s) = \sum_{n=1}^{\infty} \lambda_k(n)/n^s$, bounds for $\lambda_k(n)$ pose no problem since $\lambda_k(n) \in \{0, \pm 1\}$. However, for

$F \neq \mathbb{Q}$, we see that

$$f(s) = \sum_{\mathfrak{a} \in I_F} \frac{\lambda_k(\mathfrak{a})}{N(\mathfrak{a})^s} = \sum_{n=1}^{\infty} \frac{\sum_{N(\mathfrak{a})=n} \lambda_k(\mathfrak{a})}{n^s}.$$

The coefficients $\sum_{N(\mathfrak{a})=n} \lambda_k(\mathfrak{a})$ do not seem small to allow the use of the ordinary Perron formula. Thus, to show Theorem 1.2, we shall invoke a version of Perron’s formula due to Liu and Ye [17].

LEMMA 3.3. *With the same notation as in Lemma 3.2, we have*

$$\sum_{n \leq x} a_n = \frac{1}{2\pi i} \int_{b-iT}^{b+iT} f(s) \frac{x^s}{s} ds + O\left(\sum_{x-x/\sqrt{T} < n \leq x+x/\sqrt{T}} |a_n| \right) + O\left(\frac{x^b B(b)}{\sqrt{T}} \right).$$

Let $f(s) = \sum_{\mathfrak{a} \in I_F} \lambda_k(\mathfrak{a})/N(\mathfrak{a})^s$ and $b = 1 + 1/\log x$ in Lemma 3.3. For large positive numbers x and T , we see that

$$\begin{aligned} L_k(x) &= \frac{1}{2\pi i} \int_{1+1/\log x-iT}^{1+1/\log x+iT} \frac{\zeta_F(ks)}{\zeta_F(s)} \frac{x^s}{s} ds \\ &+ O\left(\sum_{x-x/\sqrt{T} < n \leq x+x/\sqrt{T}} \left| \sum_{N(\mathfrak{a})=n} \lambda_k(\mathfrak{a}) \right| \right) \\ &+ O\left(\frac{x^{1+1/\log x} B(1 + 1/\log x)}{\sqrt{T}} \right) \end{aligned}$$

where

$$B(\sigma) = \sum_{n=1}^{\infty} \frac{|\sum_{N(\mathfrak{a})=n} \lambda_k(\mathfrak{a})|}{n^\sigma} = \frac{\zeta_F(k\sigma)}{\zeta_F(\sigma)}.$$

To show the assertion, we take $T = \exp(\sqrt{\log x})$. This gives

$$\begin{aligned} \sum_{x-x/\sqrt{T} < n \leq x+x/\sqrt{T}} \left| \sum_{N(\mathfrak{a})=n} \lambda_k(\mathfrak{a}) \right| &\leq \left[x + \frac{x}{\sqrt{T}} \right]_F - \left[x - \frac{x}{\sqrt{T}} \right]_F \\ &= \frac{2c_F x}{\sqrt{T}} + O(x^{\frac{d-1}{d+1}}). \end{aligned}$$

Since $\zeta_F(\sigma) \ll 1/(\sigma - 1)$ for $\sigma > 1$ [13, p. 148], we find that

$$\frac{x^{1+1/\log x} B(1 + 1/\log x)}{\sqrt{T}} \ll \frac{x \log x}{\sqrt{T}}.$$

We have

$$L_k(x) = \frac{1}{2\pi i} \int_{1+1/\log x-iT}^{1+1/\log x+iT} \frac{\zeta_F(ks)}{\zeta_F(s)} \frac{x^s}{s} ds + O\left(\frac{x \log x}{\sqrt{T}} \right) + O(x^{\frac{d-1}{d+1}}).$$

It is well-known that the Dedekind zeta function $\zeta_F(s)$ has no zero for $\sigma \geq 1 - c/\log(|t| + 4)$ and $|t| \geq t_0$ for some t_0 . Moreover,

$$\frac{1}{\zeta_F(s)} \ll \log |t|$$

for $\sigma \geq 1 - c/\log(|t| + 4)$ and $|t| \geq t_0$ for some t_0 . (This is proved by a similar argument to [18, Theorem 6.7].) Thus, there are positive constants c and t_0 such that

$$\frac{\zeta_F(ks)}{\zeta_F(s)} \ll \zeta_F(k(1 - c/\log(|t| + 4))) \log |t|$$

for $\sigma \geq 1 - c/\log(|t| + 4)$ and $|t| \geq t_0$. By Cauchy's theorem, we have

$$\begin{aligned} & \int_{1+1/\log x - iT}^{1+1/\log x + iT} \frac{\zeta_F(ks)}{\zeta_F(s)} \frac{x^s}{s} ds \\ &= \left(\int_{1+1/\log x - iT}^{1-c/\log T - iT} + \int_{1-c/\log T - iT}^{1-c/\log T + iT} + \int_{1-c/\log T + iT}^{1+1/\log x + iT} \right) \frac{\zeta_F(ks)}{\zeta_F(s)} \frac{x^s}{s} ds \\ &=: S_1 + S_2 + S_3. \end{aligned}$$

We have

$$\begin{aligned} S_1 &\ll \beta_{k,T} \log T \int_{1+1/\log x}^{1-c/\log T} \left| \frac{x^\sigma}{\sigma - iT} \right| d\sigma \ll \beta_{k,T} \frac{\log T}{T} x, \\ S_3 &\ll \beta_{k,T} \log T \int_{1-c/\log T}^{1+1/\log x} \left| \frac{x^\sigma}{\sigma + iT} \right| d\sigma \ll \beta_{k,T} \frac{\log T}{T} x \end{aligned}$$

where

$$\beta_{k,T} = \zeta_F(k(1 - c/\log T)).$$

For the second integral we compute

$$\begin{aligned} S_2 &\ll \left(\int_{-T}^{-t_0} + \int_{-t_0}^{t_0} + \int_{t_0}^T \right) \left| \frac{\zeta_F(k(1 - c/\log T + it))}{\zeta_F(1 - c/\log T + it)} \frac{x^{1-c/\log T + it}}{1 - c/\log T + it} \right| dt \\ &\ll \beta_{k,T} \log T \int_{-T}^{-t_0} \frac{x^{1-c/\log T}}{t} dt + \beta_{k,T} \log T \int_{t_0}^T \frac{x^{1-c/\log T}}{t} dt \\ &\ll \beta_{k,T} (\log T)^2 x^{1-c/\log T}. \end{aligned}$$

Therefore,

$$\begin{aligned} L_k(x) &= O\left(\beta_{k,T} \frac{x \log T}{T}\right) + O(\beta_{k,T} (\log T)^2 x^{1-c/\log T}) + O\left(\frac{x \log x}{\sqrt{T}}\right) \\ &\quad + O(x^{\frac{d-1}{d+1}}). \end{aligned}$$

Since $T = \exp(\sqrt{\log x})$, we obtain

$$L_k(x) \ll \zeta_F(k(1 - c/\sqrt{\log x}))x \exp(-c'\sqrt{\log x}).$$

Thus, the assertion is proved. In the case $F \neq \mathbb{Q}$, because of the possible existence of a Siegel zero of $\zeta_F(s)$, the O -constant is not effective.

REMARK. Note that if we use a better result on the zero-free region of the zeta function, we get a better estimate. For example, by [14, Lemma 12.3], we have

$$L_k(x) \ll \zeta_F\left(k\left(1 - c/\frac{(\log x)^{3/5}}{(\log \log x)^{1/5}}\right)\right)x \exp\left(-c'\frac{(\log x)^{3/5}}{(\log \log x)^{1/5}}\right)$$

in the case $F = \mathbb{Q}$. Moreover, $L_k(x) = O(x^{1/2+\varepsilon})$ is equivalent to GRH. First, assume that $L_k(x) = O(x^{1/2+\varepsilon})$. Since

$$\sum_{n \leq x} \frac{\lambda_k(n)}{n^\sigma} = \frac{1}{x^\sigma} L_k(x) + \sigma \int_1^x \frac{L_k(u)}{u^{\sigma+1}} du = O(x^{1/2+\varepsilon-\sigma})$$

by using the partial summation formula, it follows that $\zeta_F(ks)/\zeta_F(s)$ is analytic for $\sigma > 1/2$. Thus, $\zeta_F(s)$ has no zero in this range, that is, GRH holds. Next, assume GRH. By Perron's formula,

$$L_k(x) = \frac{1}{2\pi i} \int_{2-iT}^{2+iT} \frac{\zeta_F(ks)}{\zeta_F(s)} \frac{x^s}{s} ds + O\left(\frac{x^2}{T}\right)$$

for large positive numbers x and T . Since GRH is assumed, there exists a positive number t_0 such that

$$\frac{1}{\zeta_F(s)} = O(|t|^\varepsilon)$$

for $|t| > t_0$ and $\sigma \geq 1/2$ from [23, Theorem 14.2, p. 336]. So

$$\frac{\zeta_F(ks)}{\zeta_F(s)} = O(|t|^\varepsilon)$$

for $|t| > t_0$ and $\sigma > 1/2 + \delta$ for any $\delta > 0$. By this estimate, we have

$$\begin{aligned} L_k(x) &= \frac{1}{2\pi i} \left(\int_{2-iT}^{1/2+\delta-iT} + \int_{1/2+\delta-iT}^{1/2+\delta+iT} + \int_{1/2+\delta+iT}^{2+iT} \right) \frac{\zeta_F(ks)}{\zeta_F(s)} \frac{x^s}{s} ds + O\left(\frac{x^2}{T}\right) \\ &= O(x^2 T^{\varepsilon-1}) + O(x^{1/2+\delta} T^\varepsilon) + O\left(\frac{x^2}{T}\right) \end{aligned}$$

for any small $\delta > 0$. Choosing $T = x^2$, we have

$$L_k(x) = O(x^{2\varepsilon}) + O(x^{2\varepsilon+1/2+\delta}).$$

Since ε and δ are arbitrarily small, the assertion holds.

4. Proof of Theorem 1.3. We use the following proposition.

PROPOSITION 4.1. *For an integer $k \geq 2$, we have*

$$Q_k(x) = \sum_{N(\mathfrak{d}) \leq \sqrt[k]{x}} \mu_1(\mathfrak{d}) \left[\frac{x}{N(\mathfrak{d})^k} \right]_F.$$

Proof. By Lemma 2.2,

$$\begin{aligned} Q_k(x) &= \sum_{N(A) \leq x} \sum_{\mathfrak{d}^k | \mathfrak{a}} \mu_1(\mathfrak{d}) = \sum_{N(\mathfrak{d})^k \leq x} \sum_{N(\mathfrak{b}) \leq x/N(\mathfrak{d})^k} \mu_1(\mathfrak{d}) \\ &= \sum_{N(\mathfrak{d})^k \leq x} \mu_1(\mathfrak{d}) \left[\frac{x}{N(\mathfrak{d})^k} \right]_F. \quad \blacksquare \end{aligned}$$

Let $R(x) = [x]_F - c_F x$. By Proposition 4.1, we have

$$\begin{aligned} Q_k(x) &= c_F x \sum_{N(\mathfrak{d}) \leq \sqrt[k]{x}} \frac{\mu_1(\mathfrak{d})}{N(\mathfrak{d})^k} + \sum_{N(\mathfrak{d}) \leq \sqrt[k]{x}} \mu_1(\mathfrak{d}) R\left(\frac{x}{N(\mathfrak{d})^k}\right) \\ &= \frac{c_F x}{\zeta_F(k)} - c_F x \sum_{N(\mathfrak{d}) > \sqrt[k]{x}} \frac{\mu_1(\mathfrak{d})}{N(\mathfrak{d})^k} + \sum_{N(\mathfrak{d}) \leq \sqrt[k]{x}} \mu_1(\mathfrak{a}) R\left(\frac{x}{N(\mathfrak{d})^k}\right). \end{aligned}$$

By partial summation,

$$\sum_{N(\mathfrak{d}) > \sqrt[k]{x}} \frac{\mu_1(\mathfrak{d})}{N(\mathfrak{d})^k} = O(x^{1/k-1}).$$

Thus, the second term is $O(x^{1/k})$.

Since $R(x) = O(x^{\frac{d-1}{d+1}})$, the third term is

$$\begin{aligned} \sum_{N(\mathfrak{d}) \leq \sqrt[k]{x}} \mu_1(\mathfrak{d}) R\left(\frac{x}{N(\mathfrak{d})^k}\right) &\ll x^{\frac{d-1}{d+1}} \sum_{N(\mathfrak{d}) \leq \sqrt[k]{x}} \frac{1}{N(\mathfrak{d})^{k \frac{d-1}{d+1}}} \\ &= \begin{cases} O(x^{\frac{d-1}{d+1}}) & \text{if } k \frac{d-1}{d+1} > 1, \\ O(x^{1/2}) & \text{if } (d, k) = (2, 2), \\ O(x^{1/3} \log x) & \text{if } (d, k) = (2, 3), \\ O(x^{1/2} \log x) & \text{if } (d, k) = (3, 2). \end{cases} \end{aligned}$$

Therefore, we have the assertion.

5. Proof of Theorem 1.4. First, we prove several lemmas. We denote the set of all k -free positive integers by Q_k . If $F \neq \mathbb{Q}$, then Q_k is the set of all k -free integral ideals of F .

LEMMA 5.1. *Let n be a positive integer. Then there exists a $(k - 1)$ -tuple (a_1, \dots, a_{k-1}) of positive integers such that $a_1 \cdots a_{k-1} \in Q_2$ and $n = a_1 a_2^2 \cdots a_{k-1}^{k-1}$ if and only if $n \in Q_k$.*

Proof. This is immediate. ■

For a positive integer n , the greatest square-free divisor of n is called the *core* of n , and is denoted by n_0 . If n has the prime factorization $n = \prod l^{v_l(n)}$, then $n_0 = \prod l$.

LEMMA 5.2 (Fujisaki [10, p. 133]). *Let p be an odd prime number, $n > 1$ a p -free integer, and n_0 the core of n . Then the discriminant of $F = \mathbb{Q}(\sqrt[p]{n})$ is*

$$d(F) = \begin{cases} (-1)^{(p-1)/2} p^{p-2} (n_0)^{p-1} & \text{if } n^{p-1} \equiv 1 \pmod{p^2}, \\ (-1)^{(p-1)/2} p^p (n_0)^{p-1} & \text{if } n^{p-1} \not\equiv 1 \pmod{p^2}. \end{cases}$$

Proof. The proof is based on sophisticated techniques in algebraic number theory. This formula was established by Landsberg [15]. Then Wegner [25] provided another proof. The complete details were also published in Japanese [10]. ■

LEMMA 5.3. *We have*

$$\sum_{\substack{N(\mathfrak{a}) \leq x \\ \mathfrak{a} \in Q_k}} \frac{1}{N(\mathfrak{a})} \sim \frac{c_F}{\zeta_F(k)} \log x.$$

If $\alpha \neq -1$, we have

$$\sum_{\substack{N(\mathfrak{a}) \leq x \\ \mathfrak{a} \in Q_k}} N(\mathfrak{a})^\alpha \sim \frac{c_F}{\zeta_F(k)(\alpha + 1)} x^{\alpha+1}$$

Proof. By partial summation, we see that

$$\sum_{\substack{N(\mathfrak{a}) \leq x \\ \mathfrak{a} \in Q_k}} N(\mathfrak{a})^\alpha = \sum_{N(\mathfrak{a}) \leq x} q_k(\mathfrak{a}) N(\mathfrak{a})^\alpha = Q_k(x) x^\alpha - \int_1^x Q_k(t) \frac{d}{dt} (t^\alpha) dt.$$

The assertion follows from Theorem 1.3. ■

For $k, l \geq 2$, let

$$F_k^l(x) = \#\left\{ (\mathfrak{a}_1, \dots, \mathfrak{a}_l) \in Q_k^l \mid \prod_{i=1}^l N(\mathfrak{a}_i) \leq x \right\}.$$

LEMMA 5.4. *We have*

$$F_k^l(x) \leq \left(\frac{c_F^l}{\zeta(k)^l} + \varepsilon \right) x (\log x)^{l-1} \quad \text{for any } \varepsilon > 0.$$

Proof. Let $l = 2$. Then

$$\begin{aligned}
 F_k^2(x) &= \sum_{\substack{N(\mathbf{a}_2) \leq x \\ \mathbf{a}_2 \in Q_k}} \sum_{\substack{N(\mathbf{a}_1) \leq x/N(\mathbf{a}_2) \\ \mathbf{a}_2 \in Q_k}} 1 = \sum_{\substack{N(\mathbf{a}_2) \leq x \\ \mathbf{a}_2 \in Q_k}} \left(\frac{c_F}{\zeta_F(k)} \frac{x}{N(\mathbf{a}_2)} + E\left(\frac{x}{N(\mathbf{a}_2)}\right) \right) \\
 &= c_F \frac{x}{\zeta_F(k)} \sum_{\substack{N(\mathbf{a}_2) \leq x \\ \mathbf{a}_2 \in Q_k}} \frac{1}{N(\mathbf{a}_2)} + \sum_{\substack{N(\mathbf{a}_2) \leq x \\ \mathbf{a}_2 \in Q_k}} E\left(\frac{x}{N(\mathbf{a}_2)}\right).
 \end{aligned}$$

where $E(x) = Q_k(x) - c_F x / \zeta_F(k)$. Thus

$$F_k^2(x) \leq \left(\frac{c_F^2}{\zeta_F(k)^2} + \varepsilon \right) x \log x.$$

Suppose

$$F_k^l(x) \leq \left(\frac{c_F^l}{\zeta_F(k)^l} + \varepsilon \right) x (\log x)^{l-1}.$$

Then

$$\begin{aligned}
 F_k^{l+1}(x) &= \sum_{\substack{N(\mathbf{a}_{l+1}) \leq x \\ \mathbf{a}_{l+1} \in Q_k}} F_k^l\left(\frac{x}{N(\mathbf{a}_{l+1})}\right) \\
 &\leq \sum_{\substack{N(\mathbf{a}_{l+1}) \leq x \\ \mathbf{a}_{l+1} \in Q_k}} \left(\frac{c_F^l}{\zeta_F(k)^l} + \varepsilon \right) \frac{x}{N(\mathbf{a}_{l+1})} \left(\log \frac{x}{N(\mathbf{a}_{l+1})} \right)^{l-1} \\
 &\leq \frac{c_F^l}{\zeta(k)^l} x (\log x)^{l-1} \sum_{\substack{N(\mathbf{a}_{l+1}) \leq x \\ \mathbf{a}_{l+1} \in Q_k}} \frac{1}{N(\mathbf{a}_{l+1})} \sim \frac{c_F^{l+1}}{\zeta(k)^{l+1}} x (\log x)^l. \blacksquare
 \end{aligned}$$

It is possible that $\mathbb{Q}(\sqrt[l]{n}) = \mathbb{Q}(\sqrt[l]{m})$ for $n \neq m$. From Lemma 5.2 we deduce

$$p^{p-2} n_0^{p-1} \leq |d(\mathbb{Q}\sqrt[l]{n})|$$

for p -free $n > 1$. Thus,

$$\{p\text{-free } n \text{ such that } |d(\mathbb{Q}\sqrt[l]{n})| \leq x\} \subset \{p\text{-free } n \text{ such that } p^{p-2} n_0^{p-1} \leq x\}.$$

We have

$$P_p(x) \leq \#\{p\text{-free } n \text{ such that } |d(\mathbb{Q}\sqrt[l]{n})| \leq x\}$$

by the definition of pure number fields of degree p . Therefore,

$$\begin{aligned}
 P_p(x) &\leq \#\{p\text{-free } n \text{ such that } p^{p-2} n_0^{p-1} \leq x\} \\
 &= \#\{p\text{-free } n \text{ such that } n_0 \leq (x/p^{p-2})^{1/(p-1)}\}.
 \end{aligned}$$

By Lemma 5.1, we see that

$$P_p(x) \leq \#\{(a_1, \dots, a_{p-1}) \in \mathbb{Q}_2^{p-1} \mid a_1 \cdots a_{p-1} \in Q_2 \text{ and} \\ a_1 \cdots a_{p-1} \leq (x/p^{p-2})^{1/(p-1)}\}.$$

By Lemma 5.4, we have

$$\begin{aligned} P_p(x) &\leq F_2^{p-1}((x/p^{p-2})^{1/(p-1)}) \\ &\leq \frac{1}{\zeta(2)^{p-1}}(x/p^{p-2})^{1/(p-1)}(\log((x/p^{p-2})^{1/(p-1)}))^{p-2} \\ &= \frac{1}{\zeta(2)^{p-1}} \left(\frac{1}{p^{1/(p-1)}(p-1)} \right)^{p-2} x^{1/(p-1)}(\log(x/p^{p-2}))^{p-2}. \end{aligned}$$

Thus, Theorem 1.4 is proved.

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